

# Source Coding With a Side Information “Vending Machine”

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**Abstract**—We study source coding in the presence of side information, when the system can take actions that affect the availability, quality, or nature of the side information. We begin by extending the Wyner-Ziv problem of source coding with decoder side information to the case where the decoder is allowed to choose actions affecting the side information. We then consider the setting where actions are taken by the encoder, based on its observation of the source. Actions may have costs that are commensurate with the quality of the side information they yield, and an overall per-symbol cost constraint may be imposed. We characterize the achievable tradeoffs between rate, distortion, and cost in some of these problem settings. Among our findings is the fact that even in the absence of a cost constraint, greedily choosing the action associated with the “best” side information is, in general, suboptimal. A few examples are worked out.

**Index Terms**—Actions, data acquisition, rate distortion, side information, source coding, vending machine, Wyner-Ziv coding.

## I. INTRODUCTION

THE role and potential benefit of Side Information (S.I.) in lossless and lossy data compression is a central theme in information theory. In ways that are well understood for various source coding systems, S.I. can be a valuable resource, resulting in significant performance boosts relative to the case where it is absent. In the problems studied thus far, the lack or availability of the S.I., and its quality, are a given. But what if the system can take actions that affect the availability, quality, or nature of the S.I.?

Consider a source coding system where the S.I. is a sequence of noisy measurements of the source sequence to be compressed, each S.I. symbol acquired via a sensor. The quality of each S.I. symbol may be commensurate with resources, such as power or time expended by the sensor for obtaining it, which are limited. Alternatively, or in addition, a sensor may have freedom to choose, for each source symbol, how many independent noisy measurements to observe, with a constraint

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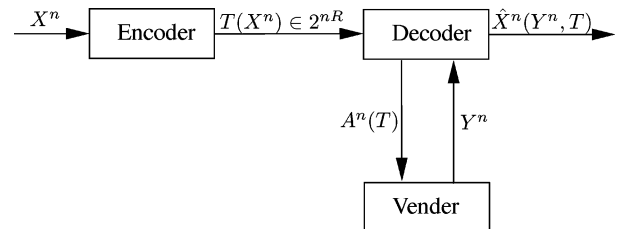


Fig. 1. Rate distortion with a side information vender at the decoder. The source  $X^n$  is i.i.d.  $\sim P_X$ , and  $Y^n$  is the output of the side information channel  $P_{Y|X,A}$  in response to the pair of sequences  $X^n$  and  $A^n$ , where  $A^n$  is the action sequence chosen by the decoder.

on the overall number of measurements. It is natural to wonder how these resources, which may or may not be limited, should best be used, and what would the corresponding optimum performance be.

We abstract this problem by assuming a memoryless source  $P_X$ , a conditional distribution of the side information given the source and an action  $P_{Y|X,A}$ , a function assigning costs to the possible actions, and a distortion measure. The first scenario we focus on is that depicted in Fig. 1, where the actions are taken at the decoder: Based on its observation of the source sequence  $X^n$ , which is i.i.d.  $\sim P_X$ , the encoder gives an index to the decoder. Having received the index, the decoder chooses the action sequence  $A^n$ . Nature then generates the side information sequence  $Y^n$  as the output of the memoryless channel  $P_{Y|X,A}$  whose input is the pair  $(X^n, A^n)$ . The reconstruction sequence  $\hat{X}^n$  is then based on the index and on the side information sequence.

The setting of Fig. 1 can be considered the source coding dual of coding for channels with action-dependent states, where the transmitter chooses an action sequence that affects the formation of the channel states, and then creates the channel input sequence based on the state sequence, as considered in [11]. We characterize the achievable tradeoff between rate, distortion, and cost in Section II. We demonstrate, by a few examples, that greedily choosing the action associated with the “best” side information may be suboptimal even in the absence of a cost constraint. Further, in the presence of a cost constraint, time-sharing between schemes that are optimal for different cost values is, in general, suboptimal. We also characterize the fundamental limits for the case where the reconstruction is confined to causal dependence on the side information sequence, and the case where the encoder observes a noisy observation of the source rather than the source itself.

The second scenario we consider is that depicted in Fig. 2, where actions are taken at the encoder: Based on its observation of the source sequence  $X^n$ , the encoder chooses a sequence of actions  $A^n$ . Nature then generates the side information sequence

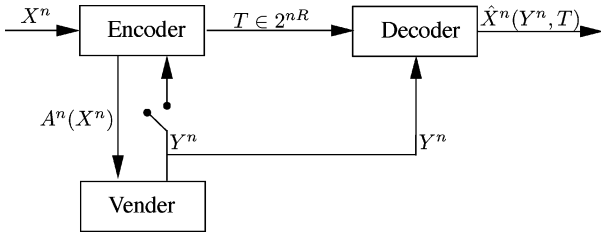


Fig. 2. Rate distortion with side information vended at the encoder, where the side information is known at the decoder and may or may not be known to the encoder. The source  $X^n$  is i.i.d.  $\sim P_X$  and side information is generated as the output of the memoryless channel  $P_{Y|X,A}$  in response to the input  $(X^n, A^n)$ , where the action sequence  $A^n$  is chosen by the encoder.

$Y^n$  as the output of the memoryless channel  $P_{Y|X,A}$  whose input is the pair  $(X^n, A^n)$ . The encoder now chooses the index to be given to the decoder on the basis of the source and possibly the side information sequence (according to whether or not the switch is closed). The reconstruction sequence  $\hat{X}^n$  is then based on the index and on the side information sequence. Though we leave the general case open, in Section III we characterize the achievable tradeoff between rate, distortion, and cost for three important special cases: the (near) lossless case, the Gaussian case (where  $Y = A + X + N$ , with  $X$  and  $N$  being independent Gaussian random variables), and the case of the Markov relation  $Y - A - X$  (i.e., when  $P_{Y|X,A}$  is of the form  $P_{Y|A}$ ). We end that section with Subsection III-D, giving lower and upper bounds on the achievable rates for the general case. We summarize the paper and related open directions in Section IV.

The family of problems we consider in this work includes scenarios arising naturally in the coding or compression of sources for which the S.I. arises from noisy measurements of the source components. The acquisition, handling, processing and storage of these measurements may require system resources that come at a cost. This premise, that the acquisition of source measurements may be costly and is to be done sparingly, is in fact central in the emerging Compressed Sensing paradigm [1], [2], [5], arising naturally in the study of an increasing array of sensing problems. In many such problems, the system has the freedom to choose how many sensors to deploy in each region of the phenomenon it is trying to gauge, subject to an overall budget of sensors. Assuming each sensor provides an independent measurement of the source region in which it was deployed, this setting corresponds to our model, with  $A_i \in \{0, 1, 2, \dots\}$  representing the number of sensors,  $P_{Y|X,A} = \prod_{j=1}^A P_{Z_j|X}$  representing  $A$  independent measurements from the ‘sensor channel’  $P_{Z|X}$ , and  $\Lambda(A_i) = A_i$  assuming all sensors are equally costly. The cost constraint  $C$  then corresponds to the budget of sensors to deploy, in number of sensors per source region. We are not aware of previous work on source coding for systems allowed to take S.I.-affecting actions from a Shannon theoretic perspective. We refer to [7] and some references therein for other recent Shannon theoretic studies of new problems involving source coding in the presence of S.I.

## II. SIDE INFORMATION VENDING MACHINE AT THE DECODER

Throughout the paper we let upper case, lower case, and calligraphic letters denote, respectively, random variables, specific or deterministic values they may assume, and their alphabets. For two jointly distributed random objects  $X$  and  $Y$ , let  $P_{X,Y}$ ,

and  $P_{X|Y}$  respectively denote the distribution of  $X$ , the joint distribution of  $X, Y$ , and the conditional distribution of  $X$  given  $Y$ . In particular, when  $X$  and  $Y$  are discrete,  $P_{X|Y}$  represents the stochastic matrix whose elements are  $P_{X|Y}(x|y) = P(X = x | Y = y)$ . The term  $X_m^n$  denotes the  $n - m + 1$ -tuple  $(X_m, \dots, X_n)$  when  $m \leq n$  and the empty set otherwise. The term  $X^n$  is shorthand for  $X_1^n$ , and  $X^{n \setminus i}$  stands for the  $n - 1$ -tuple consisting of all the components of  $X^n$  but  $X_i$ .

### A. The Setup

A source with action dependent decoder side information is characterized by the source distribution  $P_X$  and by the conditional distribution of the side information given the source and an action  $P_{Y|X,A}$ . The difference between this and previously studied scenarios is that here, after receiving the index from the encoder, the decoder may choose actions that will affect the nature of the side information it will get to observe. Specifically, a scheme in this setting for blocklength  $n$  and rate  $R$  is characterized by an encoding function  $T : \mathcal{X}^n \rightarrow \{1, 2, \dots, 2^{nR}\}$ , an action strategy  $f : \{1, 2, \dots, 2^{nR}\} \rightarrow \mathcal{A}^n$ , and a decoding function  $g : \{1, 2, \dots, 2^{nR}\} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n$  that operate as follows.

- The source  $n$ -tuple  $X^n$  is i.i.d.  $\sim P_X$
- Encoding: based on  $X^n$  give index  $T = T(X^n)$  to the decoder
- Decoding:
  - given the index, choose an action sequence  $A^n = f(T)$ ;
  - the side information  $Y^n$  will be the output of the memoryless channel  $P_{Y|X,A}$  whose input is  $(X^n, A^n)$ ;
  - let  $\hat{X}^n = g(T, Y^n)$ .

A triple  $(R, D, C)$  is said to be *achievable* if for all  $\varepsilon > 0$  and sufficiently large  $n$  there exists a scheme as above for blocklength  $n$  and rate  $R + \varepsilon$  satisfying both

$$E \left[ \sum_{i=1}^n \rho(X_i, \hat{X}_i) \right] \leq n(D + \varepsilon) \quad (1)$$

$$E \left[ \sum_{i=1}^n \Lambda(A_i) \right] \leq n(C + \varepsilon) \quad (2)$$

where  $\rho$  and  $\Lambda$  are, respectively, given distortion and cost functions. The rate distortion (and cost) function  $R(D, C)$  is defined as

$$R(D, C) = \inf \{ R' : \text{the triple } (R', D, C) \text{ is achievable} \}. \quad (3)$$

### B. The Rate-Distortion-Cost Tradeoff

Define

$$R^{(J)}(D, C) = \min [I(X; A) + I(X; U | Y, A)] \quad (4)$$

where the joint distribution of  $X, A, Y, U$  in (4) is of the form

$$P_{X,A,U,Y}(x, a, u, y) = P_X(x) P_{A,U|X}(a, u | x) P_{Y|X,A}(y | x, a) \quad (5)$$

and the minimization is over all  $P_{A,U|X}$  under which

$$E [\rho(X, \hat{X}^{\text{opt}}(U, Y))] \leq D, E[\Lambda(A)] \leq C \quad (6)$$

where  $\hat{X}^{\text{opt}}(U, Y)$  denotes the best estimate of  $X$  based on  $U, Y$ ,  $U$  is an auxiliary random variable. We show below that the cardinality of  $U$  may be restricted to  $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{A}| + 1$ . Our main result pertaining to  $R^{(I)}(D, C)$  is the following.

*Theorem 1:* The rate-distortion-cost function, as defined in (3), is given by  $R^{(I)}(D, C)$  in (4), i.e.,

$$R(D, C) = R^{(I)}(D, C). \quad (7)$$

*Remark:* Write  $R_{WZ}(P_X, P_{Y|X}, D)$  for the explicit dependence of the Wyner-Ziv rate distortion function [15] on the distribution of the source and the conditional distribution of the source given the side information. It is clear that we get (8), as shown at the bottom of the page, since the right-hand side can be achieved by letting the decoder take actions according to a pre-specified sequence with the symbol  $a$  fraction  $P_A(a)$  of the time, and performing Wyner-Ziv coding at distortion level  $D_a$  separately on each subsequence associated with each action symbol. It is natural to wonder whether the inequality in (8) can be strict. We will see through some examples below that, in general, it may very well be strict. Indeed, even in the absence of a cost constraint, we give examples showing that greedily selecting the action associated with the side information which is best in the Wyner-Ziv sense, that is the action  $a$  minimizing  $R_{WZ}(P_X, P_{Y|X, A=a}, D)$ , may be suboptimal.

The following lemma will be useful in proving Theorem 1.

*Lemma 1: Properties of the expressions defining  $R^{(I)}(D, C)$ :*

- 1) For any fixed  $P_X$  and  $P_{Y|A, X}$ , the set of distributions of the form given in (5) is a convex set in  $P_{A, U|X}$ .
- 2) For any fixed  $P_X$  and  $P_{Y|A, X}$ , the expression  $I(X; A) + I(X; U|Y, A)$  is convex in  $P_{A, U|X}$  [assuming the joint distribution given in (5)].
- 3) To exhaust  $R^{(I)}(D, C)$ , it is enough to restrict the alphabet of  $U$  to satisfy

$$|\mathcal{U}| \leq |\mathcal{X}||\mathcal{A}| + 1. \quad (9)$$

- 4) It suffices to restrict the minimization in (4) to joint distributions where  $A$  is a deterministic function of  $U$ , i.e., of the form

$$P_X(x)P_{U|X}(u|x)1_{\{a=f(u)\}}P_{Y|X, A}(y|x, a). \quad (10)$$

*Proof:*

- 1) Since the set of conditional distributions  $P_{A, U|X}$  is a convex set, and since  $P_X$  and  $P_{Y|X, A}$  are fixed, the set of distributions  $P_{X, A, U, Y}$  of the form given in (5) is a convex set. ■

- 2) Using the definition of mutual information, we have the identity shown in (11), as found at the bottom of the page. We show now that the right-hand part of (11) is convex in  $P_{U, A|X}$  for a fixed  $P_X$  and  $P_{Y|A, X}$ . The expression  $I(X; U, Y, A)$  is convex in  $P_{U, Y, A|X}$ , hence it is also convex in  $P_{U, A|X}$ . For fixed  $P_X$  and  $P_{Y|A, X}$ , the expression  $H(Y|A, X)$  is linear in  $P_{A|X}$ . Finally, we show that  $-H(Y|A)$  is convex using the log sum inequality that states that for non negative number,  $a_1, a_2$  and  $b_1, b_2$

$$a_1 \log \frac{a_1}{b_1} + a_2 \log \frac{a_2}{b_2} \geq (a_1 + a_2) \log \frac{a_1 + a_2}{b_1 + b_2}. \quad (12)$$

Now let  $P_{A|X}^2 = \alpha P_{A|X}^1 + \bar{\alpha} P_{A|X}^0$ , where  $0 \leq \alpha \leq 1$  and  $\bar{\alpha} = 1 - \alpha$ . Let us denote  $P_{Y, A, X}^i$  and  $H^i(A|X)$ , the joint distribution and the conditional entropy induced by  $P_{A|X}^i$  and the fixed pmfs  $P_X$  and  $P_{Y|A, X}$  for  $i = 0, 1, 2$ . Consider (13), shown at the bottom of the page, where (a) follows from the definition of  $P_{Y, A, X}^i$  and (b) follows from the log sum inequality. Since (13) holds for any  $a \in \mathcal{A}$  and any  $y \in \mathcal{Y}$ , we obtain that  $-H(Y|A)$  is convex, i.e.,

$$-H^2(Y|A) \leq -\alpha H^1(Y|A) - \bar{\alpha} H^0(Y|A) \quad (14)$$

It is also possible to prove the concavity of  $H(Y|A)$  in  $P_{A|X}$  for fixed  $P_X$  and  $P_{Y|A, X}$  through the fact that  $H(Y|A) \geq H(Y|A, \Theta)$  where  $\Theta \sim \text{Bernoulli}(\alpha)$  and is independent of  $X$  and defining  $P_{A|X}^i = P_{A|X, \Theta=i}$  for  $i = 0, 1$ .

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$$R(D, C) \leq \min \left\{ \sum_a P_A(a) R_{WZ}(P_X, P_{Y|X, A=a}, D_a) : \sum_a P_A(a) D_a \leq D, \sum_a P_A(a) \Lambda(a) \leq C \right\} \quad (8)$$


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$$I(X; A) + I(X; U|Y, A) = I(X; U, Y, A) + H(Y|A, X) - H(Y|A). \quad (11)$$


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$$\begin{aligned} P_{Y, A}^2(y, a) \log \frac{P_{Y, A}^2(y, a)}{P_A^2(a)} &\stackrel{(a)}{=} (\alpha P_{Y, A}^1(y, a) + \bar{\alpha} P_{Y, A}^0(y, a)) \log \frac{\alpha P_{Y, A}^1(y, a) + \bar{\alpha} P_{Y, A}^0(y, a)}{\alpha P_A^1(a) + \bar{\alpha} P_A^0(a)} \\ &\stackrel{(b)}{\leq} \alpha P_{Y, A}^1(y, a) \log \frac{P_{Y, A}^1(y, a)}{P_A^1(a)} + \bar{\alpha} P_{Y, A}^0(y, a) \log \frac{P_{Y, A}^0(y, a)}{P_A^0(a)} \end{aligned} \quad (13)$$

- 3) We invoke the support lemma [4, p. 310]. The support lemma says that given  $k$  conditions on a random variable  $U$ ,  $E[f_j(U)] = A_j$ ,  $\forall j = 1, \dots, k$ , where  $f_j$  are real-continuous functions there exists a random variable  $U'$  with finite alphabet  $\{u_1, \dots, u_k\} \in \mathcal{U}$ , with probabilities  $\{\alpha_1, \dots, \alpha_k\}$ ,  $\sum_{i=1}^k \alpha_i = 1$ , s.t.  $\sum_{i=1}^k \alpha_i f_j(u_i) = E[f_j(U)] = A_j$  for all  $j = 1, \dots, k$ .
- First we have  $|\mathcal{X}||\mathcal{A}| - 1$  conditions to preserve  $P_{X,A}$ , i.e.,  $\int_{\mathcal{U}} P(x, a | u) dF(u) = P(x, a)$ , for all  $x \in \mathcal{X}$  and  $a \in \mathcal{A}$  but one since  $\sum_{x \in \mathcal{X}, a \in \mathcal{A}} P(x, a) = 1$ . The structure of the pmf in (5) does not change and since  $P_{X,A}$  is preserved than  $P_{Y,A,X}$  is also preserved which implies  $I(A; X)$ ,  $H(X | Y, A)$  and  $E[\Lambda(A)]$  are also preserved. Two more conditions need to be preserved  $H(X | Y, A, U) = \int_{\mathcal{U}} H(X | Y, A, u) dF(u)$  and the distortion  $E[\rho(X, \hat{X}(U, Y))]$ . Therefore there are  $|\mathcal{X}||\mathcal{A}| - 1 + 2$  conditions to preserve, which implies by the support lemma [4, p. 310] that the cardinality of  $U$  may be restricted to  $|\mathcal{X}||\mathcal{A}| + 1$ .
- 4) Note that it suffices to restrict the minimization in (4) to joint distributions where  $A$  is a deterministic function of  $U$ , i.e., of the form

$$P_X(x)P_{U|X}(u|x)1_{\{a=f(u)\}}P_{Y|X,A}(y|x, a) \quad (15)$$

in lieu of (5). To see the equivalence note that a distribution of the form in (5) assumes the form in (10) by taking  $(U, A)$  as the auxiliary variable. ■

*Proof of Theorem 1: Achievability:* We briefly and informally outline the achievability part, which is based on standard arguments: A code-book of size  $2^{n(I(X;A)+\varepsilon)}$  is generated with codewords that are i.i.d.  $\sim P_A$ . For each such codeword, generate  $2^{n(I(X;U|A)+\varepsilon)}$  codewords according to  $P_{U|A}$ . Distribute these codewords uniformly at random into  $2^{n(I(X;U|Y,A)+2\varepsilon)}$  bins. Given the source realization,  $n(I(X;A)+\varepsilon)$  bits are used by the encoder to communicate the identity of a codeword from the first codebook jointly typical with it (with high probability there is at least one such codeword). The decoder now performs the actions according to the action sequence conveyed to it. The encoder now uses an additional  $n(I(X;U|Y,A)+2\varepsilon)$  number of bits to describe the bin index of the codeword from the second code-book which is jointly typical with the source and the first codeword. With high probability there is at least one such codeword (since more than  $2^{nI(X;U|A)}$  such were generated), and it is the only codeword in its bin which is jointly typical with the first codeword (which the decoder already knows) and the side information sequence that it has generated and is observed at the decoder, since the size of each bin is no larger than  $\approx 2^{n(I(X;U|A)-I(X;U|Y,A)-\varepsilon)} = 2^{n(I(Y;U|A)-\varepsilon)}$ . For the reconstruction, the decoder now employs the mapping  $\hat{X}^{\text{opt}}$  in a symbol-by-symbol fashion on the components of the pair consisting of the second codeword and the side information sequence.

**Converse:** For the converse part, fix a scheme of rate  $\leq R$  for a block of length  $n$  and consider (16), as shown at the bottom of the page. Now we get (17)–(18), also shown on the page. Com-

$$\begin{aligned} nR &\geq H(T) \\ &= H(T, A^n) \\ &= H(A^n) + H(T | A^n) \\ &\geq H(A^n) - H(A^n | X^n) + H(T | A^n, Y^n) - H(T | Y^n, A^n, X^n) \\ &= I(X^n; A^n) + I(X^n; T | A^n, Y^n) \\ &= I(X^n; A^n) + H(X^n | A^n, Y^n) - H(X^n | A^n, Y^n, T). \end{aligned} \quad (16)$$

$$\begin{aligned} I(X^n; A^n) + H(X^n | A^n, Y^n) &\geq H(X^n) - H(X^n | A^n) + H(X^n, Y^n | A^n) - H(Y^n | A^n) \\ &= H(X^n) - H(X^n | A^n) + H(X^n | A^n) + H(Y^n | A^n, X^n) - H(Y^n | A^n) \\ &= H(X^n) + H(Y^n | A^n, X^n) - H(Y^n | A^n) \\ &= \sum_{i=1}^n H(X_i) + H(Y_i | A_i, X_i) - H(Y_i | Y^{i-1}, A^n) \\ &\geq \sum_{i=1}^n H(X_i) + H(Y_i | A_i, X_i) - H(Y_i | A_i) \\ &= \sum_{i=1}^n H(X_i) - I(Y_i; X_i | A_i) \\ &= \sum_{i=1}^n I(X_i; A_i) + H(X_i | Y_i, A_i). \end{aligned} \quad (17)$$

$$\quad (18)$$

binning (16) and (18) yields (19), as shown at the bottom of the page, where (a) follows by taking  $U_i = (A^{n \setminus i}, Y^{n \setminus i}, X^{i-1}, T)$ . Noting that  $\hat{X}_i = \hat{X}_i(T, Y^n)$  is a function of the pair  $(U_i, Y_i)$ , and the Markov relation  $U_i - (A_i, X_i) - Y_i$ , the proof is now completed in the standard way upon considering the joint distribution of  $(X', A', U', Y', \hat{X}')$   $\triangleq (X_J, A_J, U_J, Y_J, \hat{X}_J)$ , where  $J$  is randomly generated uniformly at random from the set  $\{1, \dots, n\}$ , independent of  $(X^n, A^n, U^n, Y^n, \hat{X}^n)$ , and noting (20)–(23), shown at the bottom of the page, where the last inequality follows from item 2 in Lemma 1, which states that  $I(X; A) + I(X; U | Y, A)$  is convex over the set of distributions that satisfies (20). ■

It is natural to wonder whether the characterization above remains valid when the choice of the actions is allowed to depend on the side information symbols generated thus far, that is, for the  $i$ th action to be of the form  $A_i = A_i(T, Y^{i-1})$ . The converse in the proof above does not carry over to this case since the inequality  $H(Y^n | X^n, A^n) \geq \sum_{i=1}^n H(Y_i | A_i, X_i)$ , used in (17), may no longer hold. Whether the best achievable rate could, in general, be better (less) when allowing such schemes remains open.

### C. Actions Taken by the Decoder Before the Index is Seen

Consider the setting as in Fig. 1, where the actions  $A^n$  are taken by the decoder before the index  $T$  is seen. In such a case  $A^n$  is independent of  $X^n$ . For this case, the rate-distortion-cost function is similar to  $R_{A \perp X}^{(T)}(D, C)$  defined in the previous sec-

tion, but with an additional constraint that  $A$  is independent of  $X$ . Define

$$R_{A \perp X}^{(T)}(D, C) = \min I(X; U | Y, A) \quad (24)$$

where the joint distribution of  $X, A, Y, U$  is of the form shown in (25), found at the bottom of the page, and the minimization is over all  $P_A$  and  $P_{U|X,A}$  under which

$$E[\rho(X, \hat{X}^{\text{opt}}(U, Y))] \leq D, \quad E[\Lambda(A)] \leq C \quad (26)$$

where  $\hat{X}^{\text{opt}}(U, Y)$  denotes the best estimate of  $X$  based on  $U, Y$ , where  $U$  is an auxiliary random variable with a cardinality  $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{A}| + 1$ .

*Theorem 2:* The rate-distortion-cost function for the setting where actions taken by the decoder before the index is seen, is given by  $R_{A \perp X}^{(T)}(D, C)$ .

*Proof:* The proof is similar to the proof of Theorem 1, but taking into account that  $A^n$  is independent of  $X^n$ , and therefore  $A_i$  is independent of  $X_i$ . ■

If the cost is unlimited, then the greedy policy is optimal, namely the decoder blindly chooses the action  $a$  minimizing

$$R_{WZ}(P_X, P_{Y|X,A=a}, D) \quad (27)$$

and an optimal Wyner-Ziv code for the source  $P_X$  and channel  $P_{Y|X,A=a}$  is employed. For the more general case, in the presence of a cost constraint, as can be expected and is straightforward to check,  $R_{A \perp X}^{(T)}(D, C)$  in (24) coincides with the minimum on the right-hand side of (8).

$$\begin{aligned} nR &\geq \sum_{i=1}^n I(X_i; A_i) + H(X_i | Y_i, A_i) - H(X_i | X^{i-1}, A^n, Y^n, T) \\ &\stackrel{(a)}{=} \sum_{i=1}^n I(X_i; A_i) + H(X_i | Y_i, A_i) - H(X_i | Y_i, A_i, U_i) \\ &= \sum_{i=1}^n I(X_i; A_i) + I(X_i; U_i | Y_i, A_i) \end{aligned} \quad (19)$$

$$P_{X'} = P_X, \quad U' - (A', X') - Y', \quad P_{Y'|X',A'} = P_{Y|X,A} \quad (20)$$

$$\hat{X}' = \hat{X}'(U', Y') \quad (21)$$

$$E \left[ \sum_{i=1}^n \rho(X_i, \hat{X}_i) \right] = nE\rho(X', \hat{X}'), \quad E \left[ \sum_{i=1}^n \Lambda(A_i) \right] = nE\Lambda(A') \quad (22)$$

and

$$\frac{1}{n} \sum_{i=1}^n I(X_i; A_i) + I(X_i; U_i | Y_i, A_i) \geq I(X'; A') + I(X'; U' | Y', A') \quad (23)$$

$$P_{X,A,U,Y}(x, a, u, y) = P_X(x)P_A(a)P_{U|X,A}(u|x, a)P_{Y|X,A}(y|x, a) \quad (25)$$

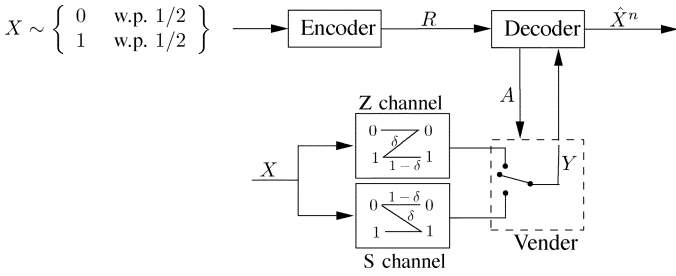


Fig. 3. Example of vending side information, where the action chooses between Z-channel and S-channel with parameter  $\delta$ .

D. Examples

1) *The Lossless Case:* As a very special case of Theorem 1 we get that, in the absence of a cost constraint on the actions, the minimum rate needed for a near lossless reconstruction at the decoder is given by

$$\min I(X; A) + H(X | Y, A) \tag{28}$$

where the joint distribution of  $X, A, Y$  in (28) is of the form

$$P_{X,A,Y}(x, a, y) = P_X(x)P_{A|X}(a|x)P_{Y|X,A}(y|x, a) \tag{29}$$

and the minimization is over all  $P_{A|X}$ . Letting  $R_{SW}(P_X, P_{Y|X})$  denote the conditional entropy  $H(X | Y)$  induced by the pair  $(P_X, P_{Y|X})$  (the subscript *SW* standing for ‘Slepian-Wolf’ [9]), it is natural to wonder whether the above minimum rate can be strictly better (smaller) than  $\min_a R_{SW}(P_X, P_{Y|X, A=a})$ , which is what would be achieved if the decoder greedily takes the one action leading to S.I. which is best in the sense of inducing lowest  $H(X | Y)$ , irrespective of any information from the encoder, and then proceeding as in Slepian-Wolf coding. The following is an example showing that this greedy strategy may be suboptimal.

Consider the case  $\mathcal{X} = \mathcal{A} = \mathcal{Y} = \{0, 1\}$  where  $X$  is a fair coin flip,  $P_{Y|X, A=0}$  is the Z-channel with crossover probability  $\delta$  from 1 to 0, and  $P_{Y|X, A=1}$  is the S-channel with crossover probability  $\delta$  from 0 to 1. The setting is depicted in Fig. 3. Symmetry implies that the  $P_{A|X}$  minimizing  $I(X; A) + H(X | Y, A)$  satisfies  $P_{A|X}(0|1) = P_{A|X}(1|0)$ , in other words, there is a BSC connecting  $X$  to  $A$  (or  $A$  to  $X$ ). Assuming this BSC has crossover probability  $\alpha$ , an elementary calculation yields (30), as shown at the bottom of the page. Thus, letting  $R_{\min}(\delta)$  denote the minimum in (28) for this scenario

$$R_{\min}(\delta) = \min_{\alpha \in [0,1]} \left[ 1 - h(\alpha) + h\left(\frac{\alpha\delta}{1-\alpha+\alpha\delta}\right) (1-\alpha+\alpha\delta) \right]. \tag{31}$$

In contrast, the minimum rate achieved by a ‘greedy’ strategy which chooses actions without regard to the information from

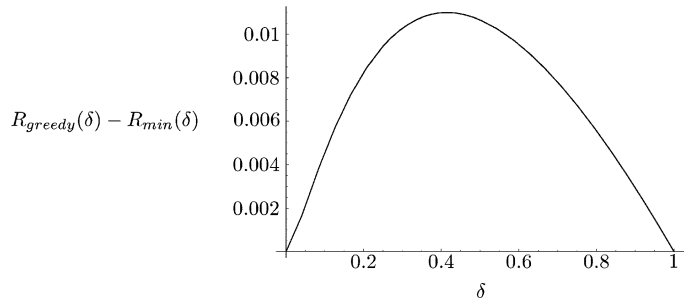


Fig. 4. Plot of  $R_{\text{greedy}}(\delta) - R_{\min}(\delta)$ .

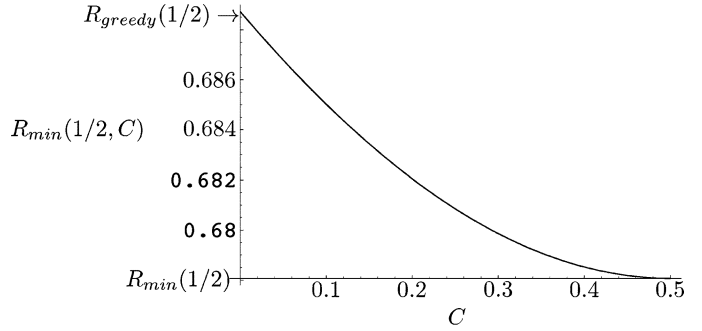


Fig. 5. Plot of  $R_{\min}(1/2, C)$  as a function of the cost  $C$ .

the encoder is given by the conditional entropy of the input given the output of the Z-channel( $\delta$ ) whose input is a fair coin flip, namely

$$R_{\text{greedy}}(\delta) = h\left(\frac{\delta}{1+\delta}\right) \frac{1+\delta}{2}. \tag{32}$$

For example, elementary calculus shows that  $R_{\min}(1/2)$  is achieved by  $\alpha^* = 2/5$ , assuming the value  $\approx 0.678072$ , which is about a 1.5% improvement over  $R_{\text{greedy}}(1/2) \approx 0.688722$ . Fig. 4 plots the difference between  $R_{\text{greedy}}(\delta)$  and  $R_{\min}(\delta)$ .

In the presence of a cost constraint, Theorem 1 implies that the minimum rate needed for a near lossless reconstruction is given by the minimum in (28), with the additional constraint  $E\Lambda(A) \leq C$ . Let  $R_{\min}(\delta, C)$  denote this minimum for our present example, assuming cost 0 for using say the first Z-channel and 1 for using the second channel. Clearly  $R_{\min}(\delta, 0) = R_{\text{greedy}}(\delta)$ ,  $R_{\min}(\delta, 1/2) = R_{\min}(\delta)$  and consequently, by a time-sharing argument

$$R_{\min}(\delta, C) \leq 2CR_{\min}(\delta) + (1-2C)R_{\text{greedy}}(\delta) \quad 0 \leq C \leq 1/2. \tag{33}$$

As it turns out, the inequality in (33) is strict, i.e., in our example one can do better than time-sharing between the respective optimum schemes for the different costs (to the level allowed by the cost constraint). Fig. 5 contains a plot of  $R_{\min}(1/2, C)$ , which is seen to be strictly convex and hence better (lower) than the straight line represented by the right side of (33).

$$I(X; A) + H(X | Y, A) = 1 - h(\alpha) + h\left(\frac{\alpha\delta}{1-\alpha+\alpha\delta}\right) (1-\alpha+\alpha\delta). \tag{30}$$

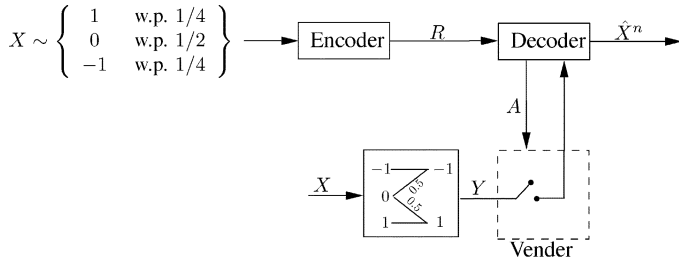


Fig. 6. Ternary example.

2) *The Lossy Case: Ternary Source and Binary Side Information of Unit Cost:* Consider a ternary  $X$  taking values in  $\{-1, 0, 1\}$ , distributed according to

$$X = \begin{cases} 1 & \text{w.p. } 1/4 \\ 0 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/4. \end{cases} \quad (34)$$

The actions are binary, taking values in  $\{0, 1\}$ , where action 0 corresponds to no S.I. while action 1 corresponds to obtaining a binary noisy measurement of  $X$ , taking values in  $\{-1, 1\}$ , which is the output of the following channel:  $P_{Y|X}(1|1) = P_{Y|X}(-1|1) = 1$  and  $P_{Y|X}(1|0) = P_{Y|X}(-1|0) = 1/2$ . Suppose that there is a unit cost for obtaining such a noisy measurement of the source, i.e.:  $\Lambda(a) = a$ ,  $a \in \{0, 1\}$ . This source coding scenario is depicted in Fig. 6.

The conditional entropy of  $X$  given  $Y$  is 1 bit. Thus, lossless compression of  $X$  is achievable at a rate of 1 bit per source symbol at a cost of 1 per source symbol with a greedy decoder who chooses to observe the noisy source measurement of all symbols. Can one do better than this greedy policy? This rate is achievable at half the cost via the following scheme: the encoder uses one bit per source symbol to describe whether or not the symbol is 0. The decoder then needs to use the noisy measurement of the source only for those symbols that are not 0 (in which case the measurement will completely determine the source symbol). This corresponds to rate  $I(X; A) + H(X|Y, A)$  under  $P_{A|X}(1|1) = P_{A|X}(1|-1) = P_{A|X}(0|0) = 1$ , which is readily verified to be the minimum of achievable rates under a cost constraint of  $1/2$ .

In the lossy case, under Hamming distortion, we note the following.

- When the S.I. is available to both encoder and decoder (at no cost) the problem is reduced to one of lossy compression for the binary symmetric source, thus  $R_{X|Y}(D) = 1 - h(D)$ .
- This rate is achievable even when the S.I. is absent at the encoder, as can be seen by letting  $W$  be the output of a

BSC( $D$ ) whose input  $Q(X)$  is the quantized version of  $X$ , defined by  $Q(0) = 0$  and  $Q(1) = Q(-1) = 1$ , where  $W = X - Y$ . It is readily seen that the optimal estimate of  $X$  based on  $(W, Y)$  satisfies  $P(X \neq \hat{X}(W, Y)) = D$  and that  $I(X; W|Y) = 1 - h(D)$ . Thus,  $R_{X|Y}^{WZ}(D) = R_{X|Y}(D) = 1 - h(D)$ .

- $R_{X|Y}^{WZ}(D)$  in the above item corresponds to a decoder that observes all of the S.I. symbols. Can the same performance be achieved with fewer observations? In other words, assuming unit cost per observation, can the same performance be achieved at a cost less than 1? We now argue that the same performance can be achieved at half the cost: letting, as before,  $A = 0$  correspond to no observation and  $A = 1$  correspond to an observation, consider a conditional distribution  $P_{A|X}$  given by  $P_{A|X}(1|1) = P_{A|X}(1|-1) = 1 - P_{A|X}(1|0) = D$  and where  $U = A$ . Then  $I(X; A) + I(X; U|Y, A) = I(X; A) = H(A) - H(A|X) = 1 - h(D)$  and the optimal estimate of  $X$  based on  $(U, Y)$  has  $P(X \neq \hat{X}(U, Y)) = D$ . The cost here is  $P(A = 1) = 1/2$ . Evidently, the rate-distortion-cost function  $R(D, C)$  in (4) satisfies  $R(D, 1/2) \leq 1 - h(D)$  and in fact  $R(D, 1/2) = 1 - h(D)$  since obviously  $R(D, 1/2) \geq R_{X|Y}^{WZ}(D)$ . Thus, the rate  $1 - h(D)$  is achievable even if the decoder is allowed to access only half of the observations.

3) *Binary Action: To Observe or Not to Observe the S.I.:* Consider a given source and side information distribution  $P_{X,Y}$ . The action is to either observe the side information symbol or not, where an observation has unit cost. Thus  $0 \leq C \leq 1$  is a constraint on the fraction of side information symbols the decoder will be allowed to observe. Let us arbitrarily take  $\mathcal{A} = \{0, 1\}$ , with  $A = 1$  corresponding to observation of the side-information symbol and  $A = 0$  to lack of it. Noting that the second mutual information term in (4) corresponds to Wyner-Ziv coding conditional on  $A$ , the specialization of Theorem 1 for this case gives (35), as shown at the bottom of the page, where  $R(P_X, D)$  denotes the rate distortion function of the source  $P_X$  and  $R_{WZ}(P_{X,Y}, D)$  denotes the Wyner-Ziv rate distortion function when source and side information are distributed according to  $P_{X,Y}$ .

A very special case is when  $Y = X$ . Thus the action is either to observe the source symbol or not. Assuming a non-negative distortion measure satisfying  $\min_{\hat{x}} \rho(x, \hat{x}) = 0$  for all  $x$ , (35) becomes When  $X$  is a fair coin flip and distortion is Hamming, (36) becomes (for  $D, C$  in the nontrivial region)

$$\begin{aligned} R(D, C) &= \min_{P_{A|X}: A-X-Y, P(A=1)=C, (1-C)D_0 + CD_1=D} I(X; A) + R(P_{X|A=0}, D_0) \cdot P(A=0) + R_{WZ}(P_{X,Y|A=1}, D_1) \cdot P(A=1) \\ &= \min_{P_{A|X}: A-X-Y, P(A=1)=C, (1-C)D_0 + CD_1=D} I(X; A) + R(P_{X|A=0}, D_0) \cdot (1-C) + R_{WZ}(P_{X,Y|A=1}, D_1) \cdot C \end{aligned} \quad (35)$$

$$R(D, C) = \min_{P_{A|X}: P(A=1)=C} I(X; A) + R\left(P_{X|A=0}, \frac{D}{1-C}\right) \cdot (1-C) \quad (36)$$

as shown in (37), found at the bottom of the page, where  $R_b(p, D) = [h_b(p) - h_b(D)]^+$  is the rate distortion function of the Bernoulli( $p$ ) source and step (a) is due to the fact that  $-h_b(P_{X=1|A=1})$  is minimized (at the value  $-1$ ) by taking  $A$  independent of  $X$ .

To see that  $R(D, C)$  can be strictly smaller than  $R_{A \perp X}(D, C)$  in the observe/not-observe binary action scenario, consider the case where  $X$  is a fair coin flip and  $Y$  is the output of an erasure channel with erasure probability  $e$  (whose input is  $X$ ). Recalling that  $R_{WZ}(P_{X,Y}, D) = eR(P_X, D/e)$  when  $Y$  is the erased version of  $X$  (cf. [8], [10]), we specialize the right-hand side of (35) for this case to obtain (38), shown at the bottom of the page, where the last minimum is over  $\max\{0, 1 - 2C\} \leq \beta \leq \min\{1, 2 - 2C\}$  and  $0 \leq D_1 \leq \min\{D/C, e\}$ . For the extreme points we get, as expected:  $R(D, 0) = R_b(\frac{1}{2}, D)$  and  $R(D, 1) = eR_b(\frac{1}{2}, D/e)$ . Fig. 7 plots the curve in (38) for  $D = 1/4$ ,  $e = 1/2$  and  $0 \leq C \leq 1$ .

### E. Causal Decoder Side Information

Consider the setting presented in Fig. 8, which is similar to that described in Section II-A, the only difference being that the reconstruction is allowed causal dependence on the side information, i.e., to be of the form  $\hat{X}_i = \hat{X}_i(T, Y^i)$  (motivation for why this might be interesting can be found in [12]).

Define

$$R_{\text{causal}}^{(I)}(D, C) = \min I(X; U, A) \quad (39)$$

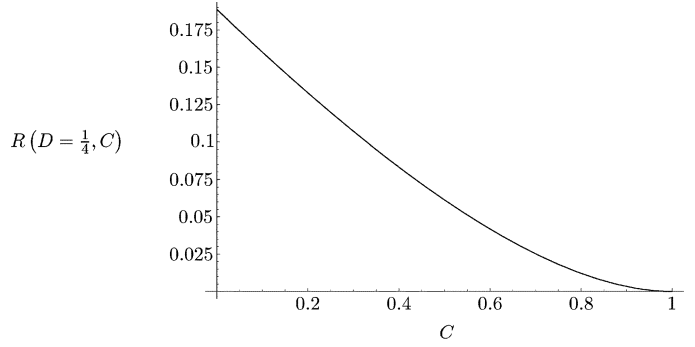


Fig. 7. Rate-distortion-cost function  $R(D, C)$ ,  $0 \leq C \leq 1$ , for the case where  $X$  is a fair coin flip,  $Y$  its erased version where  $e = 1/2$ ,  $C$  is fraction of places where decoder is allowed to observe S.I., and  $D = 1/4$ . In this case  $R(D, 0) = R_b(\frac{1}{2}, \frac{1}{4}) \approx 0.188722$ ,  $R(D, 1) = eR_b(\frac{1}{2}, D/e) = \frac{1}{2}R_b(\frac{1}{2}, \frac{1}{2}) = 0$ . The strict concavity implies suboptimality of time-sharing optimal schemes according to the available observation budget.

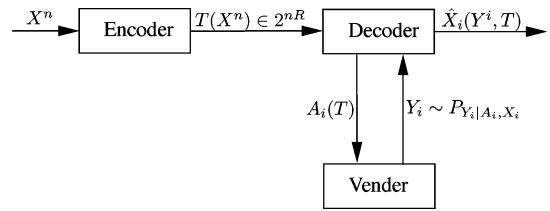


Fig. 8. Rate distortion with *causal* side information vender at the decoder.

where the joint distributions of  $X, A, Y, U$  is of the form

$$P_{X,A,U,Y}(x, a, u, y) = P_X(x)P_{A,U|X}(a, u|x)P_{Y|X,A}(y|x, a) \quad (40)$$

$$\begin{aligned} & R(D, C) \\ &= \min_{P_{A|X}: P(A=1)=C} I(X; A) + R_b\left(P_{X=1|A=0}, \frac{D}{1-C}\right) \cdot (1-C) \\ &= \min_{P_{A|X}: P(A=1)=C} 1 - [h_b(P_{X=1|A=1})C + h_b(P_{X=1|A=0})(1-C)] + h_b(P_{X=1|A=0}) - h_b\left(\frac{D}{1-C}\right) \cdot (1-C), \\ &= \min_{P_{A|X}: P(A=1)=C} 1 - h_b(P_{X=1|A=1})C - h_b\left(\frac{D}{1-C}\right) \cdot (1-C), \\ &\stackrel{(a)}{=} 1 - C - h_b\left(\frac{D}{1-C}\right) \cdot (1-C), \\ &= R_b\left(\frac{1}{2}, \frac{D}{1-C}\right) \cdot (1-C) \\ &= R_{A \perp X}(D, C) \end{aligned} \quad (37)$$

$$\begin{aligned} & R(D, C) \\ &= \min_{P_{A|X}: A \sim X - Y, P(A=1)=C, (1-C)D_0 + CD_1 = D} 1 - H(X|A) + R_b(P_{X|A=0}(1), D_0) \cdot (1-C) + eR_b(P_{X|A=1}(1), D_1/e) \cdot C \\ &= \min 1 - \left[ h_b\left(\frac{\beta}{2(1-C)}\right) (1-C) + h_b\left(\frac{1-\beta}{2C}\right) C \right] + R_b\left(\frac{\beta}{2(1-C)}, \frac{D-CD_1}{1-C}\right) (1-C) + eR_b\left(\frac{1-\beta}{2C}, D_1/e\right) C \end{aligned} \quad (38)$$



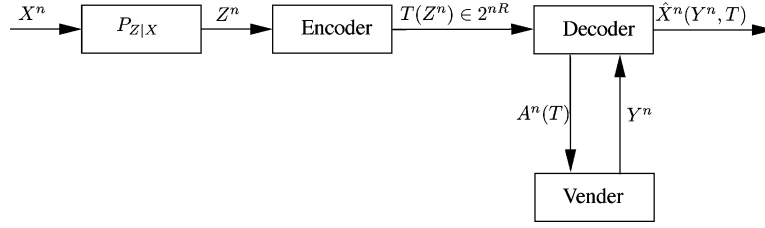


Fig. 9. Indirect rate distortion with a side information vender at the decoder. The source  $X^n$  is i.i.d.  $P_X$  and the encoder observes a noisy version of the source,  $Z^n$ , where the pairs  $(X_i, Z_i)$  are i.i.d.  $\sim P_{X,Z}$ . Side information is generated as the output of the channel  $P_{Y|X,Z,A}$  in response to the noise-free, noisy, and action sequences  $(X^n, Z^n, A^n)$ , where the action sequence  $A^n$  is generated on the basis of the index from the encoder.

and the minimization is over all  $P_{A,U|X}$  under which

$$E[\rho(X, \hat{X}^{\text{opt}}(U, Y))] \leq D, \quad E[\Lambda(A)] \leq C \quad (41)$$

where  $\hat{X}^{\text{opt}}(U, Y)$  denotes the best estimate of  $X$  based on  $U, Y$ , where  $U$  is an auxiliary random variable. The cardinality of  $U$  may be restricted to  $|U| \leq |\mathcal{X}||\mathcal{A}| + 1$  as shown in item 3, Lemma 1. One can also denote  $U, A$  as  $\tilde{U}$ , and an equivalent representation would be

$$R_{\text{causal}}^{(T)}(D, C) = \min I(X; \tilde{U}) \quad (42)$$

where  $P_{X,A,\tilde{U},Y}(x, a, \tilde{u}, y) = P_X(x)P_{\tilde{U}|X}(\tilde{u}|x)1_{\{a=f(\tilde{u})\}}P_{Y|X,A}(y|x, a)$ .

**Theorem 3:** The rate-distortion-cost function for the setting of a side information vender at the decoder with causal side information is given by  $R_{\text{causal}}^{(T)}(D, C)$ .

**Proof: Achievability:** The achievability proof is based on the fact that the encoder and decoder generate a joint type  $P_{X,A}$  using a rate that is  $I(X; A) + \epsilon$ , and since both the encoder and decoder know the sequence of actions  $a^n$ , they can time-share between  $|\mathcal{A}|$  causal schemes such that if the action is  $a$  a rate  $I(X; U|a) + \epsilon$  would achieve the distortion constraint [12]. Hence, the total rate is  $I(X; A) + \epsilon + \sum P_A(a)(I(X; U|a) + \epsilon) = I(X; A, U) + 2\epsilon$ .

**Converse:** for the converse part, fix a scheme of rate  $R$  for a block of length  $n$  and consider

$$\begin{aligned} nR &\geq H(T) \\ &\geq I(X^n; T) \\ &= \sum_{i=1}^n H(X_i) - H(X_i | X^{i-1}, T) \end{aligned}$$

$$\begin{aligned} &\stackrel{(a)}{=} \sum_{i=1}^n H(X_i) - H(X_i | X^{i-1}, T, Y^{i-1}) \\ &\geq \sum_{i=1}^n H(X_i) - H(X_i | T, Y^{i-1}) \end{aligned} \quad (43)$$

where step (a) is due to the Markov chain  $X_i - (X^{i-1}, T) - Y^{i-1}$ . Now let us denote  $\tilde{U}_i := (T, Y^{i-1})$ , and we obtain that

$$R \geq \frac{1}{n} I(X_i; \tilde{U}_i). \quad (44)$$

The proof is now completed in the standard way upon considering the joint distribution of  $(X', A', \tilde{U}', Y', \hat{X}')$   $\triangleq (X_J, A_J, (\tilde{U}_J, J), Y_J, \hat{X}_J)$ , where  $J$  is randomly generated uniformly at random from the set  $\{1, \dots, n\}$ , independent of  $(X^n, A^n, U^n, Y^n, \hat{X}^n)$ , and noting (45)–(48), as shown at the bottom of the page. ■

#### F. Indirect Rate Distortion With Action-Dependent Side Information

Consider the case shown in Fig. 9 where, rather than the source  $X$ , the encoder observes a noisy version of it,  $Z$ . The decoder, based on the index conveyed to it from the encoder, will then select an action sequence that will result in the side information  $Y$ , as output from the channel  $P_{Y|X,Z,A}$ . The reconstruction, as before, will be a function of the index and the side information. Specifically, a scheme in this setting for blocklength  $n$  and rate  $R$  is characterized by an encoding function  $T : \mathcal{Z}^n \rightarrow \{1, 2, \dots, 2^{nR}\}$ , an action strategy  $f : \{1, 2, \dots, 2^{nR}\} \rightarrow \mathcal{A}^n$ , and a decoding function  $g : \{1, 2, \dots, 2^{nR}\} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n$  that operate as follows.

- The source  $n$ -tuple  $X^n$  is i.i.d.  $\sim P_X$  goes through a DMC  $P_{Z|X}$  to yield its noisy observation sequence  $Z^n$ . Thus,

$$P_{X'} = P_X, \tilde{U}' = (A', X') - Y', \quad P_{Y'|X',A'} = P_{Y|X,A} \quad (45)$$

$$\hat{X}' = \hat{X}'(\tilde{U}', Y'), \quad A' = f(\tilde{U}') \quad (46)$$

$$E \left[ \sum_{i=1}^n \rho(X_i, \hat{X}_i) \right] = nE\rho(X', \hat{X}'), \quad E \left[ \sum_{i=1}^n \Lambda(A_i) \right] = nE\Lambda(A') \quad (47)$$

and

$$\frac{1}{n} \sum_{i=1}^n I(X_i; \tilde{U}_i) = I(X'; \tilde{U}'). \quad (48)$$

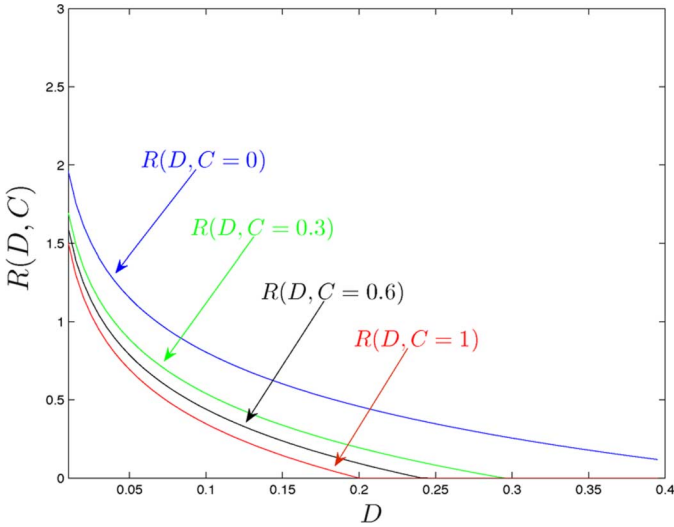


Fig. 10.  $R_e(D, C)$  in the Gaussian case, for  $\sigma_X^2 = \sigma_N^2 = 1$ . The boundary of the region where  $R_e(D, C) = 0$  is the curve  $\hat{D} = \frac{1}{(1+\sqrt{C})^2+1}$ . Indeed, this distortion level can be achieved with zero rate by estimating  $X$  on the basis of  $Y = X + N/(1 + \sqrt{C})$ .

overall the clean and noisy source are characterized by a given joint distribution  $P_{X,Z}$ .

- Encoding: based on  $Z^n$  give index  $T = T(Z^n)$  to the decoder.
- Decoding:
  - given the index, choose an action sequence  $A^n = f(T)$ ;
  - the side information  $Y^n$  will be the output of the memoryless channel  $P_{Y|X,Z,A}$  whose input is  $(X^n, Z^n, A^n)$ ;
  - let  $\hat{X}^n = g(T, Y^n)$ .

The rate-distortion-cost for this case is now defined similarly as in Subsection II-A. Let us denote it by  $R_{ID}(D, C)$ , the subscript standing for "indirect". Theorem 1 is generalized to this case as follows.

*Theorem 4:*  $R_{ID}(D, C)$  is given by

$$R_{ID}(D, C) = \min[I(Z; A) + I(Z; U | Y, A)] \quad (49)$$

where the joint distribution of  $X, Z, A, Y, U$  is of the form shown in (50), at the bottom of the page, and the minimization is over all  $P_{A,U|Z}$  under which

$$E[\rho(X, \hat{X}^{\text{opt}}(U, Y))] \leq D, \quad E[\Lambda(A)] \leq C \quad (51)$$

where  $\hat{X}^{\text{opt}}(U, Y)$  denotes the best estimate of  $X$  based on  $U, Y$ , and  $U$  is an auxiliary random variable whose cardinality is bounded as  $|\mathcal{U}| \leq |\mathcal{Z}||\mathcal{A}| + 1$ .

*Proof Outline:* The achievability part is very similar to the original. The random generation of the scheme is performed in

the same way, with the noisy source replacing the original noise-free source. This guarantees that  $(Z^n, A^n, U^n, Y^n)$  are, with high probability, jointly typical. The joint typicality also with  $X^n$ , namely the joint typicality of  $(X^n, Z^n, A^n, U^n, Y^n)$ , then follows from an application of the Markov lemma. The converse part also follows similarly to the one from the noise-free case: that

$$nR \geq I(Z^n; A^n) + H(Z^n | A^n, Y^n) - H(Z^n | A^n, Y^n, T) \quad (52)$$

follows identically as in (16) by replacing  $X^n$  by  $Z^n$ . That

$$I(Z^n; A^n) + H(Z^n | A^n, Y^n) \geq \sum_{i=1}^n I(Z_i; A_i) + H(Z_i | Y_i, A_i) \quad (53)$$

follows similarly as (18) by replacing  $X^n$  with  $Z^n$ , upon noting that  $H(Y^n | A^n, Z^n) = \sum_{i=1}^n H(Y_i | A_i, Z_i)$ , which follows from the Markov relation  $(X_i, Y_i) - (A_i, Z_i) - (A^{n \setminus i}, Z^{n \setminus i}, Y^{i-1})$  (which a fortiori implies  $Y_i - (A_i, Z_i) - (A^{n \setminus i}, Z^{n \setminus i}, Y^{i-1})$ ). Combining (52) and (53) now yields

$$nR \geq \sum_{i=1}^n I(Z_i; A_i) + I(X_i; U_i | Y_i, A_i) \quad (54)$$

similarly as in Step (a) in (19) upon defining  $U_i = (A^{n \setminus i}, Y^{n \setminus i}, Z^{i-1}, T)$ . The proof of the converse is concluded by verifying that:

- $\hat{X}_i = \hat{X}_i(T, Y^n)$  is a function of the pair  $(U_i, Y_i)$ ;
- the Markov relation  $X_i - Z_i - (A_i, U_i)$  holds (which follows from  $X_i - Z_i - (Z^n, Y^{n \setminus i})$ );
- the Markov relation  $U_i - (X_i, Z_i, A_i) - Y_i$  holds (which follows from  $(Z^n, Y^{n \setminus i}) - (X_i, Z_i, A_i) - Y_i$ );

and invoking the convexity of the informational rate distortion function defined on the right-hand side of (49), which is established similarly as in Lemma 1. ■

### III. SIDE INFORMATION VENDING MACHINE AT THE ENCODER

In this section we consider the setting where the action sequence  $A^n$  is chosen at the encoder and the side information is available at the decoder and possibly at the encoder too. The setting is depicted in Fig. 2. Specifically, a communication scheme in this setting for blocklength  $n$  and rate  $R$  is characterized by an action strategy

$$f: \mathcal{X}^n \rightarrow \mathcal{A}^n \quad (55)$$

an encoding function as shown at the bottom of the page, and a decoding function

$$g: \{1, 2, \dots, 2^{nR}\} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n. \quad (56)$$

---


$$P_{X,Z,A,U,Y}(x, z, a, u, y) = P_{X,Z}(x, z)P_{A,U|Z}(a, u|z)P_{Y|X,Z,A}(y|x, z, a) \quad (50)$$

$T: \mathcal{X}^n \times \mathcal{Y}^n \rightarrow \{1, 2, \dots, 2^{nR}\}$  (when side information is available at the encoder)

$T: \mathcal{X}^n \rightarrow \{1, 2, \dots, 2^{nR}\}$  (when side information is **not** available at the encoder)

As in the case where the actions were chosen by the decoder, the side information  $Y^n$  will be the output of the memoryless channel  $P_{Y|X,A}$  whose input is  $(X^n, A^n)$ . Furthermore, a triple  $(R, D, C)$  is said to be *achievable* if for all  $\varepsilon > 0$  and sufficiently large  $n$  there exists a scheme as above for blocklength  $n$  and rate  $R + \varepsilon$  satisfying both

$$E \left[ \sum_{i=1}^n \rho(X_i, \hat{X}_i) \right] \leq n(D + \varepsilon) \quad (57)$$

$$E \left[ \sum_{i=1}^n \Lambda(A_i) \right] \leq n(C + \varepsilon). \quad (58)$$

The rate distortion (and cost) function  $R_e(D, C)$  (The letter  $e$  stands for encoder) is defined as

$$R_e(D, C) = \inf \{ R' : \text{the triple } (R', D, C) \text{ is achievable} \}. \quad (59)$$

The general case remains open; however, we present here a characterization of three important cases: lossless case (where  $\Pr(X^n = \hat{X}^n) \rightarrow 1$ ), Gaussian case (where  $Y = A + X + N$  and  $X$  and  $N$  are independent Gaussian random variables), and a case where the Markov form  $Y - A - X$  holds. In all three cases,  $R_e(D, C)$  is independent of whether or not the S.I. is available at the encoder.

#### A. Lossless Case

Here we consider the lossless case, namely, for any  $\epsilon > 0$  there exists an  $n$  such that  $\Pr(X^n = \hat{X}^n) > 1 - \epsilon$ . Define

$$R_e^{(I)}(C) = \min [H(X|A, Y) + I(X; A) - I(Y; A)] \quad (60)$$

where  $P_X$  and  $P_{Y|A,X}$  are determined by the problem setting and the minimization is over  $P_{A|X}$  such that  $E[\Lambda(A)] \leq C$ . The term  $[H(X|A, Y) + I(X; A) - I(Y; A)]$  is convex in  $P_{A|X}$  since the term  $-I(Y; A, X)$  is convex in  $P_{A|X}$  and the identity in (61), as shown at the bottom of the page, holds.

Let us denote the minimum (operational) rate that is needed to reconstruct the source at the encoder losslessly, with an expected cost per action less than  $C$ , as  $R_e(C)$ .

**Theorem 5:** For the setting in Fig. 2 where the actions are chosen by the encoder and the side information  $Y$  is known to the decoder and may or may not be known to the encoder the minimum rate that is needed to reconstruct the source under a cost constraint  $C$  is given by

$$R_e(C) = R_e^{(I)}(C). \quad (62)$$

**Proof: Achievability:** The achievability proof is divided into two cases according to the sign of the term  $I(X; A) - I(Y; A)$ . In the first case we assume  $I(X; A) - I(Y; A) > 0$  and we use a coding scheme that is based on Wyner-Ziv coding [15] for rate distortion theory, where side information known at the decoder. In the second case, we assume  $I(Y; A) - I(X; A) > 0$  and we use a coding scheme that is based on Gel'fand-Pinsker coding [6] for channel with states where the state is known to the encoder.

*First case  $I(X; A) - I(Y; A) > 0$ :* We first generate a codebook of sequences of actions  $A^n$  that covers  $X^n$ ; hence, the size of the codebook needs to be  $2^{n(I(A;X)+\epsilon)}$ , where  $\epsilon > 0$ . Then, similarly to Wyner-Ziv coding scheme [15], we bin the codebook into  $2^{n(I(A;X)-I(Y;A)+2\epsilon)}$  bins such that into each bin we have  $2^{n(I(Y;A)-\epsilon)}$  codebooks. Similarly to Wyner-Ziv scheme, we look in the codebook for a sequence  $A^n$  that is jointly typical with  $X^n$  and transmit the number of the bin that contains the sequence. The decoder receives the bin number and looks which of the sequences of  $A^n$  in the bin that its number is received are jointly typical with the side information  $Y^n$ . Similar to the analysis in Wyner-ziv setting, with high probability there will be only one codeword that is jointly typical with  $Y^n$  (The Markov form that is needed in the analysis of Wyner-ziv setting is not needed here, since the side information  $Y^n$  is generated according to  $P_{Y|A,X}$  and therefore if  $(A^n, X^n)$  are jointly typical then with high probability the triple  $(A^n, X^n, Y^n)$  would also be jointly typical). In the final step the encoder uses a Slepian-Wolf scheme for transmitting  $X^n$  losslessly to the decoder that has side information  $(Y^n, A^n)$ ; hence additional rate of  $H(X|Y, A)$  is needed.

*Second case  $I(Y; A) - I(X; A) > 0$ :* First we notice that the expression in (62) can be written as  $H(X|A, Y) - (I(Y; A) - I(X; A))$ . The actions can be considered as input to a channel with states where the output of the channels is  $Y$  and the state is  $X$  and the conditional probability of the channel is  $P_{Y|X,A}$ . The capacity of this channel is achieved by Gel'fand-Pinsker coding scheme [6] and is given as  $I(Y; A) - I(X; A)$ . In addition the Gel'fand-Pinsker coding scheme induces a triple  $(X^n, Y^n, A^n)$  that is jointly typical. Hence, we can use the message in order to reduce the needed rate  $H(X|Y, A)$  as in Slepian-Wolf scheme to  $H(X|A, Y) - (I(Y; A) - I(X; A))$ .

**Converse:** for the converse part, fix a scheme of rate  $R$  for a block of length  $n$  with a probability of error  $\Pr(X^n \neq \hat{X}^n) = P_e^{(n)}$  and consider (63), as shown at the bottom of the next page, where  $\epsilon_n = P_e^{(n)} \log |\mathcal{X}| + \frac{1}{n}$  and step (a) follows Fano's inequality. Step (b) follows the fact that  $A^n$  and  $T$  are deterministic functions of  $X^n$ . Step (c) follows the following four relations:  $P(x^n) = \prod_{i=1}^n P(x_i)$ ,

$$\begin{aligned} H(X|A, Y) + I(X; A) - I(Y; A) &= H(X|A, Y) + H(X) - H(X|A) - H(Y) + H(Y|A) \\ &= H(X) - I(X; Y|A) - H(Y) + H(Y|A) \\ &= H(X) - H(Y|A) + H(Y|A, X) - H(Y) + H(Y|A) \\ &= H(X) - I(Y; A, X) \end{aligned} \quad (61)$$

$P(y^n | a^n, x^n) = \prod_{i=1}^n P(y_i | a_i, x_i)$   $H(A^n | X^n) = 0$ , and  $H(Y^n) \leq \sum_{i=1}^n H(Y_i)$ . The minimization in the last step is over all conditional distribution  $P_{A|X}$  that satisfy the cost constrain, namely  $E[\Lambda(A)] \leq C$ , and the inequality follows from the fact that the expression  $H(X) + H(Y | A, X) - H(Y)$  is convex in  $P_{A|X}$  for fixed  $P_X$  and  $P_{Y|A,X}$ . The converse proof is completed by invoking the fact that since  $R$  is an achievable rate there exists a sequence of codes at rate  $R$  such that  $\epsilon_n \rightarrow 0$ . ■

We have seen that, in the absence of a cost constraint on the actions, the minimum rate needed for a near lossless reconstruction at the decoder is given by

$$\min I(X; A) + H(X | Y, A) - I(Y; A) \quad (64)$$

(regardless of whether or not the side information is present at the encoder). Thus,  $I(Y; A)$  represents the saving in rate relative to the case where the actions are taken by the decoder (recall (28) for the minimum rate at that case). To see that this can be significant, recall the example  $\mathcal{X} = \mathcal{A} = \mathcal{Y} = \{0, 1\}$ , where  $X$  is a fair coin flip,  $P_{Y|X,A=0}$  is the Z-channel with crossover probability  $\delta$  from 1 to 0, and  $P_{Y|X,A=1}$  is the S-channel with crossover probability  $\delta$  from 0 to 1. It is easily seen that in this case  $I(X; A) + H(X | Y, A) - I(Y; A) = 0$  and so, a fortiori, the minimum in (64) is zero. That the source can be reconstructed losslessly with zero rate in this case is equally easy to see from an operational standpoint, since taking actions  $A_i = X_i$  ensures that  $Y_i = X_i$  with probability one.

### B. Gaussian Case

Here we consider the case where:

- the source has a Gaussian distribution with zero mean and variance  $\sigma_X^2$ , i.e.,

$$X \sim N(0, \sigma_X^2) \quad (65)$$

- the relation between  $Y, X, A$  is given by

$$Y = X + A + N \quad (66)$$

where  $N$  is a random variable independent of  $(A, X)$  and has a Gaussian distribution with zero mean and variance  $\sigma_N^2$ , i.e.,

$$W \sim N(0, \sigma_N^2) \quad (67)$$

- the distortion is a mean square error distortion, i.e.,  $E[\sum_{i=1}^n (X_i - \hat{X}_i)^2]$  and it has to be less than  $D$ ;
- the cost of the actions is  $E[\sum_{i=1}^n A_i^2]$  and has to be less than  $C$ . Without loss of generality, we assume that  $C = \alpha^2 \sigma_X^2$  where  $\alpha > 0$ .

**Theorem 6:** For the Gaussian setting of Fig. 2, as described above, we get (68), as shown at the bottom of the page.

Before proving Theorem 6, we would like to point out that the state amplification problem [16], [17] is tangent to the vending side information problem described here. In the state amplification problem, the goal is to design a communication scheme for a channel with i.i.d. states sequence,  $S^n$ , which is known to the encoder. The purpose of the scheme is to send a message through the channel, and at the same time to describe to the decoder the state sequence  $S^n$ . The case where there is no message to send, namely, the input to the channel is used only to describe the state sequence, is equivalent to the problem presented here when  $R_e(D, C) = 0$ , namely, we when are using only the actions to describe the source and no additional message is sent. If  $R_e(D, C) = 0$ , we obtain from (68) that for the Gaussian source coding problem, the minimum mean-square error satisfies

$$D \geq \frac{\sigma_X^2 \sigma_N^2}{(\sigma_X + \sqrt{C})^2 + \sigma_N^2} \quad (69)$$

---


$$\begin{aligned}
 nR &\geq H(T) \\
 &\geq H(T | Y^n) \\
 &= H(X^n, T | Y^n) - H(X^n | T, Y^n) \\
 &\stackrel{(a)}{\geq} H(X^n, T | Y^n) - n\epsilon_n \\
 &\stackrel{(b)}{=} H(X^n, A^n | Y^n) - n\epsilon_n \\
 &= H(X^n, A^n, Y^n) - H(Y^n) - n\epsilon_n \\
 &= H(X^n) + H(A^n | X^n) + H(Y^n | A^n, X^n) - H(Y^n) - n\epsilon_n \\
 &\stackrel{(c)}{\geq} \sum_{i=1}^n H(X_i) + H(Y_i | A_i, X_i) - H(Y_i) - n\epsilon_n \\
 &\geq \min[n(H(X) + H(Y | A, X) - H(Y))] - n\epsilon_n
 \end{aligned} \quad (63)$$


---

$$R_e(D, C) = \begin{cases} \frac{1}{2} \log \left[ \frac{\sigma_N^2}{(1 + \sqrt{C}/\sigma_X)^2 \sigma_X^2 + \sigma_N^2} \cdot \frac{\sigma_X^2}{D} \right] & \text{if } \left[ (1 + \sqrt{C}/\sigma_X)^2 \sigma_X^2 + \sigma_N^2 \right] \cdot D \leq \sigma_X^2 \sigma_N^2 \\ 0 & \text{otherwise.} \end{cases} \quad (68)$$

a result that was also obtained in [17, Theorem 2], where the channel is the Gaussian channel and the goal is to describe the state sequence with minimum mean square error distortion.

**Proof of Theorem 6: Achievability:** The encoder chooses the actions to be  $A = \alpha X$  and then it uses a coding for the Gaussian Wyner-Ziv with side information at the decoder [13]. The side information satisfies  $Y = X + A + N = (1 + \alpha)X + N$ , which is equivalent to having a side information  $Y = X + \frac{N}{(1 + \alpha)}$ . Denote by  $N' = \frac{N}{1 + \alpha}$ . Using the Gaussian Wyner-Ziv result, a rate

$$R = \frac{1}{2} \log \frac{\sigma_{N'}^2}{\sigma_X^2 + \sigma_{N'}^2} \frac{\sigma_X^2}{D} = \frac{1}{2} \log \frac{\sigma_N^2}{(1 + \alpha)^2 \sigma_X^2 + \sigma_N^2} \frac{\sigma_X^2}{D} \quad (70)$$

is achievable.

**Converse:** We prove the converse in two steps. First, we derive the lower bound

$$R_e(D, C) \geq \min_{P_{A|X}, P_{\hat{X}|X, A, Y}} I(X; \hat{X}) - I(Y; X, A) \quad (71)$$

which holds for any  $P_X$  and  $P_{Y|A, X}$  (not necessarily Gaussian), and then we evaluate it for the Gaussian case.

Fix a scheme at rate  $R$  for a block of length  $n$  and consider

$$\begin{aligned} nR &\geq H(T) \\ &\geq H(T|Y^n) \\ &\geq I(X^n; T|Y^n) \\ &\stackrel{(a)}{=} I(X^n; Y^n, T) - I(X^n; Y^n) \\ &\stackrel{(b)}{\geq} I(X^n; \hat{X}^n) - I(X^n, A^n; Y^n) \\ &\stackrel{(c)}{\geq} \sum_{i=1}^n I(X_i; \hat{X}_i) - I(X_i, A_i; Y_i) \end{aligned} \quad (72)$$

where (a) follows from [14, Lemma 3.2], which asserts that for arbitrary random variables  $I(X, Z; Y) = I(Z; Y) + I(X; Y|Z)$ . Step (b) follows from the facts that  $\hat{X}^n$  is a deterministic function of the pair  $(T, Y^n)$ , and  $A^n$  is a deterministic function of  $X^n$ . Step (c) follows from the facts that  $H(X^n) = \sum_{i=1}^n H(X_i)$ ,  $H(Y^n|A^n, X^n) = \sum_{i=1}^n H(Y_i|A_i, X_i)$  and conditioning reduces entropy. Since the expression in (72) is convex in  $P_{A|X}, P_{\hat{X}|X}$  for fixed  $P_X$  and  $P_{Y|A, X}$ , we obtain the lower bound

$$R_e(D, C) \geq \min I(X; \hat{X}) - I(Y; X, A) \quad (73)$$

and the minimization is over conditional distributions  $P_{A|X}, P_{\hat{X}|X}$  that satisfy the distortion and cost constraints.

Now we evaluate the lower bound for the Gaussian case, as shown in (74), found at the bottom of the page, where inequality (a) follows from the fact that  $H(X|\hat{X}) \leq H(X - \hat{X}) \leq \frac{1}{2} \log 2\pi e D$  (because of the constraint that  $E(X - \hat{X})^2 \leq D$ ) and  $H(Y) \leq \frac{1}{2} \log 2\pi e \sigma_Y^2$ , where

$$\begin{aligned} \sigma_Y^2 &= E[X^2] + 2E[AX] + E[A^2] + E[N^2] \\ &\leq E[X^2] + 2\sqrt{E[X^2]E[A^2]} + E[A^2] + E[N^2] \\ &\leq \sigma_X^2 + 2\alpha\sigma_X^2 + \alpha^2\sigma_X^2 + \sigma_N^2 \\ &= \sigma_X^2(1 + \alpha)^2 + \sigma_N^2. \end{aligned} \quad (75)$$

### C. Markov Form $Y-A-X$

Here, we consider the case where the Markov form  $X-A-Y$  holds.

**Theorem 7:** The rate distortion (and cost) function  $R_e(D, C)$  for the setting in Fig. 2 when  $P_{Y|A, X}(y|a, x) = P_{Y|A}(y|a)$  satisfies

$$R_e(D, C) = [\min(I(X; \hat{X}) - I(A; Y))]^+ \quad (76)$$

where  $[\cdot]^+$  denotes  $\min\{\cdot, 0\}$  and the minimization is over joint distributions of the form  $P_{A, X, Y, \hat{X}}(a, x, y, \hat{x}) = P_X(x)P_{A|X}(a|x)P_{Y|A}(y|a)P_{\hat{X}|X}(\hat{x}|x)$  satisfying  $E\rho(X, \hat{X}) \leq D$  and  $E\Lambda(A) \leq C$ .

It is interesting to note that the solution is the difference between a rate-distortion expression  $\min_{P_{\hat{X}|X}} I(X; \hat{X})$  and channel capacity expression  $\max_{P_A} I(A; Y)$ , i.e.,

$$R_e(D, C) = [R(P_X, D) - \text{Cap}(P_{Y|A}, C)]^+ \quad (77)$$

where  $\text{Cap}(P_{Y|A}, C)$  denotes the capacity of the channel  $P_{Y|A}$  under a cost-constraint  $C$ .

**Proof of Theorem 7: Achievability:** Design a regular rate distortion code, which needs a rate larger than  $I(X; \hat{X})$ , and then transmits part of the rate through the channel which has an input  $A$  and output  $Y$ . Therefore the total rate that is needed to be transmitted through the index  $T(X^n)$  is the difference  $I(X; \hat{X}) - I(A; Y)$ .

**Converse:** We invoke the lower bound given in (73) and obtain

$$\begin{aligned} R_e(D, C) &\geq I(X; \hat{X}) - I(Y; X, A) \\ &= I(X; \hat{X}) - I(Y; A) \end{aligned} \quad (78)$$

where the last equality is due to the Markov form  $X-A-Y$ . ■

$$\begin{aligned} I(X; \hat{X}) - I(Y; X, A) &= H(X) - H(X|\hat{X}) - H(Y) + H(Y|A, X) \\ &= H(X) - H(X|\hat{X}) - H(Y) + H(N) \\ &\stackrel{(a)}{\geq} \frac{1}{2} \log 2\pi e \sigma_X^2 - \frac{1}{2} \log 2\pi e D - \frac{1}{2} \log 2\pi e \left( (1 + \alpha)^2 \sigma_X^2 + \sigma_N^2 \right) + \frac{1}{2} \log 2\pi e \sigma_N^2 \\ &= \frac{1}{2} \log \frac{\sigma_N^2}{(1 + \alpha)^2 \sigma_X^2 + \sigma_N^2} \frac{\sigma_X^2}{D} \end{aligned} \quad (74)$$

#### D. Upper and Lower Bounds for the General Case

As mentioned, our problem setting, when specialized to rate  $R = 0$ , coincides with the state amplification problem of [16], also specialized to zero communication rate (i.e., when there is no message to communicate, only states to amplify). The latter is still open and thus, a fortiori, so is ours. We dedicate this subsection to obtaining upper and lower bounds on the achievable rate that hold for the general case.

##### 1) Achievable Rates:

- **Absence of S.I. at Encoder:** For the setting of Fig. 2 with an open switch, i.e., when the encoder has no access to the S.I., the following is an achievable rate:

$$I(U; X | A, Y) + I(X; A) - I(Y; A) \quad (79)$$

under any joint distribution of the form

$$P_X(x)P_{A|X}(a|x)P_{Y|X,A}(y|x,a)P_{U|X,A}(u|x,a)$$

such that  $E\rho(X, \hat{X}_{\text{opt}}(A, Y, U)) \leq D$  and  $E\Lambda(A) \leq C$ . The argument for why this rate is achievable is similar to that given in Subsection III-A for why the right side of (62) is achievable, the difference being that the  $H(X | A, Y)$  term in (62), corresponding to Slepian-Wolf coding of  $X^n$  conditioned on  $A^n$ , is replaced by  $I(U; X | A, Y)$ , corresponding to Wyner-Ziv coding conditioned on  $A^n$ .

- **S.I. Available at Encoder:** For the setting of Fig. 2 with a closed switch, i.e., when the encoder has access to the S.I., the following is an achievable rate:

$$I(\hat{X}; X | A, Y) + I(X; A) - I(Y; A) \quad (80)$$

under any joint distribution of the form

$$P_X(x)P_{A|X}(a|x)P_{Y|X,A}(y|x,a)P_{\hat{X}|X,A,Y}(\hat{x}|x,a,y)$$

such that  $E\rho(X, \hat{X}) \leq D$  and  $E\Lambda(A) \leq C$ . The argument for why this rate is achievable is similar to that for why the right side of (80) is achievable, the difference being that the  $I(U; X | A, Y)$  terms, corresponding to Wyner-Ziv coding conditioned on  $A^n$ , is replaced by  $I(\hat{X}; X | A, Y)$ , corresponding to standard rate distortion coding conditioned on  $A^n, Y^n$ .

2) *Lower Bound on Achievable Rate:* As pointed out in Subsection III-B, the proof of the converse part of Theorem 6 is valid for the general case (i.e., beyond the Gaussian scenario), and shows that the rate needed to achieve distortion  $D$  at cost  $C$ , regardless of whether or not S.I. is available at the encoder, is at least as large as

$$I(X; \hat{X}) - I(Y; X, A) \quad (81)$$

for some joint distribution of the form

$$P_X(x)P_{A|X}(a|x)P_{Y|X,A}(y|x,a)P_{\hat{X}|X,A,Y}(\hat{x}|x,a,y)$$

satisfying the distortion and cost constraints. It is worthwhile to note that this rate was shown to be achievable for the three special cases considered in the previous three subsections. Indeed, this fact was shown explicitly for the cases of Subsection III-B

and Subsection III-C, and in the lossless case (81) becomes  $H(X) - I(Y; X, A) = H(X | A, Y) + I(A; X) - I(A; Y)$ , which coincides with the expression on the right-hand side of (62).

To see that the lower bound in (81) may not be tight in general, even when the S.I. is available at the encoder, consider the standard case of rate distortion coding with S.I. available to both encoder and decoder. In this case  $A$  is degenerate, so the right-hand side of (81) reduces to

$$I(X; \hat{X}) - I(Y; X, A) = H(X | Y) - H(X | \hat{X}) \quad (82)$$

while the tight lower bound on the achievable rate for this scenario is well-known to be given by

$$I(X; \hat{X} | Y) = H(X | Y) - H(X | \hat{X}, Y) \quad (83)$$

which may be strictly larger than the expression in (82).

#### IV. SUMMARY AND OPEN QUESTIONS

We have studied source coding in the presence of side information, when the system can take actions that affect the availability, quality, or nature of the side information. We have given a full characterization of the rate-distortion-cost tradeoff when the actions are taken by the decoder. For the case where the actions are taken by the encoder, we have characterized this tradeoff in a few important special cases, while providing upper and lower bounds on the achievable rate for the general case.

The most significant question left open by our work is a full characterization of the rate-distortion-cost tradeoff for the setting of actions taken at the encoder (beyond the special cases considered here), with S.I. that may or may not be available at the encoder (Fig. 2). Another question left open, for the setting of actions taken by the decoder, is whether the rate-distortion-cost tradeoff can be improved when each action is allowed to depend on the side information symbols generated thus far, that is, when the  $i$ th action is allowed to be of the form  $A_i = A_i(T, Y^{i-1})$  (rather than  $A_i(T)$ ).

Finally, we mention a computational point worth future investigation. In Lemma 1 we have shown that for any fixed  $P_X$  and  $P_{Y|A,X}$ , the expression  $I(X; A) + I(X; U | Y, A)$  is convex in  $P_{A,U|X}$  (assuming the joint distribution given in (5)). Furthermore, we have shown that the set of distributions of the form given in (5) is a convex set in  $P_{A,U|X}$ . Hence, one can use convex optimization tools for finding the minimum of  $R^I(D, C)$ . However, it would be interesting to see if an algorithm similar to Willems' algorithm [18], [19] for computing the Wyner-Ziv rate distortion function can be used to compute  $R^I(D, C)$  in Equation (4) with global convergence.

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