Zero-error capacity for finite state channels with feedback and channel state information

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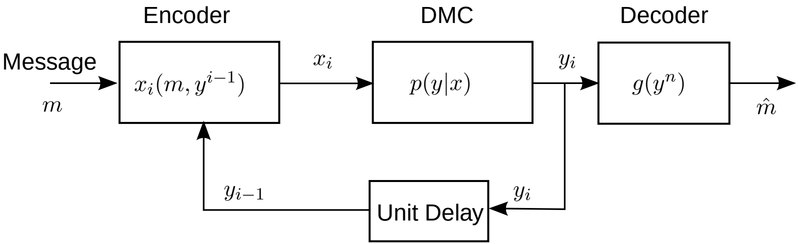
Outline:

- zero-error capacity for DMC with feedback [shannon56]
- channel model
- formulate C_0 as a dynamic programming problem;
- Bellman equation;
- conclusions.

Review: Discrete memoryless channels with feedback[Shannon56]



Definitions:

- Zero-error constraint: $Pr\{m \neq \hat{m}\} = 0$.
- A rate R is achievable: $(2^{nR}, n)$ block code, error free.
- C_0 is the supreme of all achievable rates.

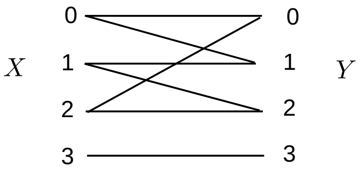
Theorem ([Shannon56])

If all pairs of input letters are adjacent, then C_0 is zero. Otherwise,

$$C_0 = \max_{P_X} \log \left[\max_y \sum_{x \in G(y)} P_X(x) \right]^{-1}$$

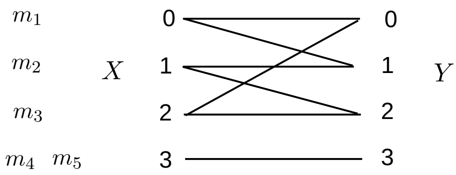
where $G(y) = \{x : p(y|x) > 0\}$.

$G(y)$ is the topology of the channel.

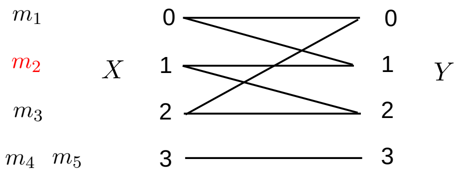


$G(0) = \{0, 2\}$, $G(1) = \{0, 1\}$, $G(2) = \{1, 2\}$, $G(3) = \{3\}$.
$P_X = \{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\}$, $C_0 = \log 2.5$.

$(5, 2)$ zero-error code.



[Shannon56]



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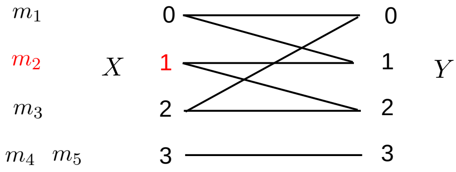

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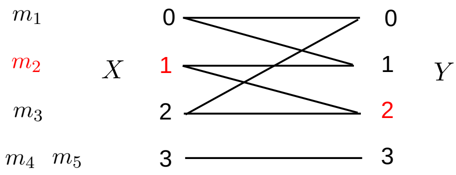

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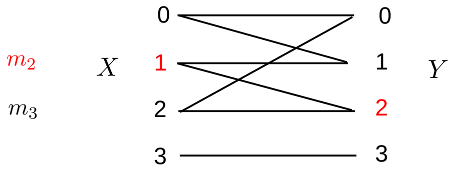

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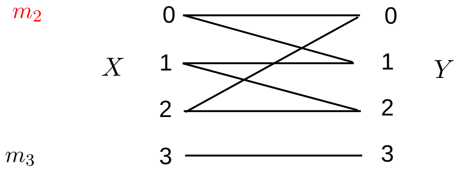

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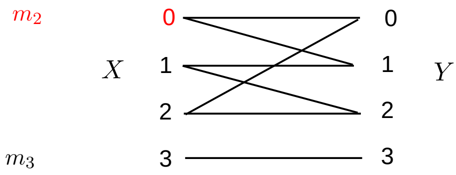

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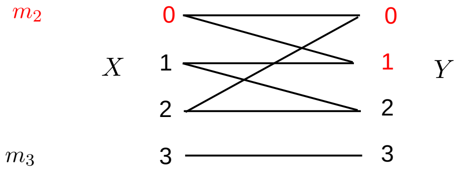

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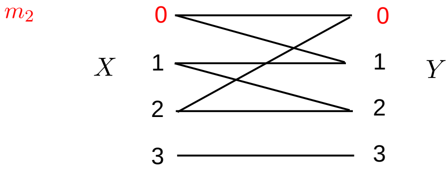
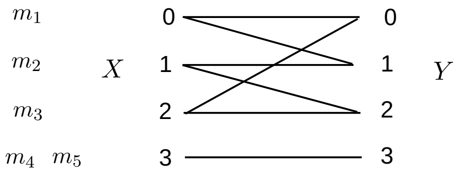


[Shannon56]

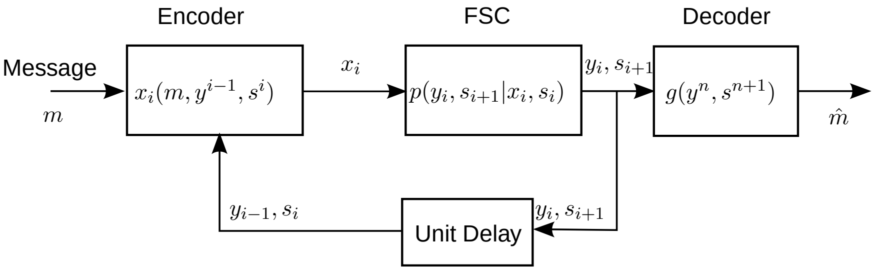
$$C_0 = \max_{P_X} \log \left[\max_y \sum_{x \in G(y)} P_X(x) \right]^{-1}$$

P_X: how the messages should be grouped.

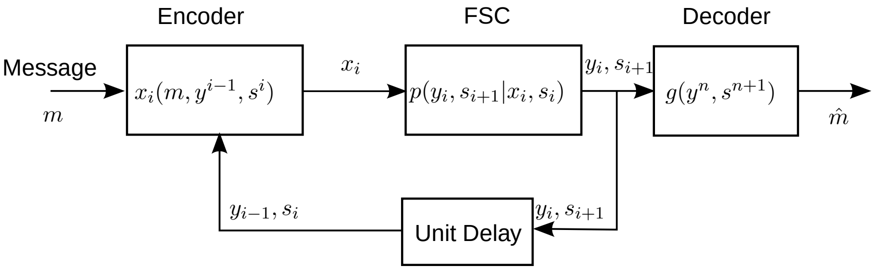


$P_X = \{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\}$, $C_0 = \log 2.5$.

FSC with feedback and CSI



FSC with feedback and CSI



Capacity solved by [Chen& Berger 2005] under similar setup.

FSC with feedback and CSI

Definitions:

- A $(2^{nR}, n)$ zero-error block code:
  2^{nR} messages, $x_i(m, y^{i-1}, s^i)$, $g(y^n, s^{n+1})$
- zero-error: $Pr\{g(y^n, s^{n+1}) \neq m | s_1, \text{Message} = m\} = 0$,
  $\forall s_1 \in \mathcal{S}$, $\forall m$.
- $N(n, s)$: maximum number of messages that can be transmitted with n transmissions and initial state s error-free.

Operational definition:

$$C_0 = \sup \frac{\min_{s\in\mathcal{S}} \log_2 N(n, s)}{n}$$
$$= \lim_n \frac{\min_{s\in\mathcal{S}} \log_2 N(n, s)}{n}$$

Main Theorem

Theorem

If $\min_{s\in\mathcal{S}} V(|\mathcal{S}|, s) > 0$*,*

$$C_0 = \lim_{n\to\infty} \frac{1}{n} \min_{s\in\mathcal{S}} \log_2 M(n,s);$$

otherwise $C_0 = 0$*.*

where

$$M(n,s) = \max_{P_{X|S}(\cdot|s)} \min_{s'\in\mathcal{S}} \left\{ M(n-1,s') \left[\max_{y\in\mathcal{Y}} \sum_{x\in G(y,s'|s)} P_{X|S}(x|s) \right]^{-1} \right\}$$

$\forall s \in \mathcal{S}$, and for $n = 1, 2, 3, \ldots$

- $G(y,s'|s) = \{x : x \in \mathcal{X}, p(y,s'|x,s) > 0\}$: the topology of the channel.
- $M(\cdot,\cdot)$: $\mathbb{Z}^+ \times \mathcal{S} \mapsto \mathbb{R}^+$; $M(0,s) = 1, \forall s \in \mathcal{S}$

DMC with feedback:

$$M(n) = \max_{P_X(\cdot)} \left\{ M(n-1) \left[\max_{y \in \mathcal{Y}} \sum_{x \in G(y)} P_X(x) \right]^{-1} \right\}$$

$$\frac{M(n-1)}{M(n)} = \sum_{x \in G(y)} P_X(x)$$

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FSC with feedback and CSI:

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$$\max_{s' \in \mathcal{S}} \frac{M(n-1,s')}{M(n,s)} = \max_{s' \in \mathcal{S}} \max_{y \in \mathcal{Y}} \sum_{x \in G(y,s'|s)} P_{X|S}(x|s)$$

Solve the dynamic programming

$$M(n,s) = \max_{P_{X|S}(\cdot|s)} \min_{s' \in \mathcal{S}} \left\{ M(n-1,s') \left[\max_{y \in \mathcal{Y}} \sum_{x \in G(y,s'|s)} P_{X|S}(x|s) \right]^{-1} \right\}$$

- Value function: $J_n(s) = \log_2 M(n,s)$
- State: s
- Action: $a = P_{X|S}(\cdot|s)$
- Reward: $r(s', a|s) = \log_2 \left[\max_{y \in \mathcal{Y}} \sum_{x \in G(y,s'|s)} P_{X|S}(x|s) \right]^{-1}$

$$J_n(s) = \max_{a \in A(s)} \min_{s' \in \mathcal{S}} \left\{ r(s', a|s) + J_{n-1}(s') \right\}$$

$$C_0 = \lim_{n \to \infty} \frac{\min_{s \in \mathcal{S}} J_n(s)}{n}$$

Two people game:

- Encoder: choose a to maximize total reward
- Channel: choose the worst y and s' to minimize total reward

Properties of $J_n(s)$:

- The sequence $\{\min_s J_n(s)\}$ is sup-additive:
  $\min_s J_{n+m}(s) \geq \min_s J_n(s) + \min_s J_m(s)$
- The sequence $\{\max_s J_n(s)\}$ is sub-additive:
  $\max_s J_{n+m}(s) \leq \max_s J_n(s) + \max_s J_m(s)$

Bounds:

$$\min_s \frac{J_n(s)}{n} \leq C_0 \leq \max_s \frac{J_n(s)}{n}$$

Exact:

$$C_0 = \lim_{n\to\infty} \min_s \frac{J_n(s)}{n}$$

Define the operator T as

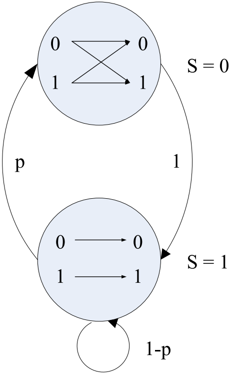
$$(T \circ J)(s) = \max_{a \in A(s)} \min_{s'} \left\{ r(s', a|s) + J(s') \right\}$$

$$J_n(s) = T \circ J_{n-1}(s)$$

Theorem

(Bellman Equation.) If there exists a bounded function $J(s)$ and a constant ρ that satisfy $J(s) + \rho = (T \circ J)(s)$ then $\lim \frac{1}{n} J_n(s) = \rho$.

Example One



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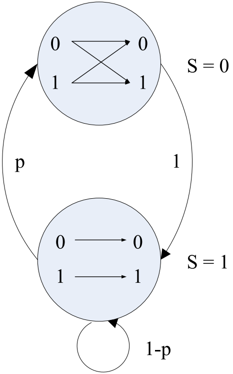

Bellman equation:

$$
\begin{aligned}
J(0) &= J(1) - \rho \\
J(1) &= 1 + J(0) - \rho
\end{aligned}
$$

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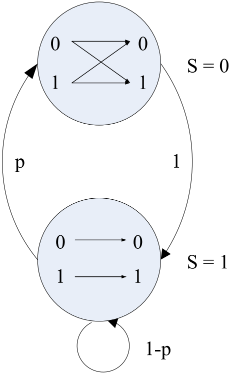


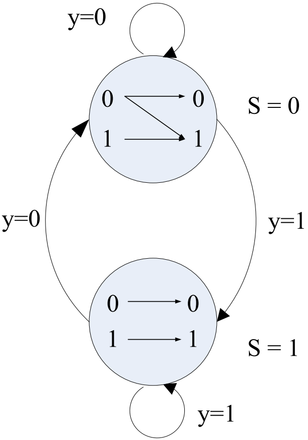
Bellman equation:

$$\begin{aligned} J(0) &= J(1) - \rho \\ J(1) &= 1 + J(0) - \rho \end{aligned}$$

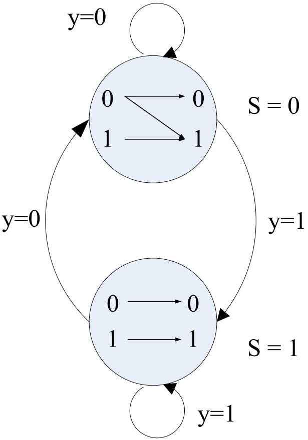
Solution:

$$\rho = \frac{1}{2}, J(0) = v, J(1) = v + \frac{1}{2}$$

Example Two

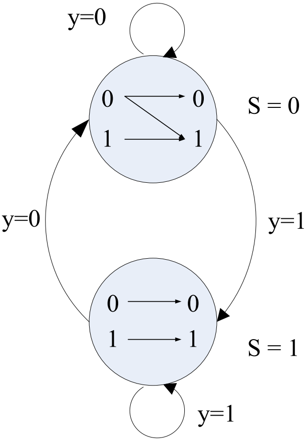


Example Two



Action $\alpha = P_{X|S}(0|0)$, $\beta = P_{X|S}(1|0)$.

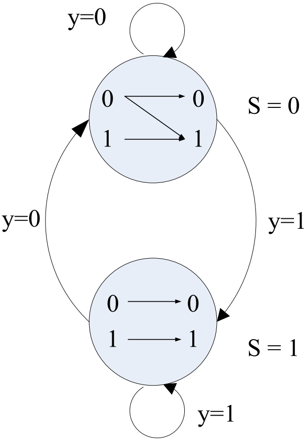
Example Two



Action $\alpha = P_{X|S}(0|0)$, $\beta = P_{X|S}(1|0)$.
Bellman equation:

$$\begin{aligned} J(0) &= J(1) - \rho \\ J(1) &= J(0) + \log_2 \frac{1}{\beta} - \rho \\ \log_2 \frac{1}{\beta} + J(0) &= \log_2 \frac{1}{1-\beta} + J(1) \end{aligned}$$

Example Two



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Solution: $\rho = \log_2 \frac{\sqrt{5}+1}{2}, \alpha = 0, \beta = \frac{3-\sqrt{5}}{2}$

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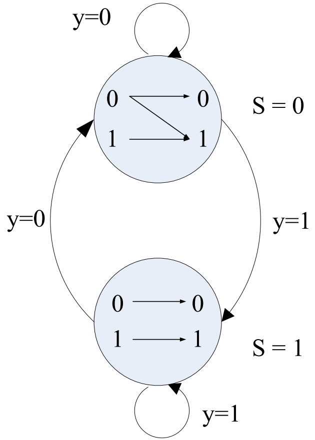

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Any binary sequence with no two consecutive 0's

Conclusion

- the zero-error capacity for FSC with feedback and CSI $\Longleftrightarrow$ a dynamic programming problem
- solve Bellman equation to calculate the zero-error capacity for some channels.

Thank you!