

Consolidating Achievable Regions of Multiple Descriptions

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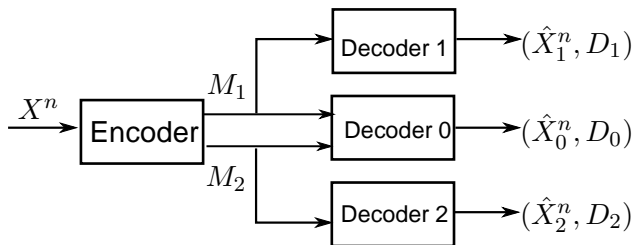
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Outline:

- Multiple Description Problem
- El Gamal-Cover (EGC) Region and EGC*
- Zhang-Berger (ZB) Region
- Venkataramani-Kramer-Goyal (VKG) Region
- Conclusions.

Multiple Description Problem



- Source: X_1, X_2, \dots i.i.d $\sim p(x)$
- Distortion Measures: $d_j(x, \hat{x}_j)$, $\hat{x}_j \in \hat{\mathcal{X}}_j$ for $j = 0, 1, 2$,

$$d_j(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^n d_j(x_i, \hat{x}_i), \quad j = 0, 1, 2$$

- A $(2^{nR_1}, 2^{nR_2}, n)$ code: $M_1 \in [1 : 2^{nR_1}]$, $M_2 \in [1 : 2^{nR_2}]$

Multiple Description Problem (cont.)

- A rate pair (R_1, R_2) is said to be achievable for distortion triple (D_1, D_2, D_0) if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes with average distortion

$$\limsup_{n \rightarrow \infty} E \left(d_j(X^n, \hat{X}_j^n) \right) \leq D_j \text{ for } j = 0, 1, 2$$

- The rate distortion region $\mathcal{R}(D_1, D_2, D_0)$ is the closure of the set of achievable rate pairs (R_1, R_2) for distortion triple (D_1, D_2, D_0) .
- The problem in general is open.

Theorem ($\mathcal{R}_{\text{EGC}^*}$ El Gamal and Cover 1979)

A rate pair (R_1, R_2) is achievable for multiple description for distortion triple (D_0, D_1, D_2) if

$$R_1 \geq I(X; U_1),$$

$$R_2 \geq I(X; U_2),$$

$$R_1 + R_2 \geq I(X; U_1, U_2) + I(U_1; U_2);$$

for some $p(u_1, u_2|x)$ and deterministic functions $\phi_1, \phi_2, \phi_{12}$ such that

$$E(d_1(X, \phi_1(U_1))) \leq D_1$$

$$E(d_2(X, \phi_2(U_2))) \leq D_2$$

$$E(d_0(X, \phi_{12}(U_1, U_2))) \leq D_0.$$

El Gamal-Cover Region (cont.)

Theorem (\mathcal{R}_{EGC} El Gamal and Cover 1982)

A rate pair (R_1, R_2) is achievable for multiple descriptions for distortion triple (D_0, D_1, D_2) if

$$R_1 \geq I(X; \hat{X}_1),$$

$$R_2 \geq I(X; \hat{X}_2),$$

$$R_1 + R_2 \geq I(X; \hat{X}_0 | \hat{X}_1, \hat{X}_2) + I(X; \hat{X}_1, \hat{X}_2) + I(\hat{X}_1; \hat{X}_2);$$

for some $p(\hat{x}_0, \hat{x}_1, \hat{x}_2 | x)$ such that

$$E(d_j(X, \hat{X}_j)) \leq D_j, \text{ for } j = 0, 1, 2$$

- $\hat{X}_1 = \phi_1(U_1), \hat{X}_2 = \phi_2(U_2), \hat{X}_0 = \phi_{12}(U_1, U_2),$
 $\implies \mathcal{R}_{\text{EGC}^*} \subseteq \mathcal{R}_{\text{EGC}}$
- Our goal: $\mathcal{R}_{\text{EGC}^*} = \mathcal{R}_{\text{EGC}}$

Lemma

For a given distribution $p(x, y)$, there exist random variables Y and W , and a deterministic function g such that $Y \sim p(y)$, $W \perp Y$ and $(g(W, Y), Y) \sim p(x, y)$. Furthermore, the cardinality of W need not be larger than $(|\mathcal{X}| - 1)|\mathcal{Y}| + 1$.

Lemma

For a given distribution $p(x, y)$, there exist random variables Y and W , and a deterministic function g such that $Y \sim p(y)$, $W \perp Y$ and $(g(W, Y), Y) \sim p(x, y)$. Furthermore, the cardinality of W need not be larger than $(|\mathcal{X}| - 1)|\mathcal{Y}| + 1$.

Proof.

$W \sim \text{Uniform}[0, 1]$, $Y \sim p(y)$ and $Y \perp W$.

$g(w, y) = F_{X|Y=y}^{-1}(w)$, where

$$F_{X|Y=y}^{-1}(w) = \inf_{x \in \mathbb{R}} \{F_{X|Y=y}(x) \geq w\}.$$

$(g(W, Y), Y) \sim p(x, y)$

$X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $F_{X|Y=(\cdot)}(\cdot)$ has at most $(|\mathcal{X}| - 1)|\mathcal{Y}| + 1$ distinguished values.



$$\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$$

EGC*

$$R_1 \geq I(X; U_1),$$

$$R_2 \geq I(X; U_2),$$

$$R_1 + R_2 \geq I(X; U_1, U_2) \\ + I(U_1; U_2);$$

$$p(u_1, u_2 | x), \phi_1, \phi_2, \phi_{12}$$

$$E(d_1(X, \phi_1(U_1))) \leq D_1$$

$$E(d_2(X, \phi_2(U_2))) \leq D_2$$

$$E(d_0(X, \phi_{12}(U_1, U_2))) \leq D_0.$$

$$\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$$

EGC*

$$\begin{aligned} R_1 &\geq I(X; U_1), \\ R_2 &\geq I(X; U_2), \\ R_1 + R_2 &\geq I(X; U_1, U_2) \\ &\quad + I(U_1; U_2); \end{aligned}$$

$$p(u_1, u_2 | x), \phi_1, \phi_2, \phi_{12}$$

$$E(d_1(X, \phi_1(U_1))) \leq D_1$$

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EGC

$$\begin{aligned} R_1 &\geq I(X; \hat{X}_1), \\ R_2 &\geq I(X; \hat{X}_2), \\ R_1 + R_2 &\geq I(X; \hat{X}_0, \hat{X}_1, \hat{X}_2) \\ &\quad + I(\hat{X}_1; \hat{X}_2); \end{aligned}$$

$$p(\hat{x}_0, \hat{x}_1, \hat{x}_2 | x)$$

$$E(d_j(X, \hat{X}_j)) \leq D_j, \text{ for } j = 0, 1, 2$$

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.

$$p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0).$$

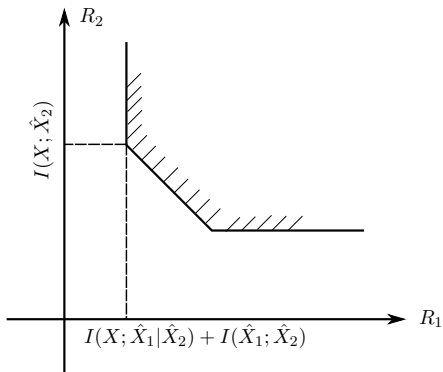
- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
 $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
- There exists $(W, \hat{X}_1, \hat{X}_2)$, g such that
 $W \perp (\hat{X}_1, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2, g(W, \hat{X}_1, \hat{X}_2)) \sim p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$
 $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
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 $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $(W, \hat{X}_1, \hat{X}_2, \hat{X}_0)$ induces $p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$

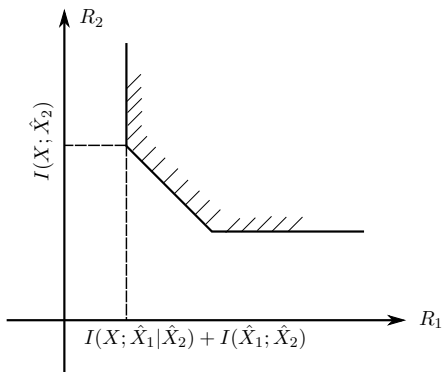
- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
 $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
- There exists $(W, \hat{X}_1, \hat{X}_2)$, g such that
 $W \perp (\hat{X}_1, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2, g(W, \hat{X}_1, \hat{X}_2)) \sim p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$
 $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $(W, \hat{X}_1, \hat{X}_2, \hat{X}_0)$ induces $p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$
- $p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
 $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
- There exists $(W, \hat{X}_1, \hat{X}_2)$, g such that
 $W \perp (\hat{X}_1, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2, g(W, \hat{X}_1, \hat{X}_2)) \sim p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$
 $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $(W, \hat{X}_1, \hat{X}_2, \hat{X}_0)$ induces $p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$
- $p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$
- $(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$
 - $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x)$,
 - $W \perp (\hat{X}_1, \hat{X}_2)$,
 - $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W$,
 - $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$ for some deterministic function g .

EGC



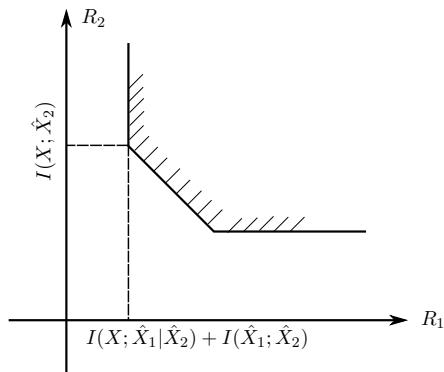
EGC



EGC*

- Set $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.

EGC



EGC*

- Set $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.
- $\phi_1(U_1) = \hat{X}_1$, $\phi_2(U_2) = \hat{X}_2$,
 $\phi_{12}(U_1, U_2) = \hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
 $\implies (X, U_1, U_2)$ valid distribution in EGC*

$$U_1 = (\hat{X}_1, W), U_2 = \hat{X}_2.$$

$$(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$$

- $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x),$
- $W \perp (\hat{X}_1, \hat{X}_2),$
- $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W,$
- $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $E(d_j(X, \hat{X}_j)) \leq D_j,$ for $j = 0, 1, 2,$

Corner point (R'_1, R'_2) in EGC^* :

$$U_1 = (\hat{X}_1, W), U_2 = \hat{X}_2.$$

$$(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$$

- $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x),$
- $W \perp (\hat{X}_1, \hat{X}_2),$
- $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W,$
- $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $E(d_j(X, \hat{X}_j)) \leq D_j,$ for $j = 0, 1, 2,$

$$\begin{aligned} R'_2 &= I(X; U_2) \\ &= I(X; \hat{X}_2) \\ &= R_2 \end{aligned}$$

Corner point (R'_1, R'_2) in EGC^* :

$$U_1 = (\hat{X}_1, W), U_2 = \hat{X}_2.$$

$$(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$$

- $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x),$
- $W \perp (\hat{X}_1, \hat{X}_2),$
- $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W,$
- $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $E(d_j(X, \hat{X}_j)) \leq D_j,$ for $j = 0, 1, 2,$

$$\begin{aligned} R'_2 &= I(X; U_2) \\ &= I(X; \hat{X}_2) \\ &= R_2 \end{aligned}$$

$$\begin{aligned} &R'_1 + R'_2 \\ &= I(X; U_1, U_2) + I(U_1; U_2) \\ &= I(X; W, \hat{X}_1, \hat{X}_2) + I(\hat{X}_1, W; \hat{X}_2) \\ &\stackrel{(a)}{=} I(X; W, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1, W; \hat{X}_2) \\ &\stackrel{(b)}{=} I(X; W, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \\ &= I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(X; W | \hat{X}_1, \hat{X}_2, \hat{X}_0) \\ &\quad + I(\hat{X}_1; \hat{X}_2) \\ &\stackrel{(c)}{=} I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \\ &= R_1 + R_2, \end{aligned}$$

Zhang-Berger Region

Theorem (\mathcal{R}_{ZB} Zhang and Berger 1987)

A rate pair (R_1, R_2) is achievable for multiple descriptions for distortion triple (D_0, D_1, D_2) if

$$R_1 \geq I(X; U_0) + I(X; U_1|U_0),$$

$$R_2 \geq I(X; U_0) + I(X; U_2|U_0),$$

$$R_1 + R_2 \geq 2I(X; U_0) + I(X; U_1, U_2|U_0) + I(U_1; U_2|U_0);$$

for some $p(u_0, u_1, u_2|x)$ and deterministic functions $\phi_1, \phi_2, \phi_{12}$ such that

$$E(d_1(X, \phi_1(U_0, U_1))) \leq D_1$$

$$E(d_2(X, \phi_2(U_0, U_2))) \leq D_2$$

$$E(d_0(X, \phi_{12}(U_0, U_1, U_2))) \leq D_0$$

$\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*} \subseteq \mathcal{R}_{\text{ZB}}$ \mathcal{R}_{ZB} is convex.

[Zhang and Berger 1987]

$$R_1 \geq I(X; U_0, U_1),$$

$$R_2 \geq I(X; U_0, U_2),$$

$$\begin{aligned} R_1 + R_2 &\geq I(X; U_0, U_1, U_2) \\ &\quad + I(U_0; X) \\ &\quad + I(U_1; U_2 | U_0); \end{aligned}$$

$$p(u_0, u_1, u_2 | x), \phi_1, \phi_2, \phi_{12}$$

$$E(d_1(X, \phi_1(U_0, U_1))) \leq D_1$$

$$E(d_2(X, \phi_2(U_0, U_2))) \leq D_2$$

$$E(d_0(X, \phi_{12}(U_0, U_1, U_2))) \leq D_0$$

[Zhang and Berger 1987]

$$R_1 \geq I(X; U_0, U_1),$$

$$R_2 \geq I(X; U_0, U_2),$$

$$\begin{aligned} R_1 + R_2 &\geq I(X; U_0, U_1, U_2) \\ &\quad + I(U_0; X) \\ &\quad + I(U_1; U_2 | U_0); \end{aligned}$$

$$p(u_0, u_1, u_2 | x), \phi_1, \phi_2, \phi_{12}$$

$$E(d_1(X, \phi_1(U_0, U_1))) \leq D_1$$

$$E(d_2(X, \phi_2(U_0, U_2))) \leq D_2$$

$$E(d_0(X, \phi_{12}(U_0, U_1, U_2))) \leq D_0$$

[Venkataramani et al. 2003]

$$R_1 \geq I(X; \hat{X}_1, U),$$

$$R_2 \geq I(X; \hat{X}_2, U),$$

$$\begin{aligned} R_1 + R_2 &\geq I(X; U, \hat{X}_1, \hat{X}_2, \hat{X}_0) \\ &\quad + I(U; X) \\ &\quad + I(\hat{X}_1; \hat{X}_2 | U); \end{aligned}$$

$$p(u, \hat{x}_1, \hat{x}_2, \hat{x}_0 | x)$$

$$E(d_0(X, \hat{X}_0)) \leq D_0,$$

$$E(d_1(X, \hat{X}_1)) \leq D_1,$$

$$E(d_2(X, \hat{X}_2)) \leq D_2.$$

$p(u, \hat{x}_0, \hat{x}_1, \hat{x}_2, x)$ from the VKG region

$(U, \hat{X}_0, \hat{X}_1, \hat{X}_2, X, W)$ such that

- $(U, \hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(u, \hat{x}_0, \hat{x}_1, \hat{x}_2, x)$
- $W \perp (U, \hat{X}_1, \hat{X}_2)$
- $X - (U, \hat{X}_0, \hat{X}_1, \hat{X}_2) - W$
- $\hat{X}_0 = g(W, U, \hat{X}_1, \hat{X}_2)$ for some deterministic function g
- $E(d_j(X, \hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$.

Set $U_0 = U$, $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$ in the ZB region.

Conclusions

- $\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$
- $\mathcal{R}_{\text{EGC}} \subset \mathcal{R}_{\text{ZB}}$
- $\mathcal{R}_{\text{ZB}} = \mathcal{R}_{\text{VKG}}$

Special thanks to Abbas El Gamal for introducing the problems to the authors.

Thanks!