Consolidating Achievable Regions of Multiple Descriptions

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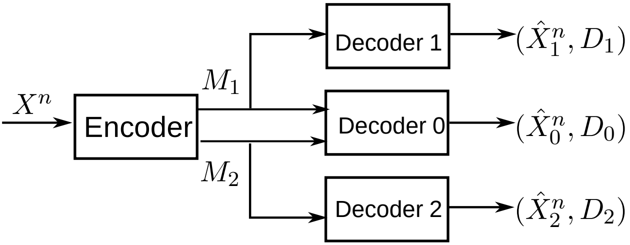
Outline:

- Multiple Description Problem
- El Gamal-Cover (EGC) Region and EGC^*
- Zhang-Berger (ZB) Region
- Venkataramani-Kramer-Goyal (VKG) Region
- Conclusions.

Multiple Description Problem



- Source: $X_1, X_2, \ldots$ i.i.d $\sim p(x)$
- Distortion Measures: $d_j(x, \hat{x}_j)$, $\hat{x}_j \in \hat{\mathcal{X}}_j$ for $j = 0, 1, 2$,

$$d_j(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d_j(x_i, \hat{x}_i), \quad j = 0, 1, 2$$

- A $(2^{nR_1}, 2^{nR_2}, n)$ code: $M_1 \in [1 : 2^{nR_1}], \quad M_2 \in [1 : 2^{nR_2}]$

Multiple Description Problem (cont.)

- A rate pair (R_1, R_2) is said to be achievable for distortion triple (D_1, D_2, D_0) if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes with average distortion

$$\limsup_{n\to\infty} E\left(d_j(X^n, \hat{X}_j^n)\right) \leq D_j \text{ for } j = 0, 1, 2$$

- The rate distortion region $\mathcal{R}(D_1, D_2, D_0)$ is the closure of the set of achievable rate pairs (R_1, R_2) for distortion triple (D_1, D_2, D_0).
- The problem in general is open.

El Gamal-Cover Region

Theorem ($\mathcal{R}_{\text{EGC}^*}$ El Gamal and Cover 1979)

A rate pair (R_1, R_2) is achievable for multiple description for distortion triple (D_0, D_1, D_2) if

$$\begin{aligned} R_1 &\geq I(X; U_1), \\ R_2 &\geq I(X; U_2), \\ R_1 + R_2 &\geq I(X; U_1, U_2) + I(U_1; U_2); \end{aligned}$$

for some $p(u_1, u_2|x)$ and deterministic functions $\phi_1, \phi_2, \phi_{12}$ such that

$$\begin{aligned} E(d_1(X, \phi_1(U_1)) &\leq D_1 \\ E(d_2(X, \phi_2(U_2)) &\leq D_2 \\ E(d_0(X, \phi_{12}(U_1, U_2)) &\leq D_0. \end{aligned}$$

El Gamal-Cover Region (cont.)

Theorem ($\mathcal{R}_{\text{EGC}}$ El Gamal and Cover 1982)

A rate pair (R_1, R_2) is achievable for multiple descriptions for distortion triple (D_0, D_1, D_2) if

$$\begin{aligned} R_1 &\geq I(X;\hat{X}_1), \\ R_2 &\geq I(X;\hat{X}_2), \\ R_1 + R_2 &\geq I(X;\hat{X}_0|\hat{X}_1,\hat{X}_2) + I(X;\hat{X}_1,\hat{X}_2) + I(\hat{X}_1;\hat{X}_2); \end{aligned}$$

for some $p(\hat{x}_0, \hat{x}_1, \hat{x}_2|x)$ such that

$$E(d_j(X,\hat{X}_j)) \leq D_j, \text{ for } j = 0,1,2$$

- $\hat{X}_1 = \phi_1(U_1), \hat{X}_2 = \phi_2(U_2), \hat{X}_0 = \phi_{12}(U_1, U_2)$,
  $\Longrightarrow \mathcal{R}_{\text{EGC}^*} \subseteq \mathcal{R}_{\text{EGC}}$
- Our goal: $\mathcal{R}_{\text{EGC}^*} = \mathcal{R}_{\text{EGC}}$

El Gamal-Cover Region (cont.)

Lemma

For a given distribution $p(x, y)$*, there exist random variables* Y *and* W*, and a deterministic function* g *such that* $Y \sim p(y)$*,* $W \perp Y$ *and* $(g(W, Y), Y) \sim p(x, y)$*. Furthermore, the cardinality of* W *need not be larger than* $(|\mathcal{X}| - 1)|\mathcal{Y}| + 1$*.*

El Gamal-Cover Region (cont.)

Lemma

For a given distribution $p(x, y)$, there exist random variables Y and W, and a deterministic function g such that $Y \sim p(y)$, $W \perp Y$ and $(g(W, Y), Y) \sim p(x, y)$. Furthermore, the cardinality of W need not be larger than $(|\mathcal{X}| - 1)|\mathcal{Y}| + 1$.

Proof.

$W \sim \mathrm{Uniform}[0, 1]$, $Y \sim p(y)$ and $Y \perp W$.
$g(w, y) = F^{-1}_{X|Y=y}(w)$, where
$F^{-1}_{X|Y=y}(w) = \inf_{x \in \mathbb{R}}\{F_{X|Y=y}(x) \geq w\}$.
$(g(W, Y), Y) \sim p(x, y)$

$X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $F_{X|Y=(\cdot)}(\cdot)$ has at most $(|\mathcal{X}| - 1)|\mathcal{Y}| + 1$ distinguished values.

□

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$

EGC*

$$\begin{aligned} R_1 &\geq I(X;U_1), \\ R_2 &\geq I(X;U_2), \\ R_1 + R_2 &\geq I(X;U_1,U_2) \\ &\quad + I(U_1;U_2); \end{aligned}$$

$p(u_1, u_2|x)$, $\phi_1, \phi_2, \phi_{12}$

$$\begin{aligned} &E(d_1(X, \phi_1(U_1)) \leq D_1 \\ &E(d_2(X, \phi_2(U_2)) \leq D_2 \\ &E(d_0(X, \phi_{12}(U_1, U_2)) \leq D_0. \end{aligned}$$

$\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$

EGC*

$$\begin{aligned} R_1 &\geq I(X;U_1), \\ R_2 &\geq I(X;U_2), \\ R_1 + R_2 &\geq I(X;U_1,U_2) \\ &\quad + I(U_1;U_2); \end{aligned}$$

$p(u_1, u_2|x)$, $\phi_1, \phi_2, \phi_{12}$

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EGC

$$\begin{aligned} R_1 &\geq I(X;\hat{X}_1), \\ R_2 &\geq I(X;\hat{X}_2), \\ R_1 + R_2 &\geq I(X;\hat{X}_0,\hat{X}_1,\hat{X}_2) \\ &\quad + I(\hat{X}_1;\hat{X}_2); \end{aligned}$$

$p(\hat{x}_0, \hat{x}_1, \hat{x}_2|x)$

$E(d_j(X,\hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
  $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
  $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
- There exists $(W, \hat{X}_1, \hat{X}_2)$, g such that
  $W \perp (\hat{X}_1, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2, g(W, \hat{X}_1, \hat{X}_2)) \sim p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$
  $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
  $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
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  $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $(W, \hat{X}_1, \hat{X}_2, \hat{X}_0)$ induces $p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

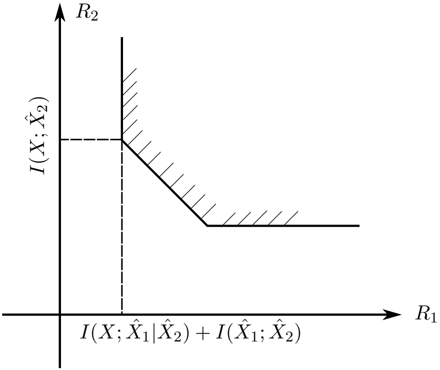
- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
  $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
- There exists $(W, \hat{X}_1, \hat{X}_2)$, g such that
  $W \perp (\hat{X}_1, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2, g(W, \hat{X}_1, \hat{X}_2)) \sim p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$
  $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $(W, \hat{X}_1, \hat{X}_2, \hat{X}_0)$ induces $p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$
- $p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$

$\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$ (cont.)

- Fix $p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x)$ in the EGC region.
  $p(x)p(\hat{x}_1, \hat{x}_2, \hat{x}_0|x) = p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$.
- There exists $(W, \hat{X}_1, \hat{X}_2)$, g such that
  $W \perp (\hat{X}_1, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2, g(W, \hat{X}_1, \hat{X}_2)) \sim p(\hat{x}_1, \hat{x}_2, \hat{x}_0)$
  $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $(W, \hat{X}_1, \hat{X}_2, \hat{X}_0)$ induces $p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$
- $p(x|\hat{x}_1, \hat{x}_2, \hat{x}_0)p(\hat{x}_1, \hat{x}_2, \hat{x}_0)p(w|\hat{x}_1, \hat{x}_2, \hat{x}_0)$
- $(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$
  - $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x)$,
  - $W \perp (\hat{X}_1, \hat{X}_2)$,
  - $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W$,
  - $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$ for some deterministic function g.

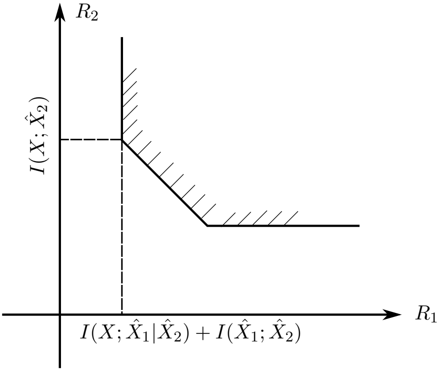
$\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$ (cont.)

EGC



$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

EGC

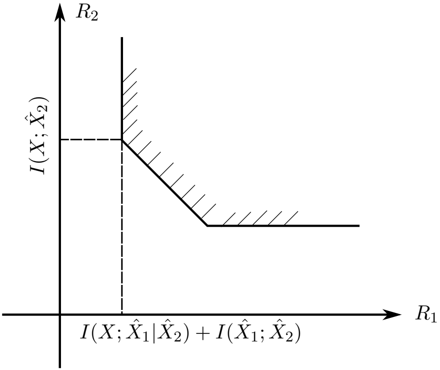


EGC*

- Set $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

EGC



EGC^*

- Set $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.
- $\phi_1(U_1) = \hat{X}_1$, $\phi_2(U_2) = \hat{X}_2$,
  $\phi_{12}(U_1, U_2) = \hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
  $\Longrightarrow (X, U_1, U_2)$ valid distribution in EGC^*

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

$U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.

$(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$

- $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x)$,
- $W \perp (\hat{X}_1, \hat{X}_2)$,
- $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W$,
- $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $E(d_j(X, \hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$,

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

$U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.

$(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$

- $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x)$,
- $W \perp (\hat{X}_1, \hat{X}_2)$,
- $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W$,
- $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $E(d_j(X, \hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$,

Corner point (R_1', R_2') in EGC^*:

$$\begin{aligned} R_2' &= I(X; U_2) \\ &= I(X; \hat{X}_2) \\ &= R_2 \end{aligned}$$

$\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$ (cont.)

$U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$.

$(X, \hat{X}_1, \hat{X}_2, \hat{X}_0, W)$

- $(\hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(\hat{x}_0, \hat{x}_1, \hat{x}_2, x)$,
- $W \perp (\hat{X}_1, \hat{X}_2)$,
- $X - (\hat{X}_0, \hat{X}_1, \hat{X}_2) - W$,
- $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$
- $E(d_j(X, \hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$,

Corner point (R_1', R_2') in EGC^*:

$$\begin{aligned} R_2' &= I(X; U_2) \\ &= I(X; \hat{X}_2) \\ &= R_2 \end{aligned}$$

$$\begin{aligned} & R_1' + R_2' \\ =& I(X; U_1, U_2) + I(U_1; U_2) \\ =& I(X; W, \hat{X}_1, \hat{X}_2) + I(\hat{X}_1, W; \hat{X}_2) \\ \overset{(a)}{=}& I(X; W, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1, W; \hat{X}_2) \\ \overset{(b)}{=}& I(X; W, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \\ =& I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(X; W | \hat{X}_1, \hat{X}_2, \hat{X}_0) \\ & \quad + I(\hat{X}_1; \hat{X}_2) \\ \overset{(c)}{=}& I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \\ =& R_1 + R_2, \end{aligned}$$

Zhang-Berger Region

Theorem ($\mathcal{R}_{\mathrm{ZB}}$ Zhang and Berger 1987)

A rate pair (R_1, R_2) *is achievable for multiple descriptions for distortion triple* (D_0, D_1, D_2) *if*

$$\begin{aligned} R_1 &\geq I(X;U_0) + I(X;U_1|U_0), \\ R_2 &\geq I(X;U_0) + I(X;U_2|U_0), \\ R_1 + R_2 &\geq 2I(X;U_0) + I(X;U_1,U_2|U_0) + I(U_1;U_2|U_0); \end{aligned}$$

for some $p(u_0, u_1, u_2|x)$ *and deterministic functions* $\phi_1, \phi_2, \phi_{12}$ *such that*

$$\begin{aligned} E(d_1(X, \phi_1(U_0, U_1)) &\leq D_1 \\ E(d_2(X, \phi_2(U_0, U_2)) &\leq D_2 \\ E(d_0(X, \phi_{12}(U_0, U_1, U_2)) &\leq D_0 \end{aligned}$$

$\mathcal{R}_{EGC} = \mathcal{R}_{EGC^*} \subseteq \mathcal{R}_{\mathrm{ZB}}$ $\quad$ $\mathcal{R}_{\mathrm{ZB}}$ is convex.

$\mathcal{R}_{\text{ZB}} = \mathcal{R}_{\text{VKG}}$

[Zhang and Berger 1987]

$$\begin{aligned} R_1 \geq & I(X; U_0, U_1), \\ R_2 \geq & I(X; U_0, U_2), \\ R_1 + R_2 \geq & I(X; U_0, U_1, U_2) \\ & + I(U_0; X) \\ & \quad + I(U_1; U_2 | U_0); \end{aligned}$$

$p(u_0, u_1, u_2 | x)$, $\phi_1, \phi_2, \phi_{12}$

$$E(d_1(X, \phi_1(U_0, U_1)) \leq D_1$$
$$E(d_2(X, \phi_2(U_0, U_2)) \leq D_2$$
$$E(d_0(X, \phi_{12}(U_0, U_1, U_2)) \leq D_0$$

$\mathcal{R}_{\text{ZB}} = \mathcal{R}_{\text{VKG}}$

[Zhang and Berger 1987]

$$\begin{aligned} R_1 &\geq I(X; U_0, U_1), \\ R_2 &\geq I(X; U_0, U_2), \\ R_1 + R_2 &\geq I(X; U_0, U_1, U_2) \\ &\quad + I(U_0; X) \\ &\qquad + I(U_1; U_2 | U_0); \end{aligned}$$

$p(u_0, u_1, u_2|x)$, $\phi_1, \phi_2, \phi_{12}$

$$E(d_1(X, \phi_1(U_0, U_1)) \leq D_1$$
$$E(d_2(X, \phi_2(U_0, U_2)) \leq D_2$$
$$E(d_0(X, \phi_{12}(U_0, U_1, U_2)) \leq D_0$$

[Venkataramani et al. 2003]

$$\begin{aligned} R_1 &\geq I(X; \hat{X}_1, U), \\ R_2 &\geq I(X; \hat{X}_2, U), \\ R_1 + R_2 &\geq I(X; U, \hat{X}_1, \hat{X}_2, \hat{X}_0) \\ &\quad + I(U; X) \\ &\quad + I(\hat{X}_1; \hat{X}_2 | U); \end{aligned}$$

$p(u, \hat{x}_1, \hat{x}_2, \hat{x}_0|x)$

$$E(d_0(X, \hat{X}_0)) \leq D_0,$$
$$E(d_1(X, \hat{X}_1)) \leq D_1,$$
$$E(d_2(X, \hat{X}_2)) \leq D_2.$$

$\mathcal{R}_{\mathrm{ZB}} = \mathcal{R}_{\mathrm{VKG}}$ (cont.)

$p(u, \hat{x}_0, \hat{x}_1, \hat{x}_2, x)$ from the VKG region

$(U, \hat{X}_0, \hat{X}_1, \hat{X}_2, X, W)$ such that

- $(U, \hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(u, \hat{x}_0, \hat{x}_1, \hat{x}_2, x)$
- $W \perp (U, \hat{X}_1, \hat{X}_2)$
- $X - (U, \hat{X}_0, \hat{X}_1, \hat{X}_2) - W$
- $\hat{X}_0 = g(W, U, \hat{X}_1, \hat{X}_2)$ for some deterministic function g
- $E(d_j(X, \hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$.

Set $U_0 = U$, $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$ in the ZB region.

Conclusions

- $\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$
- $\mathcal{R}_{\mathrm{EGC}} \subset \mathcal{R}_{\mathrm{ZB}}$
- $\mathcal{R}_{\mathrm{ZB}} = \mathcal{R}_{\mathrm{VKG}}$

Special thanks to Abbas El Gamal for introducing the problems to the authors.

Thanks!