On the Compound Finite State Channel with Feedback

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The compound channel

- A *family* of channels
- Channel $\theta \in \Theta$ is in use over all transmissions
- Transmitter and receiver know Θ but not which θ is in use
- Example: a family of discrete memoryless channels

$$\{P(y|x,\theta), x \in \mathcal{X}, y \in \mathcal{Y}, \theta \in \Theta\}$$

The memoryless compound channel

Capacity given by (Blackwell, Breiman, Thomasian, 1959; Wolfowitz, 1959)

$$\max_{Q(x)} \inf_{\theta} \mathcal{I}(Q(x); P(y|x,\theta))$$

where

$$\begin{aligned} \mathcal{I}(Q(x); P(y|x,\theta)) &= \sum_{x,y} Q(x) P(y|x,\theta) \log \frac{P(y|x,\theta)}{\sum_{x'} Q(x') P(y|x',\theta)} \\ &= I(X; Y|\theta) \end{aligned}$$

The memoryless compound channel

- If the transmitter knows the channel θ in use then capacity is (Wolfowitz, 1964)

$$\inf_{\theta} \max_{Q(x)} \mathcal{I}(Q(x); P(y|x,\theta)) = \inf_{\theta} C_{\theta}$$

  where C_θ is the capacity of the memoryless channel indexed by θ.
- If Θ is a finite set, then by use of a training sequence and feedback, the transmitter can estimate θ.
- What about compound channels with memory?

The compound finite state channel

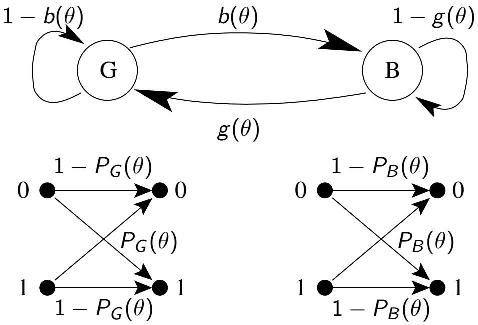
- A family of finite state channels where

$$P(y_i, s_i | x_i, s_{i-1}, \theta) = P(y_i, s_i | x^i, s^{i-1}, y^{i-1}, \theta), \quad s_i \in \mathcal{S}$$

- **Definition:** rate R is *achievable* if for a given error P_e there exists a blocklength-n rate-R code with average probability of error less than P_e for all $s_0 \in \mathcal{S}$ and $\theta \in \Theta$.
- The capacity of the compound finite state channel is given by (Lapidoth & Telatar, 1998)

$$\lim_{n\to\infty} \max_{Q(x^n)} \inf_{s_o,\theta} \frac{1}{n} \mathcal{I}(Q(x^n); P(y^n | x^n, s_0, \theta))$$

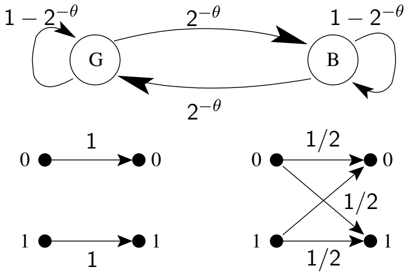
The compound Gilbert-Elliot channel
A compound finite state channel



- Used to model wireless fading channels

An example: compound Gilbert-Elliot channel

Lapidoth, Telatar, 1998



$\Theta = \{1, 2, 3, \ldots\}$

- **Compound feedback capacity = 0**: the channel $\theta = n$ will remain in the bad state with probability $(1-2^{-n})^n \geq 1/2$
- $\mathbf{C}_\theta \geq \mathbf{1} - \mathbf{h_b}(\mathbf{1}/\mathbf{4})$: use interleaver to make the channel look like BSC(1/4)

Feedback capacity of compound FSC $\neq \inf_\theta C_\theta$

The finite state channel with feedback

For time-invariant, deterministic feedback $z_i = f(y_i)$ (Permuter, Weissman, Goldsmith, 2006)

$$\lim_{n\to\infty} \max_{Q(x^n||z^{n-1})} \min_{s_0} \frac{1}{n}\mathcal{I}(Q(x^n||z^{n-1}); P(y^n||x^n, s_0)),$$

is achievable where

$$\begin{aligned} Q(x^n||z^{n-1}) &\triangleq \prod_{i=1}^{n} Q(x_i|x^{i-1}, z^{i-1}) \\ P(y^n||x^n, s_0) &\triangleq \prod_{i=1}^{n} P(y_i|y^{i-1}, x^i, s_0) \end{aligned}$$

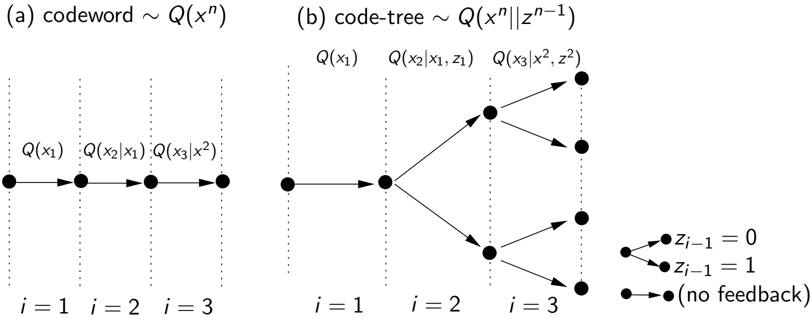
and the *directed information* is (Massey, 1990)

$$\begin{aligned} \mathcal{I}(Q(x^n||z^{n-1}); P(y^n||x^n, s_0)) &= I(X^n \to Y^n|s_0) \\ &= \sum_{i=1}^{n} I(Y_i; X^i|Y^{i-1}, s_0) \end{aligned}$$

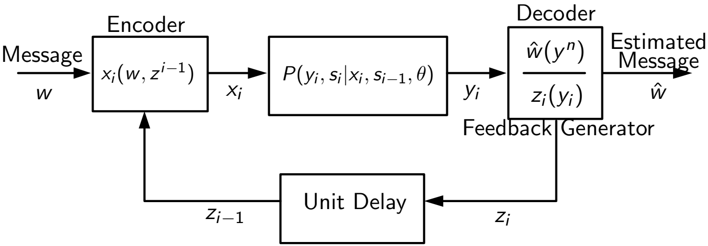
Causal conditioning distribution

$$Q(x^n||z^{n-1}) \triangleq \prod_{i=1}^{n} Q(x_i|x^{i-1}, z^{i-1})$$

An example for binary feedback



Our problem: the compound finite state channel with feedback



- Fixed blocklength n
- Need error probability $< P_e$ *for all* $s_0 \in \mathcal{S}$ and $\theta \in \Theta$.
- Includes: no feedback, quantized feedback, perfect feedback

Converse

Theorem

The feedback capacity of the compound finite state channel is upper bounded by

$$
\begin{aligned}
C &\triangleq \lim_{n\to\infty} C_n \\
C_n &= \max_{Q(x^n||z^{n-1})} \inf_{s_0,\theta} \frac{1}{n}\mathcal{I}(Q(x^n||z^{n-1}); P(y^n||x^n, s_0, \theta)).
\end{aligned}
$$

- Existence of the limit: $C_n - \log|\mathcal{S}|/n$ is super-additive
- Proof of theorem: Fano's inequality, properties of directed information
- For a memoryless channel, C reduces to Wolfowitz's result

Achievability

- For $|\Theta| < \infty$, as a consequence of (Permuter, Weissman, Goldsmith, 2006)

  $$\lim_{n\to\infty} \max_{Q(x^n||z^{n-1})} \min_{s_0,\theta} \frac{1}{n}\mathcal{I}(Q(x^n||z^{n-1}); P(y^n||x^n, s_0, \theta))$$

  is achievable.
- For $|\Theta| = \infty$, need a universal decoder, follow approach of Feder & Lapidoth, 1998
  - approximate Θ by finitely-many channels
  - *merge* the ML decoders tuned to those channels

Feedback capacity vs. capacity without feedback

- For memoryless compound channel (Wolfowitz, 1964)

$$C = 0 \iff C_{FB} = 0$$

- Using our upper bound C, the same holds for a stationary, ergodic Markovian channel

$$P(y_i, s_i | x_i, s_{i-1}, \theta) = P(s_i | s_{i-1}, \theta) p(y_i | x_i, s_{i-1}, \theta)$$

Compound Gilbert-Elliot channel

- Equivalent to additive noise channel: $Y_i = X_i + V_i$, $V_i \in \{0, 1\}$
- Feedback does not increase the capacity of the compound Gilbert-Elliot channel.
  - Maximizing input distribution: memoryless uniform Bernoulli process (for any s_i, θ)

Summary

- Capacity for compound channel with memory and feedback
  - Feedback capacity is positive iff capacity without feedback is positive
  - Feedback does not increase the capacity of the compound Gilbert-Elliot channel

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- Thanks: A. Ephremides, T. Weissman, P. Narayan, A. Goldsmith