CENTRAL LIMIT THEOREM FOR COMMUTATIVE SEMIGROUPS OF TORAL ENDOMORPHISMS

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Abstract. Let $\mathcal{S}$ be an abelian finitely generated semigroup of endomorphisms of a probability space $(\Omega, \mathcal{A}, \mu)$, with $(T_1, ..., T_d)$ a system of generators in $\mathcal{S}$. Given an increasing sequence of domains $(D_n) \subset \mathbb{N}^d$, a question is the convergence in distribution of the normalized sequence $|D_n|^{-\frac{1}{2}} \sum_{\underline{k} \in D_n} f \circ T^{\underline{k}}$, or normalized sequences of iterates of barycenters $Pf = \sum_j p_j f \circ T_j$, where $T^{\underline{k}} = T_1^{k_1} ... T_d^{k_d}$, $\underline{k} = (k_1, ..., k_d) \in \mathbb{N}^d$.

After a preliminary spectral study when the action of $\mathcal{S}$ has a Lebesgue spectrum, we consider totally ergodic d-dimensional actions given by commuting endomorphisms on a compact abelian connected group G and we show a CLT, when f is regular on G. When G is the torus, a criterion of non-degeneracy of the variance is given.

Contents

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Introduction

Let $\mathcal{S}$ be an abelian finitely generated semigroup of endomorphisms of a probability space $(\Omega, \mathcal{A}, \mu)$. Each $T \in \mathcal{S}$ is a measurable map from Ω to Ω preserving the probability measure μ. For $f \in L^1(\mu)$ a random field is defined by $(f(T.)_{T\in\mathcal{S}})$ for which limit theorems can be investigated: law of large numbers, behavior in distribution.

By choosing a system $(T_1, ..., T_d)$ of generators in $\mathcal{S}$, every $T \in \mathcal{S}$ can be represented[1] as $T = T^{\underline{k}} = T_1^{k_1}...T_d^{k_d}$, for $\underline{k} = (k_1, ..., k_d) \in \mathbb{N}^d$. Given an increasing sequence of domains $(D_n) \subset \mathbb{N}^d$, a question is the asymptotic normality of the standard normalized sequence and the "multidimensional periodogram" respectively defined by

$$(1) \qquad |D_n|^{-\frac{1}{2}} \sum_{\underline{k} \in D_n} T^{\underline{k}} f, \quad |D_n|^{-\frac{1}{2}} \sum_{\underline{k} \in D_n} e^{2\pi i \langle \underline{k}, \theta\rangle} T^{\underline{k}} f, \ f \in L^2_0(\mu), \ \theta \in \mathbb{R}^d.$$

Let us take for $(\Omega, \mathcal{A}, \mu)$ a compact abelian group G endowed with its Borel σ-algebra $\mathcal{A}$ and its Haar measure μ. In this framework, the first examples of dynamical systems satisfying a CLT in a class of regular functions are due to R. Fortet and M. Kac for endomorphisms of $\mathbb{T}^1$. In 1960 V. Leonov ([17]) showed that, if T is an ergodic endomorphism of G, then the CLT is satisfied for regular functions f on G.

The d-dimensional extension of this situation leads to the question of validity of a CLT for algebraic actions on an abelian compact group G, i.e., when $T^{\underline{k}}$ in Formula (1) is given by an action of $\mathbb{N}^d$ on G by automorphisms or more generally endomorphisms.

By composition, one obtains an action by isometries on $\mathcal{H} = L^2_0(\mu)$, the space of square integrable functions f such that $\mu(f) = 0$. The spectral analysis of this action is the content of Section 1 where the methods of summation are also discussed.

In Section 2 we consider d-dimensional actions given by commuting endomorphisms on a connected abelian compact group G. For a regular function f on G, a CLT is shown for the above normalized sequence (Theorem 2.18) and other summation methods like barycenters (Theorem 2.21), as well as a criterion of non-degeneracy of the variance when G is a torus. The barycenters yield a class of operators with a polynomial decay to zero of the iterates applied to regular functions. This contrasts with the spectral gap property for non amenable group actions by automorphisms on tori.

When (D_n) is a sequence of d-dimensional cubes, for the periodogram in (1), given a function f in $L^2_0(G)$, the CLT is obtained for almost every θ, without regularity requirement.

When G is a torus, using the exponential decay of correlation, the CLT can be shown for a class of functions with weak regularity and one can characterize the case of degeneracy in the limit theorem. In an appendix, classical results on the construction of $\mathbb{Z}^d$-actions by automorphisms are recalled.

[1] We underline the elements of $\mathbb{N}^d$ or $\mathbb{Z}^d$ to distinguish them from the scalars and write $T^{\underline{k}} f$ for $f \circ T^{\underline{k}}$.

One of our aims was to extend to a larger class of semigroups of actions by endomorphisms the CLT proved by T. Fukuyama and B. Petit ([10]) for semigroups generated by coprime integers on the circle. Their result corresponds, in our framework, to sums taken on an increasing sequence of triangles in $\mathbb{N}^2$.

After completion of a first version of this paper, we were informed by B. Weiss of the recent paper by M. Levine ([19]) in which the CLT and a functional version of it are obtained for actions by endomorphisms on the torus. The proof of the CLT for sums on d-dimensional "rectangles" is based in both approaches, as well as in [10], on results on S-units. In the present paper we use the formalism of cumulants and the result of Schmidt and Ward ([24]) on mixing of all orders for connected groups deduced from a deep result on S-units. We make use of the spectral measure which is well adapted to a "quenched" CLT and a CLT along different types of summation sequences, in particular the iterates of barycenters. The connectedness of the group G is assumed only for the CLT.

1. **Spectral analysis**

In this section we consider the general framework of the action of an abelian finitely generated semigroup $\mathcal{S}$ of isometries on a Hilbert space $\mathcal{H}$. We have in mind the example of a semigroup $\mathcal{S}$ of endomorphisms of a compact abelian group G acting on $\mathcal{H} = L^2_0(G, \mu)$, with μ the Haar measure of G.

With the notations of the introduction, every $T \in \mathcal{S}$ is represented as $T = T^{\underline{\ell}} = T_1^{\ell_1}...T_d^{\ell_d}$, where $(T_1, ..., T_d)$ is a system of generators in $\mathcal{S}$ and $\underline{\ell} = (\ell_1, ..., \ell_d) \in \mathbb{N}^d$.

Given $f \in \mathcal{H}$, for $d > 1$, there are various choices of the sets of summation D_n for the field $(T^{\underline{\ell}} f, \underline{\ell} \in \mathbb{N}^d)$. We discuss this point, as well as the behavior of the associated (by discrete Fourier transform) kernels. The second subsection is devoted to the spectral analysis of the d-dimensional action.

1.1. **Summation and kernels, barycenters.**

If $(D_n)_{n\geq 1}$ is a sequence of subsets of $\mathbb{N}^d$, the corresponding rotated sum and kernel are respectively: $\sum_{\underline{\ell}\in D_n} \mathrm{e}^{2\pi i\langle \underline{\ell},\, \theta\rangle} T^{\underline{\ell}} f$ and $\frac{1}{|D_n|} \left|\sum_{\underline{\ell}\in D_n} \mathrm{e}^{2\pi i\langle \underline{\ell},\, t\rangle}\right|^2$. The simplest choice for (D_n) is an increasing family of d-dimensional squares or rectangles.

Notation 1.1. More generally, we will call *summation sequence* a uniformly bounded sequence (R_n) of functions from $\mathbb{N}^d$ to $\mathbb{R}^+$. It could be also defined on $\mathbb{Z}^d$, but for simplicity in this section we consider summation for $\underline{\ell} \in \mathbb{N}^d$. If $T = (T^{\underline{\ell}})_{\underline{\ell}\in\mathbb{N}^d}$ is a semigroup of isometries, an associated sequence of operators on $\mathcal{H}$ can be defined by

$$R_n(T) : f \in \mathcal{H} \to R_n(T)f := \sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell}) T^{\underline{\ell}} f.$$

We will write simply R_n instead of $R_n(T)$. By introducing a rotation term, these operators extend to a family of operators R_n^θ, for $\theta \in \mathbb{R}^d$,

$$f \to R_n^\theta f := \sum_{\underline{\ell} \in \mathbb{N}^d} R_n(\underline{\ell})\, e^{2\pi i \langle \underline{\ell}, \theta \rangle}\, T^{\underline{\ell}} f.$$

We have $\| \sum_{\underline{\ell} \in \mathbb{N}^d} R_n(\underline{\ell}) e^{2\pi i \langle \underline{\ell}, . \rangle} \|^2_{L^2(\mathbb{T}^d, dt)} = \sum_{\underline{\ell} \in \mathbb{N}^d} |R_n(\underline{\ell})|^2$. Taking the discrete Fourier transform, we associate to R_n the normalized "kernel" $\tilde{R}_n$ defined on $\mathbb{T}^d$ by:

$$\tilde{R}_n(t) = \frac{|\sum_{\underline{\ell} \in \mathbb{N}^d} R_n(\underline{\ell}) e^{2\pi i \langle \underline{\ell}, t \rangle}|^2}{\sum_{\underline{\ell} \in \mathbb{N}^d} |R_n(\underline{\ell})|^2}.$$

Definition 1. We say that (R_n) is *regular* if $(\tilde{R}_n)_{n \geq 1}$ weakly converges to a measure ζ on $\mathbb{T}^d$, i.e., $\int_{\mathbb{T}^d} \tilde{R}_n\, \varphi\, dt \underset{n \to \infty}{\longrightarrow} \int_{\mathbb{T}^d} \varphi\, d\zeta$ for every continuous function φ on $\mathbb{T}^d$. If (R_n) is regular and ζ is the Dirac mass at 0, we say that (R_n) is a *Følner sequence.*

If $(R_n) = (1_{D_n})$ is associated to a sequence of sets $D_n \subset \mathbb{N}^d$, one easily proves that (R_n) is a Følner sequence if and only if (D_n) satisfies the Følner condition:

$$\lim_{n \to \infty} |D_n|^{-1} |(D_n + \underline{p}) \cap D_n| = 1, \ \forall \underline{p} \in \mathbb{Z}^d. \tag{2}$$

Examples. a) *Squares and rectangles.* Using the usual one-dimensional Fejér kernel $K_N(t) = \frac{1}{N}(\frac{\sin \pi N t}{\sin \pi t})^2$, the d-dimensional Fejér kernels on $\mathbb{T}^d$ corresponding to rectangles are defined by $K_{N_1,...,N_d}(t_1, ..., t_d) = K_{N_1}(t_1) \cdots K_{N_d}(t_d)$, $\underline{N} = (N_1, ..., N_d) \in \mathbb{N}^d$. They are the kernels associated to $D_{\underline{N}} := \{\underline{k} \in \mathbb{N}^d : \ k_i \leq N_i, 1 \leq i \leq d\}$.

b) A family of examples satisfying (2) can be obtained as follows: take a non-empty domain $D \subset \mathbb{R}^d$ with *smooth* boundary and finite area and put $D_n = \lambda_n D \cap \mathbb{Z}^d$, where (λ_n) is an increasing sequence of real numbers tending to $+\infty$.

c) *Kernels with unbounded gaps* If $(D_{\underline{n}})$ is a (non Følner) sequence of domains such that $\lim_{\underline{n}} \frac{|(D_{\underline{n}} + \underline{p}) \cap D_{\underline{n}}|}{|D_{\underline{n}}|} = 0$ for $\underline{p} \neq 0$, then $\lim_{\underline{n}} (\tilde{R}_{\underline{n}} * \varphi)(\theta) = \int_{\mathbb{T}^d} \varphi(t) dt$, for every $\theta \in \mathbb{T}^d$ and φ continuous, where $(\tilde{R}_{\underline{n}})$ is the kernel associated to (D_n).

For example, let k_j be a sequence with $k_{j+1} - k_j \to \infty$ and put $D_n = \{k_j : \ 0 \leq j \leq n-1\}$. For $p \neq 0$ the number of solutions of $k_j - k_\ell = p$, for $j, \ell \geq 0$ is finite, so that $\lim_{n \to \infty} \frac{|(D_n + p) \cap D_n|}{|D_n|} = 0$ for $p \neq 0$.

d) *Iteration of barycenter operators* Let $T_1, ..., T_d$ be d commuting unitary operators on a Hilbert space $\mathcal{H}$. If $(p_1, ..., p_d)$ is a probability vector such that $p_j > 0, \forall j$, for $\theta = (\theta_1, \theta_2, ..., \theta_d) \in \mathbb{T}^d$, we will consider the barycenter operators defined on $\mathcal{H}$ by

$$P : f \to \sum_{j=1}^d p_j\, T_j f, \quad P_\theta : f \to \sum_{j=1}^d p_j\, e^{2\pi i \theta_j}\, T_j f. \tag{3}$$

The iteration of P or P_θ gives a method of summation which is not of Følner type.

1.2. **Lebesgue spectrum, variance.**

Let $\mathcal{S}$ be a finitely generated *torsion free* commutative group of unitary operators on a Hilbert space $\mathcal{H}$. Let $(T_1, ..., T_d)$ be a system of independent generators in $\mathcal{S}$. Each element of $\mathcal{S}$ can be written in a unique way as $T^{\underline{\ell}} = T_1^{\ell_1}...T_d^{\ell_d}$, with $\underline{\ell} = (\ell_1, ..., \ell_d) \in \mathbb{Z}^d$, and $\underline{\ell} \to T^{\underline{\ell}}$ defines a unitary representation of $\mathbb{Z}^d$ in $\mathcal{H}$.

For every $f \in \mathcal{H}$, there is a positive finite measure ν_f on $\mathbb{T}^d$ such that, for every $\underline{\ell} \in \mathbb{Z}^d$, $\hat{\nu}_f(\underline{\ell}) = \langle T^{\ell_1}...T^{\ell_d} f, f\rangle$.

Definition 1.2. Recall that the action of $\mathcal{S}$ on $\mathcal{H}$ has a *Lebesgue spectrum*, if there exists $\mathcal{K}_0$, a closed subspace of $\mathcal{H}$, such that the subspaces $T^{\underline{\ell}}\mathcal{K}_0$ are pairwise orthogonal and span a dense subspace in $\mathcal{H}$.

The Lebesgue spectrum property implies mixing, i.e., $\lim_{\|\underline{n}\| \to \infty} |\langle T^{\underline{n}} f, g\rangle| = 0$, $\forall f, g \in \mathcal{H}$. With the Lebesgue spectrum property, for every $f \in \mathcal{H}$, the corresponding spectral measure ν_f of f on $\mathbb{T}^d$ has a density φ_f. A change of basis induces for the spectral density the composition by an automorphism acting on $\mathbb{T}^d$.

A family of examples of $\mathbb{Z}^d$-actions by unitary operators is provided by the action of a group of commuting automorphisms on a compact abelian group G. In the present paper, we will focus mainly on this class of examples.

Notation 1.3. For any orthonormal basis $(\psi_j)_{j\in J}$ of $\mathcal{K}_0$, the family $(T^{\underline{\ell}}\psi_j)_{j\in J, \underline{\ell}\in\mathbb{Z}^d}$ is an orthonormal basis of $\mathcal{H}$. Let $\mathcal{H}_j$ be the closed subspace (invariant by the $\mathbb{Z}^d$-action) generated by $(T^{\underline{n}}\psi_j)_{\underline{n}\in\mathbb{Z}^d}$.

We set $a_{j,\underline{n}} := \langle f, T^{\underline{n}}\psi_j\rangle$, $j \in J$. Let f_j be the orthogonal projection of f on $\mathcal{H}_j$ and γ_j an everywhere finite square integrable function on $\mathbb{T}^d$ with Fourier coefficients $a_{j,\underline{n}}$.

The spectral measure of f is the sum of the spectral measures of f_j. For f_j, the density of the spectral measure is $|\gamma_j|^2$. Therefore, by orthogonality of the subspaces $\mathcal{H}_j$, the density of the spectral measure of f is $\varphi_f(t) = \sum_{j\in J} |\gamma_j(t)|^2$.

We have: $\int_{\mathbb{T}^d} \sum_{j\in J} |\gamma_j(\theta)|^2 \, d\theta = \sum_{j\in J} \sum_{\underline{n}\in\mathbb{Z}^d} |a_{j,\underline{n}}|^2 = \int_{\mathbb{T}^d} \varphi_f(\theta) \, d\theta = \|f\|^2 < \infty$ and the set $\Lambda_0(f) := \{\theta \in \mathbb{T}^d : \sum_{j\in J} |\gamma_j(\theta)|^2 < \infty\}$ has full measure.

For θ in $\mathbb{T}^d$, let $M_\theta f$ in $\mathcal{K}_0$ (with orthogonal "increments") be defined by:

$$M_\theta f := \sum_j \gamma_j(\theta)\, \psi_j. \tag{4}$$

Under the condition $\sum_{j\in J} \left(\sum_n |a_{j,\underline{n}}|\right)^2 < +\infty$, $M_\theta f$ is defined for every θ, the function $\theta \to \|M_\theta\|_2^2$ is continuous and is equal everywhere to φ_f. For a general function $f \in L_0^2(G)$, it is defined for θ in a set $\Lambda_0(f)$ of full measure in $\mathbb{T}^d$.

Remark that the choice of the system (ψ_j) generating the orthonormal basis $(T^{\underline{n}}\psi_j)$ is not unique, so that the definition of $M_\theta f$ is not canonical. But for algebraic automorphisms of a compact abelian group G, Fourier analysis gives a natural choice for the basis.

Approximation by orthogonal increments

The rotated sums of $M_\theta f$ approximate the rotated sums of f in the following sense:

Lemma 1.4. *Let (R_n) be a summation sequence with associated kernel $(\tilde{R}_n)$.*
a) Let $\mathcal{L}$ be a space of functions on $\mathbb{T}^d$ with the property that if $0 \leq \varphi \in \mathcal{L}$ and $|\psi|^2 \leq \varphi$, then $\psi, |\psi|^2 \in \mathcal{L}$. Suppose that, for every $\varphi \in \mathcal{L}$, $\lim_n (\tilde{R}_n * \varphi)(t) = \varphi(t)$ for a.e. $t \in \mathbb{T}^d$. Then, if f in $L_0^2(\mu)$ is such that $\varphi_f \in \mathcal{L}$, we have for θ in a set $\Lambda(f) \subset \Lambda_0(f)$ of full Lebesgue measure in $\mathbb{T}^d$:

$$\lim_n \frac{\| \sum_{\underline{\ell} \in \mathbb{N}^d} R_n(\underline{\ell})\, e^{2\pi i \langle \underline{\ell}, \theta \rangle}\, T^{\underline{\ell}}(f - M_\theta f)\|_2^2}{\sum_{\underline{\ell} \in \mathbb{N}^d} |R_n(\underline{\ell})|^2} = 0. \tag{5}$$

b) If the functions φ_f, $\sum_j |\gamma_j|^2$, γ_j, for all j in J, are continuous and if $\varphi_f(\theta) = \sum_j |\gamma_j(\theta)|^2 \, \forall \theta$, then, for any Følner sequence (R_n), (5) holds for every θ.

Proof. The proof of a) is analogous to that of Proposition 1.4 in [4]. The projection of $f - M_\theta f$ on $\mathcal{H}_j$ is $f_j - \gamma_j(\theta)\psi_j$ and its spectral density is

$$\varphi_{f - M_\theta f}(t) = \sum_{j \in J} |\gamma_j(t) - \gamma_j(\theta)|^2 = \sum_j |\gamma_j(t)|^2 + \sum_j |\gamma_j(\theta)|^2 - \sum_j \gamma_j(t)\overline{\gamma}_j(\theta) - \sum_j \overline{\gamma}_j(t)\gamma_j(\theta).$$

We have

$$\frac{\| \sum_{\underline{\ell} \in \mathbb{N}^d} R_n(\underline{\ell})\, e^{2\pi i \langle \underline{\ell}, \theta \rangle}\, T^{\underline{\ell}}(f - M_\theta f)\|_2^2}{\sum_{\underline{\ell} \in \mathbb{N}^d} |R_n(\underline{\ell})|^2} = \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta)\ \varphi_{f - M_\theta f}(t)\ dt. \tag{6}$$

Observe that $|\gamma_j|^2 \leq \varphi_f$. Let $\Lambda_0'(f)$ be the set of full measure of θ's given by the hypothesis such that convergence holds at θ for $|\gamma_j|^2$, γ_j, $\forall j \in J$, and $\sum_{j \in J} |\gamma_j|^2$.

Take $\theta \in \Lambda_0(f) \cap \Lambda_0'(f)$. Let $\varepsilon > 0$ and let $J_0 = J_0(\varepsilon, \theta)$ be a finite subset of J such that $\sum_{j \notin J_0} |\gamma_j(\theta)|^2 < \varepsilon$. Since

$$\begin{aligned}
&\lim_n \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta) \sum_{j \notin J_0} |\gamma_j(t)|^2\ dt \\
= \;&\lim_n \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta) \sum_{j \in J} |\gamma_j(t)|^2\ dt - \lim_n \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta) \sum_{j \in J_0} |\gamma_j(t)|^2\ dt = \sum_{j \notin J_0} |\gamma_j(\theta)|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\limsup_n \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta)\ \varphi_{f - M_\theta f}(t)\ dt \\
&\leq \lim_n \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta) \sum_{j \in J_0} \left[|\gamma_j(t)|^2 + |\gamma_j(\theta)|^2 - \gamma_j(t)\overline{\gamma}_j(\theta) - \overline{\gamma}_j(t)\gamma_j(\theta)\right] dt \\
&\qquad + 2 \lim_n \int_{\mathbb{T}^d} \tilde{R}_n(t - \theta) \sum_{j \notin J_0} \left[|\gamma_j(t)|^2 + |\gamma_j(\theta)|^2\right] dt \\
&= \sum_{j \in J_0} \left[|\gamma_j(\theta)|^2 + |\gamma_j(\theta)|^2 - \gamma_j(\theta)\overline{\gamma}_j(\theta) - \overline{\gamma}_j(\theta)\gamma_j(\theta)\right] + 4 \sum_{j \notin J_0} |\gamma_j(\theta)|^2 \leq 0 + 4\varepsilon.
\end{aligned}$$

Therefore $\Lambda(f)$, the set for which (5) holds, contains $\Lambda_0(f) \cap \Lambda_0'(f)$ and has full measure.

The proof of b) uses the same expansion as in 1). $\square$

Variance for summation sequences

Let $(D_n) \subset \mathbb{N}^d$ be an increasing sequence of subsets. For $f \in L_0^2(\mu)$, the asymptotic variance at θ along (D_n) is, when it exists, the limit

$$\sigma_\theta^2(f) = \lim_n \frac{\| \sum_{\underline{\ell} \in D_n} e^{2\pi i \langle \underline{\ell}, \theta \rangle} T^{\underline{\ell}} f \|_2^2}{|D_n|}. \tag{7}$$

By the spectral theorem, if $\varphi_f \in L^1(\mathbb{T}^d)$ is the spectral density of f and $\tilde{R}_n$ the kernel associated to (D_n), then

$$|D_n|^{-1} \| \sum_{\underline{\ell} \in D_n} e^{2\pi i \langle \underline{\ell}, \theta \rangle} T^{\underline{\ell}} f \|_2^2 = (\tilde{R}_n * \varphi_f)(\theta). \tag{8}$$

If (D_n) is a sequence of d-dimensional cubes, we obtain, when it exists, the usual asymptotic variance at θ. By the Fejér-Lebesgue theorem, for of cubes, for every f in $\mathcal{H}$ it exists and is equal to $\varphi_f(\theta)$ for a.e. θ.

When φ_f is continuous, for Følner sequences, for every θ, the asymptotic variance at θ is $\varphi_f(\theta)$. More generally, if (R_n) is a regular summation sequence with $(\tilde{R}_n)$ weakly converging to the measure ζ on $\mathbb{T}^d$, the asymptotic variance at θ is $\int_{\mathbb{T}^d} \varphi_f(\theta - t)\, d\zeta(t)$.

Variance for barycenters

Let P and P_θ be defined by (3) for d commuting unitary operators $T_1, ..., T_d$ on a Hilbert space $\mathcal{H}$ generating a group $\mathcal{S}$ with the Lebesgue spectrum property and let $(p_1, ..., p_d)$ be a probability vector such that $p_j > 0, \forall j$. If φ_f is the spectral density of f in $\mathcal{H}$ with respect to the action of $\mathcal{S}$, we have:

$$\|P_\theta^n f\|_2^2 = \int_{\mathbb{T}^d} |\sum_{j=1}^d p_j \, e^{2\pi i t_j}|^{2n} \, \varphi_f(\theta_1 - t_1, ..., \theta_d - t_d)\, dt_1 \ldots dt_d.$$

In order to find the normalization of $P^n f$ for $f \in \mathcal{H}$, we need an estimation, when $n \to \infty$, of the integral $I_n := \int_{\mathbb{T}^d} |\sum_j p_j e^{2\pi i t_j}|^{2n} \, dt_1 ... dt_d$.

Proposition 1.5. *If $(p_1, ..., p_d)$ is a probability vector such that $p_j > 0, \forall j$, we have*

$$\lim_n n^{\frac{d-1}{2}} \int_{\mathbb{T}^d} |\sum_j p_j e^{2\pi i t_j}|^{2n} \, dt_1 ... dt_d = (4\pi)^{-\frac{d-1}{2}} \, (p_1 ... p_d)^{-\frac{1}{2}}. \tag{9}$$

Lemma 1.6. *Let r be an integer ≥ 1 and let $(q_1, ..., q_r)$ be a vector such that $q_j > 0, \forall j$ and $\sum_j q_j \leq 1$. Then the quadratic form Q on $\mathbb{R}^r$ defined by*

$$Q(\underline{t}) = \sum_{j=1}^r q_j t_j^2 - (\sum_{j=1}^r q_j t_j)^2 \tag{10}$$

is positive definite with determinant $(1 - \sum_j q_j)\, q_1 ... q_r$.

Proof. The proof is by induction on r. Let us consider the polynomial in t_1 of degree 2:

$$q_1 t_1^2 + \sum_2^r q_j t_j^2 - (q_1 t_1 + \sum_2^r q_j t_j)^2 = (q_1 - q_1^2) t_1^2 - 2q_1 (\sum_2^r q_j t_j) t_1 + \sum_2^r q_j t_j^2 - (\sum_2^r q_j t_j)^2.$$

It is always ≥ 0, since its discriminant

$$q_1^2 (\sum_2^r q_j t_j)^2 - q_1(1-q_1)(\sum_2^r q_j t_j^2 - (\sum_2^r q_j t_j)^2) = q_1(1-q_1)^2 [(\sum_2^r \frac{q_j}{1-q_1} t_j)^2 - \sum_2^r \frac{q_j}{1-q_1} t_j^2],$$

is < 0 for $\sum_{j=2}^r t_j^2 \neq 0$ by the induction hypothesis, since $\frac{q_j}{1-q_1} > 0$ and $\sum_{j=2}^r \frac{q_j}{1-q_1} \leq 1$.
The quadratic form is given by the symmetric matrix: $A = \operatorname{diag}(q_1, ..., q_r)\ B$, where

$$B \quad = \begin{pmatrix} 1-q_1 & -q_2 & . & -q_r \\ -q_1 & 1-q_2 & . & -q_r \\ . & . & . & . \\ -q_1 & -q_2 & . & 1-q_r \end{pmatrix}.$$

The determinant of B is of the form $\alpha + \sum_j \beta_j q_j$, where the coefficients $\alpha, \beta_1, ..., \beta_r$ are constant. Giving to $q_1, ..., q_r$ the values 0 except for one of them, we find $\alpha = 1$, $\beta_1 = \beta_2 = ... = \beta_r = -1$. Hence $\det A = (1 - \sum_j q_j)\, q_1 ... q_r$. □

Remark that the positive definiteness follows also from the properties of F, since Q gives the approximation of F defined below at order 2.

Proof of Proposition 1.5 Since $|\sum_j p_j e^{2\pi i t_j}|^{2n} = |p_1 + \sum_{j=2}^d p_j e^{2\pi i (t_j - t_1)}|^{2n}$, we have $I_n = \int_{\mathbb{T}^{d-1}} |p_1 + \sum_{j=2}^d p_j e^{2\pi i t_j}|^{2n}\ dt_2 ... dt_d$.

Putting $q_j := p_{j+1}$, $j = 1, ..., d-1$, $r = d-1$, we have $q_j > 0$, $\sum_1^r q_j < 1$. With the notation $F(\underline{t}) := 1 - |1 + \sum_{j=1}^r q_j (e^{2\pi i t_j} - 1)|^2$ the computation reduces to estimate:

$$I_n := \int_{\mathbb{T}^r} [|1 + \sum_{j=1}^r q_j (e^{2\pi i t_j} - 1)|^2]^n\ dt_1 ... dt_r = \int_{\mathbb{T}^r} [1 - F(\underline{t})]^n\ dt_1 ... dt_r.$$

A point $\underline{t} = (t_1, ... t_r)$ of the torus is represented by coordinates such that: $-\frac{1}{2} \leq t_j < \frac{1}{2}$. We have $F(\underline{t}) \geq 0$ and $F(\underline{t}) = 0$ if and only if $\underline{t} = 0$. Let us prove the stronger property: there is $c > 0$ such that

$$F(\underline{t}) \geq c \|\underline{t}\|^2, \ \forall \underline{t} : -\frac{1}{2} \leq t_j < \frac{1}{2}. \tag{11}$$

Indeed Inequality (11) is clearly satisfied outside a small open neighborhood V of 0, since $F(\underline{t})$ is bounded away from 0 for $\underline{t}$ in V. On V, we can replace F by a positive definite quadratic form as we will see below. This shows the result on V.

From the convergence

$$\lim_n n^{\frac{r}{2}} \int_{\{\underline{t}\in\mathbb{T}^r:\|\underline{t}\|>\frac{\ln n}{\sqrt{n}}\}} (1-F(\underline{t}))^n \, d\underline{t} \leq \lim_n n^{\frac{d-1}{2}} (1-c\frac{(\ln n)^2}{n})^n = 0,$$

it follows, with $J_n := \int_{\{\underline{t}\in\mathbb{T}^r:\|\underline{t}\|\leq\frac{\ln n}{\sqrt{n}}\}} (1-F(\underline{t}))^n \, d\underline{t}$:

$$\lim_n n^{\frac{r}{2}} \int_{\underline{t}\in\mathbb{T}^{d-1}} (1-F(\underline{t}))^n \, d\underline{t} = \lim_n n^{\frac{r}{2}} J_n.$$

By taking the Taylor approximation of order 2 at 0 of the exponential function $e^{it_j} = 1 + it_j - \frac{t_j^2}{2} + i\gamma_1(t_j) + \gamma_2(t_j)$, with $|\gamma_1(t_j) + |\gamma_2(t_j)| = o(|t_j|^2)$, we obtain:

$$F(t) = Q(2\pi t) + \gamma(t), \quad \text{with } Q(\underline{t}) = \sum q_j t_j^2 - (\sum q_j t_j)^2 \ \text{ and } \gamma(t) = o(\|t\|^2).$$

The quadratic form Q is the form defined by (10). Therefore, it is positive definite by Lemma 1.6 and there is $c > 0$ such that $Q(\underline{t}) \geq c\|\underline{t}\|^2, \forall \underline{t} \in \mathbb{R}^r$.

We have $\lim_{\delta\downarrow 0} \sup_{\|t\|\leq\delta} F(t)/Q(2\pi t) = 1$. With the notation $\underline{u} = (u_1, ..., u_r)$, $\underline{t} = (t_1, ..., t_r)$ and the change of variable $\underline{u} = \sqrt{n}\, \underline{t}$, we get:

$$n^{\frac{r}{2}} J_n \sim \int_{\{\|\underline{u}\|\leq \ln n\}} (1 - Q(\frac{2\pi \underline{u}}{\sqrt{n}}))^n \, d\underline{u} \to \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} e^{-Q(u)} \, d\underline{u}.$$

We have $\int_{\mathbb{R}^r} e^{-Q(u)} \, d\underline{u} = \pi^{\frac{r}{2}} \det(A)^{-\frac{1}{2}} = \pi^{\frac{r}{2}} \, (p_1...p_d)^{-\frac{1}{2}}$. Therefore we obtain:

$$\lim_n n^{\frac{d-1}{2}} \int_{\mathbb{T}^d} |\sum_j p_j e^{2\pi i t_j}|^{2n} \, dt_1...dt_d = (4\pi)^{-\frac{r}{2}} \, (p_1...p_d)^{-\frac{1}{2}}.$$

□

Example: With $K_n(t_1, t_2) := \sqrt{\pi n} |(\frac{e^{2\pi i t_1} + e^{2\pi i t_2}}{2})|^{2n}$, we have $\int_{\mathbb{T}^2} K_n(t_1, t_2) dt_1 dt_2 \to 1$. This can be shown also using Stirling's approximation:

$$\int_{\mathbb{T}^2} K_n(t_1, t_2) dt_1 dt_2 = \frac{\sqrt{\pi n}}{4^n} \sum_{k=0}^n \binom{n}{k}^2 = \frac{\sqrt{\pi n}}{4^n} \binom{2n}{n} \underset{n\to\infty}{\to} 1.$$

Proposition 1.7. *If φ_f is continuous, then for every $\theta \in \mathbb{T}^d$ we have*

$$\lim_{n\to\infty} (4\pi)^{\frac{d-1}{2}} \, (p_1...p_d)^{\frac{1}{2}} \, n^{\frac{d-1}{2}} \, \|P_\theta^n f\|_2^2 = \int_{\mathbb{T}} \varphi_f(\theta_1 + u, ..., \theta_d + u) \, du. \tag{12}$$

Proof. Let us put $c_n := (4\pi)^{\frac{d-1}{2}} \, (p_1...p_d)^{\frac{1}{2}} \, n^{\frac{d-1}{2}}$ for the normalization coefficient and

$$K_n(t_1, ..., t_d) := c_n \, |\sum_{j=1}^d p_j \, e^{2\pi i t_j}|^{2n}.$$

We have $c_n \, \|P_\theta^n f\|_2^2 = (K_n * \varphi_f)(\theta_1, ..., \theta_d)$ and, by (9) $\int_{\mathbb{T}^d} K_n(t_1, ..., t_d) \, dt_1...dt_d \to 1$.

Let us show that for φ continuous on $\mathbb{T}^d$, $\lim_n \int_{\mathbb{T}^d} K_n \, \varphi \, dt_1...dt_d = \int_{\mathbb{T}} \varphi(u, ..., u) \, du$. Using the density of trigonometric polynomials for the uniform norm, it is enough to prove it for characters $\chi_{\underline{k}}(\underline{t}) = \mathrm{e}^{2\pi i \sum_j k_j t_j}$, i.e., to prove that for $\varphi = \chi_{\underline{k}}$ the limit is 0 if $\sum_\ell k_\ell \neq 0$, and 1 if $\sum_\ell k_\ell = 0$. We have

$$\int_{\mathbb{T}^d} K_n(t_1, ..., t_d)\, e^{2\pi i \sum_\ell k_\ell t_\ell}\, dt_1...dt_d = c_n \int_{\mathbb{T}^d} |\sum_{j=1}^d p_j\, e^{2\pi i t_j}|^{2n}\, e^{2\pi i \sum_\ell k_\ell t_\ell}\, dt_1...dt_d$$

$$= (c_n \int_{\mathbb{T}^{d-1}} |p_1 + \sum_{j=2}^d p_j\, e^{2\pi i (t_j - t_1)}|^{2n}\, e^{2\pi i \sum_{\ell=2}^d k_\ell (t_\ell - t_1)}\, dt_2...dt_d) \int_{\mathbb{T}} e^{2\pi i (\sum_\ell k_\ell) t_1} dt_1.$$

Therefore it remains to show that the limit of the first factor when $n \to \infty$ is 1. Using the proof and the result of Proposition 1.5, we find that this factor is equivalent to

$$(4\pi)^{\frac{d-1}{2}}\, (p_1...p_d)^{\frac{1}{2}} \int_{\{\|\underline{u}\| \leq \ln n\}} (1 - Q(\frac{2\pi \underline{u}}{\sqrt{n}}))^n e^{2\pi i \sum_1^r k_{\ell+1} \frac{u_\ell}{\sqrt{n}}}\, d\underline{u},$$

which tends to 1. □

1.3. **Nullity of variance and coboundaries.**

Let $\mathcal{H}$ be a Hilbert space and let T_1 and T_2 be two commuting unitary operators acting on $\mathcal{H}$. Assuming the Lebesgue spectrum property for the $\mathbb{Z}^2$-action generated by T_1 and T_2, we study in this subsection the degeneracy of the variance. Here we consider, for simplicity, the case of two unitary commuting operators, but the results are valid for any finite family of commuting unitary operators.

Single Lebesgue spectrum

At first, let us assume that there is $\psi \in \mathcal{H}$ such that the family of vectors $T_1^k T_2^r \psi$ for $(k, r) \in \mathbb{Z}^2$ is an orthonormal basis of $\mathcal{H}$ (simplicity of the spectrum).

Lemma 1.8. *Let f be in $\mathcal{H}$ and $f = \sum_{(k,r)\in\mathbb{Z}^2} a_{k,r}\, T_1^k T_2^r \psi$ be the representation of f in the orthonormal basis $(T_1^k T_2^r \psi,\ (k, r) \in \mathbb{Z}^2)$. If*

$$A := \sum_{k,r\in\mathbb{Z}^2} (1 + |k| + |r|)\, |a_{k,r}| < +\infty, \tag{13}$$

there exists $u, v \in \mathcal{H}$ with $\|u\|, \|v\| \leq A$ such that

$$f = (\sum_{(k,r)\in\mathbb{Z}^2} a_{k,r})\, \psi + (I - T_1)u + (I - T_2)v.$$

If $\sum_{(k,r)\in\mathbb{Z}^2} a_{k,r} = 0$, then f is sum of two coboundaries respectively for T_1 and T_2:

$$f = (I - T_1)u + (I - T_2)v.$$

Proof. 1) We start with a formal computation. Let us decompose f into vectors whose coefficients are supported on disjoint quadrants of increasing dimensions. If $f = \sum_{k,r\in\mathbb{Z}^2} a_{k,r} T_1^k T_2^r \psi$, we write

$$f = f_{0,0} + f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1} + f_{1,1} + f_{-1,1} + f_{1,-1} + f_{-1,-1}, \tag{14}$$

with

$$
\begin{aligned}
&f_{0,0} = a_{0,0}\psi, \ f_{1,0} = \sum_{k>0} a_{k,0}\, T_1^k \psi, \ f_{0,1} = \sum_{r>0} a_{0,r}\, T_2^r \psi, \\
&f_{-1,0} = \sum_{k>0} a_{-k,0}\, T_1^{-k} \psi, \ f_{0,-1} = \sum_{r>0} a_{0,-r}\, T_2^{-r} \psi, \\
&f_{1,1} = \sum_{k,r>0} a_{k,r}\, T_1^k T_2^r \psi, \ f_{-1,1} = \sum_{k,r>0} a_{-k,r}\, T_1^{-k} T_2^r \psi, \\
&f_{1,-1} = \sum_{k,r>0} a_{k,-r}\, T_1^k T_2^{-r} \psi, \ f_{-1,-1} = \sum_{k,r>0} a_{-k,-r}\, T_1^{-k} T_2^{-r} \psi.
\end{aligned}
$$

For each component given by a quadrant, we solve the corresponding coboundary equation up to *constant* $\times\, \psi$.

With f decomposed as in (14), the components can be formally written in the following way, with $\varepsilon_i, \varepsilon_i' \in \{0, +1, -1\}$, for $i = 1, 2$:

$$
\begin{aligned}
f_{0,0} &= u_{0,0} = a_{0,0}\, \psi, \quad f_{\varepsilon_1,0} = u^0_{\varepsilon_1,0} + (T_1^{\varepsilon_1} - I) u^{\varepsilon_1,\,0}_{\varepsilon_1,\,0}, \quad f_{0,\varepsilon_2} = u^0_{0,\varepsilon_2} + (T_2^{\varepsilon_2} - I) u^{0,\,\varepsilon_2}_{0,\,\varepsilon_2}, \\
f_{\varepsilon_1,\varepsilon_2} &= u^0_{\varepsilon_1,\varepsilon_2} + (T_1^{\varepsilon_1} - I) u^{\varepsilon_1,0}_{\varepsilon_1,\varepsilon_2} + (T_2^{\varepsilon_2} - I) u^{0,\varepsilon_2}_{\varepsilon_1,\varepsilon_2} - (T_1^{\varepsilon_1} - I)(T_2^{\varepsilon_2} - I) u^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1,\varepsilon_2},
\end{aligned}
$$

where

$$
\begin{aligned}
u^0_{\varepsilon_1,0} &= (\sum_{t\geq 1} a_{\varepsilon_1 t,\,0})\, \psi, \quad u^{\varepsilon_1,\,0}_{\varepsilon_1,\,0} = \sum_{k\geq 0} (\sum_{t\geq k+1} a_{\varepsilon_1 t,\,0}) T_1^{\varepsilon_1 k}\, \psi, \\
u^0_{0,\varepsilon_2} &= (\sum_{s\geq 1} a_{0,\,\varepsilon_2 s})\, \psi, \quad u^{0,\,\varepsilon_2}_{0,\,\varepsilon_2} = \sum_{r\geq 0} (\sum_{s\geq r+1} a_{0,\,\varepsilon_2 s}) T_2^{\varepsilon_2 r}\, \psi, \\
u^0_{\varepsilon_1,\varepsilon_2} &= (\sum_{t,s\geq 1} a_{\varepsilon_1 t,\,\varepsilon_2 s})\, \psi, \ u^{\varepsilon_1,0}_{\varepsilon_1,\varepsilon_2} = \sum_{k0,r\geq 1} (\sum_{t\geq k+1} a_{\varepsilon_1 t,\,\varepsilon_2 r})\, T_1^{\varepsilon_1 k} T_2^{\varepsilon_2 r}\, \psi, \\
u^{0,\varepsilon_2}_{\varepsilon_1,\varepsilon_2} &= \sum_{k\geq 1, r\geq 0} (\sum_{s\geq r+1} a_{\varepsilon_1 k,\,\varepsilon_2 s})\, T_1^{\varepsilon_1 k} T_2^{\varepsilon_2 r}\, \psi, \\
u^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1,\varepsilon_2} &= \sum_{k,r\geq 0} (\sum_{t\geq k+1, s\geq r+1} a_{\varepsilon_1 t,\,\varepsilon_2 s})\, T_1^{\varepsilon_1 k} T_2^{\varepsilon_2 r}\, \psi.
\end{aligned}
$$

More explicitly we have, for instance,

$$
\begin{aligned}
&f_{1,0} = u^0_{1,0} + (T_1 - I) u^{1,0}_{1,0} = (\sum_{t\geq 1} a_{t,0})\, \psi + (T_1 - I)\, [\sum_{k\geq 0} (\sum_{t\geq k+1} a_{t,0}) T_1^k\, \psi], \\
&f_{1,1} = u^0_{1,1} + (T_1 - I) u^{1,0}_{1,1} + (T_2 - I) u^{0,1}_{1,1} + (T_1 - I)(T_2 - I) u^{1,1}_{1,1} \\
&= (\sum_{t,s\geq 1} a_{t,s})\, \psi + (T_1 - I)\, [\sum_{k\geq 0, r\geq 1} (\sum_{t\geq k+1} a_{t,r})\, T_1^k T_2^r\, \psi] \\
&+ (T_2 - I)\, [\sum_{k\geq 1, r\geq 0} (\sum_{s\geq r+1} a_{k,s})\, T_1^k T_2^r\, \psi] - (T_1 - I)(T_2 - I)\, [\sum_{k,r\geq 0} (\sum_{t\geq k+1, s\geq r+1} a_{t,s})\, T_1^k T_2^r\, \psi].
\end{aligned}
$$

By summing the previous expressions, we obtain the following representation of f:

$$\begin{aligned} f = (\sum a_{t,s})\, \psi \quad &+(T_1 - I)(u_1^1 - T_1^{-1} u_{-1}^1 + u_1^{1,2} - T_1^{-1} u_{-1}^{-1,2} + u_1^{1,-2} - T_1^{-1} u_{-1}^{-1,-2}) \\ &+(T_2 - I)(u_2^1 - T_2^{-1} u_{-2}^1 + u_2^{1,2} - T_2^{-1} u_{-2}^{1,-2} + u_2^{-1,2} - T_2^{-1} u_{-2}^{-1,-2}) \\ &+(T_1 - I)(T_2 - I)(u_{1,2}^{1,2} - T_1^{-1} u_{-1,2}^{-1,2} - T_2^{-1} u_{1,-2}^{1,-2} + T_1^{-1} T_2^{-1} u_{-1,-2}^{-1,-2}). \end{aligned}$$

The first term is the vector $a(f)\, \psi$, where $a(f)$ is the constant $\sum_{k,r \in \mathbb{Z}^2} a_{k,r}$ obtained as the sum $u_0^0 + u_1^0 + u_2^0 + u_{-1}^0 + u_{-2}^0 + u_0^{1,2} + u_0^{-1,2} + u_0^{1,-2} + u_0^{-1,-2}$. The second term is a sum of coboundaries. If $a(f) = 0$, then f reduces to a sum of coboundaries.

2) Now we examine the question of convergence in the previous computation. We need the convergence of the following series (for $\varepsilon_1, \varepsilon_2 = \pm 1$):

$$\begin{aligned} &\sum_{t,s \geq 1} a_{\varepsilon_1 t,\, \varepsilon_2 s}, \quad \sum_{t \geq k+1} a_{\varepsilon_1 t,\, \varepsilon_2 r}, \quad \sum_{s \geq r+1} a_{\varepsilon_1 k,\, \varepsilon_2 s}, \quad \sum_{t \geq k+1, s \geq r+1} a_{\varepsilon_1 t,\, \varepsilon_2 s}, \\ &\sum_{k \geq 0, r \geq 1} |\sum_{t \geq k+1} a_{\varepsilon_1 t,\, \varepsilon_2 r}|^2, \quad \sum_{k \geq 1, r \geq 0} |\sum_{s \geq r+1} a_{\varepsilon_1 k,\, \varepsilon_2 s}|^2, \quad \sum_{k,r \geq 0} |\sum_{t \geq k+1, s \geq r+1} a_{\varepsilon_1 t,\, \varepsilon_2 s}|^2. \end{aligned}$$

Sufficient conditions for the convergence are:

$$\begin{aligned} &\sum_{k,r \in \mathbb{Z}^2} |a_{k,\, r}| < +\infty, \quad \sum_{k \geq 0, r \geq 1} (\sum_{t \geq k+1} |a_{\varepsilon_1 t,\, \varepsilon_2 r}|)^2 < +\infty, \\ &\sum_{k \geq 1, r \geq 0} (\sum_{s \geq r+1} |a_{\varepsilon_1 k,\, \varepsilon_2 s}|)^2 < +\infty, \quad \sum_{k,r \geq 0} (\sum_{t \geq k+1, s \geq r+1} |a_{\varepsilon_1 t,\, \varepsilon_2 s}|)^2 < +\infty. \end{aligned}$$

We have:

$$\begin{aligned} &\sum_{k \geq 0, r \geq 0} (\sum_{t \geq k, s \geq r} |a_{t,s}|)^2 = \sum_{k \geq 0, r \geq 0} (\sum_{t,t' \geq k,\, s,s' \geq r} |a_{t,s}||a_{t',s'}|) \\ &\leq \sum_{t,t' \geq 0, s,s' \geq 0} |a_{t,s}||a_{t',s'}| \sum_k 1_{0 \leq k \leq \inf(t,t')} \sum_r 1_{0 \leq r \leq \inf(s,s')} \\ &= \sum_{t,t' \geq 0, s,s' \geq 0} |a_{t,s}||a_{t',s'}|\, (1 + \inf(t,t'))\, (1 + \inf(s,s')) \\ &\leq \sum_{t,t',s,s' \geq 0} |a_{t,s}||a_{t',s'}| (1 + t + s)\, (1 + t' + s') = (\sum_{t \geq 0, s \geq 0} (1 + t + s)\, |a_{t,s}|)^2. \end{aligned}$$

An analogous bound is valid for the indices with $\pm$ signs. Therefore, convergence holds if (13) is satisfied and we get $\sum_{\underline{t} \in \mathbb{Z}^2} (1 + \|\underline{t}\|)\, |a_{\underline{t}}|$ as a bound for the norm of the vectors $u_{\varepsilon_1,0}^0, u_{\varepsilon_1,0}^{\varepsilon_1,0}, u_{0,\varepsilon_2}^0, u_{0,\varepsilon_2}^{0,\varepsilon_2}, u_{\varepsilon_1,\varepsilon_2}^0, u_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1,0}, u_{\varepsilon_1,\varepsilon_2}^{0,\varepsilon_2}, u_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1,\varepsilon_2}$. $\square$

Countable Lebesgue spectrum

We suppose now that the action on $\mathcal{H}$ has a countable Lebesgue spectrum: there exists a countable set $(\psi_j, j \in J)$ in $\mathcal{H}$ such that the family of vectors $\{T_1^k T_2^r \psi_j, j \in J, (k,r) \in \mathbb{Z}^2\}$ is an orthonormal basis of $\mathcal{H}$. The representation of f in this orthonormal basis is given by $f = \sum_j f_j = \sum_{j \in J} (\sum_{(k,r)\in\mathbb{Z}^2} a_{j,(k,r)}\, T_1^k T_2^r \psi_j)$, with $a_{j,(k,r)} = \langle f,\, T_1^k T_2^r \psi_j\rangle$. Recall that

$$M_\theta(f) = \sum_{j\in J} \gamma_j(\theta)\psi_j, \text{ with } \gamma_j(\theta) = \sum_{\underline{k}\in\mathbb{Z}^2} a_{j,\underline{k}}\, e^{2\pi i \langle \underline{k}, \theta\rangle}.$$

Using Lemma 1.8, we have under a convergence condition:

$$\begin{aligned} f_j &= \gamma_j(\theta)\psi_j + (I - e^{2\pi i\theta_1}T_1)u_{j,\theta} + (I - e^{2\pi i\theta_2}T_2)v_{j,\theta}, \forall j \in J,\\ f &= M_\theta(f) + (I - e^{2\pi i\theta_1}T_1)\sum_{j\in J} u_{j,\theta} + (I - e^{2\pi i\theta_2}T_2)\sum_{j\in J} v_{j,\theta}. \end{aligned}$$

The result for d generators is the following:

Lemma 1.9. *Suppose that the following condition is satisfied:*

$$\sum_j \sum_{\underline{k}\in\mathbb{Z}^d} (1 + \|\underline{k}\|^d)\, |a_{j,\underline{k}}| < \infty. \tag{15}$$

Then there are $v, u_1, ..., u_d \in \mathcal{H}$ such that the family $\{T^{\underline{n}}v, \underline{n} \in \mathbb{Z}^d\}$ is orthogonal and

$$f = v + \sum_{t=1}^d (I - T_t)u_t. \tag{16}$$

The variance is 0, if and only f is a mixed coboundary.

For every θ, the rotated variance $\sigma_\theta^2(f)$ is null if and only if there are $u_{t,\theta} \in \mathcal{H}$, for $t = 1, ..., d$, such that $f = \sum_{t=1}^d (I - e^{2\pi i\theta_t}T_t)u_{t,\theta}$.

In the topological framework, when the $T^{\underline{n}}\psi_j$'s are continuous and uniformly bounded with respect to $\underline{n}$ and j, then the functions v and u_t are continuous.

2. **Multidimensional actions by endomorphisms**

In what follows we consider a finitely generated semigroup $\mathcal{S}$ of surjective endomorphisms of G, a compact abelian group with Haar measure denoted by μ. The group of characters of G will be denoted by $\hat{G}$ (or H) and the set of non trivial characters by $\hat{G}^*$ (or H^*).

After choosing a system $A_1, ..., A_d$ of generators, every element in $\mathcal{S}$ can be represented as $A^{\underline{n}}$ with the notation $A^{\underline{n}} := A_1^{n_1}...A_d^{n_d}$, $\underline{n} = (n_1, ..., n_d) \in \mathbb{N}^d$, and we obtain an action $\underline{n} \to A^{\underline{n}}$ of $\mathbb{N}^d$ by endomorphisms on G.

If f is function on G, $A^{\underline{n}}f$ stands for $f \circ A^{\underline{n}}$. We use also the notation $T^{\underline{n}}f$. For $T \in \mathcal{S}$, we denote by the same letter its action on G and on the dual H of G. The Fourier coefficients of a function f in $L^2(G)$ are $c_f(\chi) := \int_G \overline{\chi}\, f\, d\mu$.

Every surjective endomorphism A of G defines a measure preserving transformation on (G, μ) and a dual endomorphism on $\hat{G}$, the group of characters of G. For simplicity, we use the same notation for the action on G and on $\hat{G}$.

The first subsections are preparatory for the CLT.

2.1. **Preliminaries.**

Embedding of a semigroup of endomorphisms in a group

Lemma 2.1. *Let $\mathcal{S}$ be a commutative semigroup of surjective endomorphisms on a compact abelian group G with dual group $\hat{G}$. There is a compact abelian group $\tilde{G}$ such that G is a factor of G and $\mathcal{S}$ is embedded in a group $\tilde{\mathcal{S}}$ of automorphisms of $\tilde{G}$. If G is connected, then $\tilde{G}$ is also connected.*

Proof. We construct a discrete group $\tilde{H}$ such that $\hat{G}$ is isomorphic to a subgroup of $\tilde{H}$. The group $\tilde{H}$ is defined as the quotient of the group $\{(\chi, A), \chi \in \hat{G}, A \in \mathcal{S}\}$ (endowed with the additive law on the components) by the following equivalence relation R: (χ, A) is equivalent to (χ', A') if $A'\chi = A\chi'$. The transitivity of the relation R follows from the injectivity of each $A \in \mathcal{S}$ acting on $\hat{G}$. The map $\chi \in \hat{G} \to (\chi, Id)/R$ is injective. The elements $A \in \mathcal{S}$ act on $\tilde{H}$ by $(\chi, B)/R \to (A\chi, B)/R$. The equivalence classes are stable by this action. We can identify $\mathcal{S}$ and its image. For $A \in \mathcal{S}$, the automorphism $(\chi, B)/R \to (\chi, AB)/R$ is the inverse of $(\chi, B)/R \to (A\chi, B)/R$.

We obtain an embedding of $\mathcal{S}$ in a group $\tilde{\mathcal{S}}$ of automorphisms of $\tilde{H}$. If $\hat{G}$ is torsion free, then $\tilde{H}$ is also torsion free and its dual $\tilde{G}$ is a connected compact abelian group. $\square$

Our data will be a finite set of commuting surjective endomorphisms A_i of G such that the generated group $\tilde{\mathcal{S}}$ is torsion-free.

If necessary, we consider also the group of automorphisms $\tilde{\mathcal{S}}$ on the extension $\tilde{G}$. Since $\tilde{\mathcal{S}}$ is finitely generated and torsion-free, it has a system of d independent generators (not necessarily in $\mathcal{S}$) and it is isomorphic to $\mathbb{Z}^d$. The rank of the action of $\mathcal{S}$ is d. The spectral analysis for $\mathcal{S}$, as for $\tilde{\mathcal{S}}$, takes place in $\mathbb{T}^d$.

Observe that, by using the projection π from $\tilde{G}$ to G, a function f on the group G can be viewed as function on $\tilde{G}$ and a character $\chi \in \hat{G}$ as a character on $\tilde{G}$ via the composition $g \to \chi(\pi g)$. Putting $\tilde{f}(x) = f(\pi x)$, the Fourier series of $\tilde{f}$ reads: $\tilde{f} = \sum_{\chi \in \tilde{G}} c_{\tilde{f}}(\chi)\, \chi$. But, on an other side, we have $f = \sum_{\chi \in G} c_f(\chi)\, \chi$, so that by unicity of the Fourier series, the only non zero Fourier coefficients for $\tilde{f}$ are those for $\chi \in \hat{G}$.

It follows that, for a function f defined on G, the computations can be done in the group $\tilde{G}$ with the action of the group of automorphisms, but expressed in terms of the Fourier coefficients of F computed in G.

Definition 2.2. We say that a $\mathbb{Z}^d$-action by automorphisms, $\underline{n} \to A^{\underline{n}}$, is *totally ergodic* if $A_1^{n_1} ... A_d^{n_d}$ is ergodic for every $\underline{n} = (n_1, ..., n_d) \neq \underline{0}$. It is equivalent to the property:

$A^{\underline{n}}\chi \neq \chi$, for any non trivial character χ and $\underline{n} \neq \underline{0}$. For endomorphisms, we replace the semigroup $\mathcal{S}$ by the extension $\tilde{\mathcal{S}}$ defined in Lemma 2.1.

What we call "totally ergodic" in the general case of a compact abelian group G is often called "partially hyperbolic" for actions on a torus.

Lemma 2.3. *The following conditions are equivalent for a $\mathbb{Z}^d$-action T by automorphisms on a compact abelian group:*
i) T is totally ergodic;
ii) T is 2-mixing[2];
iii) T has the Lebesgue spectrum property.

Proof. The Lebesgue spectrum property (cf. Section 1) for the action of $\tilde{\mathcal{S}}$ is equivalent to the fact that the action $\tilde{\mathcal{S}}$ on $\tilde{H}^*$ is free, which is total ergodicity.

Mixing of order 2 implies total ergodicity. At last the implication $(iii) \Rightarrow (ii)$ is a general fact. $\square$

Remark 2.4. Finding the dimension of $\mathcal{S}$ and computing a set of independent generators can be very difficult in practice. For $G = \mathbb{T}^\rho = 3$, we will give explicit examples in the Appendix. Given a finite set of commuting matrices in dimension ρ with determinant 1 for $\rho > 3$, it can be difficult and even impossible to find independent generators via a computation.

In some cases the problem can be easier with endomorphisms. For instance, let $p_i, i = 1, ..., d$ be coprime positive integers and $A_i : x \to q_i x \bmod 1$ the corresponding endomorphisms acting on $\mathbb{T}^1$. Then the A_i's give a system of independent generators of the group $\tilde{\mathcal{S}}$ generated on the compact abelian group dual of $\tilde{\mathbb{Z}}^\rho := \{\underline{k}\,\Pi q_i^{\ell_i}, \underline{k} \in \mathbb{Z}^\rho, \ell_i \in \mathbb{Z}\}$.

Notation 2.5. J denote a *section* of the $\tilde{\mathcal{S}}$ action on $\tilde{H}^*$, i.e., a subset $\{\chi_j\}_{j\in J} \subset \tilde{H}\backslash\{\underline{0}\}$ such that every $\chi \in \tilde{H}^*$ can be written in a unique way as $\chi = A_1^{n_1}...A_d^{n_d}\,\chi_j$, with $j \in J$ and $(n_1, ..., n_d) \in \mathbb{Z}^d$.

Using the extension, for a function f in $L^2(\tilde{G})$, we have

$$\langle f, A^{\underline{n}} f\rangle = \sum_{\chi \in \tilde{H}} c_f(A^{\underline{n}}\chi)\,\overline{c_f(\chi)}. \tag{17}$$

For the validity of this formula we suppose $\sum_{\chi\in\tilde{H}} |c_f(A^{\underline{n}}\chi)|\,|\overline{c_f(\chi)}| < +\infty$. When f is defined on G, the formula can be written, with $\tilde{f}$ the extension of f to $\tilde{G}$ and the convention (*) that the coefficients $c_f(A^{\underline{n}}\chi)$ are replaced by 0 if $A^{\underline{n}}\chi \notin H$,

$$\langle \tilde{f}, A^{\underline{n}} \tilde{f}\rangle = \sum_{\chi \in H} c_f(A^{\underline{n}}\chi)\,\overline{c_f(\chi)}. \tag{18}$$

[2] Mixing of order 2 is defined in Definition 1.2 (here $\mathcal{H} = L_0^2(G, \mu)$) or equivalently by $\lim_{\underline{n}\to\infty} \mu(B_1 \cap T^{-\underline{n}}\, B_2) = \mu(B_1)\,\mu(B_2)$, $\forall\, B_1, B_2 \in \mathcal{A}$.

Functions with absolutely convergent Fourier series

We denote by $AC_0(G)$ the class of real functions on G satisfying $\mu(f) = 0$ and with an absolutely convergent Fourier series, i.e., such that

$$\|f\|_c := \sum_{\chi \in \hat{G}} |c_f(\chi)| < +\infty. \tag{19}$$

Theorem 2.6. *If f is in $AC_0(G)$, then $\sum_{\underline{n} \in \mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| < \infty$, the variance $\sigma^2(f)$ exists, $\sigma^2(f) = \sum_{\underline{n} \in \mathbb{Z}^d} \langle A^{\underline{n}} f, f\rangle$ and the spectral density φ_f of f is continuous.*

Moreover, if $\mathcal{N}$ is any subset of $\hat{G}$ and $f_1(x) = \sum_{\chi \in \mathcal{N}} c_f(\chi)\, \chi$, then

$$\|\varphi_f\|_\infty \leq \|f - f_1\|_c^2. \tag{20}$$

In particular, $\sigma(f - f_1) \leq \|f - f_1\|_c$.

Proof. We use the convention (*) in the notations. By total ergodicity, for every $\chi \in \tilde{H}^*$, the map $\underline{n} \in \mathbb{Z}^d \to A^{\underline{n}}\chi \in \tilde{H}^*$ is injective, and therefore $\sum_{\underline{n} \in \mathbb{Z}^d} |c_f(A^{\underline{n}}\chi)| \leq \sum_{\chi \in \hat{G}^*} |c_f(\chi)|$. The spectral density of f is

$$\varphi_f(t) = \sum_{j \in J} |\gamma_j(t)|^2 = \sum_{j \in J} |\sum_{\underline{n} \in \mathbb{Z}^d \setminus \{\underline{0}\}} c_f(A^{\underline{n}}\chi_j)\, e^{2\pi i \langle \underline{n}, t\rangle}\,|^2. \tag{21}$$

We have:

$$\begin{aligned}
\|\varphi_f\|_\infty \quad &\leq \sum_{j \in J} \|\gamma_j\|_\infty^2 \leq \sum_{j \in J} (\sum_{\underline{n} \in \mathbb{Z}^d} |c_f(A^{\underline{n}}\chi_j)|)^2 \\
&\leq \sum_{j \in J} (\sum_{\underline{n} \in \mathbb{Z}^d} |c_f(A^{\underline{n}}\chi_j)|)(\sum_{\chi} |c_f(\chi)|) \leq (\sum_{\chi} |c_f(\chi)|)^2 = \|f\|_c^2.
\end{aligned}$$

□

Approximation by M_θ

In the algebraic setting of endomorphisms of compact abelian groups, the family (ψ_j) of the general theory (Subsection 1.2) is (χ_j). We have

$$\begin{aligned}
&a_{j,n} = \langle f, T^{\underline{n}}\chi_j\rangle = c_f(T^{\underline{n}}\chi_j), \\
&\gamma_j(\theta) = \sum_{\underline{n} \in \mathbb{Z}^d} c_f(T^{\underline{n}}\chi_j)\, e^{2\pi i \langle \underline{n}, \theta\rangle},\ M_\theta f = \sum_{j} \gamma_{\underline{j}}(\theta)\, \chi_j.
\end{aligned}$$

Hence, $M_\theta f$ is defined for every θ, if $\sum_{j \in J} \left(\sum_{\underline{n} \in \mathbb{Z}^d} |c_f(T^{\underline{n}}\chi_j)|^2\right) < \infty$.

Let us assume that f has an absolutely convergent Fourier series. Since every $\chi \in \tilde{H}^*$ can be written in an unique way as $\chi = T^{\underline{n}}\chi_j$, with $j \in J$ and $\underline{n} \in \mathbb{Z}^d$, we have

$$\sum_{\underline{n} \in \mathbb{Z}^d} |c_f(T^{\underline{n}}\chi_j)| \leq \sum_{j \in J} \sum_{\underline{n} \in \mathbb{Z}^d} |c_f(T^{\underline{n}}\chi_j)| = \sum_{\chi \in \tilde{H}^*} |c_f(\chi)| = \|f\|_c.$$

Then, for every $j \in J$, the series defining γ_j is uniformly converging, γ_j is continuous and $\sum_{j\in J} \|\gamma_j\|_\infty \leq \sum_{j\in J} \sum_{\underline{n}\in\mathbb{Z}^d} |c_f(T^{\underline{n}} j)| = \|f\|_c$. The function $\sum_{j\in J} |\gamma_j|^2$ is a continuous version of the spectral density φ_f. Therefore Lemma 1.4b applies.

The following lemma allows to check the condition of Lemma 1.4a in terms of the Fourier series of f.

Lemma 2.7. *If $\sum_{\chi\in\hat{G}^*} |c_f(\chi)|^r < +\infty$ for some $1 < r \leq 2$, then the spectral density φ_f, if $\mu(f) = 0$, is in $L^p(\mathbb{T}^d)$ with $p = \frac{1}{2}\frac{r}{r-1} > 1$.*

Proof. We have $\varphi_f(t) = \sum_{j\in J} |\gamma_j(t)|^2$, with $\gamma_j(t) = \sum_{\underline{n}\in\mathbb{Z}^d\setminus\{\underline{0}\}} c_f(A^{\underline{n}}\chi_j) e^{2\pi i\langle\underline{n},t\rangle}$.

Let $p := \frac{1}{2}\frac{r}{r-1}$ be such that r and $2p$ are conjugate exponents. By the Hausdorff-Young theorem, we have:

$$\|\gamma_j\|_{2p} \leq (\sum_{\underline{n}\in\mathbb{Z}^d\setminus\{\underline{0}\}} |c_f(A^{\underline{n}}\chi_j)|^r)^{1/r}.$$

It follows, since $2/r \geq 1$:

$$\begin{aligned}\sum_{j\in J} \||\gamma_j|^2\|_p &= \sum_{j\in J} \|\gamma_j\|_{2p}^2 \leq \sum_{j\in J} (\sum_{\underline{n}\in\mathbb{Z}^d\setminus\{\underline{0}\}} |c_f(A^{\underline{n}}\chi_j)|^r)^{2/r} \\ &\leq (\sum_{j\in J} \sum_{\underline{n}\in\mathbb{Z}^d\setminus\{\underline{0}\}} |c_f(A^{\underline{n}}\chi_j)|^r)^{2/r} \leq (\sum_{\chi\in\hat{G}^*} |c_f(\chi)|^r)^{2/r}.\end{aligned}$$

Therefore we have $\varphi_f \in L^p(\mathbb{T}^d)$ and $\|\varphi_f\|_p \leq (\sum_{\chi\in\hat{G}^*} |c_f(\chi)|^r)^{2/r}$. ☐

Remark that the condition of the lemma for $r = 1$ corresponds to $f \in AC_0(G)$, which implies continuity of φ_f.

2.2. **Mixing, moments and cumulants, application to the CLT.**

Reminders on moments and cumulants

Before we continue studying actions by automorphisms, for the sake of completeness, we recall in this subsection some general results on mixing of all orders, moments and cumulants (see [12], [18] and the references given therein). Implicitly we assume existence of moments of all orders when they are used.

For a real random variable Y (or for a probability distribution on $\mathbb{R}$), the cumulants (or semi-invariants) can be formally defined as the coefficients $c^{(r)}(Y)$ of the cumulant generating function $t \to \ln \mathbb{E}(e^{tY}) = \sum_{r=0}^{\infty} c^{(r)}(Y) \frac{t^r}{r!}$, i.e.,

$$c^{(r)}(Y) = \frac{\partial^r}{\partial^r t} \ln \mathbb{E}(e^{tY})|_{t=0}.$$

Similarly the joint cumulant of a random vector $(X_1, ..., X_r)$ is defined by

$$c(X_1, ..., X_r) = \frac{\partial^r}{\partial t_1 ... \partial t_r} \ln \mathbb{E}(e^{\sum_{j=1}^r t_j X_j})|_{t_1=...=t_r=0}.$$

This definition can be given as well for a finite measure on $\mathbb{R}^r$.

One easily checks that the joint cumulant of $(Y, ..., Y)$ (r copies of Y) is $c^{(r)}(Y)$.

For any subset $I = \{i_1, ..., i_p\} \subset J_r := \{1, ..., r\}$, we put

$$m(I) = m(i_1, ..., i_p) := \mathbb{E}(X_{i_1} ... X_{i_p}),\ \ s(I) = s(i_1, ..., i_p) := c(X_{i_1}, ..., X_{i_p}).$$

The cumulants of a process $(X_j)_{j\in\mathcal{J}}$, where $\mathcal{J}$ is a set of indexes, is the family

$$\{c(X_{i_1}, ..., X_{i_r}), (i_1, ..., i_r) \in \mathcal{J}^r, r \geq 1\}.$$

The following formulas link moments and cumulants and vice-versa:

$$c(X_1, ..., X_r) \ = \ s(J_r) = \sum_{\mathcal{P}} (-1)^{p-1}(p-1)!\ m(I_1)...m(I_p), \tag{22}$$

$$\mathbb{E}(X_1 ... X_r) \ = \ m(J_r) = \sum_{\mathcal{P}} s(I_1)...s(I_p). \tag{23}$$

where in both formulas, $\mathcal{P} = \{I_1, I_2, ..., I_p\}$ runs through the set of partitions of $J_r = \{1, ..., r\}$ into $p \leq r$ non empty intervals.

Now, let be given a random process $(X_{\underline{k}})_{\underline{k}\in\mathbb{Z}^d}$, where for $\underline{k} \in \mathbb{Z}^d$, $X_{\underline{k}}$ is a real random variable, and a summation kernel R with finite support in $\mathbb{Z}^d$ and values in $\mathbb{R}^+$. (For examples of summation kernels, see Section 1, in particular Proposition 1.7). Let us consider the process defined for $\underline{k} \in \mathbb{Z}^d$ by

$$Y_k = \sum_{\underline{\ell}\in\mathbb{Z}^d} R(\underline{\ell} + \underline{k})\, X_{\underline{\ell}},\ \underline{k} \in \mathbb{Z}^d.$$

By permuting summation and integral, we easily obtain:

$$c(Y_{\underline{k}_1}, ..., Y_{\underline{k}_r}) = \sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\,\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, ..., X_{\underline{\ell}_r})\, R(\underline{\ell}_1 + \underline{k}_1)...R(\underline{\ell}_r + \underline{k}_r).$$

In particular, we have for $Y = \sum_{\underline{\ell}\in\mathbb{Z}^d} R(\underline{\ell})\, X_{\underline{\ell}}$:

$$c^{(r)}(Y) = c(Y, ..., Y) = \sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\,\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, ..., X_{\underline{\ell}_r})\, R(\underline{\ell}_1)...R(\underline{\ell}_r). \tag{24}$$

Limiting distribution and cumulants

For our purpose, we state in terms of cumulants a particular case of a theorem of M. Fréchet and J. Shohat, generalizing classical results of A. Markov. Using the formulas linking moments and cumulants, a special case of the "generalized statement of the second limit-theorem" given in [9] can be expressed follows:

Theorem 2.8. *Let $(Z^n, n \geq 1)$ be a sequence of centered r.r.v. such that*

$$\lim_n c^{(2)}(Z^n) = \sigma^2,\ \ \lim_n c^{(r)}(Z^n) = 0, \forall r \geq 3, \tag{25}$$

then (Z^n) *tends in distribution to* $\mathcal{N}(0, \sigma^2)$. *(If* $\sigma = 0$, *then the limit is* δ_0*).*

It implies the following result (cf. Theorem 7 in [17]):

Theorem 2.9. *Let* $(X_{\underline{k}})_{\underline{k}\in\mathbb{Z}^d}$ *be a random process and* $(R_n)_{n\geq 1}$ *a summation sequence on* $\mathbb{Z}^d$. *Let* $(Y^n)_{n\geq 1}$ *be the process defined by* $Y^n = \sum_{\underline{\ell}} R_n(\underline{\ell})\, X_{\underline{\ell}}, n \geq 1$. *Under the assumptions* $\lim_n \|Y^n\|_2 = +\infty$ *and*

$$\sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\,\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, ..., X_{\underline{\ell}_r})\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r) = o(\|Y^n\|_2^r), \forall r \geq 3, \tag{26}$$

$\frac{Y^n}{\|Y^n\|_2}$ *tends in distribution to* $\mathcal{N}(0, 1)$ *when* n *tends to* ∞.

Proof. Let $\beta_n := \|Y^n\|_2 = \| \sum_{\underline{\ell}} R_n(\underline{\ell})\, X_{\underline{\ell}}\|_2$ and $Z_n = \beta_n^{-1} Y^n$.

We have using (24), $c^{(r)}(Z^n) = \beta_n^{-r} \sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\,\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, ..., X_{\underline{\ell}_r})\, R(\underline{\ell}_1)...R(\underline{\ell}_r)$. The theorem follows then from the assumption (26) by Theorem 2.8 applied to (Z_n). □

Definition 2.10. *A measure preserving* $\mathbb{N}^d$- *or* $\mathbb{Z}^d$-*action* $T : \underline{n} \to T^{\underline{n}}$ *on a probability space* $(\Omega, \mathcal{A}, \mu)$ *is r-mixing,* $r > 1$, *if for all sets* $B_1, ..., B_r \in \mathcal{A}$

$$\lim_{\min_{1\leq \ell<\ell'\leq r} \|\underline{n}_\ell - \underline{n}_{\ell'}\|\to\infty} \mu(\bigcap_{\ell=1}^{r} T^{-\underline{n}_\ell}\, B_\ell) = \prod_{\ell=1}^{r} \mu(B_\ell).$$

Notation 2.11. For $f \in L_0^\infty$, the space of measurable essentially bounded functions on (Ω, μ) with $\int f\, d\mu = 0$, we apply the definition of moments and cumulants to $(T^{\underline{n}_1} f, ..., T^{\underline{n}_r} f)$ and put

$$m_f(\underline{n}_1, ..., \underline{n}_r) = \int T^{\underline{n}_1} f...T^{\underline{n}_r} f\, d\mu, \quad s_f(\underline{n}_1, ..., \underline{n}_r) := c(T^{\underline{n}_1} f, ..., T^{\underline{n}_r} f). \tag{27}$$

In order to show that the cumulants of a system which is mixing of all orders are asymptotically null, we need the following lemma.

Lemma 2.12. *For every sequence* $(\underline{n}_1^k, ..., \underline{n}_r^k)$ *in* $(\mathbb{Z}^d)^r$, *there are a subsequence with possibly a permutation of indices (still written* $(\underline{n}_1^k, ..., \underline{n}_r^k)$*), an integer* $\kappa(r) \in [1, r]$ *and a subdivision* $1 = r_1 < r_2 < ... < r_{\kappa(r)-1} < r_{\kappa(r)} \leq r$ *of* $\{1, ..., r\}$, *such that*

$$\lim_k \min_{1\leq s\neq s'\leq\kappa(r)} \|\underline{n}_{r_s}^k - \underline{n}_{r_{s'}}^k\| = \infty, \tag{28}$$

$$\underline{n}_j^k = \underline{n}_{r_s}^k + \underline{a}_j, \text{ for } r_s < j < r_{s+1},\ s = 1, ..., \kappa(r) - 1, \text{ and for } r_{\kappa(r)} < j \leq r, \tag{29}$$

where $\underline{a}_j$ *is a constant integral vector.*

If the sequence $(\underline{n}_1^k, ..., \underline{n}_r^k)$ *satisfy* $\lim_k \max_{i\neq j} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$, *then the construction can be done in such a way that* $\kappa(r) > 1$.

Remark that if $\sup_k \max_{i\neq j} \|\underline{n}_i^k - \underline{n}_j^k\| < \infty$, then $\kappa(r) = 1$ so that (28) is void and (29) is void for the indexes such that $r_{s+1} = r_s + 1$.

Proof. The proof is by induction. The result is clear for $r = 2$. Suppose we have construct the subsequence for the sequence of $r-1$-tuples $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$.

Let $1 \leq r_1 < r_2 < ... < r_{\kappa(r-1)} \leq r-1$ be the corresponding subdivision of $\{1, ..., r-1\}$, as stated above for the sequence $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$. If the sequence $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$ satisfy $\lim_k \max_{1<i<j\leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$, then $\kappa(r-1) > 1$ by construction in the induction process.

Now we consider $(\underline{n}_1^k, ..., \underline{n}_r^k)$. If $\lim_k \|\underline{n}_r^k - \underline{n}_i^k\| = +\infty$, for all $i = 1, ..., r-1$, then we have just to take $1 \leq r_1 < r_2 < ... < r_{\kappa(r-1)} < r_{\kappa(r)} = r$ as new subdivision of $\{1, ..., r\}$.

If $\liminf_k \|\underline{n}_r^k - \underline{n}_{i_s}^k\| < +\infty$, for some $s \leq \kappa(r-1)$, then along a new subsequence (still denoted with the same notation) we have $\underline{n}_r^k = \underline{n}_{i_s}^k + \underline{a}_r$, where $\underline{a}_r$ is a constant integral vector. After changing the labels, we insert n_r in the subdivision for $\{1, ..., r-1\}$ and obtain the new subdivision for $\{1, ..., r\}$.

For the last condition on κ, suppose that $\lim_k \max_{1<i<j\leq r} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$.

Then if $\liminf_k \max_{1<i<j\leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| < +\infty$, necessarily, $\kappa(r) > 1$. If, on the contrary, the sequence $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$ satisfy $\lim_k \max_{1<i<j\leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$, then $\kappa(r-1) > 1$ so that $\kappa(r) \geq \kappa(r-1) > 1$. □

Lemma 2.13. *If a $\mathbb{Z}^d$-dynamical system is mixing of order $r \geq 2$, then, for any $f \in L_0^\infty$,*

$$\lim_{\max_{i\neq j} \|\underline{n}_i - \underline{n}_j\| \to \infty} s_f(\underline{n}_1, ..., \underline{n}_r) = 0. \tag{30}$$

Proof. We give a sketch of the proof. The notation s_f was introduced in (27). Suppose that (30) does not hold. Then there is $\varepsilon > 0$ and a sequence of r-tuples $(\underline{n}_1^k = \underline{0}, ..., \underline{n}_r^k)$ such that $|s_f(\underline{n}_1^k, ..., \underline{n}_r^k)| \geq \varepsilon$ and $\max_{i\neq j} \|\underline{n}_i^k - \underline{n}_j^k\| \to \infty$ (we use stationarity).

By taking a subsequence (but keeping the same notation), we can assume that, for two fixed indexes i, j, $\lim_k \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$.

From Lemma 2.12, it follows that there is a subdivision $1 = r_1 < r_2 < ... < r_{\kappa(r)-1} < r_{\kappa(r)} \leq r$ and constant integer vectors $\underline{a}_j$ such that

$$\lim_k \min_{1\leq s\neq s'\leq\kappa(r)} \|\underline{n}_{r_s}^k - \underline{n}_{r_{s'}}^k\| = \infty, \tag{31}$$

$$\underline{n}_j^k = \underline{n}_{r_s}^k + \underline{a}_j, \text{ for } r_s < j < r_{s+1}, \ s = 1, ..., \kappa(r)-1, \text{ and for } r_{\kappa(r)} < j \leq r. \tag{32}$$

Let $d\mu_k(x_1, ..., x_r)$ denote the probability measure on $\mathbb{R}^r$ defined by the distribution of the random vector $(T^{\underline{n}_1^k} f(.), ..., T^{\underline{n}_r^k} f(.))$. We can extract a converging subsequence from the sequence (μ_k), as well as for the moments of order $\leq r$.

Let us denote $\nu(x_1, ..., x_r)$ (resp. $\nu(x_{i_1}, ..., x_{i_p})$) the limit of $\mu_k(x_1, ..., x_r)$ (resp. of its marginal measures $\mu_k(x_{i_1}, ..., x_{i_p})$ for $\{i_1, ..., i_p\} \subset \{1, ..., r\}$).

Let $\varphi_i, i = 1, ..., r$, be continuous functions with compact support on $\mathbb{R}$. Mixing of order r and condition (31) imply

$$\nu(\varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_r) = \lim_k \int_{\mathbb{R}^d} \varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_r \, d\mu_k = \lim_k \int \prod_{i=1}^r \varphi_i(f(T^{\underline{n}_i^k} x)) \, d\mu(x)$$

$$= \lim_k \int \Big[\prod_{s=1}^{\kappa(r)-1} \prod_{r_s \leq j < r_{s+1}} \varphi_j(f(T^{\underline{n}_s^k + \underline{a}_j} x))\Big] \prod_{\kappa(r) \leq j \leq r} \varphi_j(f(T^{\underline{n}_{\kappa(r)}^k + \underline{a}_j} x)) \, d\mu(x)$$

$$= \Big[\prod_{s=1}^{\kappa(r)-1} \Big(\int \prod_{r_s \leq j < r_{s+1}} \varphi_j(f(T^{\underline{a}_j} x)) \, d\mu(x)\Big)\Big] \Big[\int \prod_{\kappa(r) \leq j \leq r} \varphi_j(f(T^{\underline{a}_j} x)) \, d\mu(x)\Big].$$

Therefore ν is the product of marginal measures corresponding to disjoint subsets: at least there are $I_1 = \{i_1, ..., i_p\}, I_2 = \{i'_1, ..., i'_{p'}\} \subset J_r = \{1, ..., r\}$, two non empty subsets such that (I_1, I_2) is a partition of J_r and $d\nu(x_1, ..., x_r) = d\nu(x_{i_1}, ..., x_{i_p}) \times d\nu(x_{i'_1}, ..., x_{i'_{p'}})$.

Putting $\Phi(t_1, ..., t_r) = \ln \int e^{\sum t_j x_j} \, d\mu(x_1, ..., x_r)$ and the analogous formulas for $\nu(x_{i_1}, ..., x_{i_p})$ and $\nu(x_{i'_1}, ..., x_{i'_p})$, we obtain: $\Phi(t_1, ..., t_r) = \Phi(t_{i_1}, ..., t_{i_p}) + \Phi(t_{i'_1}, ..., t_{i'_p})$. It implies that the derivative $\frac{\partial^r}{\partial t_1 ... \partial t_r} \Phi(t_1, ..., t_r)|_{t_1 = ... = t_r = 0}$ is 0. Hence $c(\nu(x_1, ..., x_r)) = 0$.

But this contradicts $\liminf_k |s_f(\underline{n}_1^k, ..., \underline{n}_r^k)| > 0$. ☐

Application to d-dimensional actions by endomorphisms

For an action of $\mathbb{N}^d$ by commuting endomorphisms on (G, μ), a compact abelian group with Haar measure μ, the method of moments as in [17] can be used for the CLT when mixing of all orders is satisfied. It gives immediately the CLT for trigonometric polynomials.

Proposition 2.14. *Let $\underline{n} : (n_1, ..., n_d) \to T^{\underline{n}} = T_1^{n_1} ... T_d^{n_d}$ be a totally ergodic $\mathbb{N}^d$-action by commuting endomorphisms on a compact abelian group G which is mixing of all orders. Let $(R_n)_{n \geq 1}$ be a summation sequence on $\mathbb{N}^d$ and let f be a trigonometric polynomial. If $\lim_n \| \sum_{\underline{\ell}} R_n(\underline{\ell}) \, T^{\underline{\ell}} f\|_2 = \infty$, then the CLT is satisfied by the sequence $\big(\frac{\sum_{\underline{\ell}} R_n(\underline{\ell}) \, T^{\underline{\ell}} f}{\| \sum_{\underline{\ell}} R_n(\underline{\ell}) \, T^{\underline{\ell}} f\|_2}\big)_{n \geq 1}$.*

Proof. For an action by endomorphisms of compact abelian groups, the moments of the process $(f(T^{\underline{n}}.))_{\underline{n} \in \mathbb{Z}^d}$ for a trigonometric polynomial $f(x) = \sum_{\underline{k} \in \Lambda} c_{\underline{k}}(f) \, \chi_{\underline{k}}(x)$ are:

$$m_f(\underline{n}_1, ..., \underline{n}_r) = \int f(T^{\underline{n}_1} x) ... f(T^{\underline{n}_r} x) \, dx = \sum_{\underline{k}_1, ..., \underline{k}_r \in \Lambda} c_{k_1} ... c_{k_r} 1_{T^{n_1} \chi_{\underline{k}_1} ... T^{\underline{n}_r} \chi_{\underline{k}_r} = \mathbf{1}}.$$

For r fixed, the function $(\underline{k}_1, ..., \underline{k}_r) \to m_f(\underline{k}_1, ..., \underline{k}_r)$ takes a finite number of values, since by the above formula m_f is a sum with coefficients 0 or 1 of the products $c_{k_1} ... c_{k_r}$ with k_j in a finite set. The cumulants of a given order according to (22) take also a finite number of values.

Therefore, since mixing of all orders implies $\lim_{\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\| \to \infty} s_f(\underline{\ell}_1, ..., \underline{\ell}_r) = 0$ by Lemma 2.13, there is M_r such that $s_f(\underline{\ell}_1, ..., \underline{\ell}_r) = 0$ for $\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\| > M_r$.

We apply Theorem 2.9. Let us check (26). Using (24), we obtain that

$$\begin{aligned} |\sum s_f(\underline{\ell}_1, ..., \underline{\ell}_r)\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)| \quad &= |\sum s_f(\underline{0}, \underline{\ell}_2 - \underline{\ell}_1, ..., \underline{\ell}_r - \underline{\ell}_1)\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)| \\ &\leq \sum_{\|\underline{\ell}'_2\|, ..., \|\underline{\ell}'_r\| \leq M_r} |s_f(\underline{0}, \underline{\ell}'_2, ..., \underline{\ell}'_r)| \sup_n \|R_n\|_\infty^r. \end{aligned}$$

Since the summation sequences are supposed to be bounded, $\sum s_f(\underline{\ell}_1, ..., \underline{\ell}_r)\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)$ is bounded and (26) is satisfied. □

Remark that if a subsequence of $\lim_n \| \sum_{\underline{\ell}} R_n(\underline{\ell})\, T^{\underline{\ell}} f\|_2$ is bounded, then f is a double coboundary. Indeed, supposing for simplicity $d = 2$, we have the following lemma:

Lemma 2.15. *Let T_1 and T_2 generate a mixing (2-mixing) $\mathbb{N}^2$-action. For $f \in L^2_0(\mu)$, the following statements are equivalent:*
(i) $f = (I - T_1)(I - T_2)g$, for some $g \in L^2_0(\mu)$,
(ii) $\liminf_m \| \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} T_1^j T_2^k f\| < \infty$.

Proof. Let us prove $(ii) \Rightarrow (i)$. Let (m_ℓ) be an increasing subsequence for which $\| \sum_{j,\,k=0}^{m_\ell - 1} T_1^j T_2^k f\| < \infty$. We may take a subsequence of (m_ℓ), still denoted by (m_ℓ), for which $\sum_{j,\,k=0}^{m_\ell-1} T_1^j T_2^k f \underset{\ell\to\infty}{\longrightarrow} g$ weakly. Hence, the following weak convergence holds

$$(I - T_1)(I - T_2)g = \lim_\ell (I - T_1^{m_\ell})(I - T_2^{m_\ell})f = f - \lim_\ell (T_1^{m_\ell} f + T_2^{m_\ell} f - T_1^{m_\ell} T_2^{m_\ell} f) = f$$

□

2.3. CLT for compact abelian connected groups.

There is a subclass of actions by automorphisms satisfying the K-property and this is a way to prove the CLT in that case, using martingale-type property, as shown in [4]. Let us mention that, for $\mathbb{Z}^d$-action by automorphisms on zero-dimensional compact abelian groups, the K-property (or property of completely positive entropy) is equivalent to mixing of all orders (cf. [24]). We will rather focus here on an extension of the method of r-mixing used by Leonov for a single ergodic automorphism and use it for abelian groups of toral automorphisms.

The method of Leonov

The proof of the CLT given by Leonov in [17] for a single ergodic automorphism T of a compact abelian group G is based on the computation of the moments, when f is trigonometric polynomial. It uses the fact that T is mixing of all orders, a property shown by Rohlin [21], consequence of the K-property for ergodic automorphisms.

For $\mathbb{Z}^d$-actions by automorphisms on connected compact abelian groups, in particular on tori, the method of moments can also be used, since the mixing property of all orders holds (Theorem 2.16 below). First we prove a CLT for trigonometric polynomials using the mixing property, then the result is extended to regular functions by approximation.

Mixing of actions by endomorphisms (G connected)

For $\mathbb{Z}^d$-actions by automorphisms on compact abelian groups mixing of all orders is not always satisfied (cf. [15], [25]). In 1992, W. Philip [20] and K. Schmidt and T. Ward [24] used results about the number of solutions of S-units equations in the study of semigroups of endomorphisms or automorphisms on compact abelian groups. In particular, the following mixing of all orders for $\mathbb{Z}^d$-actions by automorphisms on connected groups is a consequence of algebraic results on S-units ([22]):

Theorem 2.16. *([24, Corollary 3.3]) Let $\underline{n} \to T^{\underline{n}}$ be a mixing $\mathbb{Z}^d$-action on a compact, connected, abelian group G. Then it is r-mixing for every $r \geq 2$.*

Corollary 2.17. *Let $\mathcal{S}$ be a semigroup of endomorphisms on a compact connected abelian group G. If this action is totally ergodic, it is mixing of all orders.*

Proof. We use the notations of Lemma 2.1. The group G in Lemma 2.1 is connected and Theorem 2.16 applies to the group $\tilde{\mathcal{S}}$ of automorphisms of G in which $\mathcal{S}$ is embedded. The action of $\tilde{\mathcal{S}}$ is mixing of all orders, hence also the action of $\mathcal{S}$. □

CLT for $\mathbb{N}^d$-action by endomorphisms

Theorem 2.18. *Let $\underline{n} : (n_1, ..., n_d) \to T^{\underline{n}} = T_1^{n_1}...T_d^{n_d}$ be a totally ergodic $\mathbb{N}^d$-action by commuting endomorphisms on a compact abelian connected group G. Let $(D_n)_{n\geq 1}$ be a Følner sequence in $\mathbb{N}^d$ and let f be in $AC_0(G)$. We have[3] $\sigma^2(f) = \varphi_f(0)$ and*

$$(\sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell})^2)^{-\frac{1}{2}} \sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell}) f(A^{\underline{\ell}}.) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \sigma^2(f)).$$

Proof. Let $(\mathcal{N}_s)$ be an increasing sequence of finite sets in $\hat{G}$ with union $\hat{G} \setminus \{0\}$ and let $f_s(x) := \sum_{\chi\in\mathcal{N}_s} c_f(\chi)\, \chi$ be the trigonometric polynomial obtained by restriction of the Fourier series of f to $\mathcal{N}_s$. Let Z_n^s, Z_n denote respectively

$$Z_n^s := |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}} R_n(\underline{\ell}) f_s(A^{\underline{\ell}}.),\ Z_n := |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}} R_n(\underline{\ell}) f(A^{\underline{\ell}}.).$$

By Theorem 2.6 we have $\sigma(f - f_s) \leq \|f - f_s\|_2$. It follows $\sigma^2(f) := \lim_s \sigma^2(f_s)$ and $\sigma^2(f_s) \neq 0$ for s big enough, since $\sigma^2(f) > 0$ by hypothesis.

[3]with the convention that the limiting distribution is δ_0 if $\sigma^2(f) = 0$.

For the kernel K_n associated to D_n, by the Følner property and (8), we have, if g satisfies (19):

$$|D_n|^{-1} \, \| \sum_{\underline{\ell}} R_n(\underline{\ell}) \, A^{\underline{\ell}} g \|_2^2 = \int_{\mathbb{T}^d} K_n \, \varphi_g \, dt \underset{n\to\infty}{\to} \varphi_g(0) = \sigma^2(g).$$

From Theorem 2.16 (mixing of all orders) and the result for the trigonometric polynomial f_s, it follows: $Z_n^s \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma^2(f_s))$ for every s. Moreover, since

$$\begin{aligned} \limsup_n \int |Z_n^s - Z_n|_2^2 \, d\mu \quad &= \limsup_n \int_{\mathbb{T}^d} K_n \, \varphi_{f-f_s} \, dt \\ &= \lim_n \int_{\mathbb{T}^d} K_n \, \varphi_{f-f_s} \, dt = \sigma^2(f - f_s) \leq C \|f - f_s\|_2^2, \end{aligned}$$

we have $\limsup_n \mu[|Z_n^s - Z_n| > \varepsilon] \leq \varepsilon^{-2} \limsup_n \int |Z_n^s - Z_n|_2^2 \, d\mathbb{P} \underset{s\to 0}{\to} 0$ for every $\varepsilon > 0$ and the condition $\lim_s \limsup_n \mathbb{P}[|Z_n^s - Z_n| > \varepsilon] = 0$ is satisfied.

The conclusion $Z_n \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma^2(f))$ follows from Theorem 3.2 in [2], □

We have in particular:

$$|D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} f(A^{\underline{\ell}}.) \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma^2(f)).$$

The previous result is valid for the rotated sums: if f in $AC_0(G)$, then, for every θ,

$$\sigma_\theta^2(f) = \varphi_f(\theta), \ |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} e^{2\pi i \langle \underline{\ell}, \theta \rangle} f(A^{\underline{\ell}}.) \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma_\theta^2(f)). \tag{33}$$

If f satisfies the regularity condition (35), then $\sigma_\theta^2(f) = 0$ if and only if there are continuous functions $u_{t,\theta}$ on $\mathbb{T}^\rho$, for $t = 1, ..., d$, such that $f = \sum_{t=1}^d (I - e^{2\pi i \theta} A_t) u_{t,\theta}$.

This applies in particular when (D_n) is a sequence of d-dimensional cubes in $\mathbb{Z}^d$.

A CLT for the rotated sums for a.e. θ without regularity assumptions

For the summation sequence given by d-dimensional cubes, a CLT for the rotated sums can be shown for a.e. θ without regularity assumptions on f. The proof relies on (5) which is satisfied, for any given $f \in L^2(G)$, for θ in a set of full measure. This extends results of [4].

Theorem 2.19. *Let $\underline{n} \to A^{\underline{n}}$ be a totally ergodic d-dimensional action by commuting endomorphisms on G. Let $(D_n)_{n\geq 1}$ be a sequence of cubes in $\mathbb{Z}^d$. Let $f \in L^2(G)$. For a.e. $\theta \in \mathbb{T}^d$, we have $\sigma_\theta^2(f) = \varphi_f(\theta)$ and*

$$|D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} e^{2\pi i \langle \underline{\ell}, \theta \rangle} f(A^{\underline{\ell}}.) \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma_\theta^2(f)).$$

Let us mention that, if we take for D_n triangles instead of squares, a CLT for the rotated sums is also valid for a.e. θ, provided f satisfies $\sum_{\chi \in \hat{G}} |c_f(\chi)|^r < +\infty$, for some $r < 2$.

Other examples of kernels

Theorem 2.20. *Let $(R_n)_{n\geq 1}$ be a summation sequence on $\mathbb{Z}^d$ which is regular and such that $(\tilde{R}_n(t))$ weakly converges to a measure ζ on the circle. Let f be a function in $AC_0(G)$ with spectral density φ_f. If $\zeta(\varphi_f) \neq 0$, then we have*

$$\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) f(A^{\underline{\ell}}.) / (\sum_{\underline{\ell} \in \mathbb{Z}^d} |R_n(\underline{\ell})|^2)^{\frac{1}{2}} \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \zeta(\varphi_f)).$$

Proof. The proof is the same as that of Theorem 2.18 and uses (20) and the convergence:

$$\lim_n \| \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})\, A^{\underline{\ell}} f\|_2^2 / \sum_{\underline{\ell} \in \mathbb{Z}^d} |R_n(\underline{\ell})|^2 = \lim_n \int_{\mathbb{T}^d} K_n(t)\, \varphi_f(t)\, dt = \zeta(\varphi_f).$$

□

Barycenter operators

The iterates of the barycenter operators satisfy the condition of Theorem 2.20.

Let $A_1, ..., A_d$ be to commuting endomorphisms of a connected abelian group G generating a totally ergodic action. Let P be the barycenter operator defined as in Formula (3) by:

$$Pf(x) := \sum_j p_j\, f(A_j x). \tag{34}$$

Observe that the coefficient $R_n(\underline{\ell})$ of the expansion of the summation sequence associated to $n^{\frac{d-1}{4}} P^n$ tends to 0 uniformly, when n tends to infinity (to prove it, one can use the local limit theorem for multinomial Bernoulli variables).

By Proposition 1.7 and Theorem 2.20 we obtain:

Theorem 2.21. *Let f be a function in $AC_0(G)$ with spectral density φ_f. Assume that $\sigma_P^2(f) := \int_{\mathbb{T}} \varphi_f(u, u, ..., u)\, du \neq 0$. Then we have*

$$(4\pi)^{\frac{d-1}{4}} (p_1 ... p_d)^{\frac{1}{4}}\, n^{\frac{d-1}{4}}\, P^n f(.) \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma_P^2(f)).$$

For the torus, the result holds for f satisfying Condition (38) in the next subsection.

Example: let A_1, A_2 be two commuting matrices in $\mathcal{M}^*(\rho, \mathbb{Z})$ generating a totally ergodic action on $\mathbb{T}^\rho$, $\rho \geq 3$. Let P be the barycenter operator: $Pf(x) := \frac{1}{2}(f(A_1 x) + f(A_2 x))$.

If φ_f is continuous, then we have $\lim_{n\to\infty} \sqrt{\pi n}\, \|P^n f\|_2^2 = \int_{\mathbb{T}} \varphi_f(u, u)\, du$. It follows from Theorem 2.21, for f satisfying (38) on $\mathbb{T}^\rho$:

$$(\pi n)^{\frac{1}{4}}\, P^n f \underset{n\to\infty}{\overset{distr}{\longrightarrow}} \mathcal{N}(0, \sigma_P^2(f)).$$

For any non trivial character χ on G, the spectral density is identically 1 and $\sigma_P(\chi) = 1$. The rate of convergence of $\|P^n \chi\|_2$ to 0 is the polynomial rate given by Proposition 1.7. If f is in $AC_0(G)$, we have $\sigma_P(f) = 0$ if and only if $\varphi_f(u, u) = 0$, for every $u \in \mathbb{T}^1$. In particular, by the results of Subsection 2.4, if f is not a mixed coboundary (cf. (16)), then $\sigma_P(f) \neq 0$ and the rate of convergence of $\|P^n f\|_2$ to 0 is the polynomial rate given by Proposition 1.7. A test of non degeneracy on periodic points can be deduced from it.

The condition $\sigma_P(f) = 0$ is stronger than the coboundary condition. A sufficient condition to have $\sigma_P(f) = 0$ is that f can be written $f = A_1 g - A_2 g$ with $g \in L^2_0(\mu)$.

Remarks. 1) The case of commutative or amenable actions strongly differs from the case of non amenable actions for which a "spectral gap property" is often available ([11]). For action by algebraic (non commuting) automorphisms $A_j, j = 1, ..., d$, on the torus, the existence of a spectral gap for P of the form (34) is related to the fact that the generated group has no factor torus on which it is virtually abelian ([1]).

2) For ν a discrete measure on the semigroup $\mathcal{T}$ of commuting endomorphisms of G, we can consider a barycenter of the form $Pf(x) = \sum_{T \in \mathcal{T}} \nu(T) f(Tx)$. For a barycenter with finite support, we have seen that the decay, when φ_f is continuous, is of order $n^{\frac{d-1}{2}}$. A question is to estimate the decay when ν has an infinite support and to study the asymptotic distribution of the normalized iterates.

For instance, if we $Pf(x) = \sum_{q \in \mathcal{P}} \nu(q) f(qx)$, where $\mathcal{P}$ is the set of prime numbers, $(\nu(q), q \in \mathcal{P})$ a probability vector with $\nu(q) > 0$ for every prime q and f Hölderian on the circle, what is the decay to 0 of $\|P^n f\|_2$?

A partial result is that, if ν has an infinite support, the decay is faster than Cn^{-r}, for every $r \geq 1$. Indeed, this can be deduced easily from the following observation:

Let P_1 and P_2 be two commuting contractions of $L^2(G)$, such that $\|P_1^n f\|_2 \leq Mn^{-r}$ and let $\alpha \in]0, 1], \beta = 1 - \alpha$. Then we have:

$$\|(\alpha P_1 + \beta P_2)^n f)\|_2 \leq \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \|P_1^k f\|_2.$$

Using the inequality of large deviation for the binomial law, we obtain, with $c < 1$:

$$\sum_{k \leq \frac{n}{2\alpha}} \binom{n}{k} \alpha^k \beta^{n-k} \leq c^n,$$

and therefore:

$$\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \|P_1^k f\|_2 \leq M(\frac{n}{2\alpha})^{-r} - c^n \leq M' n^{-r}.$$

2.4. **The torus case.**

Notation 2.22. Let $\mathcal{M}^*(\rho, \mathbb{Z})$ denote the semigroup of $\rho \times \rho$ non singular matrices with coefficients in $\mathbb{Z}$ and $GL(\rho, \mathbb{Z})$ the group of matrices with coefficients in $\mathbb{Z}$ and determinant ± 1.

Every A in $\mathcal{M}^*(\rho, \mathbb{Z})$ defines a surjective endomorphism of $\mathbb{T}^\rho$, hence a measure preserving transformation on $(\mathbb{T}^\rho, \mu)$ and a dual endomorphism on the group of characters of $\mathbb{T}^\rho$ identified with $\mathbb{Z}^\rho$ (action by the transposed of A). If A is in $GL(\rho, \mathbb{Z})$, it defines an automorphism of $\mathbb{T}^\rho$. For simplicity, since the matrices are commuting, we use the same notation for the matrix A, its action on the torus and the dual endomorphism, without writing transposition.

So, when G is a torus $\mathbb{T}^\rho$, $\rho \geq 1$, we consider a finite set of endomorphisms given by matrices A_i in $\mathcal{M}^*(\rho, \mathbb{Z})$. The group generated by the matrices A_i in $GL(\rho, \mathbb{Q})$ is supposed to be torsion-free.

The construction of Lemma 2.1 can be describe in the following way. Let p_i be the determinant of A_i, for each i. If $\tilde{G}$ is the compact group dual of the discrete group $\tilde{\mathbb{Z}}^\rho := \{\underline{k}\, \Pi p_i^{\ell_i}, \underline{k} \in \mathbb{Z}^\rho, \ell_i \in \mathbb{Z}\}$, then $\mathbb{Z}^\rho$ is a subgroup of $\tilde{\mathbb{Z}}^\rho$ and $\tilde{G}$ has $\mathbb{T}^\rho$ as a factor.

It is well known that ergodicity for the action of a single $A \in \mathcal{M}^*(\rho, \mathbb{Z})$ on $(\mathbb{T}^\rho, \mu)$ is equivalent to the absence of eigenvalue root of 1 for A. Recall also Kronecker's result: an integer matrix with all eigenvalues on the unit circle has all eigenvalues roots of unity.

For a torus, total ergodicity is equivalent to the property that $A^{\underline{n}}$ has no eigenvalue root of unity, for $\underline{n} \neq \underline{0}$. Replacing $\underline{n}$ by a multiple, there is no $\underline{n} \neq \underline{0}$ such that $A^{\underline{n}}$ has a fixed vector $v \neq \{0\}$. In other words, total ergodicity is equivalent to say that the $\mathbb{Z}^d$-action $\underline{n} = (n_1, ..., n_d) : (v \to A^{\underline{n}} v)$ on $\mathbb{Z}^\rho \setminus \{\underline{0}\}$ is free.

Using a common triangular representation over $\mathbb{C}$ for the commuting matrices A_j, one sees that if $\lambda_{1j}, ..., \lambda_{\rho j}$ are the eigenvalues of A_j (with multiplicity), for $j = 1, ..., d$, this is equivalent to $(\prod_{j=1}^d \lambda_{ij}^{n_j} = 1 \ \Rightarrow (n_1, ..., n_d) = \underline{0})$, $\forall i \in \{1, ..., \rho\}$.

Lemma 2.23. *Let $B \in \mathcal{M}^*(\rho, \mathbb{Z})$ be a matrix with irreducible (over $\mathbb{Q}$) characteristic polynomial P. Let $\{A_1, ..., A_d\}$ be d matrices in $\mathcal{M}^*(\rho, \mathbb{Z})$ commuting with B. They generates a commutative semigroup of endomorphisms on $\mathbb{T}^\rho$ which is totally ergodic, if and only if for any $\underline{n} \in \mathbb{Z}^d \backslash \{\underline{0}\}$, $A^{\underline{n}} \neq Id$.*

Proof. Since P is irreducible, the eigenvalues of B are distinct. It follows that (on $\mathbb{C}$) the matrices A_i are simultaneously diagonalizable, hence are pairwise commuting. Now suppose that there are $\underline{n} \in \mathbb{Z}^d \backslash \{\underline{0}\}$ and $v \in \mathbb{Z}^\rho \setminus \{\underline{0}\}$ such that $A^{\underline{n}} v = v$. Let W be the subspace of $\mathbb{R}^\rho$ generated by v and its images by B. The restriction of $A^{\underline{n}}$ to W is the identity. W is B-invariant, the characteristic polynomial of the restriction of B to W has rational coefficients and factorizes P. By the assumption of irreducibility over $\mathbb{Q}$, this implies $W = \mathbb{R}^\rho$. Therefore $A^{\underline{n}}$ is the identity. $\square$

Lemma 2.24. *Let $\mathcal{S}$ be the semigroup generated by d matrices $\{A_1, ..., A_d\}$ in $\mathcal{M}^*(\rho, \mathbb{Z})$ with the irreducibility property like in Lemma 2.23, with determinant q_i. If the numbers $\log |q_i|$ are linearly independent over $\mathbb{Q}$, then $\mathcal{S}$ is totally ergodic.*

Rate of decorrelation for automorphisms of the torus

The analysis for a general group G relies on the absolute convergence of the Fourier series of a function f on G, which is ensured, for the torus, if f satisfies the following regularity condition:

$$|c_f(\underline{k})| = O(\|k\|^{-\beta}), \text{ with } \beta > \rho. \tag{35}$$

Nevertheless, as we will see now, for the torus, a weaker regularity condition on f can be used in the study of the spectral density and the CLT. This is closely related to the rate of decorrelation for regular functions, which is based on the following lemma:

Lemma 2.25. *(D. Damjanović and A. Katok,* [6] *for automorphisms, M. Levine (*[19] *for endomorphisms) If $(A^{\underline{n}}, \underline{n} \in \mathbb{Z}^d)$ is a totally ergodic $\mathbb{Z}^d$-action on $\mathbb{T}^\rho$ by automorphisms, there are $\tau > 0$ and $C > 0$, such that for all $(\underline{n}, \underline{k}) \in \mathbb{Z}^d \times (\mathbb{Z}^\rho \setminus \{\underline{0}\})$ for which $A^{\underline{n}}\underline{k} \in \mathbb{Z}^\rho$.*

$$\|A^{\underline{n}}\underline{k}\| \geq C e^{\tau \|\underline{n}\|} \|\underline{k}\|^{-\rho}. \tag{36}$$

The proof of the previous result for automorphisms uses the fact that if B is in $\mathcal{M}^*(\rho, \mathbb{Z})$ and V a m-dimensional eigenspace of B such that $V \cap \mathbb{Z}^\rho = \{0\}$, then there exists a constant C such that, for every $\underline{j} \in \mathbb{Z}^\rho \setminus \{\underline{0}\}$, the distance $d(\underline{j}, V)$ of $\underline{j}$ to V satisfies $d(\underline{j}, V) \geq C \|\underline{j}\|^{-m}$ (cf. [18], Katznelson [14, Lemma 3]) and a result of [3]. The extension to endomorphisms was obtained by M. Levine in the recent paper [19] mentioned in the introduction.

Regularity and Fourier series

We need some results from the theory of approximation of functions by trigonometric polynomials.

For $f \in L^2(\mathbb{T}^d)$, the rectangular Fourier partial sums of f are denoted by $S_{N_1,...,N_d}(f)$. The *integral modulus of continuity* of f is defined as

$$\omega_2(\delta_1, \cdots, \delta_d, f) = \sup_{|\tau_1| \leq \delta_1, ..., |\tau_d| \leq \delta_d} \|f(x_1 + \tau_1, \cdots, x_d + \tau_d) - f(x_1, \cdots, x_d)\|_2.$$

Let $J_{N_1,...,N_d}(t_1, \cdots, t_d) = K^2_{N_1,...,N_d}(t_1, \cdots, t_d) / \|K_{N_1,...,N_d}\|^2_{L^2(\mathbb{T}^d)}$ be the d-dimensional Jackson's kernel, where $K_{N_1,...,N_d}$ is the d-dimensional Fejér kernel.

Clearly, $J_{N_1,...,N_d}(t_1, \cdots, t_d) = J_{N_1}(t_1) \cdots J_{N_d}(t_d)$. It is known that the 1-dimensional Jackson's kernel satisfies the following moment relations:

$$\int_0^{\frac{1}{2}} t^k \, J_N(t) \, dt = O(N^{-k}), \ \forall N \geq 1, \ k = 0, 1, 2. \tag{37}$$

Lemma 2.26. *There exists a positive constant C_d such that, for every $f \in L^2(\mathbb{T}^d)$, for every $N_1, \dots, N_d \geq 1$, $\|J_{N_1,\dots,N_d} * f - f\|_2 \leq C_d\, \omega_2(\frac{1}{N_1}, \cdots, \frac{1}{N_d}, f)$.*

Proof. Since $\omega_2(\delta_1, \cdots, \delta_d, f)$ is increasing and subadditive with respect to δ_i, we have for any positive numbers λ_i: $\omega_2(\lambda_1\delta_1, \cdots, \lambda_d\delta_d, f) \leq (\lambda_1+1)\cdots(\lambda_d+1)\, \omega_2(\delta_1, \cdots, \delta_d, f)$.

Using this inequality and (37), we obtain:

$$\begin{aligned}
\|J_{N_1,\dots,N_d} * f - f\|_2 &\leq \textstyle\int_{[-\frac{1}{2},\frac{1}{2}[^d} J_{N_1,\dots,N_d}(\tau_1, \cdots, \tau_d)\, \|f(. - \tau_1, \cdots, . - \tau_d) - f\|_{L^2}\, d\tau_1 \cdots d\tau_d \\
&\leq 2^d \textstyle\int_{[0,\frac{1}{2}[^d} J_{N_1,\dots,N_d}(\tau_1, \cdots, \tau_d)\, \omega_2(\tau_1, \cdots, \tau_d, f)\, d\tau_1 \cdots d\tau_d \\
&= 2^d \textstyle\int_{[0,\frac{1}{2}[^d} J_{N_1,\dots,N_d}(\tau_1, \cdots, \tau_d)\ \omega_2(\frac{N_1\tau_1}{N_1}, \cdots, \frac{N_d\tau_d}{N_d}, f)\, d\tau_1 \cdots d\tau_d \\
&\leq 2^d\omega_2(\tfrac{1}{N_1}, \cdots, \tfrac{1}{N_d}, f) \textstyle\int_{[0,\frac{1}{2}[^d} (N_1\tau_1 + 1) \cdots (N_d\tau_d + 1)\, J_{N_1,\dots,N_d}(\tau_1, \cdots, \tau_d)\, d\tau_1 \cdots d\tau_d \\
&= 2^d\omega_2(\tfrac{1}{N_1}, \cdots, \tfrac{1}{N_d}, f)\, \textstyle\prod_{i=1}^d \int_0^{\frac{1}{2}} (N_i\tau_i + 1) J_{N_i}(\tau_i)\, d\tau_i \leq C_d\, \omega_2(\tfrac{1}{N_1}, \cdots, \tfrac{1}{N_d}, f).
\end{aligned}$$

□

Proposition 2.27. *There exists a positive constant C_d, such that, for every $f \in L^2(\mathbb{T}^d)$ and $N_1, \dots, N_d \geq 1$, we have $\|f - S_{N_1,\dots,N_d}(f)\|_2 \leq C_d\, \omega_2(\frac{1}{N_1}, \cdots, \frac{1}{N_d}, f)$.*

Proof. For every trigonometric polynomial P in d variables of degree at most $N_1 \times \cdots \times N_d$, we have: $\|f - S_{N_1,\dots,N_d}(f)\|_2 \leq \|f - P\|_2$. The result follows then from Lemma 2.26. □

By Proposition 2.27, the following condition on the modulus of continuity:

(38) There are $\alpha > 1$ and $C(f) < +\infty$ such that $\omega_2(\delta, ..., \delta, f) \leq C(f)\, (\ln \dfrac{1}{\delta})^{-\alpha}, \forall \delta > 0.$

implies:

$$\|f - s_{N,\dots,N}(f)\|_2 \leq R(f)\, (\ln N)^{-\alpha}, \text{ with } \alpha > 1. \tag{39}$$

One easily checks that (35) implies (39).

In what follows in this subsection, $\underline{n} \to A^{\underline{n}}$ is a totally ergodic $\mathbb{Z}^d$-action by *endomorphisms* on $\mathbb{T}^\rho$. Recall that $\tilde{f}$ denotes the extension to $\tilde{G}$ of a function f on G (here $G = \mathbb{T}^\rho$) and that we use the convention (*) (i.e., we put $c_{A^{\underline{n}}\underline{k}}(f) = 0$ if $A^{\underline{n}}\underline{k} \notin \mathbb{Z}^\rho$). We denote simply by $|.|$ the norm of an integral vector. Recall that we do not write the transposition for the dual action of $A^{\underline{n}}$. The proof of the following proposition is like that of the analogous result in [18].

Proposition 2.28. *Let $f \in L^2_0(\mathbb{T}^\rho)$ satisfying (39) and $f_1(x) := \sum_{\underline{n} \in \mathcal{N}_1} c_{\underline{n}}(f) e^{2\pi i \langle \underline{n}, x\rangle}$, where $\mathcal{N}_1$ is a subset of $\mathbb{Z}^\rho$. Then there is a finite constant $B(f)$ depending only on $R(f)$ such that*

$$|\langle A^{\underline{n}} f_1, f_1\rangle| \leq B(f) \|f_1\|_2 \|\underline{n}\|^{-\alpha}, \ \forall \underline{n} \neq \underline{0}. \tag{40}$$

Proof. It suffices to prove the result for f, since, by setting $c_f(n) = 0$ outside N_1, we obtain (40) with the same constant $B(f)$ as shown by the proof. Let λ, b, d such that $1 < \lambda < e^\tau$, $1 < b < \lambda^{\frac{1}{\rho}}$, $\lambda b^{-\rho} = d > 1$. We have for $\underline{n} \in \mathbb{Z}^d$:

$$\langle A^{\underline{n}} f, f\rangle = \sum_{\underline{k} \in \mathbb{Z}^\rho} c_{\underline{k}}(f)\, \overline{c}_{A^{\underline{n}}\underline{k}}(f) = \sum_{|\underline{k}|<b^{|\underline{n}|}} + \sum_{|\underline{k}|\geq b^{|\underline{n}|}} . \tag{41}$$

From Inequality (36) of Lemma 2.25, we deduce that, if $|\underline{k}| < b^{|\underline{n}|}$, then $|A^{\underline{n}}\underline{k}| \geq D\lambda^{\underline{n}}\, |\underline{k}|^{-\rho} \geq D\lambda^{|\underline{n}|}\, b^{-\rho|\underline{n}|} = Dd^{|\underline{n}|}$, $\underline{n} \neq \underline{0}$. It follows, for the first sum:

$$|\sum_{|\underline{k}|<b^{|\underline{n}|}} c_{\underline{k}}(f)\ \overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq (\sum_{|\underline{k}|<b^{|\underline{n}|}} |c_{\underline{k}}(f)|^2)^{\frac{1}{2}}\ (\sum_{|\underline{k}|<b^{|\underline{n}|}} |c_{A^{\underline{n}}\underline{k}}(f)|^2)^{\frac{1}{2}} \leq \|f\|_2 \sum_{|\underline{m}|>Dd^{|\underline{n}|}} |c_{\underline{m}}(f)|^2.$$

By Parseval inequality and (39), there is a finite constant $B_1(f)$ such that, for $|\underline{n}| \neq 0$:

$$(\sum_{|\underline{m}|>Dd^{|\underline{n}|}} |c_{\underline{m}}(f)|^2)^{\frac{1}{2}} \quad \leq \|f - s_{[Dd^{|\underline{n}|}],\dots,[Dd^{|\underline{n}|}]}\|_2 \leq \frac{R(f)}{(\ln[Dd^{|\underline{n}|}])^\alpha} \leq B_1(f)|\underline{n}|^{-\alpha}. \tag{42}$$

From the previous inequalities, it follows:

$$|\sum_{|\underline{k}|<b^{|\underline{n}|}} c_n(f)\, \overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq B_1(f)\|f\|_2|\underline{n}|^{-\alpha}, \forall|\underline{n}| \neq 0. \tag{43}$$

Analogously, for the second sum in (41) we obtain

$$|\sum_{|\underline{k}|\geq b^{|\underline{n}|}} c_{\underline{k}}(f)\, \overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq B_2(f)\|f\|_2|\underline{n}|^{-\alpha},\ |\underline{n}| \neq 0. \tag{44}$$

Taking $B(f) = B_1(f) + B_2(f)$, (40) follows from (41), (43), (44). ☐

The following theorem has the same conclusion as Theorem 2.6, but requires a weaker regularity condition.

Theorem 2.29. *If f satisfies (39), (in particular if f satisfies the regularity condition (38)), then $\sum_{\underline{n}\in\mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| < \infty$, the variance $\sigma^2(f)$ exists, $\sigma^2(f) = \sum_{\underline{n}\in\mathbb{Z}^d} \langle A^{\underline{n}} f, f\rangle$, the density φ_f of the spectral measure of f is continuous.*

Moreover, there is a constant C such that, if $\mathcal{N}$ is any subset of $\mathbb{Z}^\rho$ and $f_1(x) = \sum_{\underline{k}\in\mathcal{N}} c_{\underline{k}}(f)e^{2\pi i\langle \underline{k},x\rangle}$, then $\sigma(f - f_1) \leq C\|f - f_1\|_2$.

Proof. The Fourier coefficients of φ_f are $\langle A^{\underline{n}} f, f\rangle$. The previous proposition implies [(39) $\Rightarrow$ $\sum_{\underline{n}\in\mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| < +\infty$] and the second statement. ☐

Coboundary characterization

Using the previous result on the decay of correlation, let us give a sufficient condition on the Fourier series of f for the coboundary characterization. Recall that J denotes a section of the $\mathbb{Z}^d$-action by automorphisms on $\mathbb{Z}^\rho$ (cf. Remark 2.30).

Remark 2.30. For $G = \mathbb{T}^\rho$, each character is identified to an element $\underline{k}$ of $\mathbb{Z}^\rho$. It is useful to choose the section in the following way. For a fixed $\underline{\ell}$, the set $\{A^{\underline{k}}\underline{\ell}, \underline{k} \in \mathbb{Z}^d\}$ is discrete and $\lim_{\|\underline{k}\|\to\infty} \|A^{\underline{k}}\underline{\ell}\| = +\infty$. Therefore the minimum of the norm is achieved for some value of $\underline{k}$. We can choose an element $\underline{j}$ in each class modulo the action of $\tilde{\mathcal{S}}$ on $\mathbb{Z}^\rho$, which achieves the minimum of the norm. By this choice, we have

$$\|\underline{j}\| \leq \|A^{\underline{k}}\underline{j}\|, \forall \underline{j} \in J, \underline{k} \in \mathbb{Z}^d. \tag{45}$$

The sufficient condition (15) given in Lemma 1.9 for the coboundary representation reads in the algebraic framework

$$\sum_{j\in J} \sum_{\underline{k}\in\mathbb{Z}^d} (1 + \|\underline{k}\|^d)\, |c_f(A^{\underline{k}} j)| < \infty. \tag{46}$$

Theorem 2.31. *If $|c_f(\underline{k})| = O(\|k\|^{-\beta})$, with $\beta > \rho$, we have $\sigma^2(f) = 0$ if and only if f is a mixed coboundary: there are continuous functions u_i, $i = 1, ..., d$ such that*

$$f = \sum_{i=1}^{d} (I - A_i) u_i. \tag{47}$$

Proof. Let $\varepsilon \in]0, \beta-\rho[$ and $\delta := (\beta-\rho-\varepsilon)/(\beta(1+\rho))$, we have $\delta\beta\rho - \beta(1-\delta) = -(\rho+\varepsilon)$.
There is a constant C_1 such that $\|\underline{k}\|^d\, e^{-\delta\beta\tau\|\underline{k}\|} \leq C_1$, $\forall \underline{k} \in \mathbb{Z}^d$.
According to (36), we have $|c_f(A^{\underline{k}}\underline{j})| \leq C\|A^{\underline{k}}\underline{j}\|^\beta \leq Ce^{-\beta\tau\|\underline{k}\|}\, \|\underline{j}\|^{\beta\rho}$; hence

$$e^{\delta\beta\tau\|\underline{k}\|}\, |c_f(A^{\underline{k}} j)|^\delta \leq C\|\underline{j}\|^{\delta\beta\rho}. \tag{48}$$

Recall that, for every $\underline{\ell} \in \mathbb{Z}^\rho\backslash\{\underline{0}\}$, there is a unique pair $(k, j) \in \mathbb{Z}^d \times J$ such that $A^{\underline{k}}\underline{j} = \underline{\ell}$. Therefore we have, using Inequality (45) (see Remark 2.30):

$$\begin{aligned}
&\sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|\underline{k}\|^d\, |c_f(A^{\underline{k}}\underline{j})| = \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|\underline{k}\|^d\, |c_f(A^{\underline{k}}\underline{j})|^\delta |c_f(A^{\underline{k}}\underline{j})|^{1-\delta} \\
&\quad \leq C_1 \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} e^{\delta\beta\tau\|\underline{k}\|}\, |c_f(A^{\underline{k}}\underline{j})|^\delta\, |c_f(A^{\underline{k}}\underline{j})|^{1-\delta} \;\leq C_2 \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|\underline{j}\|^{\delta\beta\rho}\, |c_f(A^{\underline{k}}\underline{j})|^{1-\delta} \\
&\leq C_2 \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|A^{\underline{k}}\underline{j}\|^{\delta\beta\rho}\, |c_f(A^{\underline{k}}\underline{j})|^{1-\delta} \text{ by (48) and (45)} \\
&\quad = C_2 \sum_{\underline{\ell}\in\mathbb{Z}^\rho\backslash\{\underline{0}\}} \|\underline{\ell}\|^{\delta\beta\rho}\, |c_f(\underline{\ell})|^{1-\delta} \leq C_3 \sum_{\underline{\ell}\in\mathbb{Z}^\rho\backslash\{\underline{0}\}} \|\underline{\ell}\|^{\delta\beta\rho-\beta(1-\delta)} \leq C_3 \sum_{\underline{\ell}\in\mathbb{Z}^\rho\backslash\{\underline{0}\}} \|\underline{\ell}\|^{-(\rho+\varepsilon)} < +\infty.
\end{aligned}$$

This implies that (46) is satisfied. Since here the functions involved in the proof of Lemmas 1.8 and 1.9 are characters, hence continuous and uniformly bounded, we have continuity of the functions u_i in the representation (16). □

Now we consider the CLT when G is the torus $\mathbb{T}^d$. As mentioned in the introduction, an analogous result for d-dimensional rectangles and a class of regular functions was recently obtained by M. Levine ([19]).

Theorem 2.32. *Let* $\underline{n} \to A^{\underline{n}}$ *be a totally ergodic* $\mathbb{N}^d$*-action by commuting matrices on* $\mathbb{T}^\rho$*. Let* $(R_n)_{n\geq 1}$ *be a Følner summation sequence.*

1) If f *satisfies (39), in particular if* f *satisfies the regularity condition (38), we have* $\sigma^2(f) = \varphi_f(0)$ *and*

$$(\sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell})^2)^{-\frac{1}{2}} \sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell})\, f(A^{\underline{\ell}}.) \overset{distr}{\underset{n\to\infty}{\longrightarrow}} \mathcal{N}(0,\sigma^2(f)).$$

2) If f *satisfies (35) (i.e.,* $|c_f(\underline{k})| = O(\|k\|)^{-\beta}$*, with* $\beta > \rho$*), then* $\sigma^2(f) = 0$ *if and only if*[4] *there are continuous functions* u_t *on* $\mathbb{T}^\rho$*, for* $t = 1, ..., d$*, such that* $f = \sum_{t=1}^d (I - A_t)u_t$*.*

Proof. 1) The proof of the first statement is like the proof of Theorem 2.18. Here we use the inequality $\sigma(f - f_s) \leq C\|f - f_s\|_2$, where the constant C does not depend on s, given by Theorem 2.29.

2) The second assertion follows from Theorem 2.31. □

2.5. **Appendix: examples of $\mathbb{Z}^d$-actions by automorphisms.** The aim of this appendix is to give explicit examples of commuting matrices generating totally ergodic $\mathbb{Z}^d$-actions on tori. We mainly recall some known facts. (See in particular [13] and [6] for the construction of $\mathbb{Z}^d$-actions by automorphisms on the torus.)

The construction of $\mathbb{Z}^d$-action by automorphisms on $\mathbb{T}^\rho$ is linked to the groups of units in number fields. Following [13], let us recall some facts.

Let $M \in GL(\rho, \mathbb{Z})$ be a matrix with an irreducible characteristic polynomial $P = P(M)$ and hence distinct eigenvalues. The centralizer of M in $\mathcal{M}(n, \mathbb{Q})$ can be identified with the ring of all polynomials in M with rational coefficients modulo the principal ideal generated by the polynomial $P(M)$, and hence with the field $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of M, by the map $\gamma : p(A) \to p(\gamma)$ with $p \in \mathbb{Q}[x]$.

By Dirichlet's theorem, if P has d_1 real roots and d_2 pairs of complex conjugate roots, then there are $d_1 + d_2 - 1$ fundamental units in the group of units in the ring of integers in $K(P)$. This provides a totally ergodic $\mathbb{Z}^{d_1+d_2-1}$-action by automorphisms on $\mathbb{T}^\rho$.

Explicit computation of examples relies on an algorithm (see [5]). The first computed examples appeared with the development of the computers. Even nowadays computations are limited to low dimensional examples.

Examples for $\mathbb{T}^3$

To give a concrete example for $\mathbb{T}^3$, we explicit a pair A, B of matrices in $SL(3, \mathbb{Z})$ such that $\{A, B\}$ generates a free action in $\mathbb{Z}^2$.

[4] This gives a test of non degeneracy of the limiting law in terms of periodic points.

We start with a integer polynomial $P(X) = -X^3 + qX + n$ which is irreducible over $\mathbb{Q}$ and its companion matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ n & q & 0 \end{pmatrix}.$$

Let $K(P)$ denote the number field associated to P. Suppose that $K(P)$ belongs to any of the tables where the characteristics of the first cubic real fields $K(P)$ are listed. Let θ be a root of P. The table gives a pair of fundamental units for the group of units in the ring of integers in $K(P)$ of the form $P_1(\theta)$, $P_2(\theta)$, where P_1, P_2 are two integer polynomials. Then the matrices $A_1 = P_1(M)$ and $A_2 = P_2(M)$ give a system of generators of the group of matrices in $GL(3, \mathbb{Z})$ commuting with M, generating a totally ergodic $\mathbb{Z}^2$-action on $\mathbb{T}^3$ by automorphisms.

Explicit examples

1) (from the table in [26]) Let us consider the polynomial $P(X) = X^3 - 12X - 10$ and its companion matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 12 & 0 \end{pmatrix}.$$

Let θ be a root of P. The table gives the pair of fundamental units for the algebraic group associated to P:

$$P_1(\theta) = \theta^2 - 3\theta - 3, \; P_2(\theta) = -\theta^2 + \theta + 11.$$

The matrices $A_1 = P_1(M)$ and $A_2 = P_2(M)$ give a system of generators of the group of matrices in $GL(3, \mathbb{Z})$ commuting with M. They generate a totally ergodic action of $\mathbb{Z}^2$ by automorphisms on $\mathbb{T}^3$. They have 3 real eigenvalues and $\det(A_1) = 1, \det(A_2) = -1$.

$$A_1 = \begin{pmatrix} -3 & -3 & 1 \\ 10 & 9 & -3 \\ -30 & -26 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 11 & 1 & -1 \\ -10 & -1 & 1 \\ 10 & 2 & -1 \end{pmatrix}.$$

2) (from tables in [26] and in [5]) Let us consider now the polynomial $P(X) = X^3 - 9X - 2$ and its companion matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 9 & 0 \end{pmatrix}.$$

Let θ be a root of P. The table gives the pair of fundamental units for the algebraic group associated to P:

$$P_1(\theta) = 3\theta^2 - 9\theta - 1, \; P_2(\theta) = 2\theta^2 - 4\theta - 1.$$

The matrices $A_1 = P_1(M)$ and $A_2 = P_2(M)$ give a system of generators of the group of matrices in $GL(3,\mathbb{Z})$ commuting with M. They generate a totally ergodic action of $\mathbb{Z}^2$ by automorphisms on $\mathbb{T}^3$. They have 3 real eigenvalues and $\det(A_1) = 1, \det(A_2) = -1$.

$$A_1 = \begin{pmatrix} -3 & -3 & 1 \\ 10 & 9 & -3 \\ -30 & -26 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 11 & 1 & -1 \\ -10 & -1 & 1 \\ 10 & 2 & -1 \end{pmatrix}.$$

Remark that in [26] a different set of generators is given. The polynomials are

$$P_1'(\theta) = 85\theta^2 - 245\theta - 59, \; P_2'(\theta) = -18\theta^2 + 4\theta + 161.$$

The matrices $A_1' = P_1'(M)$ and $A_2' = P_2'(M)$ give another pair of generators of the group of matrices in $GL(3,\mathbb{Z})$ commuting with M. The relations between the two pairs are:

$$A_1' = A_1 A_2, \; A_2' = A_1^{-1}.$$

A simple example on $\mathbb{T}^4$

If $P(X) = X^4 + aX^3 + bX^2 + aX + 1$, the polynomial P has two real roots: $\lambda_0, \lambda_0^{-1}$ and two complex conjugate roots of modulus 1: $\lambda_1, \overline{\lambda}_1$. Let $\sigma_j = \lambda_j + \overline{\lambda}_j$, $j = 0, 1$. They are roots of $Z^2 - aZ + b - 2 = 0$.

Under the conditions: $a^2 - 4b + 8 > 0$, $a > 4$, $b > 2$, $2a > b+2$, i.e., (since $2a - 2 \leq \frac{1}{4}a^2 + 2$)

$$2 < b < \frac{1}{4}a^2 + 2, \; a > 4,$$

$\lambda_0, \lambda_0^{-1}$ are solution of $\lambda^2 - \sigma_0\lambda + 1 = 0$, and $\lambda_1, \overline{\lambda}_1$ are solutions of $\lambda^2 - \sigma_1\lambda + 1 = 0$, where

$$\sigma_0 = -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4b + 8}, \; \sigma_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b + 8}.$$

The polynomial P is not factorizable over $\mathbb{Q}$. Indeed, suppose that $P = P_1P_2$ with P_1, P_2 with rational coefficients and degree ≥ 1. Since the roots of P are irrational, the degrees of P_1 and P_2 are 2. Necessarily their roots are, say, $\lambda_1, \overline{\lambda}_1$ for P_1, $\lambda_0, \lambda_0^{-1}$ for P_2. The sum $\lambda_1 + \overline{\lambda}_1$, root of $Z^2 - aZ + b - 2 = 0$, is not rational and the coefficients of P_1 are not rational. Let

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -a & -b & -a \end{pmatrix}, B = A + I.$$

From the irreducibility over $\mathbb{Q}$, it follows that, if there is a non zero fixed integral vector for A^kB^ℓ, where k, ℓ are in $\mathbb{Z}$, then we have $A^kB^\ell = Id$. This implies: $\lambda_1^k(\lambda_1 - 1)^k = 1$, hence, since we have $|\lambda_1| = 1$, it follows $|\lambda_1 - 1| = 1$ which clearly is not true.

Example: $P(X) = X^4 + 5X^3 + 7X^2 + 5X + 1$. If A is the companion matrix, then A and $A + 1$, with characteristic polynomials $P(X)$ and $X^4 + X^3 - 2X^2 + 2X - 1$ respectively, generate a $\mathbb{Z}^2$-totally ergodic action on $\mathbb{T}^4$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -5 & -7 & -5 \end{pmatrix}, \ B = A + I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -5 & -7 & -4 \end{pmatrix}.$$

This elementary example gives only a $\mathbb{Z}^2$-action on $\mathbb{T}^4$. A question is to produce an example with full dimension 3.

3) **Construction by blocks** Let M_1, M_2 be two ergodic matrices respectively of dimension d_1 and d_2. Let p_i, q_i, $i = 1, 2$ be two pairs of integers such that $p_1 q_2 - p_2 q_1 \neq 0$. On the torus $\mathbb{T}^{d_1+d_2}$ we obtain a $\mathbb{Z}^2$-totally ergodic action by taking A_1, A_2 of the following form:

$$A_1 = \begin{pmatrix} M_1^{p_1} & 0 \\ 0 & M_2^{q_1} \end{pmatrix}, \ A_2 = \begin{pmatrix} M_1^{p_2} & 0 \\ 0 & M_2^{q_2} \end{pmatrix}.$$

Indeed, if there exists $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{Z}^{d_1+d_2} \setminus \{0\}$ invariant by $A_1^n A_2^\ell$, then $M_1^{np_1+\ell p_2} v_1 = v_1$, $M_2^{nq_1+\ell q_2} v_2 = v_2$, which implies $np_1 + \ell p_2 = 0$, $nq_1 + \ell q_2 = 0$; hence $n = \ell = 0$.

This is a method to obtain explicit free $\mathbb{Z}^2$-actions on $\mathbb{T}^4$. The same method gives explicit free $\mathbb{Z}^3$-actions on $\mathbb{T}^5$ (by using a $\mathbb{Z}$-action on $\mathbb{T}^2$ and a $\mathbb{Z}^2$-action on $\mathbb{T}^3$).

We do not know explicit examples of full dimension, i.e., with 3 independent generators on $\mathbb{T}^4$, or with 4 independent generators on $\mathbb{T}^5$.

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