On random Fourier-Stieltjes transforms

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ABSTRACT. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\{\mu_k\}_{k=1}^\infty$ be a sequence of centered independent finite complex valued transition measures on $\Omega \times \mathcal{B}(\mathbb{R}^d)$, i.e., (i) for every $\omega \in \Omega$, $\{\mu_k(\omega, \cdot)\}$ is a sequence of finite complex valued measures on $\mathcal{B}(\mathbb{R}^d)$; (ii) for every $n \neq m$ and for every two Borel measurable simple functions ϕ and ψ on $\mathbb{R}^d$, the random variables $\int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_n(\omega, d\mathbf{u})$ and $\int_{\mathbb{R}^d} \psi(\mathbf{u})\mu_m(\omega, d\mathbf{u})$ are independent, and for every $k \geq 1$, $\int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_k(\omega, d\mathbf{u})$ is centered. Put $V_k(\omega) = |\mu_k|(\omega, \mathbb{R}^d)$ for the total variation norm and assume that $\{V_k\} \subset L_\infty(\mathbf{P})$. Let $\{L_k\}$ be a sequence with $L_k \geq 1$ and put $L_{n,m} = \sum_{k=n+1}^m L_k^2$. If $\sum_{n=1}^\infty \sum_{m=n+1}^\infty \frac{1}{(L_{n,m})^2} < \infty$, then there exists absolute constants $\epsilon > 0$ and $C > 0$, independent of $\{\mu_k\}$, such that (with 0/0 interpreted as 1),

$$\Big\| \sup_{m>n} \sup_{T\geq 2} \exp\Big\{\epsilon \cdot \frac{\max_{\mathbf{t}\in[0,T]^d} |\sum_{k=n+1}^m \int_{[-L_k,L_k]^d} \mathrm{e}^{i\langle \mathbf{t},\mathbf{u}\rangle} \mu_k(\omega, d\mathbf{u})|^2}{\log[(L_{n,m})^{d/2+2}T^{d+2}] \sum_{k=n+1}^m \|V_k\|_\infty^2}\Big\}\Big\|_1$$

does not exceed C. This result extends and unifies results of Weber and Cohen-Cuny. New applications are also given. For example, if $\{X_k\} \subset L_2(\Omega, \mathbf{P})$ is a sequence of centered independent complex valued random variables such that $\sum_{n=1}^\infty \frac{(\sum_{k\geq n} k^{2d}\|X_k\|_2^2)^{1/2}}{n\sqrt{\log n}}$ converges, then $\mathbf{P}$-a.s. the random series $\sum_{n=1}^\infty X_n \Pi_{j=1}^d \frac{\sin(nt_j)}{t_j}$ converges uniformly in $(t_1, \ldots, t_d) \in [0,T]^d$, for every $T > 0$.

1. Introduction

In several recent works, like Assani [**A**], Boukhari and Weber [**BW**], Cohen and Lin [**CL**], Cohen and Cuny [**CC2, CC3**] (see also the references therein),

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convergence of power series of contractions (or measure preserving transformations) with random coefficients (usually realization of independent random variables) was considered. As a typical example (see [**CC2**]) we have the following:

Theorem A. *Let $\{X_n\} \subset L_2(\Omega, \mathbf{P})$ be a sequence of centered independent random variables, and let $\{p_n\}$ be a non-decreasing sequence of natural numbers. If the series $\sum_{n=1}^{\infty} \|X_n\|_2^2 (\log n)^2 \log p_n$ converges, then there exists a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$ the series $\sum_{n=1}^{\infty} X_n(\omega) T^{p_n} g$ converges π-a.s., for every contraction T on $L_2(\pi)$ and every $g \in L_2(\pi)$.*

From [**CC3**] we take the following example:

Theorem B. *Let $\{\theta_k\}$ be a sequence of independent $\mathbb{N}$-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, and let $\{c_k\}$ be a sequence of complex numbers. Let Φ be some positive non-decreasing function on $\mathbf{R}^+$, such that for some $\eta > 0$ we have $\Phi(x) \geq x^\eta$ for every $x \geq 0$. Assume that for some $\delta > 0$ we have $\sum_{n=1}^{\infty} \mathbf{P}[\theta_n > \Phi(n)^\delta] < \infty$. Let $\{a_n\}$ be a sequence of complex numbers. If the series $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^2 \log \Phi(n)$ converges, then there exists a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$ the series $\sum_{n=1}^{\infty} a_n (T^{\theta_n(\omega)} g - \mathbb{E}[T^{\theta_n(\cdot)}] g)$ converges π-a.s., for every contraction T on $L_2(\pi)$ and every $g \in L_2(\pi)$.*

Various extensions of these two results were considered in [**CC2, CC3**], among them: series with two commuting contractions, more than two commuting isometries, or series with d-commuting (semi) flows ($d \geq 1$) with (positive) real powers.

All the above cited works based their a.s. convergence results on uniform bounds of trigonometric (almost periodic) polynomials. Especially, the results of [**BW, CL**] are based on a uniform estimate which was introduced in an important work of Weber [**W1**] (see Theorem 7 there). In [**CC2, CC3**] generalizations of this uniform estimate were given; these generalizations made it possible there to obtain the multi-dimensional cases mentioned above. The following two estimates were obtained in [**CC2**] and [**CC3**], respectively:

Theorem C. *Let $\{X_k\}$ be a sequence of complex valued, symmetric independent random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, and let $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \ldots, \lambda_k^{(d)})\} \subset \mathbb{R}^d$. Put $|\boldsymbol{\lambda}_m|^* = \max\limits_{1 \leq k \leq m} \max\{|\lambda_k^{(1)}|, \ldots, |\lambda_k^{(d)}|\}$. Then there exists absolute constants $\epsilon > 0$ and $C > 0$, independent of $\{X_k\}$, such that (with 0/0 interpreted as 1),*

$$\mathbb{E}\Big[\sup_{m>n} \sup_{T \geq 2} \exp\Big\{\epsilon \cdot \frac{\max\limits_{\mathbf{t} \in [0,T]^d} |\sum_{k=n+1}^{m} X_k e^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}|^2}{\log[(|\boldsymbol{\lambda}_m|^* + 1) m T] \sum_{k=n+1}^{m} |X_k|^2}\Big\}\Big] \leq C\,.$$

Here and throughout the paper $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$.

Theorem D. *Let $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \ldots, \lambda_k^{(d)})\}$ be a sequence of independent $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, and let $\{c_k\}$ be a sequence of complex numbers. Let Φ be some positive non-decreasing function on $\mathbf{R}^+$, such that for some $\eta > 0$ we have $\Phi(x) \geq x^\eta$ for every $x \geq 0$. Assume that for some $\delta > 0$ we have $\sum_{n=1}^{\infty} \mathbf{P}[|\boldsymbol{\lambda}_n| > \Phi(n)^\delta] < \infty$. Then there exists absolute constants $\epsilon > 0$ and $C > 0$, independent of $\{c_k\}$, such that (with 0/0 interpreted as 1),*

$$\mathbb{E}\Big[\sup_{m>n} \sup_{T \geq 2} \exp\Big\{\epsilon \cdot \frac{\max_{\mathbf{t} \in [0,T]^d} |\sum_{k=n+1}^{m} c_k (e^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle} - \mathbb{E}[e^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}])|^2}{\log[(\Phi(m) + 1) m T] \sum_{k=n+1}^{m} |c_k|^2}\Big\}\Big] \leq C\,.$$

These two results, extensions of results of [**W1**], were used in the main steps in [**CC2, CC3**] to prove a.s. convergence of random power series of contractions and its generalizations mentioned above. Furthermore, such estimates were used in order to obtain also a.s. uniform convergence of random almost periodic (or Fourier) series. This is the importance of such estimates.

The results of [**W1**] were obtained using the *metric entropy* method while the results of [**CC2, CC3**] were obtained by quite simple basic tools and a procedure of bypassing the so-called Bernstein's inequality. Bernstein's inequality was used earlier in this context by Salem and Zygmund [**SZ**].

In [**CC3**] (see also [**W1**]) Theorem D was deduced from Theorem C. A major goal of this paper is to obtain Theorem C and Theorem D from one *more general* result (see Theorem 2.8). Another goal is to obtain new applications, like Theorem 3.4, Corollary 3.5, Theorem 3.6, and Corollary 3.8, also concerning uniform convergence of more general series than trigonometric series.

2. Main results

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\mathcal{B} := \mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra of $\mathbb{R}^d$, $d \geq 1$. Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of finite complex valued transition measures on $\Omega \times \mathcal{B}$. More precisely, for every $k = 1, 2, \ldots$, we have

(i) $\mu_k(\omega.\cdot)$ is a finite complex valued measure on $\mathcal{B}$ for any fixed $\omega \in \Omega$.
(ii) $\mu_k(\cdot, B)$ is an $\mathcal{F}$-measurable function for any fixed $B \in \mathcal{B}$.

A transition measure μ_k is a *random measure* $\mu_k(\omega, \cdot)$ on $\mathcal{B}$. We always assume that the total variation norm $V_k(\omega) := |\mu_k|(\omega, \mathbb{R}^d)$ is (at least) integrable on $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition. The sequence of finite transition measures $\{\mu_k\}_{k=1}^{\infty}$ is called *independent* if for every $n \neq m$ and for every two simple functions ϕ and ψ on $\mathbb{R}^d$ (Borel measurable), the random variables $\int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_n(\omega, d\mathbf{u})$ and $\int_{\mathbb{R}^d} \psi(\mathbf{u})\mu_m(\omega, d\mathbf{u})$ are independent. The sequence $\{\mu_k\}$ is called *centered* if for every simple function ϕ on $\mathbb{R}^d$ and every $k \geq 1$, the random variable $\int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_k(\omega, d\mathbf{u})$ is centered, i.e., $\mathbb{E}[\int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_k(\omega, d\mathbf{u})] := \int_{\Omega} \int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_k(\omega, d\mathbf{u}) d\mathbf{P} = 0$.

Denote vectors in $\mathbb{R}^d$ by boldface, e.g., $\mathbf{t} = (t_1, \cdots, t_d)$ and $\mathbf{u} = (u_1, \cdots, u_d)$, and put $\langle \mathbf{t}, \mathbf{u} \rangle = t_1 u_1 + \ldots + t_d u_d$ the inner product in $\mathbb{R}^d$. By $|\mathbf{t}|$ we denote $\max\{|t_1|, \ldots, |t_d|\}$, and for a positive sequence $\{c_n\}$ we denote by c_m^* the value $\max\limits_{1 \leq n \leq m} c_n$.

Given a sequence $\mathbf{L} := \{L_n\}$ of positive numbers. For every $n \geq 1$ and for fixed $\mathbf{t} \in \mathbb{R}^d$, we define the ($\mathcal{F}$-measurable) random variable $\hat{\mu}_{n,\mathbf{L}}(\omega, \mathbf{t}) = \int_{[-L_n, L_n]^d} \mathrm{e}^{i\langle \mathbf{t}, \mathbf{u} \rangle} \mu_n(\omega, d\mathbf{u})$. To simplify the notation, we omit $\mathbf{L}$ and ω, and put $\hat{\mu}_n(\mathbf{t}) = \int_{[-L_n, L_n]^d} \mathrm{e}^{i\langle \mathbf{t}, \mathbf{u} \rangle} \mu_n(d\mathbf{u})$. If $\mu_n(\omega, \cdot)$ is supported on $[-L_n, L_n]^d$, then $\hat{\mu}_{n,\mathbf{L}}(\omega, \mathbf{t})$ is the random *Fourier-Stieltjes transform* of $\mu_n(\omega, \cdot)$ computed at $\mathbf{t}$, otherwise it is a truncated Fourier-Stieltjes transform. For every $m > n \geq 0$ we put $\hat{\mu}_{n,m} = \sum_{k=n+1}^{m} \hat{\mu}_k$, and for any $T > 0$ we put $\tilde{\mu}_{n,m}(T) = \max_{\mathbf{t} \in [0,T]^d} |\hat{\mu}_{n.m}(\mathbf{t})|$, (which all depend on $\mathbf{L}$ and ω).

Definition. We call a sequence of transition measures $\{\mu_k\}$ $\mathbf{L}$-independent, $\mathbf{L}$-centered, $\mathbf{L}$-symmetric, if for every $\mathbf{t} \in \mathbb{R}^d$ the random variables $\{\hat{\mu}_{k,\mathbf{L}}(\mathbf{t})\}$ are independent, centered, symmetric, respectively.

In particular, if the sequence of transition measures $\{\mu_k\}$ is independent and/or centered it is $\mathbf{L}$-independent and/or $\mathbf{L}$-centered for every sequence $\mathbf{L}$ (for short we might say $\mathbf{L}$-centered independent if both properties hold). It is only the random variables $\{\hat{\mu}_{k,\mathbf{L}}(\mathbf{t})\}$ that are of interest in this paper, and the above definition is more useful for our context although not so natural to start with.

The following examples of centered (or symmetric) independent transition measures are considered in the paper.

Example 2.1. Let $\{X_k\}$ be a sequence of centered independent random variables and let $\{\eta_k\}$ be a sequence of complex finite measures on $\mathcal{B}$ with finite total variation norms. Define $\mu_k(\omega, B) = X_k(\omega) \cdot \eta_k(B)$. Then $\{\mu_k\}$ forms a centered independent sequence of transition measures. For every $\mathbf{L}$ we have $\hat{\mu}_k(\mathbf{t}) = X_k \cdot \int_{[-L_k,L_k]^d} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle} \eta_k(d\mathbf{u})$ and in particular, the sequence $\{\mu_k\}$ is $\mathbf{L}$-centered and $\mathbf{L}$-independent. If $\{X_k\}$ is symmetric independent, then $\{\mu_k\}$ is $\mathbf{L}$-symmetric and $\mathbf{L}$-independent. In applications we usually take one of the following cases:

(i): Let $\{\boldsymbol{\lambda}_k\}$ be a sequence of vectors in $\mathbb{R}^d$ and define $\eta_k(B) = \delta_{\boldsymbol{\lambda}_k}(B)$. Then $\hat{\mu}_k(\mathbf{t}) = X_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle} \mathbf{1}_{[-L_k,L_k]^d}(\boldsymbol{\lambda}_k)$. Usually we take $L_k = |\boldsymbol{\lambda}_k|$.

(ii): Let $\{h_k\} \subset L_1(\mathbb{R}^d, d\mathbf{u})$ and define $\eta_k(B) = \int_B h_k(\mathbf{u}) d\mathbf{u}$. Then $\hat{\mu}_k(\mathbf{t}) = X_k \cdot \int_{[-L_k,L_k]^d} \mathrm{e}^{i\langle \mathbf{t}, \mathbf{u}\rangle} h_k(\mathbf{u}) d\mathbf{u}$.

Example 2.2. Let $\{\mu_k\}$ be a sequence of $\mathbf{L}$-symmetric and $\mathbf{L}$-independent transition measures and take a copy of $([0,1], \mathcal{B}([0,1]), dx)$, independent of $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\{\epsilon_k\}$ be a Rademacher sequence on $[0,1]$. Then $\epsilon_k(x)\mu_k(\omega, B)$ is a transition measure on $([0,1] \times \Omega) \times \mathcal{B}$. By symmetry and independence, the sequences $\{\widehat{\epsilon_k \mu_k}(\mathbf{t})\} = \{\epsilon_k \hat{\mu}_k(\mathbf{t})\}$ and $\{\hat{\mu}_k(\mathbf{t})\}$ have the same finite dimensional joint probability distributions on $[0,1] \times \Omega$ and Ω, respectively. So, a claim on $\{\epsilon_k \hat{\mu}_k(\mathbf{t})\}$ is valid a.s. if and only if a corresponding claim on $\{\hat{\mu}_k(\mathbf{t})\}$ is valid a.s. This type of symmetrization will be used in Theorem 3.4 later.

Example 2.3. Note that if $\mu(\omega, \cdot)$ is a transition measure with integrable total variation norm, i.e., $|\mu|(\omega, \mathbb{R}^d)$ is integrable, then by Lebesgue bounded convergence theorem $\int \mu(\omega, \cdot)\mathbf{P}(d\omega)$ is a deterministic measure. Let $\{\boldsymbol{\lambda}_k\}$ be a sequence of $\mathbb{R}^d$-valued independent random variables, and let $\{c_k\}$ be a sequence of complex numbers. Fix $\mathbf{L}$ and put $\Omega_k = \{|\boldsymbol{\lambda}_k| \le L_k\}$. Define

$$\mu_k(\omega, B) = c_k \delta_{\boldsymbol{\lambda}_k(\omega)}(B) - \mathbb{E}[c_k \delta_{\boldsymbol{\lambda}_k(\cdot)}(B)] = c_k \mathbf{1}_B(\boldsymbol{\lambda}_k(\omega)) - \mathbb{E}[c_k \mathbf{1}_B(\boldsymbol{\lambda}_k(\cdot))].$$

If ϕ is a Borel simple function on $\mathbb{R}^d$, then by Fubini's theorem (for transition measures, see Neveu [**N**]) we have $\int_{\mathbb{R}^d} \phi(\mathbf{u})\mu_k(\omega, d\mathbf{u}) = c_k\phi(\boldsymbol{\lambda}_k(\omega)) - c_k\mathbb{E}[\phi(\boldsymbol{\lambda}_k(\cdot))]$. Then $\{\mu_k\}$ forms a centered independent sequence of transition measures. Here $\hat{\mu}_k(\mathbf{t}) = c_k \mathbf{1}_{\Omega_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle} - \mathbb{E}[c_k \mathbf{1}_{\Omega_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle}]$ and the sequence $\{\mu_k\}$ is $\mathbf{L}$-independent and $\mathbf{L}$-centered.

Lemma 2.4. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\{\mu_k\}_{k=1}^\infty$ be a sequence of finite complex valued transition measures on $\Omega \times \mathcal{B}$. Let $\mathbf{L}$ be a sequence of positive numbers. Then for every $T > 0$, $\omega \in \Omega$, and $m > n \ge 0$, there exists a* *cube* $I = I(\omega) \subset [0,T]^d$*, with volume*

$$|I| \geq \Big[\frac{\max_{\mathbf{t}\in[0,T]^d} |\hat{\mu}_{n,m}(\mathbf{t})|}{2d(\sum_{k=n+1}^m L_k^2)^{1/2}(\sum_{k=n+1}^m V_k^2)^{1/2}}\Big]^d,$$

such that

$$\max_{\mathbf{t}\in[0,T]^d} |\hat{\mu}_{n,m}(\mathbf{t})| \leq 2|\hat{\mu}_{n,m}(\mathbf{u})| \quad \textit{for every } \mathbf{u} \in I\,.$$

Proof. Clearly, $\hat{\mu}_{n,m}(\mathbf{t})$ is differentiable with respect to the components of $\mathbf{t}$. Hence, for $j = 1, \dots, d$,

$$\Big|\frac{\partial \hat{\mu}_{n,m}}{\partial t_j}(\mathbf{t})\Big| = \Big|\sum_{k=n+1}^m \int_{[-L_k,L_k]^d} u_j i e^{i\langle \mathbf{t},\mathbf{u}\rangle}\mu_k(d\mathbf{u})\Big| \leq \sum_{k=n+1}^m L_k \int_{[-L_k,L_k]^d} |\mu_k|(d\mathbf{u}) \leq$$

$$\sum_{k=n+1}^m L_k V_k \leq (\sum_{k=n+1}^m L_k^2)^{1/2}(\sum_{k=n+1}^m V_k^2)^{1/2}\,.$$

Let $\mathbf{t}^* \in [0,T]^d$ be a point for which $\max_{\mathbf{t}\in[0,T]^d} |\hat{\mu}_{n.m}(\mathbf{t})| = |\hat{\mu}_{n,m}(\mathbf{t}^*)|$. Hence, for every $\mathbf{t} \in [0,T]^d$ we have

$$\hat{\mu}_{n,m}(\mathbf{t}^*) - \hat{\mu}_{n,m}(\mathbf{t}) = \sum_{j=1}^d (t_j^* - t_j)\frac{\partial \hat{\mu}_{n,m}}{\partial t_j}(t_1', \dots, t_d'),$$

where $(t_1', \cdots, t_s')$ is on the line segment joining $\mathbf{t} = (t_1, \cdots, t_d)$ and $\mathbf{t}^* = (t_1^*, \cdots, t_d^*)$. So,

$$|\hat{\mu}_{n,m}(\mathbf{t}^*)| - |\hat{\mu}_{n,m}(\mathbf{t})| \leq \sum_{j=1}^d |t_j^* - t_j|\Big|\frac{\partial \hat{\mu}_{n,m}}{\partial t_j}(t_1', \dots, t_d')\Big| \leq$$

$$(\sum_{k=n+1}^m L_k^2)^{1/2}(\sum_{k=n+1}^m V_k^2)^{1/2}\sum_{j=1}^d |t_j^* - t_j|\,.$$

Put

$$I = \Big\{\mathbf{t} \in [0,T]^d : |t_j^* - t_j| \leq \frac{\max_{\mathbf{t}\in[0,T]^d} |\hat{\mu}_{n,m}(\mathbf{t})|}{2d(\sum_{k=n+1}^m L_k^2)^{1/2}(\sum_{k=n+1}^m V_k^2)^{1/2}}\Big\}\,.$$

□

Remarks. 1. The above lemma is inspired by Kahane [**K1**, LEMME].

2. The assumption that μ_k, $k \geq 1$, are transition measures was not used. One can assume that $\mu_k(\omega, B)$ is independent of ω.

3. The above lemma is a kind of generalization of Lemma 2.2 in [**CC2**]. Indeed, take $\mu_k = c_k\delta_{\boldsymbol{\lambda}_k}$, where $\{\boldsymbol{\lambda}_k\} \subset \mathbb{R}^d$, $\{c_k\}$ is a sequence of complex numbers, and put $L_k = |\boldsymbol{\lambda}_k|$. It is more in the spirit of what we could have obtained using Bernstein's inequality (see e.g. Kahane [**K2**, Proposition 5]) for *periodic* trigonometric polynomials. Note that in the almost periodic case, i.e., when the vectors $\{\boldsymbol{\lambda}_k\}$ have (just) *real* coordinates, Bernstein's inequality is not applicable. In the above lemma and also in [**CC2**, Lemma 2.2], Bernstein's inequality was not used.

We recall two results that we use in the sequel. The following lemma is basically Lemma 3 in Paley and Zygumd [**PZ**, part I] (see also [**CC2**, Lemma 2.4]), and can be proved using Stirlings's approximation.

Lemma 2.5. *Let Z be a non-negative random variable on $(\Omega, \mathbf{P})$, and let C_1 and C_2 be some positive constants. If $\int Z^{2n} d\mathbf{P} \le C_1(C_2 n)^n$ for every $n \ge 1$, then $\int \exp(\delta Z^2) d\mathbf{P} \le 1 + \frac{C_1}{1-\mathrm{e}\delta C_2}$ for every $\delta < \frac{1}{\mathrm{e}C_2}$.*

The following result is a consequence of E. Rio [**R**, Théorème 2.4] (which even for bounded martingale differences gives better constants than Burkholder's inequality).

Lemma 2.6. *Let $\{Y_k\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of centered random variables. Let $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ be the σ-algebra generated by $\{Y_1, \ldots, Y_n\}$. Then for every $l \ge j \ge 1$ and for every natural $p = 1, 2, \ldots$, we have*

$$\mathbb{E}\Big[\Big|\sum_{k=j}^{l} Y_k\Big|^{2p}\Big] \le \frac{(2p)!}{p!2^p}\Big(\sum_{k=j}^{l}\|Y_k\|_\infty^2 + \sum_{k=j}^{l}\max_{k\le s\le l}\|2Y_k\sum_{v=k+1}^{s}\mathbb{E}[Y_v|\mathcal{F}_k]\|_\infty\Big)^p.$$

If for every $k \ge 1$ the total variation norm $V_k(\omega) = |\mu_k|(\omega, \mathbb{R}^d)$ is a bounded random variable, we define

$$R_{n,m} = \sum_{k=n+1}^{m}\|V_k\|_\infty^2.$$

Also, for our positive sequence $\{L_k\}$ we put $L_{n,m} = \sum_{k=n+1}^{m} L_k^2$; we also recall our notation $\tilde{\mu}_{n,m}(T) = \max_{\mathbf{t}\in[0,T]^d}|\hat{\mu}_{n,m}(\mathbf{t})|$..

Proposition 2.7. *Let $\mathbf{L} = \{L_k\}$ be a sequence of positive numbers. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\{\mu_k\}_{k=1}^\infty$ be a sequence of finite complex valued transition measures on $\Omega \times \mathcal{B}$, which is $\mathbf{L}$-centered independent. Assume that $|\mu_k|(\omega, \mathbb{R}^d)$ is a* bounded *function for every $k \ge 1$. Then for every $m > n \ge 0$ and for every $T > 0$ we have (with 0/0 interpreted as 1),*

$$\Big\|\Big[\frac{\tilde{\mu}_{n,m}(T)}{(R_{n,m})^{1/2}}\Big]^d \exp\Big\{\frac{1}{8\mathrm{e}}\cdot\frac{[\tilde{\mu}_{n,m}(T)]^2}{R_{n,m}}\Big\}\Big\|_{L_1(\mathbf{P})} \le 3\cdot 2^d d^d (L_{n,m})^{d/2} T^d\,.$$

Proof. As the summands below are independent, using Lemma 2.6 we have for every integer $p \ge 1$ and every $\mathbf{t}$,

$$\mathbb{E}\Big[\big|\hat{\mu}_{n,m}(\mathbf{t})\big|^{2p}\Big] = \mathbb{E}\Big[\Big|\sum_{k=n+1}^{m}\int_{[-L_k,L_k]^d}\mathrm{e}^{i\langle\mathbf{t},\mathbf{u}\rangle}\mu_k(d\mathbf{u})\Big|^{2p}\Big] \le \frac{(2p)!}{p!2^p}(R_{n,m})^p \tag{1}$$

$$= \frac{(p+1)(p+2)\cdots(2p)}{2^p}(R_{n,m})^p \le \frac{(2p)^p}{2^p}(R_{n,m})^p \le (p\cdot R_{n,m})^p$$

By Lemma 2.5 we obtain that

$$\int_\Omega \exp\{\delta|\hat{\mu}_{n,m}(\mathbf{t})|^2\}d\mathbf{P} \le 1 + \frac{1}{1-\mathrm{e}\delta R_{n,m}} \quad \text{for every} \quad \delta < \frac{1}{\mathrm{e}R_{n,m}} \tag{$*$}$$

By Lemma 2.4, for every $\omega \in \Omega$ there exists a cube $I = I(\omega)$, such that

$$\tilde{\mu}_{n,m}(T)\mathbf{1}_I(\mathbf{t}) \le 2|\hat{\mu}_{n,m}(\mathbf{t})| \quad \text{for every } \mathbf{t} \in [0,T]^d\,.$$

By applying $x \mapsto \exp(\delta x^2)$ and integrating over $[0,T]^d$, we obtain

$$\Big[\frac{\tilde{\mu}_{n,m}(T)}{2d(L_{n,m})^{1/2}(R_{n,m})^{1/2}}\Big]^d \exp\Big\{\delta\cdot[\tilde{\mu}_{n,m}(T)]^2\Big\} \le$$

$$\int_{[0,T]^d}\exp\Big\{4\delta\cdot|\hat{\mu}_{n,m}|^2(\mathbf{t})\Big\}d\mathbf{t}\,.$$

Integrating over Ω, applying Fubini, and using (*) we obtain

$$\mathbb{E}\Big[\Big[\frac{\tilde{\mu}_{n,m}(T)}{2d(L_{n,m})^{1/2}(R_{n,m})^{1/2}}\Big]^d \exp\Big\{\delta\cdot[\tilde{\mu}_{n,m}(T)]^2\Big\}\Big] \le$$

$$T^d\Big(1+\frac{1}{1-4\mathrm{e}\delta R_{n,m}}\Big) \quad \text{for every} \quad \delta < \frac{1}{4\mathrm{e}R_{n,m}}\,.$$

The result follows by taking $\delta = \frac{1}{8\mathrm{e}R_{n,m}}$. □

Remarks. **1.** Burkholder's inequality [**HH**, Theorem 2.10] will yield only the estimate $(Cp^2\cdot R_{n,m})^p$ in the right hand side of (1), which is not sufficient to obtain the hypothesis Lemma 2.5. Even the use of the best constants in Burkholder's inequality [**Hi**] does not help.

2. Define the σ-algebras

$$\mathcal{F}_n = \sigma\{\mu_k(\cdot,B):\, 1\le k\le n,\, B\in\mathcal{B}\}\,.$$

If instead of independence we assume that the transition measures $\{\mu_k\}$ satisfy $\mathbb{E}[\mu_{n+1}(\cdot,B)|\mathcal{F}_n]=0$, for every $B\in\mathcal{B}$ and every $n\ge 1$, then for every $\mathbf{t}\in\mathbb{R}^d$ and for every $n\ge 1$, $\mathbb{E}[\hat{\mu}_{n+1}(\mathbf{t})|\hat{\mu}_n(\mathbf{t}),\dots,\hat{\mu}_1(\mathbf{t})]=0$. Hence if $|\mu_k|(\omega,\mathbb{R}^d)$ is bounded, then $\{\hat{\mu}_k(\mathbf{t})\}$ forms a sequence of bounded martingale differences for every $\mathbf{t}$. This means that when we apply Lemma 2.6 in the proof of Proposition 2.7, we still obtain the same inequality (1) for this choice of transition measures. In this case, the conditional expectations in Lemma 2.6 vanish as in the independent case.

3. In a more general situation, where we have a control on the conditional expectations $\mathbb{E}[|\mu_n|(\cdot,B)|\mathcal{F}_k]$, for every $B\in\mathcal{B}$ and every $n>k$, Rio's inequality is still applicable. In such situations, $R_{n,m}$ in inequality (1) will contain terms related to these conditional expectations (see [**CC2**, §6] for related remarks and references).

4. Let $\{X_k\}\subset L_\infty(\mathbf{P})$ be centered (not necessarily independent) random variables, and let $\{\boldsymbol{\lambda}_k\}\subset\mathbb{R}^d$. Put $\mu_k(\omega,B)=X_k(\omega)\delta_{\boldsymbol{\lambda}_k}(B)$ and $L_k=|\boldsymbol{\lambda}_k|$. For this setup, Proposition 2.7 leads immediately to [**CC2**, Theorem 3.1] when considering there the characters $\{\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}\}$ as a σ_n-system. In this case $R_{n,m}$ involves conditional expectation terms (which can be written directly by $\{X_k\}$), which appear also in [**CC2**, Theorem 3.1].

From now on, all the logarithms will be taken with respect to the base 2. Also, the short notation $\sup_{m>n}$ means suprema over all pairs of integers $m>n\ge 0$.

Theorem 2.8. *Let $\mathbf{L}=\{L_k\}$ be a sequence of positive numbers with $L_k\ge 1$, such that $\sum_{n=1}^\infty\sum_{m=n+1}^\infty\frac{1}{(L_{n,m})^2}$ converges. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space and let $\{\mu_k\}_{k=1}^\infty$ be a sequence of finite complex valued transition measures on $\Omega\times\mathcal{B}$, which is $\mathbf{L}$-centered independent. Assume that $|\mu_k|(\omega,\mathbb{R}^d)$ is a* bounded *function for every $k\ge 1$. Then there exists absolute constants $\epsilon>0$ and $C>0$, independent of $\{\mu_k\}$, such that (with 0/0 interpreted as 1),*

$$\Big\|\sup_{m>n}\sup_{T\ge 2}\exp\Big\{\epsilon\cdot\frac{\max_{\mathbf{t}\in[0,T]^d}|\sum_{k=n+1}^m\int_{[-L_k,L_k]^d}\mathrm{e}^{i\langle\mathbf{t},\mathbf{u}\rangle}\mu_k(d\mathbf{u})|^2}{R_{n,m}\log[(L_{n,m})^{d/2+2}T^{d+2}]}\Big\}\Big\|_1\le C\,.$$

Proof. By uniform continuity, the measurable function $\max\limits_{\mathbf{t}\in[0,T]^d}|\sum_{k=n+1}^m\int_{[-L_k,L_k]^d}\mathrm{e}^{i\langle\mathbf{t},\mathbf{u}\rangle}\mu_k(d\mathbf{u})|^2$ is a continuous function of T. So, the suprema over $T\ge 2$ can be taken as a suprema over the rational numbers. Hence, the integrand is measurable. Since the numerator and the denominator are monotone increasing function of T, the suprema over the rationals $T \geq 2$ can be approximated by suprema over the naturals $T \geq 2$.

Using Proposition 2.7 we obtain

$$\Big\| \Big[\frac{\tilde{\mu}_{n,m}(T)}{(R_{n,m})^{1/2}} \Big]^2 \exp\Big\{ \epsilon \cdot \frac{[\tilde{\mu}_{n,m}(T)]^2}{R_{n,m}} - \log[(L_{n,m})^{d/2+2} T^{d+2}] \Big\} \Big\|_{L_1(\mathbf{P})} \leq \frac{3 \cdot (2d)^d}{(L_{n,m})^2 T^2}\,,$$

for some absolute $\epsilon > 0$. Put

$$I_{n,m,T} = I(L_{n,m}, T, n, m, d, \epsilon) =$$

$$\big\{ \omega \in \Omega : \epsilon [\tilde{\mu}_{n,m}(T)]^2 \geq R_{n,m} \log[(L_{n,m})^{d/2+2} T^{d+2}] \big\}\,.$$

We obtain

$$\Big(\frac{1}{\sqrt{\epsilon}}\Big)^d \Big\| \mathbf{1}_{I_{n,m,T}} \cdot \exp\Big\{ \epsilon \cdot \frac{[\tilde{\mu}_{n,m}(T)]^2}{R_{n,m} \log[(L_{n,m})^{d/2+2} T^{d+2}]} - 1 \Big\} \Big\|_1 \leq \frac{3 \cdot (2d)^d}{(L_{n,m})^2 T^2},$$

where we used the fact $\log[(L_{n,m})^{d/2+2} T^{d+2}] \geq 1$.

Hence,

$$\Big\| \sup_{m>n} \sup_{T \geq 2} \mathbf{1}_{I_{n,m,T}} \cdot \exp\Big\{ \epsilon \cdot \frac{[\tilde{\mu}_{n,m}(T)]^2}{R_{n,m} \log[(L_{n,m})^{d/2+2} T^{d+2}]} - 1 \Big\} \Big\|_1 \leq$$

$$\sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \sum_{T=2}^{\infty} \Big\| \mathbf{1}_{I_{n,m,T}} \cdot \exp\Big\{ \epsilon \cdot \frac{[\tilde{\mu}_{n,m}(T)]^2}{R_{n,m} \log[(L_{n,m})^{d/2+2} T^{d+2}]} - 1 \Big\} \Big\|_1 \leq$$

$$\sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \sum_{T=2}^{\infty} \frac{3 \cdot (2d\sqrt{\epsilon})^d}{(L_{n,m})^2 T^2} < \infty\,.$$

On the other hand, if $\omega \notin I_{n,m,T}$ for some $m > n \geq 0$ and $T \geq 2$, then

$$\frac{\epsilon \cdot [\tilde{\mu}_{n,m}(T)]^2}{R_{n,m} \log[(L_{n,m})^{d/2+2} T^{d+2}]} \leq 1\,.$$

The result now follows from a simple computation. □

Remarks. 1. The technical requirements $T \geq 2$ and $L_k \geq 1$ are used to insure $\log[(L_{n,m})^{d/2+2} T^{d+2}] \geq 1$.

2. Inspection of the proof shows that the constant C depends on d, ϵ, and $\{L_k\}$. Note that the requirement $\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{(L_{n,m})^2} < \infty$ can be replaced by $\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{(L_{n,m})^{\kappa}} < \infty$ for some $\kappa > 0$. This yields an estimation with $(L_{n,m})^{d/2+\kappa}$ at the denominator.

3. The method of proof is similar to the one in [**CC2**, Theorem 3.5].

4. When specifying $\mu_k(\omega, B) = X_k(\omega) \delta_{\boldsymbol{\lambda}_k}(B)$, for $\{X_k\} \subset L_\infty(\mathbf{P})$ centered (not necessarily independent) random variables, $\{\boldsymbol{\lambda}_k\} \subset \mathbb{R}^d$, and $L_k = |\boldsymbol{\lambda}_k|$, we obtain Theorem 3.5 in [**CC2**] for the σ_n-system of characters $\{e^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}\}$. In this case $R_{n,m}$ involves conditional expectation terms like in [**CC2**, Theorem 3.5] (see also Remark 4 after Proposition 2.7).

3. Applications

In this section we present some applications of Theorem 2.8. In particular, the first two corollaries show how Theorem 2.8 could be used in order to obtain (relatively easily) Theorem C and Theorem D.

Corollary 3.1. *Let $\{X_k\}$ be a sequence of symmetric independent complex valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \dots, \lambda_k^{(d)})\} \subset \mathbb{R}^d$. Then there exists absolute constants $\epsilon > 0$ and $C > 0$, independent of $\{X_k\}$, such that (with 0/0 interpreted as 1),*

$$\Big\| \sup_{m>n} \sup_{T\geq 2} \exp\Big\{\epsilon \cdot \frac{\max_{\mathbf{t}\in[0,T]^d} |\sum_{k=n+1}^m X_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle}|^2}{\log[(|\lambda_m|^* + 1)mT] \sum_{k=n+1}^m |X_k|^2}\Big\}\Big\|_1 \leq C\,.$$

Proof. Take a copy of $([0,1], \mathcal{B}([0,1]), dx)$ independent of $(\Omega, \mathcal{F}, \mathbf{P})$, and take a Rademacher sequence $\{\epsilon_k\}$ on it. Denote by $\mathbb{E}'$ the expectation on $[0,1]$. Let $\{c_k\}$ be an arbitrary sequence of complex numbers.

Now, define $\mu_k(x, B) = c_k \epsilon_k(x) \delta_{\boldsymbol{\lambda}_k}(B)$ and $L_k := (|\boldsymbol{\lambda}_k| + 1) \vee k$. Since $L_{n,m} \geq m^2$, we have

$$\sum_{n=1}^\infty \sum_{m=n+1}^\infty 1/(L_{n,m})^2 \leq \sum_{n=1}^\infty \sum_{m=n+1}^\infty 1/m^4 \leq \sum_{n=1}^\infty 1/n^3 < \infty\,.$$

Also we have

$$L_{n,m} \leq \sum_{k=n+1}^m (|\boldsymbol{\lambda}_k| + 1)^2 k^2 \leq m^3 (|\boldsymbol{\lambda}_m|^* + 1)^2.$$

Since the above $\{\mu_k\}$ is $\mathbf{L}$-symmetric independent for every $\mathbf{L}$ (see Example 2.1(i)), Theorem 2.8 yields

$$\mathbb{E}'\Big[\sup_{m>n} \sup_{T\geq 2} \exp\Big\{\epsilon \cdot \frac{\max_{\mathbf{t}\in[0,T]^d} |\sum_{k=n+1}^m c_k \epsilon_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle}|^2}{\log[(|\lambda_m|^* + 1)mT] \sum_{k=n+1}^m |c_k \epsilon_k|^2}\Big\}\Big] \leq C\,,$$

for some C which is *independent* of $\{c_k\}$. This prove the result for the case $X_k = c_k \epsilon_k$.

Denote by $\mathbb{E}$ the expectation in $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\{X_k\}$ be a sequence of symmetric independent random variables on Ω. For every $\omega \in \Omega$, the above inequality with $c_k(\omega) = X_k(\omega)$ yields

$$\mathbb{E}'\Big[\sup_{m>n} \sup_{T\geq 2} \exp\Big\{\epsilon \cdot \frac{\max_{\mathbf{t}\in[0,T]^d} |\sum_{k=n+1}^m X_k(\omega) \epsilon_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle}|^2}{\log[(|\lambda_m|^* + 1)mT] \sum_{k=n+1}^m |X_k(\omega) \epsilon_k|^2}\Big\}\Big] \leq C\,.$$

By construction, the sequences $\{\epsilon_k X_k\}$ and $\{X_k\}$ have the same finite dimensional joint probability distributions on $[0,1] \times \Omega$ and Ω, respectively. Hence, by taking the expectation $\mathbb{E}$ in the above inequality (note that C is independent of ω) we obtain the result. □

Remarks. 1 The symmetrization argument used in the proof above was called by Kahane [**K2**, p. 9] a *reduction principle*. This idea was already used in the proof of Corollary 3.6 in [**CC2**] (here the details are given for the sake of completeness).

2. Note that in the proof above we could obtain a more accurate estimation for $L_{n,m}$, involving only $\max_{n<k\leq m} |\boldsymbol{\lambda}_k|$ instead of $|\boldsymbol{\lambda}_m|^*$.

Corollary 3.2. *Let $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \ldots, \lambda_k^{(d)})\}$ be a sequence of independent $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, and let $\{c_k\}$ be a sequence of complex numbers. Let Φ be some positive non-decreasing function on $\mathbf{R}^+$, such that for some $\eta > 0$ we have $\Phi(x) \geq x^\eta$ for every $x \geq 0$. Assume that for some $\delta > 0$ we have $\sum_{n=1}^\infty \mathbf{P}[|\boldsymbol{\lambda}_n| > \Phi(n)^\delta] < \infty$. Then there exists absolute constants $\epsilon > 0$ and $C > 0$, independent of $\{c_k\}$, such that (with 0/0 interpreted as 1),*

$$\mathbb{E}\Big[\sup_{m>n}\sup_{T\geq 2}\exp\Big\{\epsilon\cdot\frac{\max_{\mathbf{t}\in[0,T]^d}|\sum_{k=n+1}^m c_k(\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}-\mathbb{E}[\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}])|^2}{\log[(\Phi(m)+1)mT]\sum_{k=n+1}^m|c_k|^2}\Big\}\Big] \leq C\,.$$

Proof. (i): Put $L_k = \Phi(k)^\delta$ and $\Omega_k = \{|\boldsymbol{\lambda}_k| \leq \Phi(k)^\delta\}$, and denote its complement by $\overline{\Omega}_k$. Also put $\mu_k(\omega, B) = c_k\delta_{\boldsymbol{\lambda}_k(\omega)}(B) - \mathbb{E}[c_k\delta_{\boldsymbol{\lambda}_k(\cdot)}(B)]$ (see Example 2.3). The sequence $\{\mu_k\}$ is $\mathbf{L}$-centered independent and for every $k \geq 1$, $\mu_k(\omega, B)$ as function of ω is bounded by $2c_k$ (independently of B), hence $V_k(\omega)$ is bounded.

(ii): It is easy to compute that for some $\kappa = \kappa(\eta) > 0$ large enough we have $\sum_{n=1}^\infty\sum_{m=n+1}^\infty 1/(L_{n,m})^\kappa < \infty$. As we mentioned in Remark 2 after Theorem 2.8, this does not really affect Theorem 2.8. It is also clear that $L_{n,m} \leq [(\Phi(m)+1)m]^\gamma$, for some $\gamma > 0$.

(iii) Recall that by Fubini's theorem $\hat{\mu}_k(\mathbf{t}) = c_k\mathbf{1}_{\Omega_k}\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle} - \mathbb{E}[c_k\mathbf{1}_{\Omega_k}\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}]$ (see Example 2.3).

Now, we may apply Theorem 2.8 with these settings to obtain that

$$\mathbb{E}\Big[\sup_{m>n}\sup_{T\geq 2}\exp\Big\{\epsilon\cdot\frac{\max_{\mathbf{t}\in[0,T]^d}|\sum_{k=n+1}^m c_k(\mathbf{1}_{\Omega_k}\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}-\mathbb{E}[\mathbf{1}_{\Omega_k}\mathrm{e}^{i\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}])|^2}{\log[(\Phi(m)+1)mT]\sum_{k=n+1}^m|c_k|^2}\Big\}\Big]$$

is less than some universal constant $C > 0$.

The procedure that we need to take in order to replace Ω_k by Ω (in the above inequality) is technical; we refer to the proof of Theorem 4.10 in [**CC3**]. We just say that it is an application of simple inequalities and the convergence of $\sum_{k=1}^\infty \mathbf{P}(\overline{\Omega}_k)$. □

Remarks. 1. Corollary 3.1 is Corollary 3.7 in [**CC2**] and Corollary 3.2 is Theorem 4.10 in [**CC3**]. Corollary 3.2 was deduced in [**CC3**] from Corollary 3.1; here we avoid the symmetrization procedure used in [**CC3**] (see also [**W1**]) in order to conclude from Corollary 3.1 the last inequality in the proof above. Here we see that both corollaries are (almost direct) consequences of Theorem 2.8.

2. Corollary 3.1 with $d = 1$ and $\{\lambda_k\}$ a strictly increasing sequence of natural numbers is Theorem 7 in Weber [**W1**]. Corollary 3.2 with $d = 1$, $c_k \equiv 1$, and some additional conditions on the random $\{\lambda_k\}$ is Theorem 9 in [**W1**]. Both results in [**W1**] were proved using a completely different approach – the *metric entropy* method.

3. Recently, the results of [**W1**] were re-investigated by Weber [**W2**]. Using the more precise method of *majorizing measures*, better estimates than those in [**W1**] were given when $\{\lambda_k\}$ are reals with $\{|\lambda_k|^*\}$ increasing at a rate which is faster than polynomial growth (for polynomial growth there was no improvement). Also the multi-dimensional case $d > 1$ was considered there.

As we will see later, corollaries as above can be used in order to obtain $\mathbf{P}$-a.s. uniform convergence over $[0,T]^d$ of the *random* almost periodic series $\sum_{k=1}^\infty X_k\mathrm{e}^{\langle\mathbf{t},\boldsymbol{\lambda}_k\rangle}$ (see §4 in [**CC2**]). It is also possible to show a.s. uniform convergence of the series

$\sum_{k=1}^\infty c_k(\mathrm{e}^{\langle \mathbf{t},\boldsymbol{\lambda}_k\rangle} - \mathbb{E}[\mathrm{e}^{\langle \mathbf{t},\boldsymbol{\lambda}_k\rangle}])$, where the $\{\boldsymbol{\lambda}_k\}$ are random. Here we are interested in more general series.

PROPOSITION 3.3. *Let $\mathbf{L} = \{L_k\}$ be a sequence of positive numbers with $L_k \geq 1$, such that $\sum_{n=1}^\infty \sum_{m=n+1}^\infty \frac{1}{(L_{n,m})^2}$ converges and $L_k \leq Ck^\gamma$ for some positive constants C and γ. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\{\mu_k\}_{k=1}^\infty$ be a sequence of finite complex valued transition measures on $\Omega \times \mathcal{B}$, which is $\mathbf{L}$-centered independent. Assume that $\{V_k\} \subset L_\infty(\mathbf{P})$. If*

$$\sum_{n=1}^\infty \frac{\sqrt{\sum_{k\geq n} \|V_k\|_\infty^2}}{n\sqrt{\log n}} < \infty\,,$$

then for almost every $\omega \in \Omega$, the random series

$$\sum_{n=1}^\infty \hat{\mu}_n(\mathbf{t}) = \sum_{n=1}^\infty \int_{[-L_n,L_n]^d} \mathrm{e}^{i\langle \mathbf{t},\mathbf{u}\rangle} \mu_n(d\mathbf{u})$$

converges uniformly in $\mathbf{t} \in [0,T]^d$, for every $T > 0$.

PROOF. Recall our notation $\tilde{\mu}_{n,m}(T) = \max_{\mathbf{t}\in[0,T]^d} |\hat{\mu}_{n,m}(\mathbf{t})|$. Using Theorem 2.8 we obtain a subset of full $\mathbf{P}$-measure $\Omega_1 \subset \Omega$, such that for every $\omega \in \Omega_1$ there exists a finite constant C_ω, such that for every $m > n \geq 0$ and for every $T \geq 2$, we have

$$\max_{\mathbf{t}\in[0,T]^d} |\sum_{k=n+1}^m \int_{[-L_k,L_k]^d} \mathrm{e}^{i\langle \mathbf{t},\mathbf{u}\rangle} \mu_k(d\mathbf{u})|^2 \leq C_\omega \log[(L_{n,m})^{d/2+2} T^{d+2}] \sum_{k=n+1}^m \|V_k\|_\infty^2\,.$$

Hence using $L_k \leq Ck^\gamma$, for every $\omega \in \Omega_1$ we have

$$\sum_{n=1}^\infty \max_{2^{2^n} < l \leq 2^{2^{n+1}}} \tilde{\mu}_{2^{2^n},l}(T) \leq$$

$$\sqrt{C_\omega} \sum_{n=1}^\infty (\log[(L_{2^{2^n},2^{2^{n+1}}})^{d/2+2} T^{d+2}])^{\frac{1}{2}} \Big(\sum_{k=2^{2^n}+1}^{2^{2^{n+1}}} \|V_k\|_\infty^2 \Big)^{\frac{1}{2}} \leq$$

$$C' \sqrt{C_\omega} \sum_{n=1}^\infty \sqrt{2^n} \Big(\sum_{k=2^{2^n}+1}^{2^{2^{n+1}}} \|V_k\|_\infty^2 \Big)^{\frac{1}{2}},$$

for some absolute positive constant C', depends on γ, d, C, and T (but not on ω, n or m).

Our assumption implies that the series $\sum_{n=1}^\infty \frac{(\sum_{k\geq n} V_k^2)^{1/2}}{n\sqrt{\log n}}$ converges a.s. Denote this full $\mathbf{P}$-measure set of convergence by Ω_2. By change of variables, this convergence implies the a.s. convergence of the series $\sum_{n=1}^\infty \sqrt{2^n} \big(\sum_{k=2^{2^n}+1}^{2^{2^{n+1}}} V_k^2 \big)^{\frac{1}{2}}$ (see the proof of Theorem 5.1.5 in Salem and Zygmund [**SZ**] or in [**CC2**]). Hence, for every $\omega \in \Omega_1 \cap \Omega_2$ we obtain that $\sum_{n=1}^\infty \max_{2^{2^n} < l \leq 2^{2^{n+1}}} \tilde{\mu}_{2^{2^n},l}(T)$ converges. This implies the result. □

THEOREM 3.4. *Let $\mathbf{L} = \{L_k\}$ be a sequence of positive numbers with $L_k \geq 1$, such that $\sum_{n=1}^\infty \sum_{m=n+1}^\infty \frac{1}{(L_{n,m})^2}$ converges and $L_k \leq Ck^\gamma$ for some positive constants C and γ. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\{\mu_k\}_{k=1}^\infty$ be a sequence* *of finite complex valued transition measures on* $\Omega\times\mathcal{B}$*, which is* $\mathbf{L}$*-centered independent. Assume that* $\{V_k\}\subset L_2(\mathbf{P})$*. If*

$$\sum_{n=1}^{\infty}\frac{\sqrt{\sum_{k\ge n}\|V_k\|_2^2}}{n\sqrt{\log n}}<\infty\,, \tag{2}$$

then $\mathbf{P}$*-a.s. the random series* $\sum_{n=1}^{\infty}\hat{\mu}_n(\mathbf{t})$ *converges uniformly in* $\mathbf{t}\in[0,T]^d$*, for every* $T>0$*.*

Proof. First we prove the result for the case that $\{\mu_k\}$ is a sequence of $\mathbf{L}$-*symmetric* independent transition measures (see Example 2.2 for the symmetrization procedure below).

Take a copy of $([0,1],\mathcal{B}([0,1]),dx)$ independent of $(\Omega,\mathcal{F},\mathbf{P})$, and take $\{\epsilon_k\}$ a Rademacher sequence on it. For fixed $\omega\in\Omega$ apply Proposition 3.3 to the sequence of transition measures $\epsilon_k(\cdot)\mu_k(\omega,\cdot):[0,1]\times\mathcal{B}\mapsto\mathbb{C}$, in order to conclude that if ω satisfies $\sum_{n=1}^{\infty}\frac{\sqrt{\sum_{k\ge n}|V_k(\omega)|^2}}{n\sqrt{\log n}}<\infty$, then for dx-almost every $x\in[0,1]$ the series $\sum_{n=1}^{\infty}\epsilon_n(x)\hat{\mu}_n(\omega,\mathbf{t})$ converges uniformly in $\mathbf{t}\in[0,T]^d$, for every $T>0$.

Since the square root is a concave function, we conclude by (2) and by Beppo Levi that $\sum_{n=1}^{\infty}\frac{\sqrt{\sum_{k\ge n}|V_k(\omega)|^2}}{n\sqrt{\log n}}$ converges $\mathbf{P}$-a.s. Hence, by Fubini's theorem for $dx\times\mathbf{P}$-almost every $(x,\omega)\in[0,1]\times\Omega$ the series $\sum_{n=1}^{\infty}\epsilon_n(x)\hat{\mu}_n(\omega,\mathbf{t})$ converges uniformly in $\mathbf{t}\in[0,T]^d$, for every $T>0$.

Since $\{\mu_k\}$ is $\mathbf{L}$-symmetric independent, by construction the sequences $\{\epsilon_k\hat{\mu}_k(\mathbf{t})\}$ and $\{\hat{\mu}_k(\mathbf{t})\}$ have the same finite dimensional joint probability distributions on $[0,1]\times\Omega$ and Ω, respectively. So, we conclude that for $\mathbf{P}$-almost every $\omega\in\Omega$ the series $\sum_{n=1}^{\infty}\hat{\mu}_n(\omega,\mathbf{t})$ converges uniformly in $\mathbf{t}\in[0,T]^d$, for every $T>0$. This establishes the symmetric case.

Now we prove the general centered case. We build two independent copies of $\{\mu_k\}$ as follow: on $(\Omega\times\Omega,\mathcal{F}\otimes\mathcal{F},\mathbf{P}\times\mathbf{P})$ we define $\mu_k^{(1)}(\omega_1,\omega_2,B)=\mu_k(\omega_1,B)$ and $\mu_k^{(2)}(\omega_1,\omega_2,B)=\mu_k(\omega_2,B)$. We have the following properties: (i) for every $k\ge 1$ and $\mathbf{t}\in\mathbb{R}^d$ the random variables $\hat{\mu}_k^{(1)}(\mathbf{t})$ and $\hat{\mu}_k^{(2)}(\mathbf{t})$ are independent and have the *same* probability distribution as $\hat{\mu}_k(\mathbf{t})$; (ii) each of the sequences $\{\mu_k^{(1)}\}$ and $\{\mu_k^{(2)}\}$ is $\mathbf{L}$-centered independent. We conclude that the sequence of random variables $\{\hat{\mu}_k^{(1)}(\mathbf{t})-\hat{\mu}_k^{(2)}(\mathbf{t})\}$ is symmetric independent, which means that $\{\mu_k^{(1)}-\mu_k^{(2)}\}$ is $\mathbf{L}$-symmetric independent.

Using condition (2) we obtain

$$\|\sup_{n\ge1}|\mu_n^{(1)}-\mu_n^{(2)}|(\cdot,\cdot,\mathbb{R}^d)\|_{L_2(\mathbf{P}\times\mathbf{P})}\le 2\|\sup_{n\ge1}|\mu_n|(\cdot,\mathbb{R}^d)\|_{L_2(\mathbf{P})}<\infty\,, \tag{$*$}$$

and we conclude the general centered case as done (in a simpler situation on the torus) in [**CC1**, Theorem 2.2]. The proof of the current case goes as follows: by what we have shown above, we can already conclude the $\mathbf{P}\times\mathbf{P}$-a.s. uniform convergence of the symmetric version $\sum_{n=1}^{\infty}[\hat{\mu}_n^{(1)}(\mathbf{t})-\hat{\mu}_n^{(2)}(\mathbf{t})]$. Since $(*)$ holds, we conclude by Hoffman-Jørgensen [**H-J**, Corollary 3.3] that $\sum_{n=1}^{\infty}[\hat{\mu}_n^{(1)}(\mathbf{t})-\hat{\mu}_n^{(2)}(\mathbf{t})]$ converges in $L_1(\Omega\times\Omega,\mathbf{P}\times\mathbf{P},C([0,T]^d))$, the Banach space of $C([0,T]^d)$-valued random variables, i.e., random variables with values in the space of continuous functions on $[0,T]^d$ with finite $\mathbb{E}[\|\cdot\|_\infty]$-seminorm. Since $\{\mu_k^{(2)}\}$ is $\mathbf{L}$-centered, by independence and convexity of the seminorm we conclude that $\sum_{n=1}^{\infty} \hat{\mu}_n^{(1)}(\mathbf{t})$ converges in $L_1(\Omega \times \Omega, \mathbf{P} \times \mathbf{P}, C([0,T]^d))$, which is equivalent to the convergence of $\sum_{n=1}^{\infty} \hat{\mu}_n(\mathbf{t})$ in $L_1(\Omega, \mathbf{P}, C([0,T]^d))$. Now the result follows by Itô and Nisio theorem [**IN**] (see also Ledoux and Talagrand [**LT**, Theorem 6.1]) □

Remarks. 1. The technique of the proof of Proposition 3.3 goes back to Salem and Zygmund [**SZ**, Theorem 5.1.5].

2. The assumption $L_k = O(k^\gamma)$ is only technical. One can remove this assumption from Proposition 3.3 (and hence from Theorem 3.4) like it is done in [**CC2**, Theorem 4.2]. If we do it, we will obtain the following (more precise) sufficient conditions

$$\sum_{n=1}^{\infty} 2^{n/2} \Big(\sum_{k:\, 2^{2^n} \le L_{1,k} \le 2^{2^{n+1}}} \|V_k\|_2^2 \Big)^{1/2} < \infty \,,$$

or in particular

$$\sum_{n=1}^{\infty} \frac{\Big(\sum_{k:\, L_{1,k} \ge n} \|V_k\|_2^2 \Big)^{1/2}}{n\sqrt{\log n}} < \infty \,.$$

3. It is possible to extend Theorem 3.4 to $\{V_k\} \subset L_p$, for $1 < p < 2$. This can be done using the Talagrand-Fernique tool which was re-investigated in [**CC3**].

Corollary 3.5. *Let $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \ldots, \lambda_k^{(d)})\}$ be a sequence of independent $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\sum_{k=1}^{\infty} \mathbf{P}(|\boldsymbol{\lambda}_k| > k^\gamma)$ converges for some $\gamma > 0$. Let $\{c_k\}$ be a sequence of complex numbers. If the series $\displaystyle\sum_{n=1}^{\infty} \frac{(\sum_{k \ge n} |c_k|^2)^{1/2}}{n\sqrt{\log n}}$ converges, then a.s. the series with random powers $\displaystyle\sum_{k=1}^{\infty} c_k (\mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle} - \mathbb{E}[\mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}])$ converges uniformly in $\mathbf{t} \in [0,T]^d$, for every $T > 0$.*

Proof. Define $\mu_k(\omega, B) = c_k \delta_{\boldsymbol{\lambda}_k(\omega)}(B) - \mathbb{E}[c_k \delta_{\boldsymbol{\lambda}_k(\cdot)}(B)]$ and $L_k = k^{\gamma \vee 1}$, and put $\Omega_k = \{|\boldsymbol{\lambda}_k| \le k^{\gamma \vee 1}\}$ (see also Example 2.3). Now, we apply Proposition 3.3 to conclude the $\mathbf{P}$-a.s. uniform convergence of $\sum_{k=1}^{\infty} [c_k \mathbf{1}_{\Omega_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle} - \mathbb{E}[c_k \mathbf{1}_{\Omega_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}]]$. Since by our assumption $\sum_{k=1}^{\infty} \mathbf{P}(\overline{\Omega}_k) < \infty$, by Borel Cantelli lemma we have $\mathbf{P}$-a.s. uniform convergence of $\sum_{k=1}^{\infty} [c_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle} - \mathbb{E}[c_k \mathbf{1}_{\Omega_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}]]$. Also by our assumptions,

$$\sum_{k=1}^{\infty} |\mathbb{E}[c_k \mathbf{1}_{\overline{\Omega}_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}]| \le \sum_{k=1}^{\infty} |c_k| \mathbb{E}[\mathbf{1}_{\overline{\Omega}_k}] \le$$

$$\Big(\sum_{k=1}^{\infty} |c_k|^2 \Big)^{1/2} \Big(\sum_{k=1}^{\infty} (\mathbf{P}(\overline{\Omega}_k))^2 \Big)^{1/2} \le \Big(\sum_{k=1}^{\infty} |c_k|^2 \Big)^{1/2} \Big(\sum_{k=1}^{\infty} \mathbf{P}(\overline{\Omega}_k) \Big)^{1/2} < \infty.$$

The result now follows by

$$\sum_{k=1}^{\infty} [c_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle} - \mathbb{E}[c_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}]] =$$

$$\sum_{k=1}^{\infty} [c_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle} - \mathbb{E}[c_k \mathbf{1}_{\Omega_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}]] - \sum_{k=1}^{\infty} \mathbb{E}[c_k \mathbf{1}_{\overline{\Omega}_k} \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k \rangle}].$$

□

Remarks. 1. If in Corollary 3.5 we assume that $\{\boldsymbol{\lambda}_k\}$ is an i.i.d. sequence, our assumption becomes $\mathbb{E}[|\boldsymbol{\lambda}_1|^{1/\gamma}] < \infty$. Even in this particular case the result is new, as far as we know.

2. One can assume a more general condition $\sum_{k=1}^\infty \mathbf{P}(|\boldsymbol{\lambda}_k| > \Phi(k)^\delta) < \infty$, for Φ like in Corollary 3.2.

In the following corollary $\hat{\eta}_k(\mathbf{t})$ denotes the *usual* Fourier-Stieltjes transform of η_k.

Theorem 3.6. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\{X_k\} \subset L_2(\mathbf{P})$ be a sequence of centered independent complex valued random variables. Let $\{L_k\}$ be a sequence with $L_k \ge 1$, such that $\sum_{n=1}^\infty \sum_{m=n+1}^\infty \frac{1}{(L_{n,m})^2} < \infty$ and $L_k \le Ck^\gamma$ for some positive constants C and γ. Let $\{\eta_k\}_{k=1}^\infty$ be a sequence of finite complex measures on $\mathcal{B}$, with the support of η_k contained in $[-L_k, L_k]^d$. Assume that the total variation norms $\|\eta_k\| = |\eta_k|(\mathbb{R}^d)$, $k \ge 1$, are finite. If*

$$\sum_{n=1}^\infty \frac{(\sum_{k\ge n} \|X_k\|^2_{L_2(\mathbf{P})} \cdot \|\eta_k\|^2)^{1/2}}{n\sqrt{\log n}} < \infty\,,$$

then $\mathbf{P}$-a.s. the random series $\sum_{n=1}^\infty X_n \hat{\eta}_n(\mathbf{t})$ converges uniformly in $\mathbf{t} \in [0,T]^d$, for every $T > 0$.

Proof. Define the transition measures $\mu_k(\omega, B) = X_k(\omega)\eta_k(B)$ (see Example 2.1). Clearly $V_k(\omega) = |\mu_k|(\omega, \mathbb{R}^d) = |X_k(\omega)| \cdot \|\eta_k\|$ and $\hat{\mu}_k(\mathbf{t}) = X_k \hat{\eta}_k(\mathbf{t})$. Now, we may apply Theorem 3.4. □

Remark. By considering Remark 4 after Proposition 2.7 and Remark 4 after Theorem 2.8, we see that we may extend Theorem 3.6 to the case that $\{X_k\}$ are centered and bounded, but not necessarily independent.

Corollary 3.7. *Let $\{X_k\} \subset L_2(\Omega, \mathbf{P})$ be a sequence of centered independent complex valued random variables, such that the series $\displaystyle\sum_{n=1}^\infty \frac{(\sum_{k\ge n} \|X_k\|_2^2)^{1/2}}{n\sqrt{\log n}}$ converges. Then for every sequence $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \dots, \lambda_k^{(d)})\} \subset \mathbb{R}^d$ with $|\boldsymbol{\lambda}_k| = O(k^\gamma)$ for some $\gamma > 0$, a.s. the random series $\displaystyle\sum_{k=1}^\infty X_k \mathrm{e}^{i\langle \mathbf{t}, \boldsymbol{\lambda}_k\rangle}$ converges uniformly in $\mathbf{t} \in [0,T]^d$, for every $T > 0$.*

Proof. Put $\eta_k = \delta_{\boldsymbol{\lambda}_k}$ and $L_k = (|\boldsymbol{\lambda}_k|+1) \vee k$. Now we apply Theorem 3.6. □

Remark. Corollary 3.7 completely recovers Theorem 5.1.5 of Salem and Zygmund [**SZ**]. By considering Remark 2 after Theorem 3.4 and the above remark we conclude that Corollary 3.7 yields the results of [**CC2**, §4.1].

Corollary 3.8. *Let $\{X_k\} \subset L_2(\Omega, \mathbf{P})$ be a sequence of centered independent complex valued random variables. Let $\{\boldsymbol{\lambda}_k = (\lambda_k^{(1)}, \dots, \lambda_k^{(d)})\} \subset (\mathbb{R}^+)^d$, with $|\boldsymbol{\lambda}_k| = O(k^\gamma)$, for some $\gamma > 0$. If*

$$\sum_{n=1}^\infty \frac{[\sum_{k\ge n} (\Pi_{j=1}^d \lambda_k^{(j)})^2 \|X_k\|_2^2]^{1/2}}{n\sqrt{\log n}} < \infty\,, \tag{3}$$

then $\mathbf{P}$-a.s. the random series $\sum_{n=1}^\infty X_n \, \Pi_{j=1}^d \frac{\sin(\lambda_k^{(j)} t_j)}{t_j}$ converges uniformly in $\mathbf{t} \in [0,T]^d$, for every $T > 0$.

Proof. Put $h_k = \Pi_{j=1}^d \mathbf{1}_{[-\lambda_k^{(j)},\lambda_k^{(j)}]}$ and $L_k = (|\boldsymbol{\lambda}_k| + 1) \vee k$. Define $\eta_k(B) = \int_B h_k(\mathbf{u})\, d\mathbf{u}$, so $\hat{\eta}_k(\mathbf{t}) = 2^d \cdot \Pi_{j=1}^d \sin(\lambda_k^{(j)} t_j)/t_j$. Now we apply Theorem 3.6. $\square$

Remarks. 1. Note that $\sum_{n=1}^\infty (\Pi_{j=1}^d \lambda_n^{(j)})^2 \|X_n\|_2^2 (\log n)^{1+\epsilon} < \infty$, for some $\epsilon > 0$ implies condition (3).

2. When $d = 1$, the random series of the corollary is nothing but

$$\frac{1}{2}\int_{-1}^{1} \sum_{n=1}^{\infty} X_n \lambda_n \cos(u\lambda_n t) du\,,$$

so Corollary 3.8 is also a consequence of Corollary 3.7. When $\lambda_k = k$ one can apply directly Theorem 5.1.5 of [**SZ**].

3. Put $\lambda_k^{(j)} = k$ for every $k \geq 1$ and $j = 1, \ldots, d$. In that case, Corollary 3.8 yields a.s. convergence of $\sum_{n=1}^\infty X_n \Pi_{j=1}^d \frac{\sin(nt_j)}{t_j}$ uniformly in $\mathbf{t} \in [0,T]^d$. Condition (3) becomes $\sum_{n=1}^\infty \frac{(\sum_{k\geq n} k^{2d}\|X_k\|_2^2)^{1/2}}{n\sqrt{\log n}}$, which is similar to that of Salem and Zygmund [**SZ**, Theorem 5.1.5], except the k^{2d} in the square root. This fact can be explained as follow: since $\lim_{t\to 0} \frac{\sin(kt)}{t} = k$, this means that the system $\{\Pi_{j=1}^d \sin(kt_j)/t_j\}_{k\geq 1}$ is not uniformly bounded, but has maximal amplitudes $\{k^d\}$. These amplitudes enter inside the square root as $\{k^{2d}\}$. Although the convergence of $\sum_{k=1}^\infty k^{2d}\|X_k\|_2^2$ implies the a.s. convergence of $\sum_{k=1}^\infty |X_k|$ (in fact of $\sum_{k=1}^\infty k^{d/2}|X_k|$ for $d > 1$), the uniform convergence is not a trivial consequence of this absolute convergence, since the system $\{\Pi_{j=1}^d \sin(kt_j)/t_j\}_{k\geq 1}$ is not uniformly bounded. As far as we know Corollary 3.8 is new.

4. Additional types of series may be considered. For example, convergence of $\sum_{n=1}^\infty \frac{(\sum_{k\geq n} k^{4d}\|X_k\|_2^2)^{1/2}}{n\sqrt{\log n}}$ implies a.s. convergence of $\sum_{n=1}^\infty X_n \big(\frac{\sin(nt_j)}{t_j}\big)^2$ uniformly in $\mathbf{t} \in [0,T]^d$ (take h_k to be the product of functions with graph an appropriate triangle). In this case, the procedure of Remark 2 above will not work.

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