RATES OF CONVERGENCE OF POWERS OF CONTRACTIONS

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Abstract. We prove that if the numerical range of a Hilbert space contraction T is in a certain closed convex set of the unit disk which touches the unit circle only at 1, then $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ with $\beta \in [\frac{1}{2}, 1)$. For normal contractions the condition is also necessary. Another sufficient condition for $\beta = \frac{1}{2}$, necessary for T normal, is that the numerical range of T be in a disk $\{z : |z-\delta| \le 1-\delta\}$ for some $\delta \in (0,1)$. As a consequence of results of Seifert, we obtain that a power-bounded T on a Hilbert space satisfies $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ with $\beta \in (0,1]$ if and only if $\sup_{|\lambda|>1} |\lambda-1|^{1/\beta}\|R(\lambda,T)\| < \infty$. When T is a contraction on L_2 satisfying the numerical range condition, it is shown that $T^n f/n^{1-\beta}$ converges to 0 a.e. with a maximal inequality, for every $f \in L_2$. An example shows that in general a positive contraction T on L_2 may have an $f \ge 0$ with $\limsup T^n f/\log n\sqrt{n} = \infty$ a.e.

1. Introduction

Let T be a power-bounded operator on a complex Banach space X. The Katznelson-Tzafriri theorem [17] says that $\|T^n(I-T)\| \to 0$ if and only if the peripheral spectrum $\sigma(T)\cap\mathbb{T}$ is at most the point 1. In Hilbert spaces, it follows from Léka's work [21] that when $\|T^n(I-T)\| \to 0$, then we also have $\|T^n(I-T)^\gamma\| \to 0$ for every $\gamma \in (0,1)$ (where $(I-T)^\gamma = I - \sum_{k=1}^\infty a_k T^k$, with $\{a_k\}_{k\ge1}$ the coefficients of $(1-t)^\gamma = 1 - \sum_{k=1}^\infty a_k t^k$ for $t \in [-1,1]$, which satisfy $a_k > 0$ and $\sum_{k=1}^\infty a_k = 1$).

Nagy and Zemánek [28] and Lyubich [25] proved that the powers of the operator T have the rate of convergence $\|T^n(I-T)\| = \mathcal{O}(1/n)$ if and only if T satisfies the Ritt resolvent condition

$$\sup_{|\lambda|>1} \|(\lambda-1)R(\lambda,T)\| < \infty.$$

It follows from Nevanlinna's work [30, Theorem 9] that if T is power-bounded and satisfies, for some $\alpha \in [1,2)$,

$$\sup_{1<|\lambda|<2} |\lambda-1|^\alpha \|R(\lambda,T)\| < \infty, \tag{1}$$

then $\|T^n(I-T)\| = \mathcal{O}(1/n^{(2-\alpha)/\alpha})$. (The case $\alpha = 1$ is Ritt's condition).

Dungey [11] obtained several characterizations of the property $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$, and in [10] he gave several sufficient conditions for a contraction T on a Hilbert space to satisfy this estimate.

Léka [22] has recently constructed, for any $\beta \in (\frac{1}{2}, 1)$, a contraction T in a complex Hilbert space with $\sigma(T) = \{1\}$ and $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$. Earlier, Nevanlinna [29, Example 4.5.2] had constructed contractions on $C[0,1]$ with the above rates (but

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with larger spectra), and Paulauskas [31, Theorem 6] showed how to obtain normal contractions on a (separable) Hilbert space with the above rates.

Cachia and Zagrebnov [5] called a contraction T on a complex Hilbert space *quasi-sectorial* if its numerical range $W(T) := \{\langle Tf, f\rangle : \|f\| = 1\}$ is included in a Stolz region (the closed convex hull of the point 1 and a disk centered at 0 with radius less than 1). They proved [5, Lemma 3.1] that if T is quasi-sectorial, then $\|T^n(I-T)\| = \mathcal{O}(1/n)$; see also [7, Proposition 2.3].

Paulauskas [31] defined generalized quasi-sectorial contractions by the inclusion of their numerical ranges in a certain convex subset of the closed unit disk, larger than a Stolz region (see definition below), and proved that $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ for an appropriate $\beta \in (\frac{1}{2}, 1)$. We offer here a different proof, which under the assumptions of [31] yields a better (larger) value of β as a function of the parameters.

2. A limit theorem for generalized quasi-sectorial contractions

We start this section by defining certain convex subsets of the closed unit disk. The geometric construction of a Stolz region is by taking a circle of radius $r < 1$ centered at 0 and drawing two tangent line segments from the point 1 to this circle. Paulauskas [31] suggests a similar construction, but replacing the tangent line segments by arcs of a *tangent* "parabola-like" curve $x = 1 - b|y|^\alpha$, $1 < \alpha < 2$, $b > 0$, or $\alpha = 2$ and $b > \frac{1}{2}$ (with $|y| \le |y_0| < 1$); we call such a curve a *quasi-parabola*. We denote the obtained convex set by $D(\alpha, b)$, and call it a *quasi-Stolz set*. For a drawing see [31, p. 2078]. The actual construction of $D(\alpha, b)$ is by starting with the parameters α and b, and finding the radius of the corresponding circle; see Lemma 10 of [31]. Whenever we refer to a quasi-Stolz set $D(\alpha, b)$, it is implied that $1 < \alpha \le 2$. An operator with numerical range contained in a quasi-Stolz set is called in [31] *generalized quasi-sectorial.* Note that the numerical radius of a generalized quasi-sectorial T is at most 1, so necessarily T is power-bounded with $\sup_n \|T^n\| \le 2$ [36]. Note that curves of the form $x = 1 - b|y|^\alpha$ with $\alpha > 2$ and $b > 0$ are outside the unit disk in a neighborhood of $(1, 0)$, so cannot be used.

Lemma 2.1. *Let $D(\alpha, b)$ be a quasi-Stolz set. Then there exists $K > 0$ such that*

$$(n+1)^{1/\alpha} \sup_{\lambda \in D(\alpha,b)} |\lambda^n(1-\lambda)| \le K \quad \forall n \ge 1. \tag{2}$$

The proof of Proposition 6 of [31] actually shows (2), with a value for K.

Corollary 2.2. *Let $D(\alpha, b)$ be a quasi-Stolz set and let $\{\lambda_k\} \subset D(\alpha, b)$. Let T be the "diagonal" operator T defined on ℓ_p, $1 \le p < \infty$, by $Te_k = \lambda_k e_k$, where $\{e_k\}$ is the standard unit basis. Then $\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$.*

Proof. We have $\|T^n(I-T)\| = \sup_k |\lambda_k^n(1-\lambda_k)|$, and apply (2). □

Defintion. A compact set A is called a *K-spectral set for T* if there exists a $K_A > 0$ such that for every rational function $u(z)$ with poles outside A we have

$$\|u(T)\| \le K_A \sup_{z \in A} |u(z)|. \tag{3}$$

As noted in [7], an easy adaptation of Lebow's lemma [20, p. 66] yields that if a compact set A does not separate the plane, then it is K-spectral for T as soon as $\|p(T)\| \leq K_A \sup_{z\in A} |p(z)|$ for every polynomial $p(z)$.

Proposition 2.3. *let T be a bounded operator in a Banach space for which a quasi-Stolz set $D(\alpha, b)$ is a K-spectral set. Then T is power-bounded and $\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$.*

Proof. Since $D(\alpha, b)$ is a subset of the closed unit disk, (3) yields power-boundedness. Combining (3) with (2) we obtain

$$(n+1)^{1/\alpha}\|T^n(I-T)\| \leq K_{\bar{D}}K \quad \forall n \geq 1.$$

□

Theorem 2.4. *Let T be a contraction on a complex Hilbert space with numerical range contained in a quasi-Stolz set $D(\alpha, b)$. Then $\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$.*

Proof. The set $\bar{D} := \overline{D(\alpha, b)}$ is a compact convex set containing the numerical range $W(T)$, so by Delyon and Delyon [8, Theorem 3] $\bar{D}$ is a K-spectral set for T. The theorem now follows from Proposition 2.3. □

Remark. The theorem improves [31, Theorem 4], where Paulauskas obtained only the rate $(2-\alpha)/\alpha$ (for $\alpha < 2$, of course; the same rate as in [30] when (1) holds). He obtained the rate $1/\alpha$ only for diagonal operators (which are necessarily normal).

The next lemma shows that in Theorem 2.4 $n^{1/\alpha}\|T^n(I-T)\|$ need not converge to zero, so the rate obtained there is optimal.

Lemma 2.5. *Let T be a power-bounded operator on a complex Banach space X and assume that for some $\alpha \in (1, 2]$ and $b > 0$ ($b > \frac{1}{2}$ when $\alpha = 2$) there exist reals $0 \neq t_n \to 0$, such that $(1 - b|t_n|^\alpha + it_n)_{n\geq 1} \subset \sigma(T)$. Then*

$$\limsup_{n\to\infty} n^{1/\alpha}\|T^n(I-T)\| \geq (b^{1/\alpha}\,\mathrm{e})^{-1}.$$

Proof. Assume the statement fails. Then, for some fixed $0 < \varepsilon < (b^{1/\alpha}\mathrm{e})^{-1}$ and for every large k we have $k^{1/\alpha}\|T^k(I-T)\| < (b^{1/\alpha}\,\mathrm{e})^{-1} - \varepsilon$. The spectral mapping theorem then yields

$$k^{1/\alpha} \sup_{\lambda\in\sigma(T)} |\lambda|^k|\lambda - 1| < (b^{1/\alpha}\,\mathrm{e})^{-1} - \varepsilon.$$

Taking the given sequence $\lambda_j = 1 - b|t_j|^\alpha + it_j \in \sigma(T)$ and choosing $k_j = [\frac{1}{b|t_j|^\alpha}] + 1$ we obtain

$$k_j^{1/\alpha}|\lambda_j|^{k_j}|\lambda_j - 1| \geq \frac{|t_j|}{b^{1/\alpha}|t_j|}(1 - b|t_j|^\alpha)^{1+1/(b|t_j|^\alpha)} \underset{j\to\infty}{\to} (b^{1/\alpha}\,\mathrm{e})^{-1}$$

which is a contradiction. □

Remark. Nevanlinna [29, Theorem 4.5.1] proved that if $\sigma(T) \cap \mathbb{T} = \{1\}$ and 1 is not isolated in $\sigma(T)$, then $\limsup n\|T^n(I-T)\| > 1/\mathrm{e}$. Lemma 2.5 improves this result when additional information is given; however, with $\alpha = 1$, the lemma, though true, is in fact weaker than [29].

Proposition 2.6. *Let T be a power-bounded operator on a complex Banach space X and assume that for some $\frac{1}{2} < \beta < 1$ we have $\sup_{n\geq 1} n^\beta \|T^n(I-T)\| = M < \infty$. Then there exists $b > 0$, such that the sepctrum $\sigma(T)$ is contained in a quasi-Stolz region $D(1/\beta, b)$.*

Proof. Step 1: We denote $\alpha = 1/\beta$, and prove that there exists $\varepsilon > 0$ such that $\sigma(T) \cap \{\lambda : 0 < |\lambda - 1| < \varepsilon\}$ is contained in the "interior" of some quasi-parabola $x = 1 - b|y|^\alpha$ with $b > 0$. Assume the statement is false. Since the "interior" of the quasi-parabola $x = 1 - b|y|^\alpha$ increases to the half plane $\{x < 1\}$ as b decreases to 0, the assumption yields that there exist a positive sequence $\{b'_j\}$ decreasing to zero and a sequence $\{\lambda_j = x_j + iy_j\} \subset \sigma(T)$ with $1 \neq \lambda_j \to 1$ such that $x_j > 1 - b'_j|y_j|^\alpha$. By continuity in b of the quasi-parabolas, there are $0 < b_j < b'_j$ with $x_j = 1 - b_j|y_j|^\alpha$; by the construction $0 \neq y_j \to 0$ and $b_j \to 0$, and we may assume (by taking a subsequence) that $y_j > 0$ for every j (the case of $y_j < 0$ for every j being treated similarly). Clearly (for j large) $|\lambda_j| \geq x_j = 1 - b_j y_j^\alpha > 0$ and $|\lambda_j - 1| \geq y_j$. Putting $k_j = [1/b_j y_j^\alpha] + 1$ and remembering that $\alpha = 1/\beta$, we obtain

$$k_j^\beta |\lambda_j|^{k_j} |\lambda_j - 1| \geq \frac{y_j}{b_j^\beta y_j^{\alpha\beta}} (1 - b_j y_j^\alpha)^{1/(b_j y_j^\alpha)} \geq C \frac{1}{b_j^\beta} \to \infty,$$

which contradicts the given rate $\sup_{n\geq 1} n^\beta \|T^n(I-T)\| \leq M$, since by the spectral mapping theorem

$$\sup_{\lambda\in\sigma(T)} k^\beta |\lambda|^k |\lambda - 1| = \sup_{\lambda\in\sigma(k^\beta T^k(I-T))} |\lambda| \leq \|k^\beta T^k(I-T)\|.$$

Step 2: By Step 1 there is $b > 0$ and an open neighborhood of 1, say V, such that $\sigma(T) \cap V$ is included in the interior of the quasi-parabola $x = 1 - b|y|^\alpha$. Since $\|T^n(I-T)\| \to 0$, The spectral mapping theorem implies that $\sigma(T)$ intersects the unit circle at most at the point 1, so $\sigma(T) \cap V^c$ is included in some disk of radius $\rho < 1$ centered at the origin. If the disk is in the interior of the quasi-parabola, we can increase ρ till the quasi-parabola is tangent to it, and still $\rho < 1$ since everything takes place in the open unit disk. If the quasi-parabola intersects the disk in two points, we decrease b till the quasi-parabola is tangent to the disk of radius ρ. In either case, we end up with a quasi-Stolz region $D(1/\beta, b)$ which contains $\sigma(T)$. □

Corollary 2.7. *Let T be a normal contraction on a complex Hilbert space, and let $1 < \alpha < 2$. Then $\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$ if and only if $\sigma(T)$ is contained in a quasi-Stolz region $D(\alpha, b)$ for some $b > 0$.*

Proof. Since T is normal, $\overline{W(T)}$ is the convex hull of $\sigma(T)$ (e.g. [4]), so by convexity of quasi-Stolz regions the "if" part follows from Theorem 2.4. The converse follows from the previous proposition. □

Remark. For $\alpha = 1$ we replace a quasi-Stolz region by a Stolz region, and then the corollary holds by [7, Prposition 2.5]. The "if" part in this case is due to Bellow, Jones and Rosenblatt [3, p. 111].

3. Resolvent conditions for convergence rates

Recall that for T bounded on a complex Banach space X the resolvent is defined by $R(\lambda,T) := (\lambda I - T)^{-1}$ whenever $\lambda I - T$ is one-to-one onto X (and then $R(\lambda,T)$ is also bounded). If $|\lambda| > r(T)$, then $R(\lambda,T) = \lambda^{-1}\sum_{n=0}^\infty (T/\lambda)^n$, so when T is power-bounded, the series representation of $R(\lambda,T)$ holds for every $|\lambda| > 1$.

Proposition 3.1. *Let T be a power-bounded operator in X. Assume that for some $1 < \alpha < 2$ we have*

$$\|R(\lambda,T)\| \le \frac{C}{|\lambda-1|^\alpha} \qquad \text{for} \quad |\lambda| > 1. \tag{4}$$

Then every $\lambda \in S(\alpha,1/C) := \{z = x+iy : \, x > 1 - \frac{1}{C}|y|^\alpha\}$ is in $\rho(T)$. For $b < 1/C$, every $\lambda \in S(\alpha,b)$ satisfies $\|R(\lambda,T)\| \le \frac{C_b}{|\lambda-1|^\alpha}$.

Proof. Every $1 \ne \lambda_0 \in \mathbb{T}$ is in $\rho(T)$, since by (4) $\limsup_{\lambda\to\lambda_0, |\lambda|>1} \|R(\lambda,T)\| < \infty$. We follow ideas of [28, p. 146]. Let $A = T - I$, so $\sigma(A) = \sigma(T) - 1 \subset \mathbb{D} \cup \{0\}$ and by (4)

$$\|R(\lambda,A)\| = \|R(\lambda+1,T)\| \le \frac{C}{|\lambda|^\alpha} \qquad \forall\ |\lambda+1| > 1. \tag{5}$$

This holds in particular when $\operatorname{Re}\lambda > 0$ and when $\lambda = it$ with $t \ne 0$. Fix $\lambda_0 = it_0$, $t_0 \ne 0$. If $|\lambda - \lambda_0| \cdot \|(A - \lambda_0 I)^{-1}\| < 1$, then $\lambda \in \rho(A)$ and

$$(A - \lambda I)^{-1} = \sum_{n=0}^\infty (A - \lambda_0)^{-n-1}(\lambda - \lambda_0)^n. \tag{6}$$

Thus, if $|\lambda - \lambda_0| \frac{C}{|\lambda_0|^\alpha} < 1$, then $\lambda \in \rho(A)$, which implies that $\lambda = x_0 + it_0$ is in $\rho(A)$ if $|x_0| \le \frac{1}{C}|t_0|$. This yields that $x + iy \in \rho(T)$ if $x \ge 1 - \frac{1}{C}|y|^\alpha$ (when $y = 0$ we have $x > 1$).

We now prove the estimate for the resolvent for $z \in S(\alpha,b)$ when $b < 1/C$; put $q = bC < 1$. In view of (4), we need to prove the estimate only for $z \in S(\alpha,b)$ with $\operatorname{Re} z < 1$ and $|\operatorname{Im} z| \le 1$. Let $\lambda = z - 1 = x + iy \in S(\alpha,b) - 1$ with $0 > x > -b|y|^\alpha$ and $|y| \le 1$, so $|x|/|y|^\alpha \le b = q/C$. Let $\lambda_0 = iy$. Then $|\lambda - \lambda_0| = -x$, and by (5)

$$|\lambda - \lambda_0| \cdot \|R(\lambda_0,A)\| \le \frac{q}{C}|y|^\alpha \frac{C}{|\lambda_0|^\alpha} = q < 1.$$

Now $|y| \le 1$ and $\alpha \ge 1$ imply $|\lambda|^2 = x^2 + y^2 < (|y|^{2\alpha}/C^2) + y^2 \le |y|^2(C^2+1)/C^2$, and by (6) and (5) we have

$$\|(A - \lambda I)^{-1}\| \le \|(A - \lambda_0)^{-1}\| \sum_{n=0}^\infty q^n \le \frac{C}{|y|^\alpha(1-q)} \le \frac{C_b}{|\lambda|^\alpha},$$

which for $z = \lambda + 1$ yields $\|R(z,T)\| = \|R(\lambda,A)\| \le C_b/|z-1|^\alpha$.

But $C_b = \frac{C}{1-q} \cdot \left(\frac{C^2+1}{C^2}\right)^{\alpha/2} > C$, so the estimate holds for every $z \in S(\alpha,b)$. □

Corollary 3.2. *Let T be a normal contraction on a complex Hilbert space, and assume that for some $\alpha \in (1,2)$ the resolvent condition (4) holds. Then*
$\|T^n(I-T)\| = \mathcal{O}(1/n^{1/\alpha})$.

Proof. For $b > 0$ small enough, Proposition 3.1 yields that $\sigma(T) \subset D(\alpha, b)$. Since T is normal, $\overline{W(T)}$ is the convex hull of $\sigma(T)$ [4], so is also contained in $D(\alpha, b)$. Now apply Theorem 2.4. □

We now show that Corllary 3.2 extends to all power-bounded operators in Hilbert space, using the recent work of Seifert [35]. For this we need the following Lemma.

Lemma 3.3. *Let T be a power-bounded operator in a complex Banach space X, and let $\alpha \geq 1$. Then the following are equivalent:*
(i) $\|R(\lambda, T)\| \leq C|\lambda - 1|^{-\alpha}$ *for* $|\lambda| > 1$.
(ii) $\|R(\mathrm{e}^{i\theta}, T)\| \leq c|\theta|^{-\alpha}$ *for* $0 < |\theta| \leq \pi$.

Proof. As noted at the beginning of the proof of Proposition 3.1, (i) implies that $\sigma(T) \cap \mathbb{T} \subset \{1\}$. The same is implicit in (ii). If $1 \notin \sigma(T)$ the equivalence is obvious by continuity of the norm $\|R(\lambda, T)\|$ at 1, so we may assume $\sigma(T) \cap \mathbb{T} = \{1\}$.

The first step of the proof of [35, Lemma 3.9] shows that (ii) implies (i).

Assume (ii). Let $|\lambda| > 1$ and write $\lambda = r\mathrm{e}^{i\theta}$, $|\theta| \leq \pi$. For $\theta \neq 0$ we have by continuity of the resolvent

$$\|R(\mathrm{e}^{i\theta}, T)\| = \lim_{r \to 1^+} \|R(r\mathrm{e}^{i\theta}, T)\| \leq \lim_{r \to 1^+} \frac{C}{|r\mathrm{e}^{i\theta} - 1|^{\alpha}} = \frac{C}{|\mathrm{e}^{i\theta} - 1|^{\alpha}}.$$

Since $|\mathrm{e}^{i\theta} - 1| \sim |\theta|$ as $|\theta| \to 0$, condition (ii) holds. □

The next two remarkable results are due to Seifert [35].

Theorem 3.4 ([35], Corollary 3.1)**.** *Let T be power-bounded on a Banach space X with $\sigma(T) \cap \mathbb{T} = \{1\}$ and let $\alpha \geq 1$. If there exist $\epsilon > 0$ and constants $C_1 > c_1 > 0$ such that $c_1|\theta|^{-\alpha} \leq \|R(\mathrm{e}^{i\theta}, T)\| \leq C_1|\theta|^{-\alpha}$ for $0 < |\theta| < \epsilon$, then there exist constants $C > c > 0$ such that*

$$\frac{c}{n^{1/\alpha}} \leq \|T^n(I - T)\| \leq C\left(\frac{\log n}{n}\right)^{1/\alpha}, \qquad n \geq 1.$$

Remark. Theorem 3.4 and Lemma 3.3 yield that if $\|R(\lambda, T)\| \leq C|\lambda - 1|^{-\alpha}$ for $|\lambda| > 1$, then $\|T^n(I-T)\| \leq C\left(\frac{\log n}{n}\right)^{1/\alpha}$, an improvement of Nevanlinna's [30, Theorem 9], where the rate obtained, for $1 < \alpha < 2$, is only $1/n^{(2-\alpha)/\alpha}$.

Theorem 3.5 ([35], Theorem 3.10)**.** *Let T be power-bounded on a Hilbert space H with $\sigma(T) \cap \mathbb{T} = \{1\}$ and let $\alpha \geq 1$. Then the following are equivalent:*
(i) There exist $\epsilon > 0$ and $C_1 > 0$ such that $\|R(\mathrm{e}^{i\theta}, T)\| \leq C_1|\theta|^{-\alpha}$ for $0 < |\theta| < \epsilon$,
(ii) There exists $C > 0$ such that

$$\|T^n(I - T)\| \leq \frac{C}{n^{1/\alpha}} \quad n \geq 1.$$

Combining Theorem 3.5 with Lemma 3.3 we obtain

Theorem 3.6. *Let T be power-bounded on a Hilbert space H with $\sigma(T) \cap \mathbb{T} = \{1\}$ and let $\alpha \geq 1$. Then the following conditions are equivalent:*
(i) There exists $C > 0$ such that $\|R(\lambda, T)\| \leq C|\lambda - 1|^{-\alpha}$ for $|\lambda| > 1$.

$$(ii) \qquad \|T^n(I - T)\| = \mathcal{O}(1/n^{1/\alpha}).$$

Remarks. 1. The case of $\alpha = 1$ in Theorem 3.6 is the known case of Ritt operators.

2. Let $Vf(t) := \int_0^t f(s)ds$ be the Volterra operator on $L_2[0,1]$ (see Example 1 below). It is known that $T = I - V$ is power-bounded [1] with $\sigma(T) = \{1\}$, and by [37] $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$. Paulauskas [32] showed that $\sup_{|\lambda|<1} |\lambda - 1|^2 \|R(\lambda, T)\| = \infty$ (see also [31, p. 2082]); this shows that even when 1 is isolated in the spectrum, the resolvent estimate need not hold if we approach 1 from within the open unit disk.

4. On conditions for $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$

Dungey [11] gave several equivalent conditions for the rate $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$ when T is power-bounded on a Banach space. We look at additional conditions when T is a contraction on a Hilbert space.

Lemma 4.1. *Let $\delta \in (0,1)$ and define the disk $\bar{D}_\delta := \{z : |z - \delta| \le 1 - \delta\}$. Then there exists $C_\delta > 0$ such that $\sup_{z\in\bar{D}_\delta} |z^n(1-z)| \le C_\delta/\sqrt{n}$ for $n \ge 1$.*

Proof. Since $z^n(1-z)$ is continuous on $\bar{D}_\delta$ and holomorphic inside the disk, by the maximum principle

$$\sup_{z\in\bar{D}_\delta} |z^n(1-z)| = \sup_{\{|z-\delta|=1-\delta\}} |z^n(1-z)|.$$

Points on the circle $\{|z-\delta| = 1-\delta\}$ are represented by $z_\theta = \delta + (1-\delta)\mathrm{e}^{i\theta}$, $0 \le \theta < 2\pi$. Applying Lemma 2.1 of Foguel-Weiss [13] to the numbers 1 and $\mathrm{e}^{i\theta}$ (note that its proof is valid for elements of norm at most 1, and the estimate there should be $K/(\sqrt{n}\cdot\alpha\cdot\delta)$), we obtain

$$|z_\theta^n(1-z_\theta)| = (1-\delta)|z_\theta^n(1-\mathrm{e}^{i\theta})| \le (1-\delta)K/(\sqrt{n}\delta(1-\delta)) = K/(\sqrt{n}\delta).$$

Thus the assertion of the lemma holds with $C_\delta = K/\delta$ (where K is an absolute constant, obtained in [13]). □

Proposition 4.2. *let T be a bounded operator in a Banach space for which a closed disk $\bar{D}_\delta$ is a K-spectral set. Then $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$.*

Proof. By Lemma 4.1, $\|T^n(I-T)\| \le K_{\bar{D}_\delta} C_\delta/\sqrt{n}$ for every $n \ge 1$. □

Proposition 4.3. *Let T be a contraction on a complex Hilbert space. Then each condition in the list below implies the next one:*

(i) There exist a contraction S and some $\delta \in (0,1)$ such that $T = \delta I + (1-\delta)S$.

(ii) For some $\delta \in (0,1)$, the numerical range of T is contained the closed disk $\bar{D}_\delta$.

(iii) $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$.

(iv) There exist a power-bounded operator S and a number $\delta \in (0,1)$ such that $T = \delta I + (1-\delta)S$.

(v) For some $\delta \in (0,1)$ we have $\sigma(T) \subset \bar{D}_\delta$.

Proof. Assume (i). Then $|\langle Sf, f\rangle| \le 1$ for $\|f\| = 1$, and we obtian

$$\langle Tf, f\rangle = \delta + (1-\delta)\langle Sf, f\rangle \in \bar{D}_\delta.$$

Assume (ii). Then by Delyon and Delyon [8, Theorem 3], $\bar{D}_\delta$ is a K-spectral set for T. By Proposition 4.2 we have $\|T^n(I-T)\| \le K_{\bar{D}_\delta} C_\delta/\sqrt{n}$ for every $n \ge 1$.

(iii) implies (iv) by [11, Theorem 1.2] (for any power-bounded T in a Banach space).

If (iv) holds, then $\sigma(T) = \{\lambda = \delta + (1-\delta)z : z \in \sigma(S)\}$. Hence $\lambda \in \sigma(T)$ satisfies $|\lambda - \delta| = |1-\delta||z| \le 1-\delta$. □

Remarks. 1. (i) implies (ii) if we assume only that the numerical radius of S, $w(S) := \sup\{|\lambda| : \lambda \in W(S)\}$, is not more than 1 (which implies $\sup_n \|S^n\| \le 2$ [36]).

2. When (iv) holds (in any Banach space), $|||x||| := \sup_{n\ge 0} \|S^n x\|$ is an equivalent norm for which $|||S||| \le 1$, and then also $|||T||| \le 1$; (iii) then follows from [13].

3. If (i) holds, then $\|T - \delta I\| = \|(1-\delta)S\| \le 1-\delta$ implies that $\bar{D}_\delta$ is a spectral set for T (i.e. K-spectral with $K_{\bar{D}_\delta} = 1$), by von-Neumann's inequality [33, Section 154]. Conversely, if $\bar{D}_\delta$ is a spectral set for T, then $\|T - \delta I\| \le \sup\{|z-\delta| : z \in \bar{D}_\delta\} = 1-\delta$, so (i) holds with $S = (1-\delta)^{-1}(T - \delta I)$.

Corollary 4.4. *Let T be a normal contraction in a complex Hilbert space. Then all the conditions of Proposition 4.3 are equivalent.*

Proof. Since T is normal, when (iv) holds the power-bounded S is also normal, hence $\|S\| \le 1$ and (i) holds, so all first four conditions of Proposition 4.3 are equivalent.

By normality, $\overline{W(T)}$ is the convex hull of $\sigma(T)$ [4], so (v) is equivalent to (ii). □

Remark. Given a non-normal contraction S, the operator $T := \frac{1}{2}(I+S)$ is a non-normal contraction which satisfies all the conditions of Proposition 4.3.

Example 1. *Some power-bounded operators obtained from the Volterra operator.*

Let $Vf(t) := \int_0^t f(s)ds$ be the Volterra operator on $L_2[0,1]$. It is known that $T = (I+V)^{-1}$ is a contraction [15, Problem 150]. Since $\sigma(V) = \{0\}$, we have $\sigma(T) = \{1\}$, so $T \ne I$ shows T is not normal. From the work of Sarason [34] it follows that $T = \frac{1}{2}(I+S)$ for some contraction S, so T satisfies all the conditions of Proposition 4.3, hence the closed disk $\bar{D}_{1/2}$ is a K-spectral set for T. The fact that $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$, which follows from [13], was first observed by Tsedenbayar [37] (for $I - V$, which is similar to T [1]). The rate $\mathcal{O}(1/\sqrt{n})$ is precise, by [27].

By the similarity of $I - V$ and T, the closed disk $\bar{D}_{1/2}$ is a K-spectral set for $I - V$ (since by [20] it is enough to prove (3) for polynomials). Thus $I - V$ is an example of a non-contractive power-bounded operator in H satisfying the assumptions of Proposition 4.2. Moreover, it is not difficult to deduce from the work of Foiaş and Williams [14, Proposition 2] that all the closed disks $\bar{D}_\delta$, $0 < \delta < 1$, are K-spectral sets for $I - V$.

In [12, Proposition 2.5] Dungey proved (among other things) the following:

Proposition 4.5. *Let $\mu := \{a_k\}_{k\in\mathbb{Z}}$ be a probability distribution on $\mathbb{Z}$. If μ is strictly aperiodic, i.e. its support $\mathbb{S} := \{k : a_k > 0\}$ is not contained in a translate of a proper subgroup of $\mathbb{Z}$ (equivalently, $\mathbb{S} - \mathbb{S}$ generates $\mathbb{Z}$), then the convolution powers of μ satisfy $\sup_n \sqrt{n}\|\mu^n - \mu^{n+1}\|_{L_1(\mathbb{Z})} < \infty$.*

Corollary 4.6. *Let $\mu := \{a_k\}_{k\in\mathbb{Z}}$ be a strictly aperiodic probability on $\mathbb{Z}$ and let S be an invertible operator on a Banach space which is bilaterally power-bounded (i.e. $\sup_{n\in\mathbb{Z}} \|S^n\| = K < \infty$). Then $T := \sum_{k\in\mathbb{Z}} a_k S^k$ satisfies $\|T^n - T^{n+1}\| = \mathcal{O}(1/\sqrt{n})$.*

Proof. Let $\mu^n = \{a_k^{(n)}\}_{k\in\mathbb{Z}}$. Since T is a $\mathbb{Z}$-representation average, we have that $T^n = \sum_{k\in\mathbb{Z}} a_k^{(n)} S^k$ and

$$\|T^n - T^{n+1}\| \le K \sum_{k\in\mathbb{Z}} |a_k^{(n)} - a_k^{(n+1)}| = K\|\mu^n - \mu^{n+1}\|_{L_1(\mathbb{Z})} = \mathcal{O}(1/\sqrt{n})$$

by the previous proposition. □

Remarks. 1. When μ is supported on $\mathbb{N}$, the result of the corollary holds for any S power-bounded, without requiring invertibility [12].

2. Dungey [12] showed that for certain probabilities μ supported on $\mathbb{N}$ with infinite support, we have $\|T^n(I-T)\| = \mathcal{O}(1/n)$.

3. Let P be a Markov operator with invariant probability. Then for μ strictly aperiodic supported on $\mathbb{N}$, some asymptotic properties of the Markov chain generated by the operator $T := \sum_{k=0}^{\infty} a_k P^k$, called in [19] a *time-sampled* Markov chain, were studied in [19].

4. If μ is strictly aperiodic and symmetric and S is unitary, then by Stein's theorem $\|T^n(I-T)\| = \mathcal{O}(1/n)$.

5. Let U be the unitary operator induced by a probability preserving invertible transformation τ. Bellow, Jones and Rosenblatt [3] showed that if μ strictly aperiodic satisfies $\sum_{k\in\mathbb{Z}} k^2 a_k < \infty$ and $\sum_{k\in\mathbb{Z}} k a_k = 0$, then the Markov operator $Q := \sum_{k\in\mathbb{Z}} a_k U^k$ on L_2 satisfies $\|Q^n(I-Q)\| = \mathcal{O}(1/n)$, and deduced a.e. convergence of $Q^n f$ for every $f \in L_2$. For such a μ, a "quenched" central limit theorem was proved in [9] for the Markov chain generated by the above Q. A d-dimensional analogue is studied in [6].

Example 2. *Convex combinations of powers of non-normal contractions*
Let $\mu := \{a_k\}_{k\in\mathbb{Z}}$ be a strictly aperiodic probability distribution on $\mathbb{Z}$. Let S be a contraction on H, and define

$$T = \sum_{k\ge 0} a_k S^k + \sum_{k<0} a_k S^{*|k|}.$$

Note that if S is not normal, then T is not normal. Let U be the unitary dilation of S, defined on a larger Hilbert space H_1 of which H is a subspace with orthogonal projection P from H_1 onto H, and define

$$Q = \sum_{k\ge 0} a_k U^k + \sum_{k<0} a_k U^{*|k|} = \sum_{k\in\mathbb{Z}} a_k U^k.$$

Then Q is a normal operator (on H_1), and by Proposition 4.6 $\|Q^n(I-Q)\| = \mathcal{O}(1/\sqrt{n})$. By Corollary 4.4, there exists a contraction R on H_1 such that $Q = \delta I + (1-\delta)R$ for some $\delta \in (0,1)$. Then for $x \in H$ we have $Tx = PQx = \delta x + (1-\delta)PRx$. Since $R_0 := (PR)_{|H}$ is a contraction on H, we obtain that $T = \delta I + (1-\delta)R_0$, so the contraction T satisfies condition (i) (and therefore all the other conditions) of Proposition 4.3. Note that if μ is symmetric, or satisfies the conditions of [3], then $\|T^n(I-T)\| = \mathcal{O}(1/n)$.

Remarks. 1. If in the previous example $0 < a_0 < 1$, then condition (i) of Proposition 4.3 holds without requiring the strict aperiodicity.

2. If μ in the example is supported on $\mathbb{N}$, then $T := \sum_{k\ge 0} a_k S^k$ satisfies all the conditions of Proposition 4.3.

5. A pointwise convergence theorem for some L_2 operators

Let T be a power-bounded operator on $L_p(\Omega,\Sigma,\mu)$ of a σ-finite measure space, $1<p<\infty$. For $\gamma>1/p$, the convergence of $\sum_{n=1}^\infty \frac{\|T^nf\|^p}{n^{\gamma p}}$ and Beppo Levi's theorem yield that $\frac{T^nf}{n^\gamma}\to 0$ a.e., for every $f\in L_p(\mu)$. In fact, there is also a maximal inequality [2, Proposition I(iii)].

If T is a Markov operator induced by a transition probability $P(x,A)$ with invariant measure μ, then it induces a contraction on all the $L_p(\mu)$ spaces. The pointwise ergodic theorem for L_1 functions and the inequality $|T^nf|^p\le T^n(|f|^p)$, which holds a.e., yield that $\frac{|T^nf|}{n^{1/p}}\to 0$ a.e. for every $f\in L_p(\mu)$. A similar result holds if T is a Dunford-Schwartz operator (a contraction of L_1 and L_∞), by applying Lemma 7.4 of [18, p. 65] to the linear modulus of T. Example 4 below shows that in general, a positive contraction T on L_p may have some $f\in L_p$ with $\limsup \frac{T^n|f|}{n^{1/p}}=\infty$ a.e.

In this section we look at conditions on a power-bounded T on $L_2(\mu)$ which will yield, for an appropriate $\gamma\in[0,\frac12]$, the a.e. convergence $\frac{T^nf}{n^\gamma}\to 0$ for every $f\in L_2(\mu)$.

Lemma 5.1. *Let $1<p<\infty$, and let T be a power-bounded operator on $L_p(\Omega,\Sigma,\mu)$ which satisfies $\sup_n n^\beta\|T^n(I-T)\|<\infty$ for some $\beta>1/p$. Then $T^nf\to 0$ a.e. for every $f\in(I-T)L_p(\mu)$.*

Proof. $\sum_{n=1}^\infty \|T^n(I-T)g\|^p<\infty$, so $\sum_{n=1}^\infty |T^n(I-T)g|^p<\infty$ a.e. □

Lemma 5.2. *let $D(\alpha,b)$ be a quasi-Stolz region, with $1\le\alpha\le 2$, $b>0$ when $\alpha<2$ and $b>\frac12$ when $\alpha=2$. Then $\sup_{1\ne z\in D(\alpha,b)}\frac{|1-z|^2}{(1-|z|^2)^{2/\alpha}}<\infty$.*

Proof. By the construction of quasi-Stolz domains, 1 is the only unimodular point in $\bar D(\alpha,b)$, so $\frac{|1-z|^2}{(1-|z|^2)^{2/\alpha}}$ is bounded on $D(\alpha,b)\cap\{\Re\mathrm{e}\, z\le 0\}$, and boundedness depends on the behaviour near 1. Take a point $z=x+iy\in D(\alpha,b)$, and put $u=u(z)=x+i(\frac{1-x}{b})^{1/\alpha}$, which is a point on the upper half of the quasi-parabola $x=1-b|y|^\alpha$. It is clear that $|1-z|\le|1-u|$ and $|u|\ge|z|$, so

$$\frac{|1-z|^2}{(1-|z|^2)^{2/\alpha}}\le\frac{|1-u|^2}{(1-|u|^2)^{2/\alpha}}=\frac{(1-x)^2+(\frac{1-x}{b})^{2/\alpha}}{(1-x^2-(\frac{1-x}{b})^{2/\alpha})^{2/\alpha}}.$$

After dividing by $(1-x)^{2/\alpha}$ and letting $x\uparrow 1$, we conclude that the limit is $\frac{1/b^{2/\alpha}}{2^{2/\alpha}}$ for $1<\alpha<2$, the limit is $\frac{b^2+1}{4b^2}$ for $\alpha=1$, and the limit is $\frac{1}{2b-1}$ in the case $\alpha=2$ (in which $b>1/2$). Thus in all cases $|1-z|^2/(1-|z|^2)^{2/\alpha}$ is bounded near 1. □

The proof of our Theorem 5.4 below is inspired by a method of E. Stein (see [3]), and will require the following lemma.

Lemma 5.3. *Let $0\le\beta\le 1$. Then there exists $C>0$ such that $\sum_{n=1}^\infty n^\beta t^n\le\frac{C}{(1-t)^{\beta+1}}$ for all $0\le t<1$.*

Proof. From the theory of hypergeometric functions we have the representation (see formula (1.9) in [38, p. 76]) $\frac{1}{(1-t)^{\beta+1}}=\sum_{n=0}^\infty\binom{n+\beta}{n}t^n$, with the following estimate for the coefficients (see formula (1.18) in [38, p. 77]):

$$\binom{n+\beta}{n}=\frac{n^\beta}{\Gamma(\beta+1)}\Big[1+\mathcal{O}(\frac1n)\Big]. \tag{7}$$

We then write

$$\frac{1}{\Gamma(\beta+1)}\sum_{n=0}^\infty n^\beta t^n = \sum_{n=0}^\infty \Big[\frac{n^\beta}{\Gamma(\beta+1)} - \binom{n+\beta}{n}\Big]t^n + \sum_{n=0}^\infty \binom{n+\beta}{n} t^n. \tag{8}$$

Using (7) we estimate the first series on the right hand side of (8):

$$\sum_{n=1}^\infty \Big|\frac{n^\beta}{\Gamma(\beta+1)} - \binom{n+\beta}{n}\Big| t^n \le C\sum_{n=1}^\infty \frac{1}{n^{1-\beta}}t^n \le \frac{C}{1-t} \le \frac{C}{(1-t)^{1+\beta}},$$

which together with the last series in (8) yields the assertion. □

Theorem 5.4. *Let $D(\alpha,b)$ be a quasi-Stolz region, with $1<\alpha\le 2$ and $b>0$ ($b\ge 1/2$ for $\alpha=2$). If $\bar D(\alpha,b)$ is a K-spectral set for a power-bounded operator T on $L_2(\mu)$, in particular (by [8]) if the numerical range of T is included in $\bar D(\alpha,b)$, then for every $f\in L_2(\mu)$ we have*
(i) $\|\sup_n \frac{|T^nf|}{n^{1-1/\alpha}}\|_2<\infty$.
(ii) $\frac{T^nf}{n^{1-1/\alpha}}\to 0$ a.e.

Proof. By Proposition 2.3 we have $\sup_n n^{1/\alpha}\|T^n(I-T)\|<\infty$, so when $\alpha<2$ Lemma 5.1 yields that T^nf converges a.e. for f in the dense subspace $\{Tg=g\}+(I-T)L_2$, hence $\frac{T^nf}{n^{1-1/\alpha}}\to 0$ a.e. for f in a dense subspace. When $\alpha=2$, for every $\epsilon>0$ we obtain $\frac{T^n(I-T)g}{n^\epsilon}\to 0$ a.e. for every g, similarly to the proof of Lemma 5.1.

Hence we always have $\frac{T^nf}{n^{1-1/\alpha}}\to 0$ a.e. for f in a dense subspace, so (ii) for every $f\in L_2$ follows from (i) by the Banach principle.

It is well-known (and easy to check) that

$$T^n = \frac{1}{n}\sum_{k=0}^{n-1}T^k + \frac{1}{n}\sum_{k=1}^n k(T^k-T^{k-1}).$$

Hence for every $f\in L_2(\mu)$ we have

$$\Big\|\sup_n \frac{|T^nf|}{n^{1-1/\alpha}}\Big\|_2 \le$$

$$\big\|\sup_n \frac{1}{n^{2-1/\alpha}}\big|\sum_{k=0}^n T^kf\big|\,\big\|_2 + \big\|\sup_n \frac{1}{n^{2-1/\alpha}}\big|\sum_{k=1}^n k(T^k-T^{k-1})f\big|\,\big\|_2. \tag{9}$$

We first deal with the second term. The Cauchy-Schwarz inequality yields

$$\frac{1}{n^{2-1/\alpha}}\big|\sum_{k=1}^n k(T^k-T^{k-1})f\big| \le \frac{1}{n^{2-1/\alpha}}\big(\sum_{k=1}^n k^{\frac{3\alpha-2}{\alpha}}\big)^{\frac12}\big(\sum_{k=1}^n k^{\frac{2-\alpha}{\alpha}}|(T^k-T^{k-1})f|^2\big)^{\frac12}$$

$$\le C_\alpha\big(\sum_{k=1}^n k^{\frac{2-\alpha}{\alpha}}|(T^k-T^{k-1})f|^2\big)^{\frac12} \le C_\alpha\big(\sum_{k=1}^\infty k^{\frac{2-\alpha}{\alpha}}|(T^k-T^{k-1})f|^2\big)^{\frac12}.$$

We use the fact that $\bar D(\alpha,b)$ is K-spectral, and then apply Lemma 5.3 to obtain

$$\sum_{k=1}^n k^{\frac{2-\alpha}{\alpha}}\|(T^k-T^{k-1})f\|_2^2 \le K_{D(\alpha,b)}\sup_{z\in D(\alpha,b)}\sum_{k=1}^\infty k^{\frac{2-\alpha}{\alpha}}|z^k-z^{k-1}|^2 \le$$

$$\kappa_\alpha K_{D(\alpha,b)} \sup_{1\neq z\in D(\alpha,b)} \frac{|1-z|^2}{(1-|z|^2)^{2/\alpha}}.$$

Lemma 5.2 now implies $\sum_{k=1}^\infty k^{\frac{2-\alpha}{\alpha}}\|(T^k-T^{k-1})f\|_2^2 < \infty$, so we conclude that the second term of (9) is finite.

Since $1<\alpha\le 2$, the series $\sum_{n=1}^\infty \frac{\|T^nf\|_2}{n^{2-\frac{1}{\alpha}}}$ converges, and the finiteness of the first term in (9) follows from the modified Kronecker's lemma in the claim below.

Claim: If $\{b_n\}$ is positive increasing and $\{f_k\}\subset L_2$ such that $\sum_{n=1}^\infty \frac{\|f_n\|_2}{b_n}$ converges, then $\|\sup_n \frac{1}{b_n}\sum_{k=1}^n f_k\|_2<\infty$.

Proof. We may and do assume that $f_k\ge 0$ for every k. Put $S_0=0$ and $S_n=\sum_{k=1}^n \frac{f_k}{b_k}$, so $b_k(S_k-S_{k-1})=f_k$. Then

$$\frac{1}{b_n}\sum_{k=1}^n f_k = \frac{1}{b_n}\sum_{k=1}^n b_k(S_k-S_{k-1}) = \frac{1}{b_n}[b_nS_n - \sum_{k=1}^{n-1}(b_{k+1}-b_k)S_k].$$

Since S_n is increasing with $\|S_n\|_2\le\sum_{k=1}^\infty \frac{\|f_k\|_2}{b_k}$, it converges a.e. as well as in L_2-norm, say to S, and we obtain $\frac{1}{b_n}\sum_{k=1}^n f_k\le 2S$ a.e., which proves the claim. $\square$

When $\alpha=1$, the above proof shows the finiteness of the second term of (9) (this is Stein's argument, as in [3]). However, finiteness of the first term in (9) does not always hold, even for contractions, unless we assume T to be a positive contraction and refer to Akcoglu's theorem (e.g. [18, p. 189]). We then have the following extension of [3].

Theorem 5.5. *If a Stolz region $\bar{D}(1,b)$ is a K-spectral set for a positive contraction T on $L_2(\mu)$, in particular (by [8]) if the numerical range of T is included in $\bar{D}(1,b)$, then for every $f\in L_2(\mu)$ we have*
(i) $\|\sup_n |T^nf|\|_2<\infty$.
(ii) $T^nf\to 0$ a.e.

Remark. Le Merdy and Xu [24] proved that if T is a positive Ritt contraction on L^p, $1<p<\infty$, then T has a bounded $H^\infty(D)$ functional calculus for some Stolz region D (so D is a K-spectral set for T), and used it to prove the above theorem also for positive contractions of L^p.

Theorem 5.6. *Let $\delta\in(0,1)$ and put $D_\delta=\{z:|z-\delta|<1-\delta\}$. If $\bar{D}_\delta$ is a K-spectral set for a power-bounded operator T on $L_2(\mu)$, in particular (by [8]) if the numerical range of T is included in $\bar{D}_\delta$, then for every $f\in L_2(\mu)$ we have*
(i) $\|\sup_n \frac{|T^nf|}{n^{1/2}}\|_2<\infty$.
(ii) $\frac{T^nf}{n^{1/2}}\to 0$ a.e.

The proof is similar to that of Theorem 5.4 (with $\alpha=2$), except that instead of Lemma 5.2 we use the following lemma.

Lemma 5.7. *For $\delta\in(0,1)$ put $D_\delta=\{z:|z-\delta|<1-\delta\}$. Then* $\sup_{z\in D_\delta}\frac{|1-z|^2}{1-|z|^2}<\infty$.

Proof. Since the closed disk $\bar{D}_\delta$ touches the unit circle only at 1, the boundedness depends on the behaviour in D_δ near 1. Take $z=x+iy\in D_\delta$ and let $u=u(z):=$

$x + i\sqrt{(1-\delta)^2 - (x-\delta)^2}$ be on the boundary of D_δ above z. Clearly $|1-z| \le |1-u|$ and $|z| \le |u|$, so

$$\frac{|1-z|^2}{1-|z|^2} \le \frac{|1-u|^2}{1-|u|^2} = \frac{(1-x)^2 + (1-\delta)^2 - (x-\delta)^2}{1-x^2-(1-\delta)^2+(x-\delta)^2} =$$

$$\frac{(1-x)^2 + (1+x-2\delta)(1-x)}{(1-x)(1+x) - (1+x-2\delta)(1-x)},$$

which tends to $(1-\delta)/\delta$ as $x \uparrow 1$. Thus $|1-z|^2/(1-|z|^2)$ is bounded on D_δ. □

Remark. When T is a normal contraction on $L_2(\mu)$, by [4] Theorems 5.4 and 5.6 apply if $\sigma(T)$ is in $\bar{D}(\alpha, b)$ or in $\bar{D}_\delta$, respectively.

Corollary 5.8. *Let T be a normal contraction on $L_2((\mu)$. If $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ for some $\frac{1}{2} \le \beta < 1$, then for every $f \in L_2(\mu)$ we have $\|\sup_n \frac{|T^n f|}{n^{1-\beta}}\| < \infty$ and $\frac{T^n f}{n^{1-\beta}} \to 0$ a.e.*

Proof. We apply Corollary 2.7 or Corollary 4.4, and then [4], and use Theorem 5.4 or Theorem 5.6. □

Example 3 *Non-normal operators to which Theorem 5.6 applies.*
Let V be the Volterra operator on $L_2[0,1]$. Then, as discussed in Example 1, the operator $T = (I+V)^{-1}$ is a non-normal contraction for which $\bar{D}_{1/2}$ is a K-spectral set, and $I - V$, which is similar to T, is a power-bounded operator for which $\bar{D}_{1/2}$ is a K-spectral set. Thus Theorem 5.6 applies to these operators.

Theorem 5.6 is justified since, even for positive contractions in L_2, property (ii) need not hold in general, as shown by the following example, based on ideas of Irmisch [16, p. 37], which was suggested by Y. Derriennic.

Example 4. *A positive contraction S on L_2 and $g \in L_2$ with* $\limsup \frac{S^n g}{\sqrt{n \log n}} \equiv \infty$.
In 1964, Chacon constructed a positive contraction T on L_1 for which there is some $0 \le f_0 \in L_1$ with $\limsup_{n\to\infty} \frac{T^n f_0}{n} = \infty$ a.e. (see [18, p. 151]). Mesiar [26] modified Chacon's construction to obtain a positive contraction T on L_1 for which there is a function $0 \le f_0 \in L_1$ with $\limsup_{n\to\infty} \frac{T^n f_0}{n \log n} = \infty$ a.e. We use the following notations (as presented in [18, p. 151]): τ is an invertible non-singular transformation of (Ω, μ) and T on $L_1(\mu)$ is defined by $T(d\nu/d\mu) = d(\nu\tau^{-1})/d\mu$ for $\nu << \mu$, so $T^* h = h \circ \tau$. We then obtain, with $\theta = \tau^{-1}$, that

$$Tf = \frac{d(\mu\theta)}{d\mu} \cdot (f \circ \theta), \qquad f \in L_1(\mu). \tag{10}$$

Since for any $h \in L_1(\mu)$ we have $\int_{\theta A} h d\mu = \int_A (h \circ \theta) d(\mu\theta)$, for $\nu << \mu$ we obtain

$$\frac{d\nu}{d\mu} \circ \theta = \frac{d(\nu\theta)}{d(\mu\theta)}. \tag{11}$$

This yields $T^2 f = \frac{d(\mu\theta)}{d\mu} \cdot \left(\frac{d(\mu\theta)}{d\mu} \circ \theta\right) \cdot (f \circ \theta^2) = \frac{d(\mu\theta^2)}{d\mu} \cdot (f \circ \theta^2)$, and by induction

$$T^n f(x) = \frac{d(\mu\theta^n)}{d\mu}(x) \cdot f(\theta^n x), \qquad f \in L_1(\mu). \tag{12}$$

Fix $1 < p < \infty$, and define S on $L_p(\mu)$ by $Sg := \big(\frac{d(\mu\theta)}{d\mu}\big)^{1/p} \cdot (g \circ \theta)$. Then S is a positive isometry of $L_p(\mu)$. Using (11) we obtain by induction that

$$S^n g(x) = \left(\frac{d(\mu\theta^n)}{d\mu}(x)\right)^{1/p} \cdot g(\theta^n x) \qquad g \in L_p(\mu). \tag{13}$$

Let $0 \le f_0 \in L_1(\mu)$ be the function of Mesiar's example, with $\limsup_n \frac{T^n f_0}{n \log n} = \infty$ a.e., and put $g_0 = {f_0}^{1/p}$. Then (13) and (12) yield

$$\left(\frac{S^n g_0}{(n\log n)^{1/p}}\right)^p = \frac{1}{n\log n}\frac{d(\mu\theta^n)}{d\mu} \cdot (g_0^p \circ \theta^n) = \frac{T^n f_0}{n \log n},$$

which shows that $\limsup_n \frac{S^n g_0}{(n\log n)^{1/p}} = \infty$ a.e. The asserted example is for $p = 2$.

Since T is invertible with $(T^{-1})^* h = h \circ \theta$, we obtain that $\frac{d(\mu\tau)}{d\mu} \cdot \frac{d(\mu\theta)}{d\mu} \circ \tau = 1$. Hence S is invertible on $L_p(\mu)$, with $S^{-1} g := \big(\frac{d(\mu\tau)}{d\mu}\big)^{1/p} \cdot (g \circ \tau)$. Thus for $p = 2$, S is unitary.

Remarks. 1. Note that $\frac{\|T^n g\|}{(n\log n)^{1/p}} \to 0$ for any T power-bounded on L_p and $g \in L_p$, which implies that there is $\{n_k\}$ (which depends on g) with $\frac{T^{n_k} g}{(n_k \log n_k)^{1/p}} \to 0$ a.e.

2. Assani and Mesiar [2] studied the a.e. convergence of $\frac{T^n f}{n^\gamma}$ for power-bounded positive operators of L_p. In their Theorem III they show that there exists a probability preserving transformation such that for every $p \ge 1$ and $\gamma \in (0, 1/p)$ there is a function $f = f_{p,\gamma} \in L_p$ such that $\limsup_n \frac{T^n f}{n^\gamma} \ge 1$ a.s. In their Theorem V they show the existence of a probability preserving transformation such that for $p > 1$ there is a function $f = f_p \in L_p$ for which $\sup_n \frac{T^n |f|}{n^{1/p}}$ is not in L_p (though $T^n f / n^{1/p} \to 0$ a.e.).

6. Problems

In this section we discuss some problems raised by our results.

Problem 1. *Let T be a contraction on H with $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$, $\frac{1}{2} \le \beta < 1$. Is there a quasi-Stolz $D(1/\beta, b)$ which is K-spectral for T?*
This question deals with the converse of Proposition 2.3 for contractions in H, and is about a weak converse to Theorem 2.4. By Proposition 2.6 $\sigma(T)$ is contained in some quasi-Stolz region $D(1/\beta, b)$, so when T is normal the answer is positive. For a Ritt contraction ($\beta = 1$) Le Merdy [23, Theorem 8.1] proved that the answer is positive. However, it is not known if the numerical range of a (non-normal) Ritt contraction in H must be in a Stolz region.

Problem 2. *Are all the conditions of Proposition 4.3 equivalent for every contraction T on H?*
Corollary 4.4 yields a positive answer for normal contractions. Of particular interest is the question whether $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$ implies (ii) in Proposition 4.3, which in turn yields that the disk $\bar{D}_\delta$ is a K-spectral set for T; a positive answer will allow the use of Theorem 5.6 for any contraction on $L_2(\mu)$ which satisfies $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$. This problem is related to the case $\beta = \frac{1}{2}$ of Problem 1 above.

Problem 3. *Given $\frac{1}{2} < \beta < 1$, find a strictly aperiodic $\mu = \{a_k\}$ on $\mathbb{Z}$ such that for every unitary U the operator $T = \sum_k a_k U^k$ satisfies $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$.*
Corollary 4.6 yields the rate $1/\sqrt{n}$ for any μ strictly aperiodic. Of course, if μ is such that for every unitary the operator T satisfies $\|T^n(I-T)\| = \mathcal{O}(1/n)$ (e.g. μ is symmetric, or as in the results of [3] or [12]), then it has also the rate $1/n^\beta$ for every $\beta < 1$. Thus the question is about finding μ such that $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ for every unitary and, in addition, for *some* unitary U the rate $1/n^\beta$ is precise.

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