ON THE KOMLÓS–RÉVÉSZ ESTIMATION PROBLEM FOR RANDOM VARIABLES WITHOUT VARIANCES

GUY COHEN

Ben-Gurion University, Israel
e-mail: guycohen@ee.bgu.ac.il

ABSTRACT. Let $\{X_n\} \subset L_p(\mathbf{P})$, $1 < p \le 2$, $q = p/(p-1)$, be a sequence of martingale differences. We prove that the Komlós–Révész type weighted averages $\frac{\sum_{k=1}^n (X_k/\|X_k\|_p^q)}{\sum_{k=1}^n (1/\|X_k\|_p^q)}$ converge a.s. and in the L_p-norm, and the limit is 0 if and only if $\sum_{n=1}^\infty (1/\|X_n\|_p^q) = \infty$. We show also that convergence need not hold when we deal with a centered uncorrelated sequence (whether the series $\sum_{n=1}^\infty (1/\|X_n\|_2^2)$ converges or not). Furthermore, for $1 < p < 2$ all the results of Komlós–Révész are extended to symmetric independent p-stable random variables.

1. Introduction

Let $\{X_n\}$ be a sequence of independent random variables with constant mean m and finite variances. It was proved by Komlós and Révész [12, Theorem 1] that the weighted averages

$$\frac{\sum_{k=1}^n \big(X_k/\mathrm{var}(X_k)\big)}{\sum_{k=1}^n \big(1/\mathrm{var}(X_k)\big)} \tag{1}$$

converge a.s. to m if and only if $\sum_{n=1}^\infty \big(1/\mathrm{var}(X_n)\big) = \infty$. It was noted there, using Cauchy's inequality, that among all the possible weighted averages of $X_1, \cdots, X_n$ the above gives the minimal variance of error, and this minimum is attained only for the weighted average (1). It was also proved in Theorem 2 there that for any sequence of positive numbers σ_n for which $\sum_{n=1}^\infty (1/\sigma_n^2)$ converges there exists a sequence of centered independent random variables $\{\xi_n\} \subset L_2(\mathbf{P})$ with $\|\xi_n\|_2^2 = \sigma_n^2$ such that no sequence of weighted averages of $\{\xi_n\}$, in particular (1), converges to m (even) in *probability*.

It is worth mentioning that there is another approach to the concept of weighted averages where the weights are pre-specified independently of the

---

Date: 24 October 2007.

1991 *Mathematics Subject Classification.* Primary: 60F15, 60F25; Secondary: 60G42, 60G52, 62F12.

Key words and phrases. independent random variables, martingale differences, p-stable random variables, weighted averages, a.s. convergence, norm convergence, consistent estimation of a common mean.

random variables under consideration. In that case we look for conditions on the weights such that the weighted averages a.s. converge for *any* sequence of random variables in a *certain* class. We mention the work of Jamison, Orey and Pruitt [10] for integrable i.i.d. random variables (extended in the recent work of Etemadi [7] to identically distributed pairwise independent integrable random variables). Beyond independence, we mention the work of Azuma [1] for uniformly bounded martingale differences (see also [14] for recent results and extended references).

In this note we define Komlós–Révész type weighted averages for martingale difference sequences, even without variances. For $\{X_n\} \subset L_p(\mathbf{P})$, $1 < p \leq 2$ with dual index $q = p/(p-1)$, they have the form

$$\frac{\sum_{k=1}^n (X_k/\|X_k\|_p^q)}{\sum_{k=1}^n (1/\|X_k\|_p^q)}.$$

We prove that these weighted averages converge a.s. and in the corresponding norm whether the series $\sum_{n=1}^\infty (1/\|X_n\|_p^q)$ converges or diverges. The limit is 0 if and only if $\sum_{n=1}^\infty (1/\|X_n\|_p^q)$ diverges. Then Theorem 1 of [12] follows as a corollary. We discuss the limitation of the theory when we replace the independent random variables by centered uncorrelated random variables. Finally, for $p < 2$ and symmetric independent p-stable random variables we obtain complete analogs of Theorem 1 and Theorem 2 of [12].

2. Komlós–Révész estimation for martingale differences

Theorem 2.1. *Let $\{X_n\} \subset L_p(\mathbf{P})$, $1 < p \leq 2$, be a sequence of martingale differences with respect to the increasing sequence of σ-algebras $\{\mathcal{F}_n\}$. Let $q = p/(p-1)$ and assume that $\|X_n\|_p \neq 0$ for every $n \geq 1$. Then the weighted averages*

$$\frac{\sum_{k=1}^n (X_k/\|X_k\|_p^q)}{\sum_{k=1}^n (1/\|X_k\|_p^q)} \tag{2}$$

converge a.s. and in the L_p-norm. The limit is 0 a.s. if and only if the series $\sum_{n=1}^\infty (1/\|X_n\|_p^q)$ diverges.

Proof. We distinguish between two cases. In the first case we assume that $\sum_{n=1}^\infty (1/\|X_n\|_p^q)$ diverges. We quote the Abel–Dini result as it appears in Hildebrandt [9].

If $\{d_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^\infty d_n$ diverges, then for every $\alpha > 1$, the series $\sum_{n=1}^\infty \big(d_n/(\sum_{k=1}^n d_k)^\alpha\big)$ converges.

Now, put $Y_n = X_n/\big(\|X_n\|_p^q \sum_{k=1}^n (1/\|X_k\|_p^q)\big)$. By our assumption and by the Abel–Dini theorem we obtain that

$$\sum_{n=1}^\infty \|Y_n\|_p^p = \sum_{n=1}^\infty \frac{1/\|X_n\|_p^q}{\big(\sum_{k=1}^n (1/\|X_k\|_p^q)\big)^p} < \infty\,.$$

Using Chow's extension [3, Corollary 5] of the Marcinkiewicz-Zygmund result for martingales, we conclude that $\sum_{n=1}^{\infty} Y_n$ converges a.s. Kronecker's lemma then yields the a.s. convergence to 0 of (2). Using Theorem 2 of Bahr and Esseen [2], we obtain

$$\Big\| \sum_{n=j}^{l} Y_n \Big\|_p^p \leq 2 \sum_{n=j}^{l} \|Y_n\|_p^p = 2 \sum_{n=j}^{l} \frac{1/\|X_n\|_p^q}{\big(\sum_{k=1}^{n}(1/\|X_k\|_p^q)\big)^p} \underset{j,l\to\infty}{\longrightarrow} 0\,.$$

Hence $\sum_{n=1}^{\infty} Y_n$ converges in the L_p-norm. Using a Banach space version of Kronecker's lemma (see [5]), the averages in (2) converge to (necessarily) 0 in the L_p-norm.

Assume that $\sum_{n=1}^{\infty}(1/\|X_n\|_p^q)$ converges. Put $Z_n = X_n/\|X_n\|_p^q$. By assumption, $\sum_{n=1}^{\infty} \|Z_n\|_p^p$ converges. So by Chow's theorem the series $\sum_{n=1}^{\infty} Z_n$ converges a.s. and the averages in (2) converge a.s.

Again, using the Bahr–Esseen result,

$$\Big\| \sum_{k=j}^{l} Z_k \Big\|_p^p \leq 2 \sum_{k=j}^{l} \|Z_k\|_p^p = 2 \sum_{k=j}^{l} (1/\|X_k\|_p^q).$$

So, the series $\sum_{n=1}^{\infty} Z_n = \sum_{n=1}^{\infty}(X_n/\|X_n\|_p^q)$ converges in the L_p-norm and so the weighted averages (2) converge in the L_p-norm (necessarily to the same a.s. limit). We will show that it is a non-zero limit.

Using Jensen's inequality for conditional expectations (e.g., [6, p. 33]), we have

$$\mathbf{E}\Big(\big|\sum_{k=1}^{n+1} Z_k\big|^p \big| \mathcal{F}_n \Big) \geq \Big| \mathbf{E}\Big(\sum_{k=1}^{n+1} Z_k \big| \mathcal{F}_n \Big) \Big|^p = \Big| \sum_{k=1}^{n} Z_k \Big|^p .$$

Hence,

$$\Big\| \sum_{k=1}^{n+1} Z_k \Big\|_p \geq \Big\| \sum_{k=1}^{n} Z_k \Big\|_p \geq \cdots \geq \|Z_1\|_p > 0.$$

So,

$$0 < \|Z_1\|_p \leq \lim_{n\to\infty} \Big\| \sum_{k=1}^{n} Z_k \Big\|_p = \Big\| \sum_{k=1}^{\infty} Z_k \Big\|_p .$$

Hence $\sum_{n=1}^{\infty} Z_n$ is non-zero and the weighted averages (2) do not converge to zero, neither a.s. nor in norm.

$\square$

Corollary 2.2. *Let $\{X_n\} \subset L_p(\mathbf{P})$, $1 < p \leq 2$, be a sequence of independent random variables with $\mathbf{E}[X_n] = m$ for every $n \geq 1$. Let $q = p/(p-1)$ and assume that $\|X_n - m\|_p \neq 0$ for every $n \geq 1$. Then the weighted averages*

$$\frac{\sum_{k=1}^{n}(X_k/\|X_k - m\|_p^q)}{\sum_{k=1}^{n}(1/\|X_k - m\|_p^q)}$$

converge a.s. and in the L_p-norm. The limit is a.s. m if and only if the series $\sum_{n=1}^{\infty}(1/\|X_n - m\|_p^q)$ diverges.

Proof. Apply the previous theorem to $\{X_n - m\}$. $\square$

Remarks. 1. The above corollary extends Theorem 1 in Komlós–Révész [12] to the case where none of the random variables has a finite variance (so for $p = 2$ the averages (2) are not defined).

2. The above theorem is a generalization of Komlós–Révész beyond the scope of independence.

3. The statistical meaning of the above corollary is the following: if you have noisy measurements of an unknown quantity m by n independent devices, such that the imperfection of each device is measured by the p-th norm of its deviation (which is assumed to be known, or can be estimated in advance for each device), then the suggested sequence of weighted averages is a strong consistent estimator of m if the Komlós–Révész type condition holds. For $p < 2$, $\|X_n - m\|_p$ is the p-th analog of the standard deviation of measurement by the n-th device. The Komlós–Révész type condition means that the sequence of p-deviations should not grow too fast.

Example 1. *A sequence of centered independent random variables $\{X_n\}$ which for every $1 < p \le 2$ have weighted averages (2) that converge a.s., while their usual Cesàro averages fail to converge a.s.*

We construct a centered independent sequence by the following law: $\mathbf{P}(X_n = \pm n) = (2n)^{-1}$ and $\mathbf{P}(X_n = 0) = 1 - n^{-1}$. Hence by construction the series $\sum_{n=1}^{\infty}\mathbf{P}(|X_n| = n)$ diverges and by the Borel–Cantelli lemma we have $\limsup_n X_n/n = 1$. So, the usual Cesàro averages fail to converge a.s. On the other hand, $\|X_n\|_p = n^{(p-1)/p}$, so $\sum_{n=1}^{\infty}(1/\|X_n\|_p^q) = \sum_{n=1}^{\infty}(1/n)$ diverges and the weighted averages (2) converge to 0 a.s.

Example 2. *For every $1 < p < 2$ there exits a sequence of centered independent random variables $\{X_n\}$ without variances such that the series $\sum_{n=1}^{\infty}(1/\|X_n\|_p^q)$ diverges.*

Take a sequence $\{\xi_n\}$ of centered i.i.d. random variables such that $\|\xi_1\|_p < \infty$ and $\|\xi_1\|_2 = \infty$. Put $X_n = n^{1/q}\xi_n$ with $q = p/(p-1)$. Clearly, the assertions of the example hold so our theorem applies, while the result of [12] cannot be applied.

Example 3. *For every $1 < p < 2$ there exits a sequence of symmetric independent* bounded *random variables $\{X_n\}$ such that the series $\sum_{n=1}^{\infty}(1/\|X_n\|_p^q)$ diverges but $\sum_{n=1}^{\infty}(1/\|X_n\|_2^2)$ converges.*

Take a symmetric independent sequence $\{\xi_n\} \subset L_p$ with infinite variances for which $\sum_{n=1}^{\infty}(1/\|\xi_n\|_p^q)$ diverges. Since $\|\xi_n\|_2 = \infty$, for every n there exists $\alpha_n > 0$ such that $\mathbf{E}[|\xi_n|^2 \mathbf{1}_{\{|\xi_n| \le \alpha_n\}}] \ge n^2$. Now, define the symmetric independent sequence $X_n = \xi_n \mathbf{1}_{\{|\xi_n| \le \alpha_n\}}$. By construction $\sum_{n=1}^{\infty}(1/\|X_n\|_2^2)$ converges and since $\|X_n\|_p \le \|\xi_n\|_p$, the series $\sum_{n=1}^{\infty}(1/\|X_n\|_p^q)$ diverges.

Example 4. *A sequence of symmetric independent bounded random variables $\{X_n\}$ exists such that for every $1 < p < 2$ the series $\sum_{n=1}^{\infty}(1/\|X_n\|_p^q)$ converges but $\sum_{n=1}^{\infty}(1/\|X_n\|_2^2)$ diverges.*

Define the independent sequence by the following rule $\mathbf{P}(X_n = \pm\sqrt{n}) = 1/2$. So, $\|X_n\|_p = \sqrt{n}$ and since $q > 2$, the sequence $\{X_n\}$ satisfies the assertions.

It is a natural question to ask whether the above theorem and its corollary hold for other classes of random variables. Natural classes are the centered uncorrelated or pairwise independent random variables. The extension of Theorem 1 of [12] to the latter class was explored by Rosalsky [18]. Using the method of N. Etemadi, he proved that if $\{X_n\}$ is a sequence of pairwise independent random variables with finite variances and a constant mean m, then under the following three conditions

$$(i) \quad \sum_{n=1}^{\infty} \big(1/\mathrm{var}(X_n)\big) = \infty,$$

$$(ii) \quad \lim_{n\to\infty} \mathrm{var}(X_n) \sum_{k=1}^{n} \big(1/\mathrm{var}(X_k)\big) = \infty, \quad (iii) \quad \sup_n \|X_n\|_1 < \infty,$$

the weighted averages (1) converge a.s. to m. On the other hand, we prove that for martingale differences the weighted averages (2) converge a.s. whether (i) above holds or not — it is only the identification of the limit which depends on (i). We will show that for the class of uncorrelated random variables a.s. convergence of the weighted averages (1) may fail, with or without the Komlós–Révész condition (i); in the following two examples (ii) is satisfied.

Example 5. *A sequence of centered uncorrelated random variables can be constructed such that the series $\sum_{n=1}^{\infty}(1/\|X_n\|_2^2)$ converges and the weighted averages (2) diverge a.s.*

K. Tandori [19] proved the following result: *if $\{|a_n|\}$ is a non-increasing sequence for which $\sum_{n=1}^{\infty}|a_n|^2\log^2 n$ diverges, then there exists a centered uncorrelated sequence $\{\phi_n\}$ in $(0,1)$ for which the orthogonal series $\sum_{n=1}^{\infty} a_n\phi_n$ diverges a.s.* Hence there exists a centered uncorrelated sequence $\{\phi_n\}$ for which the orthogonal series $\sum_{n=1}^{\infty}\big(\phi_n/(\sqrt{n}\log n)\big)$ diverges a.s. Now, put $X_n = \sqrt{n}\log n\phi_n$. So, $\sum_{n=1}^{\infty}(1/\|X_n\|_2^2) = \sum_{n=1}^{\infty}\big(1/(n\log^2 n)\big)$ converges and $\sum_{n=1}^{\infty}(X_n/\|X_n\|_2^2) = \sum_{n=1}^{\infty}\big(\phi_n/(\sqrt{n}\log n)\big)$ diverges a.s. Hence, the weighted averages (2) fail to converge a.s.

Example 6. *A sequence of centered uncorrelated random variables exists such that the series* $\sum_{n=1}^{\infty}(1/\|X_n\|_2^2)$ *diverges and the weighted averages* (2) *diverge a.s.*

L. Csernyák [4] has proved (see also [17]) the following result:

If $\{|a_n|\}$ *is a non-increasing sequence for which* $\sum_{n=1}^{\infty}|a_n|^2$ *diverges, then there exists a centered uncorrelated sequence* $\{\phi_n\}$ *in* $(0,1)$ *so that* $\limsup_{n\to\infty}|\sum_{k=1}^{n}a_k\phi_k|/\log n=\infty$ *a.s.* Hence, there exists a centered uncorrelated sequence $\{\phi_n\}$ such that $\limsup_{n\to\infty}|\sum_{k=1}^{n}(\phi_k/\sqrt{k})|/\log n=\infty$ a.s. Now, put $X_n=\sqrt{n}\phi_n$. So,

$$\frac{\sum_{k=1}^{n}(X_k/\|X_k\|_2^2)}{\sum_{k=1}^{n}(1/\|X_k\|_2^2)}=\frac{\sum_{k=1}^{n}(\phi_k/\sqrt{k})}{\sum_{k=1}^{n}(1/k)}\approx\frac{\sum_{k=1}^{n}(\phi_k/\sqrt{k})}{\log n},$$

and the weighted averages a.s. diverge.

Remarks. 1. Examples 5 and 6 show that even the convergence part of Theorem 2.1 does not hold in general for centered uncorrelated random variables. We do not know if it does for centered pairwise independent random variables with finite variances with no additional conditions.

2. According to Tandori's works the constructed centered uncorrelated sequence $\{\phi_n\}$ which we refer to in Example 5 can be taken to be uniformly bounded. The centered uncorrelated sequence $\{\phi_n\}$ constructed by Csernyák [4] is unbounded. The questions whether one could construct $\{\phi_n\}$ to be real unimodular, i.e., ± 1 a.s., (in either construction) are still open. I would like to thank professor Ference Móricz for clarifying this point. Using a result of M. Kac, affirmative answer(s) to this (these) question(s) will imply that $\{X_n\}$ in Example 5 and/or Example 6 could be taken to be centered and pairwise independent. This would show that Theorem 2.1 can not be extended even to the pairwise independent case without the additional assumption(s) of Rosalsky (ii) and/or (iii).

3. Symmetric p-stable independent random variables

Definition. A real random variable X is called a symmetric *p-stable* random variable (r.v.) with parameter $\sigma=\sigma_p(X)>0$ and index $0<p<2$ if for all $t\in\mathbb{R}$ its characteristic function satisfies

$$\mathbf{E}[\exp(itX)]=\exp(-\sigma^p|t|^p/2)\,.$$

It is known that (see Feller [8, XVII, §4], and for more specific calculation see Marcus and Pisier [16, §1]),

$$\lim_{t\to\infty}t^p\mathbf{P}(|X|>t)=c_p\sigma^p$$

for some c_p which depends only on p.

It is also known that a p-stable random variable might not have an absolute p-th moment, but according to the above property it does have an absolute r-th moment for every $r < p$.

Now, if $\{Z_n\}$ are i.i.d. symmetric p-stable random variables, then for all complex sequences $\{a_n\}$, by stability we have

$$\sum_{k=1}^{n} a_k Z_k \stackrel{\mathcal{D}}{=} Z_1 \Big(\sum_{k=1}^{n} |a_k|^p \Big)^{1/p}.$$

Hence, for every $r < p$, there exists a positive constant $c_{p,r}$, which depends only on p and r such that

$$\Big\| \sum_{k=1}^{n} a_k Z_k \Big\|_r = c_{p,r} \Big(\sum_{k=1}^{n} |a_k|^p \Big)^{1/p}.$$

Lemma 3.1. *Let $\{\alpha_k\}$ be a positive sequence with $\sum_{k=1}^{n} \alpha_k = 1$. If $1 < p < \infty$ and $q = p/(p-1)$, then for every sequence of numbers $\{\sigma_k\}$ we have*

$$\sum_{k=1}^{n} \alpha_k^p \sigma_k^p \geq 1 / \Big(\sum_{k=1}^{n} (1/\sigma_k^q) \Big)^{p-1}.$$

Proof. Using Hölder's inequality we obtain

$$1 = \Big(\sum_{k=1}^{n} \alpha_k \Big)^p = \Big(\sum_{k=1}^{n} \alpha_k \, \sigma_k (1/\sigma_k) \Big)^p \leq \Big(\sum_{k=1}^{n} \alpha_k^p \, \sigma_k^p \Big) \Big(\sum_{k=1}^{n} (1/\sigma_k^q) \Big)^{p-1}$$

□

From now on we always assume that $1 < p < 2$.

Theorem 3.2. *Let $1 < r < p < 2$. Let $\{Z_n\}$ be symmetric p-stable independent random variables. Let $q = p/(p-1)$ and for every $n \geq 1$ put $\sigma_n = \sigma_p(Z_n)$. Assume that $\sigma_n > 0$ for every $n \geq 1$. Then the weighted averages*

$$\frac{\sum_{k=1}^{n} (Z_k/\sigma_k^q)}{\sum_{k=1}^{n} (1/\sigma_k^q)} \tag{3}$$

converge in the L_r-norm and a.s. The limit is a.s. 0 if and only if the series $\sum_{n=1}^{\infty} (1/\sigma_n^q)$ diverges. Among all the weighted averages of $\{Z_1, \ldots, Z_n\}$ the weighted average (3) has the minimal L_r-norm and this minimum is attained only with (3).

Proof. Clearly, if X is a symmetric p-stable r.v. with parameter σ, then X/σ is a symmetric p-stable r.v. with parameter 1. Therefore, the sequence $\{Z_n/\sigma_n\}$ is a sequence of i.i.d. symmetric p-stable random variables with common parameter 1. Hence for $r < p$ we have

$$\Big\| \sum_{n=j}^{l} \frac{Z_n/\sigma_n^q}{\sum_{k=1}^{n} (1/\sigma_k^q)} \Big\|_r = \Big\| \sum_{n=j}^{l} \frac{(1/\sigma_n^{q-1})(Z_n/\sigma_n)}{\sum_{k=1}^{n} (1/\sigma_k^q)} \Big\|_r =$$

(*)

$$c_{p,r}\Big(\sum_{n=j}^{l}\frac{1/\sigma_n^q}{\big(\sum_{k=1}^n(1/\sigma_k^q)\big)^p}\Big)^{1/p}$$

If $\sum_{n=1}^\infty(1/\sigma_n^q)$ diverges, the Abel–Dini theorem yields convergence of the series $\sum_{n=1}^\infty\big[(1/\sigma_n^q)/\big(\sum_{k=1}^n(1/\sigma_k^q)\big)^p\big]$. So, the right hand side of (*) tends to zero as $\min(j,l)\to\infty$. This means that $\sum_{n=1}^\infty\big[(Z_n/\sigma_n^q)/\big(\sum_{k=1}^n(1/\sigma_k^q)\big)\big]$ converges in the L_r-norm, hence in probability. By the Lévy–Itô–Nisio theorem (see Ledoux and Talagrand [13, Theorem 6.1]) the series converges a.s. So, Kronecker's lemma (also its Banach space version [5]) yields the a.s. (L_r-norm) convergence to 0.

Now, we assume that $\sum_{n=1}^\infty(1/\sigma_n^q)$ converges. Using the same idea as above, for $r<p$,

$$\Big\|\sum_{k=j}^{l}(Z_k/\sigma_k^q)\Big\|_r=c_{p,r}\Big(\sum_{k=j}^{l}(1/\sigma_k^q)\Big)^{1/p}\underset{j,l\to\infty}{\longrightarrow}0\,.$$

Therefore, the series $\sum_{k=j}^l(Z_k/\sigma_k^q)$ converges in the L_r-norm. Again using the Lévy–Itô–Nisio theorem, it converges a.s. So do the weighted averages converge a.s. and in the L_r-norm. The equality

$$\Big\|\sum_{k=1}^{n}(Z_k/\sigma_k^q)\Big\|_r^p=c_{p,r}^p\sum_{k=1}^{n}(1/\sigma_k^q)$$

shows that the L_r-limit (hence the a.s. limit) is not zero.

Now for the last assertion, for any sequence of weights $w_k^{(n)}$ such that $\sum_{k=1}^n w_k^{(n)}=1$, using the Lemma above, we have,

$$\Big\|\sum_{k=1}^{n}w_k^{(n)}Z_k\Big\|_r^p=\Big\|\sum_{k=1}^{n}(w_k^{(n)}\sigma_k)(Z_k/\sigma_k)\Big\|_r^p=c_{p,r}^p\sum_{k=1}^{n}(w_k^{(n)}\sigma_k)^p\geq$$

$$\frac{c_{p,r}^p}{\big(\sum_{k=1}^n(1/\sigma_k^q)\big)^{p-1}}=\frac{c_{p,r}^p\sum_{k=1}^n(1/\sigma_k^q)}{\big(\sum_{k=1}^n(1/\sigma_k^q)\big)^p}=\frac{\Big\|\sum_{k=1}^n(Z_k/\sigma_k^q)\Big\|_r^p}{\big(\sum_{k=1}^n(1/\sigma_k^q)\big)^p}.$$

Under the assumption $\sum_{k=1}^n w_k^{(n)}=1$, equality in Hölder's inequality holds only when $w_k^{(n)}=(1/\sigma_k^q)/\sum_{k=1}^n(1/\sigma_k^q)$. □

Corollary 3.3. *Let $1<p<2$ and put $q=p/(p-1)$. Let $\{\sigma_n\}$ be a sequence of positive numbers such that the series $\sum_{n=1}^\infty(1/\sigma_n^q)$ converges. Then there exists a sequence of symmetric independent p-stable random variables with $\sigma_n=\sigma_p(Z_n)$ such that for any sequence of positive weights $\{w_k^{(n)}:k\geq1\}_{n\geq1}$ with $\sum_{k=1}^n w_k^{(n)}=1$, the weighted averages $\sum_{k=1}^n w_k^{(n)}Z_k$ do not converge to zero in* probability.

Proof. Construct a sequence of symmetric independent p-stable random variables $\{Z_n\}$ with parameters $\sigma_k = \sigma_p(Z_k)$, $k = 1, 2, \ldots$. By stability we have

$$\sum_{k=1}^{n} w_k^{(n)} Z_k = \sum_{k=1}^{n} (w_k^{(n)} \sigma_k)(Z_k/\sigma_k) \stackrel{\mathcal{D}}{=} (Z_1/\sigma_1)\Big(\sum_{k=1}^{n} (w_k^{(n)} \sigma_k)^p\Big)^{1/p}.$$

Put $M = \sum_{n=1}^{\infty}(1/\sigma_n^q)$. By the lemma above, we obtain

$$\mathbf{P}\Big(\Big|\sum_{k=1}^{n} w_k^{(n)} Z_k\Big| > t\Big) = \mathbf{P}\Big(\Big|(Z_1/\sigma_1)\Big(\sum_{k=1}^{n} (w_k^{(n)} \sigma_k)^p\Big)^{1/p}\Big| > t\Big) \geq$$

$$\mathbf{P}\Big(\big|Z_1/\sigma_1\big| M^{-(p-1)/p} > t\Big) > 0\,.$$

□

Remarks. 1. The above corollary is the analog, for the p-stable case, of Theorem 2 of Komlós–Révész [12].

2. The results in this section could be done for symmetric *complex* independent p-stable random variables.

ACKNOWLEDGEMENTS. I would like to thank Michael Lin for suggesting this problem and for many helpful discussions.

References

[1] Azuma, K. (1967). Weighted sums of certain dependent random variables. *Tôhoku Math. J.* 2, 19, 357-367.

[2] von Bahr, B. and Esseen, C.G. Inequalities for the r-th absolute moment of a sum of random variables, $1 \leq r \leq 2$, Ann. Math. Stat. 36 (1965), 299-303.

[3] Chow, Y.S. (1965). Local convergence of martingales and the law of large numbers, *Ann. Math. Statist.* 36 552-558.

[4] Csernyák, L. (1975). On the convergence of orthogonal series, *Period. Math. Hungar.* 6, no. 3, 267-272.

[5] Derriennic, Y. and Lin, M. (2001). Fractional Poisson equations and ergodic theorems for fractional coboundaries. *Israel J. Math.* 123, 93-130.

[6] Doob, J.L. *Stochastic Processes.* John Wiley & sons, New York 1953.

[7] Etemadi, N. (2006). Convergence of weighted averages of random variables revisited. *Proc. Amer. Math. Soc.* 134, no. 9, 2739-2744.

[8] Feller, W. An introduction to probability theory and its applications. Vol. II. *John Wiley and Sons, Inc.* 1966.

[9] Hildebrandt, T. H. (1942). Remarks on the Abel–Dini theorem. *Amer. Math. Monthly* 49 441-445.

[10] Jamison, B., Orey, S., and Pruitt, W. (1965). Convergence of weighted averages of independent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 4 40-44.

[11] Kac. M. (1936). Sur les fonctions indépendantes I (Propriétés générales). *Stud. Math.* 6, 45-58.

[12] Komlós, J. and Révész, P. (1965). On the weighted averages of independent random variables. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 9, 583-587.

[13] Ledoux, M. and Talagrand, M. Probability in Banach spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 23. *Springer-Verlag*, Berlin, 1991.

[14] Lin, M. and Weber, M. (2007). Weighted ergodic theorems and strong laws of large numbers. *Ergodic Theory Dynam. Systems* 27, no. 2, 511-543.

[15] Marcinkiewicz, J. and Zygmund, A. (1937). Sur les fonctions indépendentes. *Fund. Math.* 29, 60-90 (currently available at http://matwbn.icm.edu.pl).

[16] Marcus, M.B. and Pisier, G. (1984). Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes, *Acta Math.* 152, 245-301.

[17] Móricz, F. and Tandori, K. (1987). Counterexamples in the theory of orthogonal series, *Acta Math. Hungar.* 49, no. 1-2, 283-290.

[18] Rosalsky, A. (1986). A strong law for weighted averages of random variables and the Komlós–Révész estimation problem, *Calcutta Statist. Assoc. Bull.* 35, no. 137-138, 59-66.

[19] Tandori, K. (1957). Über die orthogonalen Funktionen. I. *Acta Sci. Math. (Szeged)* 18, 57-130.

Ben-Gurion University of the Negev, Israel
E-mail address: guycohen@ee.bgu.ac.il