Decoupled Linear Estimation of Affine Geometric Deformations and Non-Linear Intensity Transformations of Images

Shahar Z. Kovalsky, Guy Cohen, Rami Hagege and Joseph M. Francos

***Abstract*—We consider the problem of registering two observations on an arbitrary object, where the two are related by a geometric affine transformation of their coordinate systems, and by a non-linear mapping of their intensities. More generally, the framework is that of jointly estimating the geometric and radiometric deformations relating two observations on the same object. We show that the original high-dimensional, non-linear, non-convex search problem of simultaneously recovering the geometric and radiometric deformations can be represented by an equivalent sequence of two linear systems. A solution of this sequence yields an exact, explicit, and efficient solution to the joint estimation problem.**

***Index Terms*—Affine transformations, image registration, linear estimation, parameter estimation, domain registration, nonlinear range registration.**

I. Introduction

UNDERSTANDING different appearances of an object is an elementary problem in various fields. Since acquisition conditions vary (*e.g.*, pose, illumination, acquisition system), the set of possible observations on a particular object is immense; therefore, the complicated task of characterizing and determining the relation between a pair of observations is crucial whether one is after the differences themselves (*e.g.*, change detection) or whether the interest is restricted to determining if two observations are on the same object (*e.g.*, face recognition).

Image registration, and in general the problem of estimating transformations of observed objects, has been intensively studied for several decades [1], [2]. As evidently seen in these comprehensive surveys, the vast majority of the study focuses on geometric-only registration; that is, alignment of the domain (coordinates) of images. Correspondingly, the term "registration" is commonly associated with geometry, which indeed is a challenging problem for itself [3]. On the other hand, various recent studies have focused on radiometry-only registration; that is, alignment of the range (values/intensities) of geometrically-aligned images. This problem is much simpler since images are geometrically pre-aligned, and thus pointwise correspondence inherently exists. Thus, straightforward methods may be used to combine images captured in different optical settings into a single image of high dynamic range (HDR) [4].

The authors are with the Department of Electrical and Computer Engineering, Ben-Gurion University, Beer-Sheva 84105, Israel, Tel: +972 8 6461842, FAX: +972 8 6472949, email: shaharko,guycohen,hagege,francos@ee.bgu.ac.il

Nonetheless, it seems that the problem of deriving a registration procedure in the presence of combined radiometric and geometric variations has received much less attention. The derivation of a registration procedure in the presence of combined radiometric and geometric variations (*i.e.*, both domain and range) is crucial in many applications and especially for *change detection*. In order to detect *local* variations in the appearance of the object, both its global geometric and global radiometric deformations have to be estimated; next, local changes may be detected, for example by eliminating the estimated global effects and inspecting the resulting difference image between the reference and the geometrically-radiometrically aligned observation. In this case, employing geometric registration methods *without* taking the radiometric variation into account is suboptimal in the following sense: featureless methods (area methods [2]), such as those based on "correlation-like" [5], Fourier [6], mutual information [7] and optical flow [8] principles, directly employ the intensity information of the image in order to establish an estimate of the geometric deformation; as such, these methods are inherently sensitive to intensity variations and typically fail in the presence of some non-negligible radiometric transformation relating the intensities of the observations. In many cases, this limitation eventually leads to the need to restrict the registration procedures to employ only a *small* fraction of the information available in the observations by considering salient features of various types [2], which however are less sensitive to radiometric mismatches.

For example, in medical imaging, cross-modality geometric registration procedures are prone to failure due to significant, difficult-to-model, differences in pixel (voxel) intensities; this has led researchers to the employment of various computationally demanding variational methods for performing the registration [9]–[11]. In the geometric registration of optical images, it is customary to *evade* the radiometric effects by using "radiometry-invariant" procedures (usually feature-based, as previously mentioned). The physical understandings and invariancy principles of the *color constancy* framework [12] have also been utilized in attempt to minimize the effects of such radiometric variations (these, however, usually assume linear radiometric effects) [13]. Another approach, that relies on the principles of the aforementioned radiometry-only registration (*e.g.*, HDR), overcomes radiometric phenomena in the registration of optical images (and, in fact, benefits from it) by using special optical apparatus (spatially varying filters attached to the camera) [14].

The work presented in this paper is concerned with the problem of estimating joint geometric and radiometric image deformations. Observations of an object are assumed to simultaneously undergo an affine transformation of coordinates and a non-linear mapping of the intensities.

In the geometric aspect, the case of affine transformations of coordinates is basic and provides a "first-order" approximation to more complex cases (such as "small" projective deformations, etc.). In the radiometric aspect, the type of global variability we consider is often referred to as *intensity mapping*. It naturally appears in the important case of single-modal registration, where non-linearities are typically introduced by an image acquisition system as the overall non-linearity of its various components (sensors, amplifiers, etc.). In optical imaging, for example, such non-linearities are characterized by the *camera response function* (CRF) [15]–[17]. Notably, such non-linearities are sometimes deliberately introduced in hardware design for various reasons [18]. It has also been suggested that this type of global mapping may model the intensity variations in certain medical imaging multi-modal schemes (see "monofunctional dependence assumption" in [9]).

As previously mentioned, the difficulties associated with the joint geometric-radiometric estimation problem have led to the current state where only a few attempts have been made to solve it. The lack of point-wise correspondence (due to the geometric transformation) and the lack of intensity-wise alignment (due to the radiometric mapping) does not allow for a simple direct usage of the intensity information of the images. Seemingly, the geometric and radiometric problems are strongly coupled and may not be answered separately. As such, straightforward approaches for solving this problem typically lead to a high-dimensional non-linear non-convex optimization problem. Only a few works have *explicitly* modeled joint geometric-radiometric deformations. Indeed, among these, most evade the inherent non-linearity of this estimation problem through linear approximation and/or variational optimization-based approaches [9], [19]–[23].

An exception is the work of Candocia [24], based on a previous work of Mann [25], whereby the joint registration of images in domain as well as range is accomplished using piecewise linear comparametric analysis; the camera's non-linear comparametric function [26] is approximated with a constrained piecewise linear model; the registration model is then expanded using a first order Taylor expansion (of the image), resulting in a linear estimation problem. Extensions and enhancement to this work appeared in more recent papers. Being based on first order Taylor expansion, such solutions are restricted by the (implicit) assumptions of small geometric deformations and differentiability of the images, required for the approximation to be reasonable. Although such assumptions may sometimes be acceptable (for example, in the mosaicing of an image sequence shown as an example in Candocia's work [24]), they are restrictive in general; an attempt to employ a Taylor based solution in the presence of a large geometric transformation (*e.g.*, the simple case of a 180 degrees rotation) will certainly fail. Furthermore, even when such methods may be employed, the solution is *approximated*.

In this paper we propose a method for solving the joint estimation problem in terms of an *equivalent* linear estimation problem. In contrast to many featureless estimation methods, and in contrast with the joint registration method proposed in [24], the method we propose is neither approximation-based nor does it have strong assumptions on the model (*e.g.*, differentiability of the images). Also, no assumptions are made as to the magnitude of the geometric and radiometric deformations relating the observations. The solution we propose is *explicit* and thus computationally efficient. Moreover, in the absence of noise it is shown to be *exact* (not approximated), regardless of the deformations' magnitudes.

In section II, we rigorously formulate the problem addressed in this paper. We then discuss the problem in the absence of noise in section III; the underlying algebraic structure of the problem and the notion of *sample distribution* are exploited in order to decouple the problem into two estimation problems, where each is *equivalently* reformulated as a linear estimation problem. Sections V and VI conclude the work with an example and a brief summary.

II. Problem Statement

Let us begin by informally stating the problem studied in this paper. Suppose we are given a single observation h, about a *known* m-dimensional signal g, of the form

$$h(\mathbf{x}) = Q(g(\mathcal{A}(\mathbf{x}))),$$

where Q is invertible and $\mathcal{A}$ is affine. The right-hand composition of g with $\mathcal{A}$ (composition from within) can be thought of as a spatial/time deformation (*i.e.*, a deformation of the domain - the coordinate system), while its left-hand composition with Q (composition from without) can be seen as a memoryless non-linear input/output system applied to the signal's amplitude. Hence, in terms of image formation, such model physically corresponds to a simultaneous deformation of both geometry and radiometry.

We study the problem of estimating the joint radiometric and geometric deformations, which amounts to the estimation of Q and $\mathcal{A}$, as illustrated in Fig. 1.

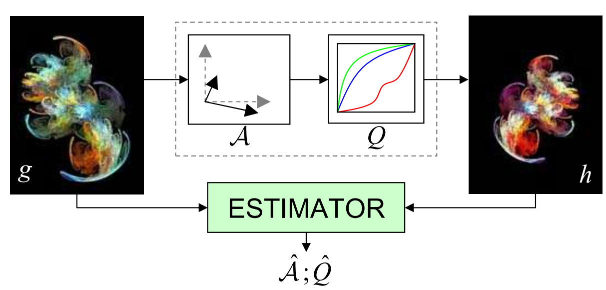

Fig. 1. Illustration of the problem description (where different non-linear intensity mappings are associated with each of the color channels of the image).

More formally, let $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^m$ be an affine transformation of coordinates, that is, $\mathcal{A} : \mathbf{x} \mapsto \mathbf{Ax}+\mathbf{c}$ where $\mathbf{A} \in \mathbb{R}^{m\times m}$ is non-singular and $\mathbf{c} \in \mathbb{R}^m$. $\mathcal{A}$ shall represent the geometric deformation. Let $Q : \mathbb{R} \to \mathbb{R}$ be an increasing invertible function, representing the non-linear radiometric deformation. Let us further assume that $Q(0) = 0$.

Denote by $B_c(\mathbb{R}^m)$ the space of bounded, compactly supported, Lebesgue measurable functions from $\mathbb{R}^m$ to $\mathbb{R}$ and let $g \in B_c(\mathbb{R}^m)$.

Throughout, we shall use $\circ$ to denote the composition of functions, and $\text{supp}\{f\}$ shall be used to denote the support of a function f, *i.e.*, the closure of the set where f does not vanish. With these notations, the problem addressed in this paper is the following:

> *Given the* known *function g and a* single *measurement (observation) h of the form*
>
> $$h(\mathbf{x}) = [Q \circ g \circ \mathcal{A}](\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^m, \tag{1}$$
>
> *find an* estimate *for the left-hand composition Q and the affine transformation $\mathcal{A}$.*

Remark 1: Since Q is invertible, the recovery (estimation) of Q or of its inverse Q^{-1} are equivalent. Similarly, estimating $\mathcal{A}$ or $\mathcal{A}^{-1}$ are equivalent.

III. Proposed Method

In this section, we propose an algorithmic solution to the above problem. Adopting a functional notation and omitting the argument of the functions, the problem model becomes

$$h = Q \circ g \circ \mathcal{A}. \tag{2}$$

In the following, we show that the joint estimation problem may be decoupled into two simpler problems in the unknown left-hand composition Q and affine transformation $\mathcal{A}$, respectively.

In order to do so, let us first introduce the basic transformation studied throughout this paper. Denote by λ the Lebesgue measure on $\mathbb{R}^m$, the m-dimensional Euclidean space. We define the *sample distribution transformation* T on $B_c(\mathbb{R}^m)$ by

$$[Tg](t) = \frac{\lambda\{\mathbf{x} \in \text{supp}\{g\} : g(\mathbf{x}) \le t\}}{\lambda\{\text{supp}\{g\}\}}, \qquad g \in B_c(\mathbb{R}^m). \tag{3}$$

The sample distribution may be thought of as a continuous "cumulative histogram" of a function. The next simple lemma demonstrates the role of T as a "distribution" transformation and elaborates on some of its properties with respect to certain right- and left-hand compositions.

Lemma 1: For a given function $g \in B_c(\mathbb{R}^m)$, the following statements hold:

(i) The function $G(t) = [Tg](t)$ is a distribution function. Furthermore, the support of the distribution $G(t)$ is bounded, in the following sense:
  a) $G(t) = 0$ for $t < \inf_{\mathbf{x}} g(\mathbf{x})$.
  b) $G(t) = 1$ for $t > \sup_{\mathbf{x}} g(\mathbf{x})$.

(ii) T is invariant under right-hand affine compositions: $T(g \circ \mathcal{A}) = Tg$ for any non-singular affine transformation $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^m$.

(iii) $T(W \circ g) = [Tg] \circ W^{-1}$ for any strictly increasing continuous function $W : \mathbb{R} \to \mathbb{R}$ such that $W(0) = 0$.

Proof: Part (i) is immediate. Notice that $\text{supp}\{g \circ \mathcal{A}\} = \mathcal{A}^{-1}(\text{supp}\{g\})$; this is simply since $\{\mathbf{x} : g(\mathcal{A}(\mathbf{x})) = 0\} = \{\mathcal{A}^{-1}(\mathbf{y}) : g(\mathbf{y}) = 0\}$. Thus, using the properties of the Lebesgue measure, we have

$$\lambda\{\text{supp}\{g \circ \mathcal{A}\}\} = \lambda\left\{\mathcal{A}^{-1}(\text{supp}\{g\})\right\} = |\mathcal{A}^{-1}|\lambda\{\text{supp}\{g\}\},$$

where $|\mathcal{A}^{-1}|$ denotes the determinant of the transformation $\mathcal{A}^{-1}$. Next, let $\chi_{(-\infty,t]} = \begin{cases} 1 & , x \le t \\ 0 & , otherwise \end{cases}$. Then, T admits the following equivalent integral form

$$[Tg](t) = \frac{1}{\lambda\{\text{supp}\{g\}\}} \int_{\text{supp}\{g\}} \left[\chi_{(-\infty,t]} \circ g\right](\mathbf{x}) d\lambda(\mathbf{x}). \tag{4}$$

Set $\mathbf{y} = \mathcal{A}(\mathbf{x})$, thus $\mathbf{x} = \mathcal{A}^{-1}(\mathbf{y})$ and $d\lambda(\mathbf{x}) = |\mathcal{A}^{-1}|d\lambda(\mathbf{y})$. Hence, by (4) and a change of variables, we have

$$\begin{aligned} [T(g \circ \mathcal{A})](t) &= \frac{\int_{\text{supp}\{g\circ\mathcal{A}\}} \chi_{(-\infty,t]}(g(\mathcal{A}(\mathbf{x})))d\lambda(\mathbf{x})}{\lambda\{\text{supp}\{g \circ \mathcal{A}\}\}} \\ &= \frac{\int_{\text{supp}\{g\}} \chi_{(-\infty,t]}(g(\mathbf{y}))|\mathcal{A}^{-1}|d\lambda(\mathbf{y})}{|\mathcal{A}^{-1}|\lambda\{\text{supp}\{g\}\}} = [Tg](t) \end{aligned}$$

for all t, and thus (ii) is proved. Lastly, since W is strictly increasing and $W(0) = 0$ we have that $\text{supp}\{W \circ g\} = \text{supp}\{g\}$. Hence, for all t we have

$$\begin{aligned} [T(W \circ g)](t) &= \frac{\lambda\left\{\mathbf{x} \in \text{supp}\{W \circ g\} : [W \circ g](\mathbf{x}) \le t\right\}}{\lambda\{\text{supp}\{W \circ g\}\}} \\ &= \frac{\lambda\left\{\mathbf{x} \in \text{supp}\{g\} : g(\mathbf{x}) \le W^{-1}(t)\right\}}{\lambda\{\text{supp}\{g\}\}} = \left[[Tg] \circ W^{-1}\right](t), \end{aligned}$$

thus (iii) is proved. ■

In the following we show that properties (ii) and (iii) above are the key in enabling an elegant solution to the joint registration problem. The next two subsections present two complementary approaches, in which either the radiometry, Q, or the geometry, $\mathcal{A}$, is first estimated.

A. Radiometry-First Approach

In this subsection, by directly employing the properties of the transformation T, a *radiometry*-first estimation scheme is derived. By simply applying T to the relation (2) and using the above properties, we obtain the following functional relation

$$Th = T(Q \circ g \circ \mathcal{A}) = T(Q \circ g) = [Tg] \circ Q^{-1}. \tag{5}$$

Hence, the following corollary may be stated:

Corollary 1: Let $H(t) = [Th](t)$ and $G(t) = [Tg](t)$. Then, for all $t \in \mathbb{R}$ the following relation holds

$$H(t) = [G \circ Q^{-1}](t) = G(Q^{-1}(t)). \tag{6}$$

We thus conclude that the functions H and G are related by a right-hand composition Q^{-1}. Hence, using the transformation T we have "converted" a functional relation expressed by a left-hand composition (*i.e.*, "radiometric deformation") to a new functional relation expressed by a right-hand composition (*i.e.*, "geometric deformation"). In fact, equation (6) describes a new, one-dimensional, "time-domain" registration problem. Hence, any suitable (parametric or non-parametric) registration method may be used to recover (estimate) Q^{-1}.

"Time-domain" registration problems of the type (6) have gained substantial interest in many research fields. Considered as key problems (for example in speech or hand-writing recognition), many solution methods have been proposed , *e.g.* [27]. As such, these methods (*e.g.*, DTW and modified-DTW) may be used as is, or with minimal adaptation, in order to solve for Q^{-1} in the above case. However, these methods typically lead to *optimization-based techniques*.

Nevertheless, based on the work presented in [28], moment-like *non-linear* functionals may be utilized to produce *linear* constraints on Q. Thus enabling the explicit recovery of Q^{-1} by solving a linear system of equations.

More specifically, suppose that Q is also *continuously differentiable*. In this case, (6) is an estimation problem of the form considered in [28]. However, since H and G are distribution functions, they are not compactly supported (as functions). Thus, further "compactification" is required in order to employ the method proposed in [28] for the estimation of homeomorphic deformations of compactly supported signals.

Let $\varepsilon > 0$ be some arbitrarily small number and define

$$c_{\varepsilon}(t) = \begin{cases} t & , t < 1 - \varepsilon \\ 0 & , elsewhere \end{cases} \tag{7}$$

Next, let $\tilde{H} = c_{\varepsilon} \circ H$ and $\tilde{G} = c_{\varepsilon} \circ G$. By left composing c_{ε} on both sides of (6), we find that

$$\begin{aligned} \tilde{H} = c_{\varepsilon} \circ H &= c_{\varepsilon} \circ (G \circ Q^{-1}) \\ &= (c_{\varepsilon} \circ G) \circ Q^{-1} = \tilde{G} \circ Q^{-1}. \end{aligned} \tag{8}$$

Thus, $\tilde{H}$ and $\tilde{G}$ are bounded, compactly supported, Lebesgue measurable functions from $\mathbb{R}$ to itself, related by the right-hand composition Q^{-1}, which, by assumption, has a continuously differentiable inverse.

Let $\{e_i\}$ be a countable basis of $L_2(\mathrm{supp}\{\tilde{H}\})$. Since, by assumption Q' is continuous, it is in $L_2(\mathrm{supp}\{\tilde{H}\})$ and can be represented as

$$Q'(t) = \sum_i b_i e_i(t). \tag{9}$$

Using the estimation algorithm proposed in [28], any finite order model of the type (9) can now be solved for the coefficients $\{b_i\}$ by means of solving a system of linear equations.

Namely, let $w : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function that vanishes at zero. By applying w to (8) as left composition and integrating we obtain

$$\int_{\mathbb{R}} w(\tilde{H}(t))dt = \int_{\mathbb{R}} w(\tilde{G}(Q^{-1}(t)))dt. \tag{10}$$

By the change of variables $\tau = Q^{-1}(t)$, we have $dt = Q'(\tau)d\tau$. Hence, using the parametrization (9), under the finite sum assumption, we find that

$$\begin{aligned} \int_{\mathbb{R}} w(\tilde{G}(Q^{-1}(t)))dt &= \int_{\mathbb{R}} w(\tilde{G}(\tau))Q'(\tau)d\tau \\ &= \sum b_i \int_{\mathbb{R}} e_i(\tau)w(\tilde{G}(\tau))d\tau. \end{aligned} \tag{11}$$

Thus, for any choice of function $w : \mathbb{R} \to \mathbb{R}$, a linear constraint in the unknown parameters $\{b_i\}$ may be constructed as follows

$$\int_{\mathbb{R}} w \circ \tilde{H} = \left[\int_{\mathbb{R}} e_1(w \circ \tilde{G}) \ \cdots \ \int_{\mathbb{R}} e_M(w \circ \tilde{G}) \right] \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}, \tag{12}$$

where M is the model order of the expansion of Q' in (9).

Choosing w's that produce linearly independent constraints lets us establish a linear system of equations in the parameters $\{b_i\}$. It is possible to show that different w's are "almost always" linearly independent, yielding an exact solution for Q' in the absence of noise.

Lastly, since $Q(0) = 0$, Q can be easily obtained by integration, which completes the estimation of the mapping Q.

Based on the conclusions in [28], we conclude that if the derivative of Q admits a *finite* order representation in the form of (9), the solution for the radiometric deformation Q is completely determined and *exact*.

Once the radiometric deformation Q (or equivalently its inverse Q^{-1}) is recovered, (2) may be rewritten as

$$h = \underbrace{(Q \circ g)}_{known} \circ \mathcal{A}. \tag{13}$$

Hence, the problem reduces to a strictly geometric problem, where $\mathcal{A}$ may be recovered (see further discussion in section III-B, below).

B. Geometry-First Approach

In this subsection, the properties of the transformation T given in Lemma 1 are used to derive a *geometry*-first estimation scheme. As previously shown, T converts the joint problem (2), in the unknowns Q and $\mathcal{A}$, to a "new" problem in a single unknown, Q^{-1}. In order to obtain a symmetric result in $\mathcal{A}$, let us define an auxiliary operator R on $B_c(\mathbb{R}^m)$ by

$$Rh = [Th] \circ h - [Th](0).$$

Next, apply R to the basic relation $h = Q \circ g \circ \mathcal{A}$ given in (2). Notice that $[Th](0)$ is a constant function (over all of $\mathbb{R}^m$), and thus $[Th](0) \circ \mathcal{A} = [Th](0)$. Hence, by employing Lemma 1 and since $Q(0) = 0$ we have

$$\begin{aligned} Rh &= [Th] \circ h - [Th](0) \\ &= ([Tg] \circ Q^{-1}) \circ (Q \circ g \circ \mathcal{A}) - [Tg](Q^{-1}(0)) \\ &= [Tg] \circ g \circ \mathcal{A} - [Tg](0) \\ &= \big([Tg] \circ g - [Tg](0)\big) \circ \mathcal{A} \\ &= [Rg] \circ \mathcal{A}. \end{aligned} \tag{14}$$

Thus, the following corollary may be stated:

Corollary 2: Let $\mathcal{H}(\mathbf{x}) = [Rh](\mathbf{x})$ and $\mathcal{G}(\mathbf{x}) = [Rg](\mathbf{x})$. Then, for all $\mathbf{x} \in \mathbb{R}^m$ the following relation holds

$$\mathcal{H}(\mathbf{x}) = [\mathcal{G} \circ \mathcal{A}](\mathbf{x}) = \mathcal{G}(\mathcal{A}(\mathbf{x})). \tag{15}$$

Hence, R (which has been defined in terms of T) has converted the joint problem (2), in the unknowns Q and $\mathcal{A}$, to a "new" problem in a single unknown, $\mathcal{A}$.

We thus conclude that the functions $\mathcal{H}$ and $\mathcal{G}$ are related by a right-hand composition $\mathcal{A}$. Hence, by applying the operator R we have eliminated the left composition Q, representing the radiometric deformation, in the basic relation (2). Thus, equation (15) describes a new, strictly geometric, affine registration problem. Hence, any suitable registration method may be used to recover (estimate) $\mathcal{A}$.

As mentioned in the introduction, geometric-only registration, *i.e.*, the problem of registering signals that vary by transformations of the domain, is a field of active research. In particular, a vast number of methods have been proposed in the (fundamental) case of affine geometric transformations [1], [2]. As with the case of "time-domain" registration problems, discussed in section III-A, each of these methods may be used, as is or with minimal adaptation, in order to solve for $\mathcal{A}$ in the above case.

Whereas the majority of the solution methods are either approximate or optimization-based, only a few explicit linear methods have been proposed [29], [30]. In particular, in [29] it is shown that (15) may be reformulated as a linear system of equations in $\mathcal{A}$. The method derived in [29] resembles the method discussed in section III-A for the parametric estimation of homeomorphisms; for the completeness of the discussion, its outline may be summarized as follows:

Let $\mathcal{A}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{c}$. For now, let us assume that $|\mathbf{A}| = det(\mathbf{A}) = k$ is known. Let $w : \mathbb{R} \to \mathbb{R}$ be a measurable function that vanishes at zero. By applying w to (15) as a left composition, followed by multiplication by x and integration (as in the calculation of a first order moment) we obtain

$$\int_{\mathbb{R}^m} \mathbf{x} w(\mathcal{H}(\mathbf{x})) d\mathbf{x} = \int_{\mathbb{R}^m} \mathbf{x} w(\mathcal{G}(\mathbf{A}\mathbf{x} + \mathbf{c})) d\mathbf{x}. \tag{16}$$

Denote $\mathbf{b} = -\mathbf{A}^{-1}\mathbf{c}$. By the change of variables $\mathbf{y} = \mathbf{A}\mathbf{x}+\mathbf{c}$, we have $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} + \mathbf{b}$ and $d\mathbf{x} = |\mathbf{A}^{-1}| d\mathbf{y} = k^{-1} d\mathbf{y}$ (recall that $|\mathbf{A}| = k$), yielding

$$\begin{aligned} \int_{\mathbb{R}^m} \mathbf{x} w(\mathcal{G}(\mathbf{A}\mathbf{x})) d\mathbf{x} &= \int_{\mathbb{R}^m} (\mathbf{A}^{-1}\mathbf{y} + \mathbf{b}) w(\mathcal{G}(\mathbf{y})) k^{-1} d\mathbf{y} \\ &= k^{-1}\mathbf{A}^{-1} \int_{\mathbb{R}^m} \mathbf{y} w(\mathcal{G}(\mathbf{y})) d\mathbf{y} \qquad (17) \\ &\quad + k^{-1}\mathbf{b} \int_{\mathbb{R}^m} w(\mathcal{G}(\mathbf{y})) d\mathbf{y}. \qquad (18) \end{aligned}$$

Putting (17) in matrix form, it is clear that for any choice of function $w : \mathbb{R} \to \mathbb{R}$ a linear constraint in the unknown entries of $\mathbf{A}^{-1}$ and $\mathbf{b}$ may be constructed as follows

$$\underbrace{k \int_{\mathbb{R}^m} \mathbf{x}(w \circ \mathcal{H})}_{\substack{m\times 1 \\ \text{known}}} = \underbrace{\left[\mathbf{A}^{-1} \,\vdots\, \mathbf{b} \right]}_{\substack{m\times(m+1) \\ \text{unknown}}} \cdot \underbrace{\begin{bmatrix} \int_{\mathbb{R}^m} \mathbf{x}(w \circ \mathcal{G}) \\ \hdashline \int_{\mathbb{R}^m} (w \circ \mathcal{G}) \end{bmatrix}}_{\substack{(m+1)\times 1 \\ \text{known}}}. \tag{19}$$

Choosing w's that produce linearly independent constraints lets us establish a linear system of equations in the entries of $\mathbf{A}^{-1}$ and $\mathbf{b}$. It is possible to show that different w's are "almost always" linearly independent, yielding an exact solution in the absence of noise; in fact, as mentioned before, a method for optimal selection of such constraints in the presence of model mismatch (with respect to prescribed optimality criteria) is derived in [31].

Relaxing the assumption that $|\mathbf{A}| = k$ is known is straightforward: again, by simply changing variables, it is easy to see that for any measurable function $w : \mathbb{R} \to \mathbb{R}$ that vanishes at zero we have

$$|\mathbf{A}| = \frac{\int_{\mathbb{R}^m} w \circ \mathcal{G}}{\int_{\mathbb{R}^m} w \circ \mathcal{H}}. \tag{20}$$

Hence, $|\mathbf{A}| = k$ may be calculated in advance.

Once the geometric transformation $\mathcal{A}$ is determined, (2) may be rewritten as

$$h = Q \circ \underbrace{(g \circ \mathcal{A})}_{known}. \tag{21}$$

Hence, the problem reduces to a strictly radiometric problem, where Q may be recovered (see further discussion in section III-C).

C. *Joint Registration - Concluding Results*

This short subsection concludes the results derived for solving the problem of joint registration in the case of noiseless measurement. In sections III-A and III-B we have presented two complementary approaches for decoupling and recovering the unknown radiometric and geometric transformations relating the two given observations on the object. These are summarized as follows:

> Recall that the noiseless joint registration problem is
> $$h = Q \circ g \circ \mathcal{A},$$
> where the unknowns are Q and $\mathcal{A}$. Using the transformations T and R (which is defined in terms of T) it is possible to symmetrically decouple this problem into two strictly geometric problems in each of the unknowns (Corollaries 1 and 2):
> $$\begin{aligned} [Th] &= [Tg] \circ Q^{-1}, \\ [Rh] &= [Rg] \circ \mathcal{A}. \end{aligned}$$

Hence, a solution to the joint problem may be obtained, by separately solving the latter two problems for the unknowns Q^{-1} and $\mathcal{A}$, respectively. Accordingly, *explicit* parametric solution methods for estimating Q^{-1} and $\mathcal{A}$, have been presented. It has been also noted that once one of these new problems is solved, the original problem may be reduced to a radiometric- or geometric-only problem - again solvable by the same means.

Moreover, based on the conclusions in [28], [29], [31], if the conditions in sections III-A and III-B are satisfied, the overall solution for both the geometric and the radiometric deformations, Q and $\mathcal{A}$, is completely determined and *exact* (*i.e.*, not approximated), regardless of the magnitudes of these deformations.

IV. Implementation Remarks

Throughout, we have considered images as functions over a continuous domain (coordinates). Due to the inherent physical properties of the problem, it is natural to model and solve it in the continuous domain. Inherently, the mapping $\mathcal{A}$ of $\mathbb{R}^m$ into itself is of a continuous nature, as is the physical phenomenon of geometric deformation of real-life objects it represents. Thus, if we impose a discrete spatial model (*e.g.*, $\mathbf{x} \in \mathbb{Z}^m$), we find that, in general, the natural $\mathcal{A}$ to consider is incompatible (as for "almost all" $\mathbf{x} \in \mathbb{Z}^m$, $\mathcal{A}(\mathbf{x}) \notin \mathbb{Z}^m$). In practice, in order to apply the proposed methods to digital images, we utilized an approximated discrete form of each expression, by a straightforward replacement of the integrals by their corresponding finite sums.

For example, in practice, the sample distribution transformation T, defined in (3), is replaced with the following discrete approximation:

$$[Th](t) = \frac{\#\{(i,j) : 0 < h(i,j) \leq t\}}{\#\{(i,j) : h(i,j) > 0\}}. \tag{22}$$

$[Th](t)$ is then only calculated on the finite range of h (*e.g.*, $t = 0, 1, \ldots, 255$ in the common case of 8-bit images). Other expressions are similarly discretized.

Obviously, using such discrete approximation introduces errors and model mismatches. However, the typical high resolution of (optical) digital imaging allows for these errors to be reasonably small, and satisfactory estimation results are obtained. In [32], a rigorous analysis of the errors introduced in the estimation of the geometric deformation due to sampling and quantization (in (12) and (19)) is given, and methods for minimizing these errors are proposed. Since the problem of joint geometric-radiometric estimation has been decoupled into two strictly geometric estimation problems (section III-C), these methods may be directly exploited to improve upon the performance of the joint estimation algorithm proposed in this paper.

In the computational aspect, carefully going through the derivations in section III shows that the overall complexity of the proposed joint geometric-radiometric estimation method is *linear* in image size. This is due to the fact that the estimation procedure is essentially comprised of pointwise operations and summations applied to, at most, the image pixels themselves, and the solution of low-dimensional linear systems of equations; other computations (such as these employed to the sample distributions) are computationally negligible for large images.

V. Example

As a demonstration of the proposed method, we tested its applicability to a controlled sequence of real optical images. Images of a doll's head were acquired using a computer controlled digital camera (Canon S2IS). The acquired RGB images were of dimensions 2592×1944 at 8-bit per channel. *A total of over 10,000 images were collected at different combinations of geometry (rotation angle) and radiometry (illumination power and shutter speed).* Sample images are shown in Fig. 2 and Fig. 3.

A computer controlled motorized arm has rotated the doll's head in a plane approximately perpendicular to the camera's optical axis; rotation angles ranged from -150 to 150 degrees. The illumination power and camera's shutter speed were also controlled; illumination power ranged from 35% to 100%, and shutter speed ranged from $1/1600$ to 0.6 seconds. Changing the illumination power and shutter speed globally

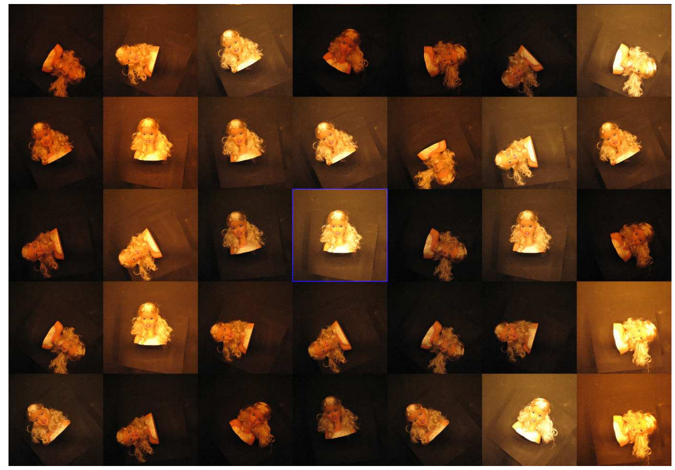

Fig. 2. A sample of the experiment dataset. A total of over 10,000 images were collected at different combinations of geometry (rotation angle) and radiometry (illumination power and shutter speed). The chosen template image is shown in the center.

Fig. 3. A sample pair of images taken at the same angle but at different illumination power and shutter speed settings. $78\%@0.01$sec. (left) $48\%@0.3$sec. (right).

affect intensity levels in the acquired image, where overall measurement non-linearities are inherently introduced by the camera. Moreover, changing the illumination power also alters the light's spectrum, thus intensifying the global non-linear effects. In practice, we separately associated a non-linear mapping with each of the image color channels, denoted by Q_R, Q_G and Q_B, respectively.

To validate our modeling assumption on the non-linear radiometric model, we examined several pairs of images taken at the same rotation angle, but with different illumination power and shutter speed settings (*e.g.*, see Fig. 3). We then plotted the joint histogram (comparagram) of each pair (for each color channel). *The joint histogram of two images is defined as the matrix whose (n, m) entry is the number of pixels that simultaneously assume the value n in the first image and m in the second one.* Recall that each pair of images is geometrically registered; hence, had there been a non-linear function Q radiometrically relating each pair, the joint histogram of the pair should exactly follow the graph of Q. Conversely, if the joint histogram of such pair seems to follow a graph (a curve) then there is (approximately) a global non-linear intensity mapping relating the pair. Next, we applied the method derived in section III to estimate Q_R, Q_G and Q_B for each channel separately. Finally, we overlaid the

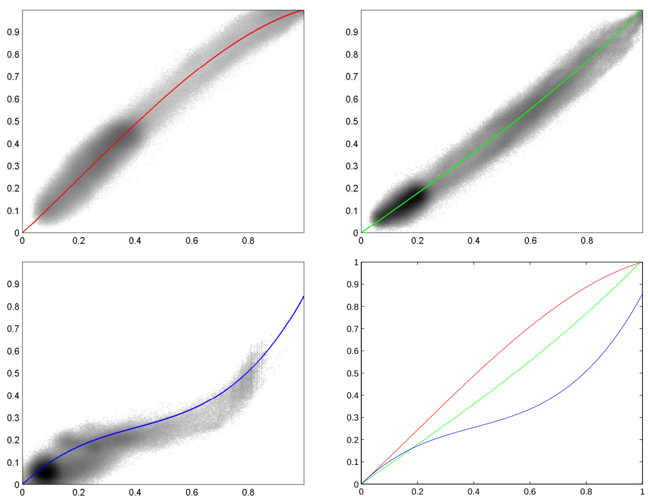

Fig. 4. The joint histograms of the Red, Green and Blue channels of a sample pair of images taken at different illumination power and shutter speed settings. Joint histograms are represented by logarithmic scale graylevels. The corresponding estimates of Q_R, Q_G and Q_B are overlaid as solid lines, and are also separately depicted in the lower-right plot.

estimates on the joint histograms of each channel (see Fig. 4). The results strongly suggest that the type of global radiometry considered in this paper is indeed a good approximation to the radiometric effects evident in the collected data.

Finally, we used the joint estimation method proposed in this paper to estimate the rotation angle. First, a reference template image has been arbitrarily selected from the dataset (see the marked image in the center of Fig. 2). For *every* image in the dataset, the radiometric functions Q_R, Q_G and Q_B were estimated, and a radiometrically-registered image was calculated (see section III-A). Finally, the affine estimation procedure was applied, and the corresponding rotation angle calculated.

The overall angular estimation, *over the entire dataset*, had a bias of $3.2 \cdot 10^{-6}$ degrees with a standard deviation of 1.41 degrees. Angular estimation error is shown in Fig. 5.

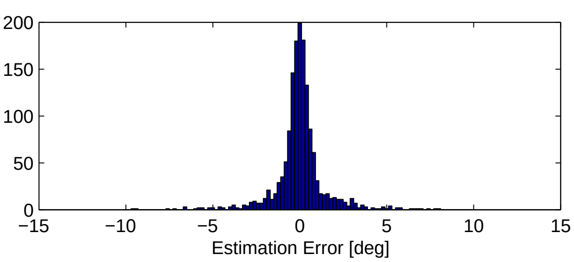

Fig. 5. Angular estimation error.

Employing the estimation procedure for image pairs that closely match the fundamental model (1) yields negligible estimation errors; for example, when applied to images taken at similar angles but at different illumination conditions (see Fig. 3). However, the majority of the dataset has shown substantial model mismatches due to real-life phenomena ignored by the basic model (1); among such are projective geometry, non-uniform illumination (due to the light sources),

Fig. 6. Robustness to model mismatches - a sample case of an angular estimation error of 2.42 degrees. Top row: a pair of images taken at different angles and illumination conditions; Middle Row: the same pair after applying the radiometry and geometry estimates; Bottom row: error image - primarily due to model mismatch (*e.g.*, projective geometry, hair).

shadows, occlusions and other un-modeled perturbations (such as movements of the doll's hair, easily noticeable in Fig. 2 and 6). Nevertheless, due to the linear setting of the equivalent problems derived in the previous sections, the overall estimation procedure has shown robustness and yielded reasonable estimates despite the significant mismatches (for example see Fig. 6). On the other hand, attempts to directly use geometry estimation procedures on the original data, without first eliminating the radiometric effect, have failed.

VI. Concluding Remarks

We have proposed a novel method for jointly estimating the geometric and radiometric deformations relating two observations on the same object. The case of a geometric affine transformation relating the images' coordinate systems, and a non-linear function relating the intensities of the two images has been modeled.

In section III, we derived an estimation method for the joint problem. We have shown that by using the sample distribution transformation T, the joint estimation problem may be decoupled into two simpler problems in the unknown radiometry and unknown geometry. More specifically, we have shown that the original high-dimensional non-linear non-convex search problem of simultaneously recovering the geometric and radiometric deformations can be represented by an equivalent sequence of two linear systems. In the absence of noise, solution of this sequence yields an *exact*, *explicit* and *efficient* solution to the joint estimation problem, regardless of the deformations' magnitudes. In a forthcoming paper, we further elaborate on the performance of the proposed method in the presence of model mismatches. (See [33] for a preliminary exposition of the extended solution).

Results obtained in experiments with real images show the applicability of the methods proposed in this paper. The proposed linear framework of the solution was shown to lead to an efficient method, of linear computational complexity, which is extremely robust to model mismatches. Thus, the proposed solution may be used as is, or in conjunction with other techniques in case improved performance is required; for example, estimation results may be refined by efficiently exploiting the robustness of the proposed method in order to initialize an optimization based estimation method. Such an approach will considerably reduce the probability of wrong convergence due to local minima, as well as the involved computational load.

The principles of the proposed method, and the general framework into which the problem of joint estimation of geometric and radiometric deformations has been casted to, may be further exploited to address generalizations of the model described in this paper. These include the cases of an elastic (non-affine) geometric deformation as well as the case of non-invertible intensity mappings.

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