The one-sided ergodic Hilbert transform of normal contractions

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Dedicated to the memory of Moshe Livšic

Abstract. Let T be a normal contraction on a Hilbert space H. For $f \in H$ we study the one-sided ergodic Hilbert transform $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{T^k f}{k}$. We prove that weak and strong convergence are equivalent, and show that the convergence is equivalent to convergence of the series $\sum_{n=1}^{\infty} \frac{\log n \| \sum_{k=1}^{n} T^k f\|^2}{n^3}$. When $H = \overline{(I-T)H}$, the transform is shown to be precisely minus the infinitesimal generator of the strongly continuous semi-group $\{(I-T)^r\}_{r\geq 0}$.

The equivalence of weak and strong convergence of the transform is proved also for T an isometry or the dual of an isometry.

For a general contraction T, we obtain that convergence of the series $\sum_{n=1}^{\infty} \frac{\langle T^n f, f\rangle \log n}{n}$ implies strong convergence of $\sum_{n=1}^{\infty} \frac{T^n f}{n}$.

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1. Introduction

Let θ be a measure preserving invertible transformation of a probability space $(\mathcal{S}, \Sigma, m)$, and let U be the unitary operator induced on $L_2(m)$. For θ ergodic, Izumi [I] raised the question of almost everywhere (a.e.) convergence of $\sum_{k=1}^{\infty} \frac{U^k f}{k}$ for all functions $f \in L_2(m)$ with zero integral. Halmos [H] proved that when the probability space $(\mathcal{S}, \Sigma, m)$ is non-atomic, there is always a function $f \in L_2(m)$ with zero integral for which the above series fails to converge in L_2-norm. For additional background and references see [AL].

For T power-bounded on a Banach space X we have $\|\frac{1}{n}\sum_{k=1}^n T^k f\| \to 0$ if and only if $f \in \overline{(I-T)X}$, and it is known that weak and strong convergence of the averages are equivalent (e.g., [Kr, §2.1]). Hence, by Kronecker's lemma, *weak* convergence of the *one-sided ergodic Hilbert transform* $\sum_{k=1}^\infty \frac{T^k f}{k}$ strengthens the strong convergence to zero of the averages.

Theorem 1.1. *Let T be a power-bounded operator on a Banach space X, put $Y := \overline{(I-T)X}$, and denote by S the restriction of T to Y. Then the following are equivalent:*

(i) $(I-T)X$ is closed in X.

(ii) The series $\sum_{k=1}^\infty \frac{S^k}{k}$ converges in operator norm.

(iii) The series $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges in norm for every $f \in Y$.

(iv) The series $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges weakly for every $f \in Y$.

Proof. (i)$\Longrightarrow$(ii): It is easy to compute that we always have operator-norm convergence of $\sum_{k=1}^\infty \frac{T^k}{k}(I-T)$. By [L], condition (i) implies that $(I-S)$ is invertible on Y, so (ii) holds.

Clearly (ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv).

By [AL, Proposition 4.1], (iv) implies that $Gf := -\sum_{k=1}^\infty \frac{T^k f}{k}$ (weak convergence) is a bounded operator on Y which is the infinitesimal generator of a semi-group. Now the proof of [DL, Theorem 2.23] yields (i). $\square$

Remarks. 1. Condition (i) implies that $\frac{1}{n}\sum_{k=1}^n T^k$ converges in operator norm, even for non-reflexive spaces [L].

2. The equivalence of the first three conditions is implicit in [DL], since (iii)$\Longrightarrow$(i) by [DL, Theorem 2.23].

3. The result of [H] follows from the theorem, since condition (i) is not satisfied by unitary operators induced by aperiodic probability preserving transformations (for which the spectrum is the whole unit circle).

Since for a contraction T in H the fixed points of T and T^* are the same [RN, §144], we have $\overline{(I-T)H} = \overline{(I-T^*)H}$, so $\frac{1}{n}\sum_{k=1}^n T^k f \to 0$ if and only if $\frac{1}{n}\sum_{k=1}^n T^{*k} f \to 0$.

Proposition 1.2. *Let T be a contraction in a Hilbert space H. Then $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges (weakly) if and only if $\sum_{k=1}^\infty \frac{T^{*k} f}{k}$ converges (weakly).*

Proof. Using the unitary dilation of T, Campbell [Ca] proved that for every $f \in H$ the series $\sum_{k=1}^\infty \frac{T^k f - T^{*k} f}{k}$ converges in norm. $\square$

For a power-bounded operator T on a Banach space X, Derriennic and Lin [DL] defined the operator $(I-T)^\alpha$ for $0<\alpha<1$ by the series $I - \sum_{k=1}^\infty a_k^{(\alpha)} T^k$, where $a_k^{(\alpha)} > 0$ with $\sum_{k=1}^\infty a_k^{(\alpha)} = 1$ are the coefficients of the power-series $(1-t)^\alpha = 1 - \sum_{k=1}^\infty a_k^{(\alpha)} t^k$ for $|t| \le 1$. They proved that $(I-T)X \subset (I-T)^\alpha X \subset \overline{(I-T)X}$, and when $(I-T)X$ is not closed both inclusions are strict. For T mean ergodic (e.g., X is reflexive) we have $f \in (I-T)^\alpha X$ if and only if $\displaystyle\sum_{k=1}^{\infty} \frac{T^k f}{k^{1-\alpha}}$ converges strongly ([DL, Theorem 2.11]), and then $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converges strongly.

When $\overline{(I-T)X} = X$ we have that $\{(I-T)^r : r \geq 0\}$ is a strongly continuous one-parameter semi-group [DL, Theorem 2.22], and the domain of its infinitesimal generator G contains all f for which $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converges weakly, and then the sum of the series is $-Gf$ [AL, Proposition 4.1].

For fixed $t \in [-1,1)$ the infinitesimal generator of $(1-t)^r = e^{r\log(1-t)}$ is obviously $\log(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k}$, so two natural questions arise (when $(I-T)X$ is not closed, but dense):

(i) If f is in the domain of G, does the series $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converge weakly?

(ii) If $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converges weakly, does it converge strongly?

We answer both questions positively for normal contractions in a (complex) Hilbert space; for T unitary or self-adjoint this was proved in [AL].

2. Preliminaries

Lemma 2.1. *Let $\{f_k\}$ be a sequence in a Banach space. If the series $\displaystyle\sum_{k=1}^{\infty} \frac{f_k}{k}$ converges, then for every $\alpha > 0$ the series $\displaystyle\sum_{k=1}^{\infty} \frac{f_k}{k^{1+\alpha}}$ converges and we have*

$$\lim_{\alpha\to 0^+} \sum_{k=1}^{\infty} \frac{f_k}{k^{1+\alpha}} = \sum_{k=1}^{\infty} \frac{f_k}{k}.$$

Proof. We put $S_n = \sum_{k=1}^{n} \frac{f_k}{k}$. By Abel's summation by parts we have

$$\sum_{k=1}^{n} \frac{f_k}{k^{\alpha+1}} = \frac{S_n}{n^\alpha} + \sum_{k=1}^{n-1} S_k\Big[\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha}\Big]$$

Since S_n converges we have $\sup_n \|S_n\| < \infty$, so the first term on the right-hand side above tends to zero as n tends to infinity. The second term is absolutely summable as the factor of S_k there behaves like $1/k^{1+\alpha}$. Hence we obtain the first assertion. In particular we obtain

$$\sum_{k=1}^{\infty} \frac{f_k}{k^{\alpha+1}} = \sum_{k=1}^{\infty} S_k\Big[\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha}\Big] \tag{1}$$

It remains to prove the second assertion. We are going to define a Toeplitz summability matrix.

Let $\alpha_j \to 0^+$ be an arbitrary sequence and define a summability matrix (with positive entries!) $a_{j,k} = \frac{1}{k^{\alpha_j}} - \frac{1}{(k+1)^{\alpha_j}}$. Clearly, (i): $\lim_{j\to\infty} a_{j,k} = 0$ for every $k \geq 1$ and (ii): $\sum_{k=1}^\infty a_{j,k} = 1$ for every j. Put $S = \lim_{n\to\infty} S_n$.

Using (1) above we have

$$\Big\| \sum_{k=1}^\infty \frac{f_k}{k^{1+\alpha_j}} - S \Big\| = \Big\| \sum_{k=1}^\infty a_{j,k}(S_k - S) \Big\| \leq \sum_{k=1}^\infty a_{j,k} \|S_k - S\|$$

Since $\|S_k - S\| \underset{k\to\infty}{\longrightarrow} 0$, properties (i) and (ii) yield

$$\lim_{j\to\infty} \Big\| \sum_{k=1}^\infty \frac{f_k}{k^{1+\alpha_j}} - S \Big\| = 0.$$

Since $\{\alpha_j\}$ is arbitrary, the assertion follows. $\square$

Corollary 2.2. *Let (X,μ) be a measure space and let T be an operator on $L_p(X)$ $(p \geq 1)$. If for some $f \in X$, the series $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges a.e., then*

$$\lim_{\alpha\to 0^+} \sum_{k=1}^\infty \frac{T^k f}{k^{1+\alpha}} = \sum_{k=1}^\infty \frac{T^k f}{k} \quad a.e.$$

Proof. For a.e. $x \in X$ we put $f_k = [T^k f](x)$. Now, apply Lemma 2.1 in the normed space $\mathbb{C}$. $\square$

Corollary 2.3. *For every $|z| \leq 1$, with $z \neq 1$, we have*

$$\lim_{\alpha\to 0^+} \sum_{n=1}^\infty \frac{z^n}{n^{1+\alpha}} = \sum_{n=1}^\infty \frac{z^n}{n}.$$

Corollary 2.4. *Let T be an operator in a Banach space X and let $f \in X$. If the series $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges, then $\lim_{\alpha\to 0^+} \sum_{k=1}^\infty \frac{T^k f}{k^{1+\alpha}} = \sum_{k=1}^\infty \frac{T^k f}{k}$.*

Remarks. 1. When the series $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges weakly, the proof of the lemma still yields norm convergence of $\sum_{k=1}^\infty \frac{T^k f}{k^{1+\alpha}}$ for each $\alpha > 0$, but the lemma yields only weak convergence of these series, as $\alpha \to 0^+$.

2. Combining Proposition 4.1 and Corollary 4.5 of [AL], we obtain the more difficult result (not used in the sequel) that for T power-bounded, *weak* convergence of the series $\sum_{k=1}^\infty \frac{T^k f}{k}$ implies the full conclusion of Corollary 2.4.

By considering the power series $\sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z)$ for $|z|<1$, we conclude that for $z = re^{ix}$, with $r<1$, $0 \le x < 2\pi$, we have (see [Z, Ch. I, p. 2]):

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = \frac{1}{2} \log \frac{1}{1-2r\cos x + r^2} = -\log|1-z|$$

$$\sum_{n=1}^{\infty} \frac{r^n \sin nx}{n} = \arctan \frac{r \sin x}{1 - r\cos x} = -\arg(1-z).$$

On the other hand, since the series $\sum_{n=1}^{\infty} n^{-1} \cos nx$ and $\sum_{n=1}^{\infty} n^{-1} \sin nx$ converge for $x \neq 0$ (the latter even everywhere and both converge uniformly for $\epsilon \le x \le 2\pi - \epsilon$), we have continuity at $r = 1^-$ by Abel's summability, so

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \log \frac{1}{|2 \sin \frac{1}{2}x|} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{1}{2}(\pi - x), \tag{2}$$

for $0 < x < 2\pi$ (see [Z, Ch. I, p. 5]).

We also have (see [Z, Ch. II, p. 61] and [Z, Ch. V, p. 191], respectively)

$$\sup_{n \ge 1} \max_{0 \le x \le 2\pi} |\sum_{k=1}^{n} \frac{\sin kx}{k}| < \infty. \tag{3}$$

$$\text{and} \quad \sup_{n\ge 1} |\sum_{k=1}^{n} \frac{\cos kx}{k}| \le \log \frac{1}{x} + C \text{ for } 0 < x \le \pi$$

Put $S_n(x) = \sum_{k=1}^{n} \frac{\cos kx}{k}$. Abel's summation by parts (with $S_0 \equiv 0$) yields

$$\sum_{k=1}^{n} \frac{r^k \cos kx}{k} = r^n S_n(x) + \sum_{k=1}^{n-1} (r^k - r^{k+1}) S_k(x) =$$

$$r^n S_n(x) + r(1-r) \sum_{k=1}^{n-1} r^{k-1} S_k(x).$$

Hence for $0 \le r \le 1$ and $0 < x \le \pi$ we have

$$\sup_{n \ge 1} |\sum_{k=1}^{n} \frac{r^k \cos kx}{k}| \le 2r \log \frac{1}{x} + C \tag{4}$$

Similar summation by parts for $\sum_{k=1}^{n} \frac{r^k \sin kx}{k}$ yields

$$\sup_{0 \le r \le 1} \sup_{n \ge 1} \max_{0 \le x \le 2\pi} |\sum_{k=1}^{n} \frac{r^k \sin kx}{k}| < \infty. \tag{5}$$

We also notice that for $0 \le r < 1$ and any x we have

$$\sup_{n \ge 1} |\sum_{k=1}^{n} \frac{r^k \cos kx}{k}| \le \sum_{k=1}^{\infty} \frac{r^k}{k} = -\log(1-r)\,,$$

so we obtain by (4) that for every $0 \le r \le 1$ and $0 \le |x| \le \pi$ we have

$$\sup_{n\ge 1} \Big|\sum_{k=1}^{n} \frac{r^k \cos kx}{k}\Big| \le C + 2\min\{\log\frac{1}{|x|}, -\log(1-r)\} \tag{6}$$

Note that only when $x = 0$ *and* $r = 1$ both sides of (6) are infinite; in all other cases they are finite.

3. The ergodic Hilbert transform for normal contractions

Let T be a normal operator on a complex Hilbert space H with resolution of the identity $E(dz)$. For $f \in H$ denote by $\sigma_f(dz) = \langle E(dz)f, f\rangle$ the spectral measure of T with respect to f. By the mean ergodic theorem, $f \in \overline{(I-T)H}$ if and only if $\sigma_f(\{1\}) = 0$.

Theorem 3.1. *Let T be a normal contraction on H and let $0 \ne f \in H$ with spectral measure σ_f. Put $D = \{z : |z| \le 1\}$. The following conditions are equivalent:*

(i): $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$ *converges strongly;*

(ii): $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$ *converges weakly;*

(iii): $\displaystyle\sup_N \Big\|\sum_{j=1}^{N} \frac{T^j f}{j}\Big\| < \infty$;

(iv): $\displaystyle\int_D \log^2|1-z|\sigma_f(dz) < \infty$.

If either condition holds, then $f \in \overline{(I-T)H}$,

$$\langle\sum_{n=1}^{\infty} \frac{T^n f}{n}, g\rangle = -\int_D \log(1-z)\langle E(dz)f, g\rangle \quad \textit{for every} \quad g \in H, \quad \textit{and}$$

$$\Big\|\sum_{n=1}^{\infty} \frac{T^n f}{n}\Big\|^2 = \int_D |\log(1-z)|^2\sigma_f(dz).$$

Proof. Clearly, $(i) \Rightarrow (ii)$ and by the uniform boundedness principle $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (iv)$. Clearly (iii) implies that f is orthogonal to the fixed points of T^*, so $f \in \overline{(I-T)H}$, and we have $\sigma_f(\{1\}) = 0$. Hence all integrals below with respect to σ_f are in fact over $\tilde{D} = \{z : |z| \le 1,\ z \ne 1\}$.

The spectral theorem gives us the equality

$$\Big\|\sum_{j=1}^{N} \frac{T^j f}{j}\Big\|^2 = \int_{\tilde{D}} \Big[\Big(\Re\Big\{\sum_{j=1}^{N} \frac{z^j}{j}\Big\}\Big)^2 + \Big(\Im\Big\{\sum_{j=1}^{N} \frac{z^j}{j}\Big\}\Big)^2\Big]\sigma_f(dz).$$

The imaginary part is uniformly bounded on the whole closed unit disk $D = \{|z| \le 1\}$, so we just need to take care of the real part. By Fatou's lemma and the previous equality we have

$$\int_{\tilde{D}} \log^2 |1-z| \sigma_f(dz) =$$

$$\int_{\tilde{D}} \liminf_{N\to\infty} \Big(\Re\Big\{\sum_{n=1}^{N} \frac{z^n}{n}\Big\}\Big)^2 \sigma_f(dz) \le \sup_N \Big\| \sum_{j=1}^{N} \frac{T^j f}{j} \Big\|^2 < \infty.$$

$(iv) \Rightarrow (i)$. The convergence of the integral yields $\sigma_f(\{1\}) = 0$. Hence all integrals with respect to σ_f are actually over $\tilde{D}$. By the spectral theorem, we have

$$\Big\| \sum_{j=N}^{M} \frac{T^j f}{j} \Big\|^2 = \int_{\tilde{D}} \Big[\Big(\Re\Big\{\sum_{j=N}^{M} \frac{z^j}{j}\Big\}\Big)^2 + \Big(\Im\Big\{\sum_{j=N}^{M} \frac{z^j}{j}\Big\}\Big)^2\Big] \sigma_f(dz).$$

We will show that $\lim_{N,M\to\infty} \| \sum_{j=N}^{M} \frac{T^j f}{j} \| = 0$.

The series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at each point of $\tilde{D}$, so we conclude that $\lim\limits_{k\to\infty} \sup\limits_{N,M\ge k} | \sum_{n=N}^{M} \frac{z^n}{n}| = 0$. Furthermore, $\Im\{\sum_{n=1}^{N} \frac{z^n}{n}\}$ is uniformly bounded on $\tilde{D}$, hence $\sup\limits_{k\ge 1} \sup\limits_{N,M\ge k} |\Im\{\sum_{n=N}^{M} \frac{z^n}{n}\}| < \infty$ uniformly on $\tilde{D}$.

Using Lebesgue's monotone convergence theorem (by considering $\sup\limits_{k\le N,M\le K}(\cdot)$ and letting $K\to\infty$), we obtain

$$\sup_{N,M\ge k} \int_D \Big(\Im\Big\{\sum_{n=N}^{M} \frac{z^n}{n}\Big\}\Big)^2 \sigma_f(dz) \le \int_D \sup_{N,M\ge k} \Big(\Im\Big\{\sum_{n=N}^{M} \frac{z^n}{n}\Big\}\Big)^2 \sigma_f(dz)$$

Using Lebesgue's dominated convergence theorem we conclude that

$$\lim_{k\to\infty} \sup_{N,M\ge k} \int_{\tilde{D}} \Big(\Im\Big\{\sum_{j=N}^{M} \frac{z^j}{j}\Big\}\Big)^2 \sigma_f(dz) = 0\,.$$

So, it only remains to check the assertion for $\Re\{\sum_{n=1}^{N} \frac{z^n}{n}\}$. We split $\tilde{D}$ into two disjoint parts by putting

$$D' = \{z \in \tilde{D} : |\arg z| > 1\} \quad \text{and} \quad D'' = \{z \in \tilde{D} : |\arg z| \le 1\}.$$

Using (6) we conclude that

$$\sup_N \max_{z\in D'} \Big|\Re\Big\{\sum_{n=1}^{N} \frac{z^n}{n}\Big\}\Big| \le C.$$

Again, the same arguments and using Lebesgue's dominated convergence theorem we conclude that

$$\lim_{N,M\to\infty} \int_{D'} \Big(\Re\Big\{\sum_{j=N}^{M} \frac{z^j}{j}\Big\}\Big)^2 \sigma_f(dz) = 0\,.$$

On D'' we have the following consideration. By (6) we have

$$\int_{D''} \sup_{N\ge 1} \Big(\Re\Big\{\sum_{n=1}^{N} \frac{z^n}{n}\Big\}\Big)^2 \sigma_f(dz) \le$$

$$\le C_1\|f\|^2 + C_2 \int_{D''} \min\{\log^2[|\log(z/|z|)|], \log^2(1-|z|)\}\sigma_f(dz)\,.$$

Now, let $z = |z|e^{ix}$ with $z \in D''$. Since $|x| \le 1$ and $1-|z| \le 1$ we have,

$$\min\{\log^2|x|, \log^2(1-|z|)\} = \log^2[\max\{|x|, 1-|z|\}]\,.$$

On the other hand, since $4|z|\sin^2\frac{x}{2} \le x^2$ we obtain

$$2[\max\{|x|, 1-|z|\}]^2 \ge (1-|z|)^2 + x^2 \ge (1-|z|)^2 + 4|z|\sin^2\frac{x}{2} = |1-z|^2.$$

Since all the arguments of the logarithms below are less than or equal 1, this yields using our assumption,

$$\int_{D''} \min\{\log^2[|\log(z/|z|)|], \log^2(1-|z|)\}\sigma_f(dz) \le$$

$$\int_{D''} \log^2[\frac{1}{\sqrt{2}}|1-z|]\sigma_f(dz) \le C\|f\|^2 + C' \int_{D''} \log^2|1-z|\sigma_f(dz) < \infty\,.$$

Hence, using the same arguments as we have considered in the case of the imaginary part and applying Lebesgue's dominated convergence theorem, we conclude the implication $(iv) \Rightarrow (i)$.

If any of the conditions in the theorem holds, then the last assertion follows from what we have done and the convergence $\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$ on $\tilde{D}$. $\square$

Remarks. 1. The equivalence of conditions (i) and (iv) in Theorem 3.1 is implicit (without proof) in [G4]; an explicit statement is given and proved there for T unitary.

2. For the particular cases of T unitary or self-adjoint, the equivalence of the four conditions in the theorem was proved in [AL].

Proposition 3.2. *Let T be a normal contraction on H and let $0 \ne f \in H$ with spectral measure σ_f. If*

$$\sum_{n=1}^{\infty} \frac{\|\sum_{k=1}^{n} T^k f\|^2 \log n}{n^3} < \infty\,,$$

then $\displaystyle\int_D \log^2|1-z|\sigma_f(dz) < \infty$.

Proof. For every $n \ge 1$ put

$$D_n := \{z = re^{2i\pi\theta} :\ 1-\frac{1}{n} \le r \le 1,\ -\frac{1}{2n} \le \theta \le \frac{1}{2n}\}.$$

Then $\{D_n\}$ is decreasing, $D_1 = D$, and $\bigcup_{n=1}^{\infty}(D_n - D_{n+1}) = D - \{1\}$.

Let $n \geq 2$. Since $(1-\frac{1}{n})^{n-1}$ decreases to 1/e, for $1-1/n \leq r \leq 1$ we have

$$r^n \geq r(1-\frac{1}{n})^{n-1} > r/3$$

$$1-r^n = (1-r)\sum_{k=0}^{n-1} r^k \geq (1-r)nr^{n-1} \geq n(1-r)/3.$$

For $|\theta| \leq \frac{1}{2n}$ we have $|\sin(\pi n\theta)| \geq 2n|\theta| \geq \frac{2n}{\pi}|\sin(\pi\theta)|$.

For $z = re^{2i\pi\theta} \in D_n$, $n \geq 2$, since $r \geq \frac{1}{2}$, we thus obtain

$$|\sum_{k=1}^{n} z^k|^2 = |z|^2\left|\frac{1-z^n}{1-z}\right|^2 = r^2\frac{1-2r^n\cos(2\pi n\theta)+r^{2n}}{1-2r\cos(2\pi\theta)+r^2} \geq$$

$$\frac{1}{4}\frac{(1-r^n)^2+4r^n\sin^2(\pi n\theta)}{(1-r)^2+4r\sin^2(\pi\theta)} \geq \frac{n^2}{36}\,.$$

So, by the spectral theorem we obtain

$$\sigma_f(D_n) \leq \frac{36}{n^2}\int_{D_n}|\sum_{k=1}^{n} z^k|^2\sigma_f(dz) \leq \frac{36}{n^2}\Big\|\sum_{k=1}^{n} T^k f\Big\|^2 \tag{7}$$

For $j \geq 2$ and $z \in D_j - D_{j+1}$ we have $\frac{j}{4} \leq \frac{1}{|1-z|} \leq j+1$, so $\int_D \log^2|1-z|\sigma_f(dz) < \infty$ if and only if $\sum_{n=1}^{\infty}(\sigma_f(D_n)-\sigma_f(D_{n+1}))\log^2 n < \infty$.

Assume that $\sum_{n=1}^{\infty}\frac{\|\sum_{k=1}^{n} T^k f\|^2\log n}{n^3} < \infty$. Then (7) yields

$$\sum_{n=1}^{\infty}\frac{\log n\,\sigma_f(D_n)}{n} \leq 36\sum_{n=1}^{\infty}\frac{\log n\|\sum_{k=1}^{n} T^k f\|^2}{n^3} < \infty.$$

Abel's summation by parts yields

$$\sum_{n=1}^{N-1}(\sigma_f(D_n)-\sigma_f(D_{n+1}))\log^2 n \leq C'\sum_{n=1}^{N}\frac{\log n\,\sigma_f(D_n)}{n}$$

So $\int_D \log^2|1-z|\sigma_f(dz) < \infty$. □

Remarks. 1. The proposition, suggested by Christophe Cuny, leads (see below) to a characterization of the convergence of the transform by a condition on the norms of the sums (or of the averages).

2. The computations leading to (7) (and (10) below) were made in [CL], and are included for the sake of completeness. Computations of this type on the unit circle (for unitary operators) appear in [G2] and [G3].

Theorem 3.3. *Let T be a normal contraction on H and let $0 \neq f \in H$. Then the following are equivalent:*

(i): $\displaystyle\int_D \log^2|1-z|\sigma_f(dz) < \infty$

(ii): $\displaystyle\sum_{n=1}^{\infty} \frac{\langle T^n f, f\rangle \log n}{n}$ *converges.*

(iii): $\displaystyle\sum_{n=1}^{\infty} \frac{\| \sum_{k=1}^n T^k f\|^2 \log n}{n^3} < \infty\,.$

Proof. Proposition 3.2 shows $(iii) \Longrightarrow (i)$.

$(i) \Longrightarrow (ii)$: Assume $\displaystyle\int_D \log^2 |1-z| \sigma_f(dz) < \infty$. By the spectral theorem,

$$\sum_{k=1}^{n} \frac{\langle T^k f, f\rangle \log k}{k} = \int_D \sum_{k=1}^{n} \frac{z^k \log k}{k} \sigma_f(dz). \tag{8}$$

We continue to denote $\tilde{D} = \{z \,:\, |z| \le 1,\ z \neq 1\}$. For every $n \ge 1$ and $z \in \tilde{D}$ we have $|\sum_{k=1}^n z^k| \le 2/|1-z|$. Since the sequence $\{\log n/n\}_{n>2}$ is decreasing to zero, Abel's summation by parts yields that the series $\sum_{n=1}^{\infty} \frac{z^n \log n}{n}$ converges for every $z \in \tilde{D}$. Actually, the partial sums are uniformly bounded on $\{z \in D : |1-z| \ge \epsilon > 0\}$. By our assumption $\sigma_f(\{1\}) = 0$, so $\sum_{n=1}^{\infty} \frac{z^n \log n}{n}$ converges σ_f-a.e. To prove convergence in (8), we will prove σ_f-integrability of $\sup_{n\ge1} |\sum_{k=1}^n \frac{z^k \log k}{k}|$.

Recall that using (3) and (6) we have already shown in the proof of Theorem 3.1 that for every $z \in \tilde{D}$,

$$\sup_{n\ge1} \Big| \sum_{k=1}^{n} \frac{z^k}{k} \Big| \le C + \Big| \log \frac{1}{|1-z|} \Big| .$$

Now, we majorize $\sup_{n\ge1} |\sum_{k=1}^n \frac{z^k \log k}{k}|$ for $z \in \tilde{D}$ with $0 < |1-z| \le \frac{1}{3}$. We fix z and put $n' = [1/|1-z|]$. For $n > n'$ write

$$\sum_{k=1}^{n} \frac{z^k \log k}{k} = \sum_{k=1}^{n'} \frac{z^k \log k}{k} + \sum_{k=n'+1}^{n} \frac{z^k \log k}{k} = P_1 + P_2$$

We deal with two cases: (i) $n \le n'$ and (ii) $n > n'$.

Case (i): put $S_j = \sum_{k=1}^j \frac{z^k}{k}$. Since $\log n \le \log n' \le \log(1/|1-z|)$, we have

$$\Big| \sum_{k=1}^{n} \frac{z^k \log k}{k} \Big| \le \sum_{k=1}^{n} \frac{\log k}{k} \le C \log^2 n \le C \log^2(1/|1-z|).$$

Case (ii): put $S'_j = \sum_{k=1}^j z^k$. We use the decomposition $P_1 + P_2$, with P_1 estimated in case (i). Using Abel's summation, we obtain

$$\Big| \sum_{k=n'+1}^{n} \frac{z^k \log k}{k} \Big| \le$$

$$\frac{\log n}{n}|S'_n| + \sum_{k=n'+1}^{n-1}\Big(\frac{\log(k)}{k} - \frac{\log(k+1)}{k+1}\Big)|S'_k| + \frac{\log(n'+1)}{n'+1}|S'_{n'}|.$$

Since $n \ge n'+1 > 1/|1-z|$ and $\log x/x$ is decreasing, we obtain

$$|P_2| \le \frac{\log n}{n}\frac{2}{|1-z|} + 2\frac{\log(n'+1)}{n'+1}\frac{2}{|1-z|} \le$$

$$3\frac{\log(1/|1-z|)}{1/|1-z|}\frac{2}{|1-z|} = 6\log\Big(\frac{1}{|1-z|}\Big)$$

Putting the two cases together, for $z \in \tilde{D}$ we obtain

$$\sup_{n\ge 1}\Big|\sum_{k=1}^{n}\frac{z^k\log k}{k}\Big| \le C'|\log(1/|1-z|)| + C\log^2(1/|1-z|) \qquad (*)$$

Now we prove our claim. For z close to 1, the dominant part in (*) is $\log^2(1/|1-z|)$. Hence by our assumption, the convergence in (8) follows from the Lebesgue bounded convergence theorem.

(ii) implies (iii): We first prove the implication for T unitary. In this case, it follows from the general Lemma 3 of [G2] (see also [V], [G4]), but the proof of its applicability to our case is omitted; for the sake of completeness we give the full proof.

$$\sum_{n=1}^{N}\frac{\log n\Big\|\sum_{k=1}^{n}T^kf\Big\|^2}{n^3} = \sum_{n=1}^{N}\frac{\log n\big(n\|f\|^2 + 2\Re\sum_{k=1}^{n-1}(n-k)\langle T^kf,f\rangle\big)}{n^3} =$$

$$\sum_{n=1}^{N}\frac{\log n\|f\|^2}{n^2} + 2\Re\sum_{n=1}^{N}\frac{\log n}{n^3}\sum_{k=1}^{n}(n-k)\langle T^kf,f\rangle.$$

The first series converges, and we show convergence of the second. Write

$$\sum_{n=1}^{N}\frac{\log n}{n^3}\sum_{k=1}^{n}(n-k)\langle T^kf,f\rangle = \sum_{k=1}^{N}\langle T^kf,f\rangle\Big[\sum_{n=k}^{N}\frac{\log n}{n^2} - k\sum_{n=k}^{N}\frac{\log n}{n^3}\Big].$$

For $h(x) > 0$ non-increasing we have

$$\int_N^{N+1}h(x)dx \le \sum_{n=k}^{N}h(k) - \int_k^N h(x)dx \le \int_{k-1}^{k}h(x)dx \le h(k-1).$$

We fix K large, and approximating sums by integrals we obtain

$$\sum_{k=K}^{N}\langle T^kf,f\rangle\sum_{n=k}^{N}\frac{\log n}{n^2} = \sum_{k=K}^{N}\langle T^kf,f\rangle\Big(\frac{\log k}{k} + \frac{1}{k} - \frac{\log N}{N} - \frac{1}{N} + \mathcal{O}(\frac{\log k}{k^2})\Big).$$

$$\sum_{k=K}^{N}\langle T^kf,f\rangle k\sum_{n=k}^{N}\frac{\log n}{n^3} = \sum_{k=K}^{N}\langle T^kf,f\rangle k\Big(\frac{\log k}{2k^2} + \frac{1}{4k^2} - \frac{\log N}{2N^2} - \frac{1}{4N^2} + \mathcal{O}(\frac{\log k}{k^3})\Big).$$

By Abel's summation (ii) implies convergence of $\sum_{k=2}^\infty \frac{\langle T^k f,f\rangle}{k}$, and by Kronecker's lemma, (ii) implies

$$\frac{\log N}{N}\sum_{k=K}^N \langle T^k f,f\rangle \to 0 \quad \text{and} \quad \frac{\log N}{N^2}\sum_{k=K}^N k\langle T^k f,f\rangle \to 0.$$

Letting $N\to\infty$ we obtain (iii) for T unitary.

Now let T be a contraction, and let U be its unitary dilation, defined on a larger space H_1. Since $\langle U^n f,f\rangle = \langle T^n f,f\rangle$ for $f\in H$, condition (ii) for T implies the same for U, so by the above we have that $\sum_{n=1}^\infty \frac{\|\sum_{k=1}^n U^k f\|^2\log n}{n^3}$ converges. Continuity of the projection from H_1 onto H yields convergence of $\sum_{n=1}^\infty \frac{\|\sum_{k=1}^n T^k f\|^2\log n}{n^3}$. □

Remarks. 1. The proof shows that the implication $(ii) \Longrightarrow (iii)$ is in fact true for any contraction.

2. It follows from Theorems 3.1 and 3.3 that, when T is normal, a sufficient condition for the convergence of the one-sided ergodic Hilbert transform is $\|\frac{1}{n}\sum_{k=1}^n T^k f\| = \mathcal{O}\big(1/\log n(\log\log n)^\delta\big)$ for some $\delta > \frac{1}{2}$; this is weaker than the general assumption in [AL, remark to Corollary 2.2] (for arbitrary contractions in Banach spaces), which requires $\delta > 1$.

3. For T a normal contraction, the equality $\|T^n f\| = \|T^{*n} f\|$ yields that the series in (ii) converges absolutely when

$$\sum_{n=1}^\infty \frac{\|T^n f\|^2\log n}{n} < \infty$$

(by separating the series in (ii) to summations on odd and even integers).

When T is self-adjoint non-negative definite, the converse implication also holds.

Proposition 3.4. *Let T be a normal contraction in H. If $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges, then*

$$\Big\|\frac{1}{n}\sum_{k=1}^n T^k f\Big\| = \mathcal{O}\Big(\frac{1}{\log n}\Big)\ . \tag{9}$$

Proof. For $z\in D_j - D_{j+1}$ we have $1-|z| \ge \frac{1}{j+1}$ or $|1-z| \ge |z|\sin\frac{\pi}{j+1} \ge \frac{2|z|}{j+1}$, so

$$|\sum_{k=1}^n z^k| \le \sum_{k=0}^\infty |z|^k = \frac{1}{1-|z|} \le j+1 \ \text{ or } \ |\sum_{k=1}^n z^k| \le \frac{2|z|}{|1-z|} \le j+1\,.$$

For $n\ge 2$ we obtain

$$\Big\|\sum_{k=1}^n T^k f\Big\|^2 = \int_D |\sum_{k=1}^n z^k|^2\sigma_f(dz) =$$

$$\int_{D_n} |\sum_{k=1}^n z^k|^2\sigma_f(dz) + \sum_{j=1}^{n-1}\int_{D_j-D_{j+1}} |\sum_{k=1}^n z^k|^2\sigma_f(dz) \le$$

$$n^2\sigma_f(D_n) + \sum_{j=1}^{n-1}(j+1)^2(\sigma_f(D_j) - \sigma_f(D_{j+1})) \le$$

$$n^2\sigma_f(D_n) + \sum_{j=2}^{n-1}\sigma_f(D_j)((j+1)^2 - j^2) - n^2\sigma_f(D_n) + 4\sigma_f(D_1)$$

Hence, for $n \ge 2$, we have

$$\Big\| \sum_{k=1}^{n} T^k f \Big\|^2 \le 4\sum_{j=1}^{n-1} j\sigma_f(D_j)\,. \tag{10}$$

Since for $z \in D_n$ we have $|1-z| \le \frac{4}{n}$, condition (iv) of Theorem 3.1 yields $\sup_n \sigma_f(D_n)\log^2 n < \infty$. Using (10) we obtain

$$\Big\| \sum_{k=1}^{n} T^k f \Big\|^2 \le 4\sum_{j=1}^{n-1} j\sigma_f(D_j) \le K\sum_{j=1}^{n-1}\frac{j}{\log^2 j} \sim K'\frac{n^2}{\log^2 n}.$$

□

Remarks. 1. For unitary operators the proposition is proved in [AL].

2. Assani [A] has constructed a unitary operator T, induced on L_2 by an ergodic probability preserving transformation, and a function f satisfying (9) for which $\sum_{k=1}^{n}\frac{T^k f}{k}$ does not converge.

Theorem 3.5. *Let T be a normal contraction on H and let $0 \ne f \in H$. The following assertions are equivalent*

(i): $\displaystyle\sum_{n=1}^{\infty}\frac{T^n f}{n}$ *converges strongly*

(ii): $\displaystyle\lim_{\alpha\to 0^+}\sum_{n=1}^{\infty}\frac{T^n f}{n^{1+\alpha}}$ *converges strongly*

(iii): $\displaystyle\lim_{\alpha\to 0^+}\sum_{n=1}^{\infty}\frac{T^n f}{n^{1+\alpha}}$ *converges weakly*

(iv): $\sup_{0<\alpha<1/2}\Big\| \sum_{j=1}^{\infty}\frac{T^j f}{j^{\alpha+1}}\Big\| < \infty$.

If either condition holds, then $f \in \overline{(I-T)H}$ and

$$\lim_{\alpha\to 0^+}\sum_{n=1}^{\infty}\frac{T^n f}{n^{1+\alpha}} = \sum_{n=1}^{\infty}\frac{T^n f}{n} \quad \textit{strongly.}$$

Proof. $(i) \Rightarrow (ii)$ follows from Corollary 2.4. Clearly, $(ii) \Rightarrow (iii)$. The uniform boundedness principle yields $(iii) \Rightarrow (iv)$.

Now we prove $(iv) \Rightarrow (i)$. By (iv) we have

$$\sup_{0<\alpha<1/2}\Big\| \sum_{j=1}^{\infty}\frac{T^j f}{j^{\alpha+1}}\Big\| \le M < \infty,$$

so by the spectral theorem we have

$$\sup_{0<\alpha<1/2} \int_{\tilde{D}} \left(\Re\Big\{ \sum_{k=1}^{\infty} \frac{z^k}{k^{1+\alpha}} \Big\} \right)^2 \sigma_f(dz) \le M^2 < \infty.$$

Corollary 2.3 and Fatou's lemma yield

$$\int_{\tilde{D}} \log^2 |1-z| \sigma_f(dz) =$$

$$\int_{\tilde{D}} \left(\Re\Big\{ \sum_{k=1}^{\infty} \frac{z^k}{k} \Big\} \right)^2 \sigma_f(dz) = \int_{\tilde{D}} \liminf_{\alpha\to 0^+} \left(\Re\Big\{ \sum_{k=1}^{\infty} \frac{z^k}{k^{1+\alpha}} \Big\} \right)^2 \sigma_f(dz) \le M^2 < \infty\,.$$

This proves (i) via Theorem 3.1.

Clearly if either condition holds, then $f \in \overline{(I-T)H}$. The last assertion follows from Corollary 2.4. □

Theorem 3.6. *Let T be a normal contraction or an isometry on H with $\overline{(I-T)H} = H$. For $0 \ne f \in H$ the following assertions are equivalent:*

(i): $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$ *converges strongly*

(ii): $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$ *converges weakly*

(iii): $\displaystyle\sum_{n=1}^{\infty} \frac{\|\sum_{k=1}^{n} T^k f\|^2 \log n}{n^3} < \infty\,.$

(iv): $\displaystyle\sum_{n=1}^{\infty} \frac{\langle T^n f, f\rangle \log n}{n}$ *converges.*

(v): f is in the domain of G, the infinitesimal generator of the semi-group $\{(I-T)^r : r \ge 0\}$.

If either condition holds, then $Gf = -\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$.

Proof. Assume first that T is a normal contraction. We already know by Theorem 3.1 and Theorem 3.3 that the first four conditions are equivalent. By Theorem 3.5 (i) is equivalent to the convergence of $\lim\limits_{\alpha\to 0^+} \sum_{n=1}^{\infty} \frac{T^n f}{n^{1+\alpha}}$ and by Corollary 4.5 in [AL] this last convergence is equivalent to (v).

When either condition holds, we apply [AL, Proposition 4.1] (or [DL, Theorem 2.22(ii)]) to obtain $Gf = -\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$.

Assume now that T is an isometry, and let U be its unitary dilation (on a larger space H_1). By the construction, $T^n f = EU^n f$ for $f \in H$ and $n > 0$, where E is the orthogonal projection from H_1 onto H, and since T is an isometry we have $T^n f = U^n f$. An application of Theorem 3.5 to U yields that it is in fact valid also for the isometry T. Similarly, by Theorem 3.1 and Theorem 3.3, the first four conditions of the theorem are equivalent for the isometry T. Now the first part of the proof yields the result. $\square$

Corollary 3.7. *Let T be a contraction on H such that T^* is an isometry. Then* $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ *converges weakly if and only if it converges strongly.*

Proof. We may restrict ourselves to $\overline{(I-T)H} = \overline{(I-T^*)H}$. If $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ converges weakly, then by Proposition 1.2 and the previous theorem applied to T^* we have strong convergence. $\square$

Remark. Similarly, if T^* is an isometry, $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{\langle T^n f, f\rangle \log n}{n}$ converges.

Corollary 3.8. *Let T be a normal contraction on H such that $\overline{(I-T)H} = H$ (so also $\overline{(I-T^*)H} = H$). Then the infinitesimal generators of $\{(I-T)^r : r \geq 0\}$ and $\{(I-T^*)^r : r \geq 0\}$ have the same domain of definition.*

Proof. Use Proposition 1.2 and the characterization of the domain of the generator given by Theorem 3.6. $\square$

4. The ergodic Hilbert transform for general contractions

For any contraction T on H, weak convergence of $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ implies convergence of the series

$$\sum_{k=1}^{\infty} \frac{\langle T^k f, f\rangle}{k} \tag{11}$$

Convergence of (11) yields $\|\frac{1}{n}\sum_{k=1}^{n} T^k f\| \to 0$, by Kronecker's lemma and the next proposition.

Proposition 4.1. *Let T be a contraction on a Hilbert space H and $f \in H$. If* $\frac{1}{n}\sum_{k=1}^{n}\langle T^k f, f\rangle \to 0$, *then* $\|\frac{1}{n}\sum_{k=1}^{n} T^k f\| \to 0$.

Proof. By the mean ergodic theorem, $\frac{1}{n}\sum_{k=1}^{n} T^k f$ converges to some $g \in H$, and $Tg = g$, so also $T^*g = g$ [RN, §144]. Hence the assumption yields

$$\|g\|^2 = \lim_{n\to\infty} \langle g, \frac{1}{n}\sum_{k=1}^{n} T^k f\rangle = \lim_{n\to\infty} \langle \frac{1}{n}\sum_{k=1}^{n} T^{*k} g, f\rangle = \langle g, f\rangle = 0.$$

$\square$

Remark. Foguel [F] proved that if $\langle T^n f, f\rangle \to 0$, then $T^n f \to 0$ weakly. As mentioned above, for the averages weak and strong convergence are the same.

Since the condition of Theorem 3.3(ii) is stronger than convergence of the series (11), the latter convergence is not expected to imply convergence of $\sum_{k=1}^\infty \frac{T^k f}{k}$. This will be exhibited in the examples below.

Theorem 4.2. *Let T be a contraction on a complex Hilbert space H and $f \in H$. If* $\displaystyle\sum_{k=1}^\infty \frac{\langle T^k f, f\rangle \log k}{k}$ *converges, then* $\displaystyle\sum_{k=1}^\infty \frac{T^k f}{k}$ *converges strongly.*

Proof. For T unitary the assertion follows from theorem 3.6.

Now let T be a contraction, and let U be its unitary dilation, defined on a larger space H_1. Since $\langle U^n f, f\rangle = \langle T^n f, f\rangle$ for $f \in H$, by Theorem 3.6 applied to U the assumption yields strong convergence of $\sum_{k=1}^\infty \frac{U^k f}{k}$, and continuity of the projection from H_1 onto H yields convergence of $\sum_{k=1}^\infty \frac{T^k f}{k}$. □

When $H = L_2(\mathcal{S}, \Sigma, m)$ of a σ-finite measure space, it is of interest to investigate also the almost everywhere (a.e.) convergence of the one-sided ergodic Hilbert transform of a contraction T. For T unitary there are extensive studies by Gaposhkin ([G2], [G3], [G4]). Gaposhkin assumes m to be a probability, but this is not a restriction, since (e.g., [Kr, p. 189]) when m is not finite we take an equivalent probability m' and the map $Vf := f/\sqrt{\psi}$, with $\psi = dm'/dm$, is an order-preserving linear isometry of $L_2(m)$ onto $L_2(m')$ which preserves also pointwise convergence.

Theorem 4.3. *Let T be a contraction of $L_2(\mathcal{S}, m)$ of a σ-finite measure space, and $f \in L_2(m)$. If*

$$\sum_{n=1}^\infty \frac{\langle T^n f, f\rangle \log n (\log\log\log n)^2}{n} \quad \textit{converges} \tag{12}$$

then $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges a.e. (and in norm).

Proof. The norm convergence follows from Theorem 4.2.

If T is unitary, this is Theorem 3a of [G4] (see also [G2, Theorem 7]).

Now let T be a contraction of $L_2(\mathcal{S}, m)$. We may assume that m is a probability. We will use Schäffer's construction of the unitary dilation [Sc]: Let $(\mathcal{S}_n, m_n)$ be disjoint copies $(\mathcal{S}, m)$, put $\Omega = \bigcup_{n\in\mathbb{Z}} \mathcal{S}_n$ with the obvious σ-algebra, and define $\mu(A) = \sum_{n\in\mathbb{Z}} m_n(A \cap \mathcal{S}_n)$. Then $L_2(\Omega, \mu) = \sum_n \oplus L_2(\mathcal{S}_n, m_n)$, and the unitary dilation U is defined on $L_2(\mu)$. The orthogonal projection on $L_2(\mathcal{S}_0, m_0)$ is in fact multiplication by the indicator function $1_{\mathcal{S}_0}$. If (12) is satisfied, then also $\tilde f$, the extension by zero to Ω of f on $\mathcal{S}_0$, satisfies (12) with T replaced by U. Now we apply Gaposhkin's result to obtain μ-a.e. convergence of $\sum_{n=1}^\infty \frac{U^n \tilde f}{n}$ on Ω, which yields m_0-a.e. convergence on $\mathcal{S}_0$ of $\sum_{n=1}^\infty \frac{T^n f}{n}$. □

Note that there are contractions on L_2 for which even the averages may fail to converge a.e. ([B, p. 128]; for examples of unitary operators see [G1] or [Kr, p. 191]). The proof of [Kr, Lemma 5.2.1] can be adapted to show that if T is power-bounded on L_2 and $f \in (I-T)^\alpha L_2$ with $\alpha > \frac{1}{2}$, then $\frac{1}{n}\sum_{k=1}^n T^k f \to 0$ a.e. The next proposition shows that for contractions we can do better.

Proposition 4.4. *Let T be a contraction of $L_2(\mathcal{S}, m)$ of a σ-finite measure space, and $f \in L_2(m)$. If the series* (11) *converges, in particular if $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges weakly, then $\frac{1}{n}\sum_{k=1}^n T^k f \to 0$ a.e.*

Proof. For T unitary, this is due to Gaposhkin [G1, Theorem 2]. In the general case, we use the unitary dilation of [Sc] as in the previous proof. □

Theorem 4.5. *Let T be a contraction of $L_2(\mathcal{S}, m)$ of a σ-finite measure space, and $f \in L_2(m)$. If*

$$\sum_{n=1}^\infty \frac{\|\frac{1}{n}\sum_{k=1}^n T^k f\|}{n} \quad \textit{converges} \tag{13}$$

then $\sum_{k=1}^\infty \frac{T^k f}{k}$ converges a.e. (and in norm).

Proof. We may assume that m is a probability. Denote $S_n f := \sum_{k=1}^n T^k f$. The mean ergodic theorem, (13), and the identity

$$\sum_{k=1}^n \frac{T^k f}{k} = \frac{S_n f}{n} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} S_k f$$

yield that $\sum_{k=1}^n \frac{T^k f}{k}$ converges strongly in L_2. By Proposition 4.4, $\frac{1}{n} S_n f \to 0$ a.e.; since $\|S_n f\|_1 \le \|S_n f\|_2$ in a probability space, by (13) and Beppo Levi's theorem $\sum_{k=1}^\infty \frac{1}{k(k+1)} S_k f$ converges a.e. Thus $\sum_{k=1}^n \frac{T^k f}{k}$ converges a.e. □

Remark. The previous theorem is true also for isometries or *order-preserving* contractions of L_p, $1 < p < \infty$. The proof is the same, except that instead of Proposition 4.4, we use Kan's pointwise ergodic theorem [Kn, Corollary 5.1] for isometries of L_p, $p \ne 2$, and Akcoglu's pointwise ergodic theorem (e.g., [Kr, p. 190]) for order-preserving contractions.

For T unitary on L_2, Gaposhkin [G4] proved that

$$\sum_{n=1}^\infty \frac{\|\frac{1}{n}\sum_{k=1}^n T^k f\|^2 \log n (\log\log\log n)^2}{n} < \infty \tag{14}$$

is sufficient for a.e. (and norm) convergence of the ergodic Hilbert transform, and showed that (12) implies (14). We do not know if this latter condition implies a.e. convergence of the transform for general contractions on L_2. We even do not know if $\left\|\frac{1}{n}\sum_{k=1}^n T^k f\right\| = \mathcal{O}\left(\frac{1}{\log n (\log\log n)^\delta}\right)$ for some $\delta > \frac{1}{2}$, which implies (14), is sufficient for a.e. convergence of the transform for general contractions in L_2.

Example 1. *U unitary on L_2, with $f \in L_2$ satisfying* (14)*, but not* (13)*.*

Put $H = L_2([0,1), dt)$ and for $h \in H$ define $Uh(t) = \mathrm{e}^{2\pi i t} h(t)$ (the operator induced by the shift). Denote $\log_2 x := \log(\log x)$, and for $k \geq 4 > \mathrm{e} + 1$ put $c_k := (\sqrt{k \log^3 k} \log_2 k)^{-1}$, $c_k = 0$ for $k < 4$. Let $f := \sum_{k=1}^{\infty} c_k \mathrm{e}^{2\pi i k t}$. Clearly,

$$\Big\| \sum_{j=1}^{n} U^j f \Big\|^2 = \Big\| \sum_{k=1}^{\infty} \mathrm{e}^{2\pi i k t} \sum_{j=1}^{n} c_{k-j} \Big\|^2 = \sum_{l=0}^{\infty} \sum_{k=ln+1}^{(l+1)n} \big(\sum_{j=1}^{n} c_{k-j} \big)^2 =$$

$$\sum_{k=1}^{n} \big(\sum_{j=1}^{n} c_{k-j} \big)^2 + \sum_{k=n+1}^{2n} \big(\sum_{j=1}^{n} c_{k-j} \big)^2 + \sum_{l=2}^{\infty} \sum_{k=ln+1}^{(l+1)n} \big(\sum_{j=1}^{n} c_{k-j} \big)^2 = \Sigma_I + \Sigma_{II} + \Sigma_{III}.$$

We start with Σ_{III}. For $n \geq 4$, the monotonicity of $\{c_k\}_{k \geq 4}$ yields

$$\Sigma_{III} \leq \sum_{l=2}^{\infty} \sum_{k=ln+1}^{(l+1)n} (n \cdot c_{(l-1)n})^2 = \sum_{l=2}^{\infty} n^3 (c_{(l-1)n})^2 =$$

$$n^2 \sum_{l=1}^{\infty} \frac{1}{l \log^3(ln) \log_2^2(ln)} \leq n^2 C \int_{n-1}^{\infty} \frac{dx}{x \log^3 x \log_2^2 x} \leq \frac{C_1 n^2}{(\log n \log_2 n)^2}.$$

For Σ_{II} we have, using monotonicity,

$$\Sigma_{II} \leq n \Big(\sum_{k=4}^{n+3} c_k \Big)^2 \leq nC \Big(\int_3^{n+3} \frac{dx}{\sqrt{x \log^3 x} \log_2 x} \Big)^2 \leq \frac{C_2 n^2}{\log^3 n (\log_2 n)^2}.$$

The same estimate holds for Σ_I, since $\Sigma_I \leq n \big(\sum_{k=4}^{n-1} c_k \big)^2 \leq n \Big(\sum_{k=4}^{n+3} c_k \Big)^2$.

On the other hand, $\big\| \sum_{j=1}^{n} U^j f \big\|^2 \geq \Sigma_{III} \geq \frac{C'' n^2}{(\log n \log_2 n)^2}$, by a similar computation. Hence

$$\frac{C'' n^2}{(\log n \log_2 n)^2} \leq \Big\| \sum_{j=1}^{n} U^j f \Big\|^2 \leq \frac{C' n^2}{(\log n \log_2 n)^2}$$

and the assertion clearly follows. In fact, also (12) holds, since for $n \geq 4$

$$\langle U^n f, f \rangle = \sum_{l=0}^{\infty} \sum_{k=ln+1}^{(l+1)n} c_k c_{k+n} \leq c_n \sum_{k=1}^{n} c_k + \sum_{l=1}^{\infty} n (c_{ln})^2.$$

Definition. A *Dunford-Schwartz operator* is a contraction T of $L_1(\mathcal{S}, m)$ which is also a contraction of L_∞ (if m is σ-finite infinite, T is extended to L_∞ from $L_1 \cap L_\infty$). By the Riesz-Thorin theorem (see also [Kr, p. 65]), T is also (extendable to) a contraction of $L_2(\mathcal{S}, m)$.

Measure preserving transformations, and more generally Markov operators with a subinvariant measure, induce order-preserving Dunford-Schwartz operators. If T is a Dunford-Schwartz operator, then $\frac{1}{n} \sum_{k=1}^{n} T^k f$ converges a.e. for any $f \in L_p$, $1 \leq p < \infty$ (e.g., [DuS, p. 675]).

Example 2. *A self-adjoint Dunford-Schwartz operator T, $f \in L_2$ with* (11) *convergent,* $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ *converges a.e. but not weakly in L_2.*

Let m be the finite measure on the Borel sets of $[0,1)$ with density $\frac{dm}{dt} = 1/(1-t)|\log(1-t)|^3$ for $t > \frac{1}{2}$ and $\frac{dm}{dt} = c$ for $0 \leq t \leq \frac{1}{2}$. On $L_1([0,1),m)$ define the operator $Th(t) = th(t)$, which is obviously Dunford-Schwartz. Since $\langle T^n h, h\rangle = \int t^n |h|^2 dm$ for $h \in H = L_2([0,1),m)$, the function $f \equiv 1$ has the spectral measure $\sigma_f = m$, and by Beppo Levi

$$\sum_{k=1}^{\infty} \frac{\langle T^k f, f\rangle}{k} = \sum_{k=1}^{\infty} \frac{\int_{[0,1)} t^k dm}{k} = \int_{[0,1)} |\log(1-t)| dm < \infty.$$

However, the one-sided ergodic Hilbert transform does not converge weakly by Theorem 3.1, since

$$\int_{[0,1)} |\log(1-t)|^2 dm \geq \int_{1/2}^{1} \frac{1}{(1-t)\,|\log(1-t)|} dt = \infty \ .$$

Remark. In this example $H = \overline{(I-T)H}$, and $\sum_{k=1}^{\infty} \frac{T^k h}{k}$ converges a.e. for *every* $h \in H$, although $(I-T)H$ is not closed. Note that T is order-preserving.

Example 3. *U unitary on L_2, $f \in L_2$ with* $\sum_{k=1}^{\infty} \frac{U^k f}{k}$ *convergent a.e., but not weakly in L_2.*

Let T be the operator on $L_2([0,1),m)$ described in the previous example, and let U be the unitary dilation constructed by Schäffer [Sc], which is defined on $L_2(\mathbb{R},\mu)$ (see the proof of Theorem 4.3). For f of the previous example we define $\tilde{f}$ on $\mathbb{R}$ by $\tilde{f}(x) = f(x)$ for $0 \leq x < 1$ and $\tilde{f}(x) = 0$ otherwise. Schäffer's definition of U yields that $U^n \tilde{f}$ is zero on $[1,\infty)$ and on $(-\infty,-n)$, on the interval $[-j,-j+1)$ $(1 \leq j \leq n)$ we have $U^n \tilde{f}(x) = \sqrt{1-(x+j)^2}(x+j)^{n-j} f(x+j)$, and $U^n \tilde{f}(x) = x^n f(x)$ on $[0,1)$. This yields that $\sum_{n=1}^{\infty} \frac{U^n \tilde{f}}{n}$ converges a.e. However, L_2-weak convergence of $\sum_{k=1}^{\infty} \frac{U^k \tilde{f}}{k}$ would imply that of $\sum_{k=1}^{\infty} \frac{T^k f}{k}$, a contradiction.

Note that since $\langle U^n \tilde{f}, \tilde{f}\rangle = \langle T^n f, f\rangle$, the series $\sum_{n=1}^{\infty} \frac{\langle U^n \tilde{f}, \tilde{f}\rangle}{k}$ converges.

Remarks. 1. In the example convergence a.e. of the transform does not imply weak (norm) convergence, which shows that for unitary operators, Gaposhkin's sufficient condition (14) for a.e. convergence of the transform is not necessary; neither is condition (13), as either condition implies also norm convergence.

2. Gaposhkin [G4, pp. 253-254] constructed an example of U unitary on $L_2[0,1]$ and a function f such that $\sum_{k=1}^{\infty} \frac{U^k f}{k}$ converges in norm, but not a.e. Thus, for unitary operators on L_2, a.e. and norm convergence of the series are not comparable in general. It is worth to mention that in his example the weighted averages do converge a.e. to 0, since the sufficient conditions of Theorem 3A in [G3] are satisfied, and also the *two sided Hilbert transform* converges a.e.

3. Almost everywhere convergence of the transform for every function (for which the averages converge to 0), as in Example 2, cannot occur for U induced

by an invertible ergodic measure preserving transformation of a separable non-atomic probability space: Kakutani and Petersen [KP] proved that there always exists a bounded function of zero integral for which the one-sided ergodic Hilbert transform is a.e. non-convergent; for references to earlier related results see [AL].

Let T be a contraction in H such that $(I-T)H$ is not closed. By Theorem 1.1 there exists $f \in \overline{(I-T)H}$ such that $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ does not converge. A natural question (raised in the context of Fourier series – see [Z, Theorem V(8.12)] with remarks and references on [Z, p. 380]), is the convergence of the one-sided ergodic Hilbert transform for almost every random choice of signs, i.e. the convergence of $\sum_{n=1}^{\infty} \pm \frac{T^n f}{n}$ for every $f \in \overline{(I-T)H}$. This is made precise in the following theorem, when we take for $\{\xi_n\}$ the Rademacher functions.

Theorem 4.6. *Let $\{\xi_n\}$ be independent identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbb{E}(|\xi_1| \log^+ |\xi_1|) < \infty$ and $\mathbb{E}\xi_1 = 0$. Then there exists a set $\Omega_1 \in \mathcal{F}$ with $\mathbf{P}(\Omega_1) = 1$ such that when $\omega \in \Omega_1$, for every contraction T on a Hilbert space H and any $f \in H$ the "modulated" transform $\sum_{n=1}^{\infty} \xi_n(\omega) \frac{T^n f}{n}$ converges in norm.*

Proof. Cuzick and Lai [CuLa, Theorem 1(iv)] proved that for $\{\xi_n\}$ as in the theorem there exists $\Omega_1 \in \mathcal{F}$ with $\mathbf{P}(\Omega_1) = 1$ such that for $\omega \in \Omega_1$ the "random Fourier series"

$$\sum_{n=1}^{\infty} \frac{\xi_n(\omega)}{n} \lambda^n$$

converges uniformly for complex $|\lambda| = 1$. We fix $\omega \in \Omega_1$.

For a unitary operator U on H and $f \in H$ the spectral theorem yields

$$\Big\| \sum_{n=j}^{k} \frac{\xi_n(\omega) U^n f}{n} \Big\|^2 = \int_{\{|\lambda|=1\}} \Big| \sum_{n=j}^{k} \frac{\xi_n(\omega)\lambda^n}{n} \Big|^2 d\sigma_f \le \|f\|^2 \sup_{|\lambda|=1} \Big| \sum_{n=j}^{k} \frac{\xi_n(\omega)\lambda^n}{n} \Big|^2$$

which converges to 0 as $k > j \to \infty$ by the choice of ω. Hence $\sum_{n=1}^{\infty} \xi_n(\omega) \frac{U^n f}{n}$ converges in norm.

For T a contraction on H let U be its unitary dilation on H_1 containing H. Then

$$\Big\| \sum_{n=j}^{k} \frac{\xi_n(\omega) T^n f}{n} \Big\|^2 \le \Big\| \sum_{n=j}^{k} \frac{\xi_n(\omega) U^n f}{n} \Big\|^2 \underset{k>j\to\infty}{\longrightarrow} 0$$

so $\sum_{n=1}^{\infty} \xi_n(\omega) \frac{T^n f}{n}$ converges in norm. □

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