On random almost periodic series and random ergodic theory

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Abstract

In this paper we obtain different types of random ergodic theorems for dynamical systems or continuous semi-flows. These results recover and extend previous works on dynamical systems and are completely new in case of semi-flows. The proofs are based on uniform estimates on random almost periodic polynomials that we obtained recently [8] and on an improvement of a tool introduced by Talagrand [28] and further developed by Fernique [14]. In the course, we partially recover results of Marcus and Pisier [18] on almost sure uniform convergence of random almost periodic series.

1 Introduction

In the last decades, many authors (see, e.g., Rosenblatt [26], Assani [1], Schneider [27], Weber [29], Boukhari and Weber [4], Cohen and Lin [5] [6], Assani [2], Cuny [9] and Cohen and Cuny [8]) worked on ergodic theorems with random modulation (sometimes called "randomly weighted ergodic theorems"). One important matter may be formulated as follows: Given a sequence of random variables $\{X_n\}$ on a probability space $(\Omega, \mathbf{P})$, find a measurable set $\Omega^* \subset \Omega$ with $\mathbf{P}(\Omega^*) = 1$, such that for any $\omega \in \Omega^*$ the sequence $a_n := X_n(\omega)$ is a universally good sequence for the weighted ergodic theorem for all functions in some specified class. More precisely, one wants that for any measure preserving transformation T on a probability space (Y, Σ, π), and any function f on Y with a certain integrability property (e.g., $f \in L_p$), the sequence $\frac{1}{n}\sum_1^n a_k f \circ T^k$ converges π-a.s.

Another important matter may be formulated as follows: Let $\{a_n\}$ be a fixed sequence of complex numbers. Given a sequence $\{\theta_n\}$ on a probability space $(\Omega, \mathbf{P})$, with values in $\mathbf{N}$ (a sequence of "random times"), find a measurable set $\Omega^* \subset \Omega$ with $\mathbf{P}(\Omega^*) = 1$, such that for any $\omega \in \Omega^*$ the sequence $p_n := \theta_n(\omega)$ is a universally good sequence of powers for the weighted ergodic theorem for all functions in some specified class. More precisely, one wants that for any measure preserving transformation T on a probability space (Y, Σ, π), and any function f on Y with a certain integrability property, the sequence $\frac{1}{n}\sum_1^n a_k f \circ T^{p_k}$ converges π-a.s.

This paper is devoted to the study of these two different types of random ergodic theorems, in a universal and more general setting. Moreover, we obtain results in the context of continuous d-parameter semi-flows or commuting families of measure preserving transformations (which induce continuous representations by L_2-isometries of $(\mathbf{R}^+)^d$ or $\mathbf{N}^d$, respectively). Let us describe briefly some of our main results.

Let μ_f be the spectral measure (on $\mathbf{R}^d$) of an L_2 function f associated to a representation of $(\mathbf{R}^+)^d$ by isometries (see §4 for more details). For a vector $\mathbf{t} := (t^{(1)}, \ldots, t^{(d)}) \in \mathbf{R}^d$ we write $|\mathbf{t}| = \max\{|t^{(1)}|, \ldots, |t^{(d)}|\}$. We write $\langle \mathbf{t}, \mathbf{s} \rangle$ for the inner product in $\mathbf{R}^d$.

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In the case of continuous semi-flows we obtain the following (see §4):

Theorem A. *Let $\{X_n\} \subset L_p(\Omega, \mathbf{P})$, $1 < p \le 2$, be a sequence of complex valued centered independent random variables, and let $\mathbf{s}_n = (s_n^{(1)}, \dots, s_n^{(d)})$ be a sequence in $(\mathbf{R}^+)^d$. Assume that either of the following conditions hold:*

$$\sum_{n=2}^{\infty} \|X_n\|_2^2 (\log n)^2 \log(n \vee \max_{1\le k\le n} |\mathbf{s}_k|) < \infty \qquad \text{and} \quad p = 2,$$

$$\sum_{n=2}^{\infty} \|X_n\|_p^p (\log(n \vee \max_{1\le k\le n} |\mathbf{s}_k|))^{p-1+\delta} < \infty \quad \text{for some} \quad \delta > 0 \quad \text{and} \quad 1 < p < 2$$

Then there exists a measurable set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any probability space (Y, Σ, π), any continuous semi-flow $\{T_{\mathbf{s}}\}_{\mathbf{s}\in(\mathbf{R}^+)^d}$ on $L_2(Y, \Sigma, \pi)$, and any $f \in L_2(\pi)$ such that $\int_{\mathbf{R}^d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t}) < \infty$, the series $\sum_{n=1}^{\infty} X_n(\omega) T_{\mathbf{s}_n} f$ converges a.s., and $\| \sup_{n\ge1} |\sum_{k=1}^{n} X_k(\omega) T_{\mathbf{s}_k} f|\|_2 \le C\big(\int_{\mathbf{R}^d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t})\big)^{1/2}$, for a constant $C = C_\omega > 0$ which depends only on $\{X_n(\omega)\}$ and $\{\mathbf{s}_n\}$.

For a commuting family of measure preserving transformations, we obtain in §4 a more general result (see [8] for $p = 2$):

Theorem B. *Let $\{X_n\} \subset L_p(\Omega, \mathbf{P})$, $1 < p \le 2$, be a sequence of complex centered independent random variables, and let $\mathbf{s}_n = (s_n^{(1)}, \dots, s_n^{(d)})$ be a sequence of vectors in $\mathbf{N}^d$. Assume either of the conditions of* Theorem *A hold. Then there exists a measurable set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any probability space (Y, Σ, π), any pairwise commuting measure preserving transformations $\tau_1, \dots, \tau_d$ on (Y, π), and any $f \in L_r(Y, \pi)$, $1 < r \le 2$, the series*

$$\sum_{n=1}^{\infty} \frac{X_n(\omega) f \circ \tau_1^{s_n^{(1)}} \circ \cdots \circ \tau_d^{s_n^{(d)}}}{n^{(2-r)(p-1)/pr}}$$

converges π-a.s. Moreover, for every $\omega \in \Omega^*$, there exists $K_\omega > 0$ such that

$$\Big\| \sup_{N\ge1} \Big| \sum_{n=1}^{N} \frac{X_n(\omega) f \circ \tau_1^{s_n^{(1)}} \circ \cdots \circ \tau_d^{s_n^{(d)}}}{n^{(2-r)(p-1)/pr}} \Big| \Big\|_r \le K_\omega.$$

For an ergodic theorem with "random times" we have the following (see §4):

Theorem C. *Let $\{\boldsymbol{\theta}_n\}$ be a sequence of independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, with values in $(\mathbf{R}^+)^d$. Let Φ be some positive non-decreasing function on $\mathbf{R}^+$, such that there exists $\eta > 0$, for which $\Phi(x) \ge x^\eta$ for every $x \ge 0$. Assume that there exists $\delta > 0$, such that $\sum_{n=1}^{\infty} \mathbf{P}[|\boldsymbol{\theta}_n| > \Phi(n)^\delta] < \infty$. Let $\{a_n\}$ be a sequence of complex numbers satisfying*

$$\sum_{n=1}^{\infty} |a_n|^2 (\log n)^2 \log \Phi(n) < \infty\,.$$

Then there exists a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any probability space (Y, Σ, π), any representation $\{V_{\mathbf{s}}\}_{s\in(\mathbf{R}^+)^d}$ on $L_2(\pi)$ by isometries, and any $f \in L_2(\pi)$ with $\int_{H_d} \log(2 + |\mathbf{t}|) d\mu_f(\mathbf{t}) < \infty$, the series

$$\sum_{n=1}^{\infty} a_n (V_{\boldsymbol{\theta}_n}(f) - \mathbf{E}[V_{\boldsymbol{\theta}_n}](f))$$

converges π-a.s.

As far as we know, Theorem A is the first one of this kind concerning semi-flows.

For previous results similar to Theorem B, one can refer to Assani [1], Boukhari and Weber [4], Cohen and Lin [5], Durand and Schneider [12], and Cohen and Cuny [8].

Although a centered i.i.d. sequence $\{Y_n\} \subset L_p(\Omega, \mathbf{P})$ does not satisfy either condition of Theorem A, in the case of polynomial growth of $\{|s_n|^*\}$ Theorem B can be applied to $X_n = Y_n/n^\alpha$ for some $\alpha > 1/p$, to obtain π-a.s. convergence of

$$\sum_{n=1}^{\infty} \frac{Y_n(\omega) f \circ \tau_1^{s_n^{(1)}} \circ \cdots \circ \tau_d^{s_n^{(d)}}}{n^\beta}$$

for $\beta = \alpha + (2-r)(p-1)/pr$. Applying Kronecker's lemma we obtain a randomly weighted ergodic theorem with rate $o(1/n^{1-\beta})$.

We also obtain, as a consequence of Theorem C, a result for the special flow generated by the rotations of the circle. Let m be the Lebesgue measure on $[0,1)$, and for every $t \in \mathbf{R}$ define the rotation $\tau_t : [0,1) \mapsto [0,1)$ by $\tau_t(x) = x + t$ modulo 1. For any f in $L_2([0,1), m)$, we write its Fourier expansion $f(x) = \sum_{n \in \mathbf{Z}} \hat{f}(n) e^{2in\pi x}$. We have (see Corollary 4.13):

Theorem D. *Let $\{\theta_n\}$ be a sequence of i.i.d. random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, with values in $[0,1)$. There exists a measurable set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any $f \in L_2([0,1), m)$, such that $\sum_{n=2}^{\infty} |\hat{f}(n)|^2 \log n < \infty$, the sequence*

$$\frac{1}{n} \sum_{k=1}^{n} f \circ \tau_{\theta_k(\omega)} \qquad \textit{converges } m\textit{-a.s.}$$

A. Bellow asked whether the above convergence holds for every f in $L_2([0,1), m)$ in the special case that the $\{\theta_n\}$ are uniformly distributed on $[0,1)$. It was proved by Bergelson, Boshernitzan, and Bourgain [3] that in general, this convergence does not hold, even for bounded f. Theorem D gives a simple sufficient condition on f to ensure the above convergence.

The methods of proof (which come from general convergence results, see also [6] and [8]) are somewhat classical. We first prove convergence results along a suitable subsequence and then show that we can control the maximal function along blocks generated by the given subsequence (see Theorem 3.1 and Proposition 4.7). In order to prove such convergences, by mean of the spectral theorem and the unitary dilation, it suffices to control random (d-dimensional) almost periodic polynomials uniformly on compact sets (see Proposition 4.2). Such a control (estimation) was obtained in [8, Theorem 1.1]. For previous works, see [29] and Durand and Schneider [12].

Such an approach to proving convergence of series has already been used previously in [6] and [8]. The proof of Theorem A, for a sequence of weights $\{X_n\}$ in L_p, $1 < p < 2$, needs the development of a tool of Talagrand-Fernique [28], [14]. The proof of Theorem B relies on a version of Theorem A for commuting families of measure preserving transformations and on Stein's complex interpolation theorem. The proof of Theorem C is based on a generalization of our estimate, given in [8, Theorem 1.1], to the case of random powers (see Theorem 4.10).

One of the consequences of the Talagrand-Fernique tool mentioned above, is the following result (see [16] for $d = 1$ and $p = 2$) on a.s. uniform convergence of random almost periodic series.

Theorem E. *Let $\{X_n\} \subset L_p(\Omega, \mathbf{P})$, $1 < p \le 2$, be a sequence of centered independent complex valued random variables. Let $\{\boldsymbol{\gamma}_n\} \subset (\mathbf{R}^+)^d$ be such that each of the sequences $\{\gamma_n^{(i)}\}_{1 \le i \le d}$ is unbounded and non-decreasing. If*

$$\sum_{n=1}^{\infty} \frac{\big(\sum_{|\boldsymbol{\gamma}_k| \ge n} \mathbf{E}|X_k|^p\big)^{1/p}}{n(\log n)^{1/p}} < \infty\,,$$

then the series $\sum_{n=1}^{\infty} X_n \mathrm{e}^{i\langle \boldsymbol{\gamma}_n, \mathbf{t}\rangle}$ is a.s. uniformly convergent on any bounded set in $\mathbf{R}^d$.

We also obtain similar convergence results for non-integrable random variables by imposing a condition on their tails (see, e.g., Corollary 3.4 and Corollary 3.5). These results extend results of [14] in several directions, and also partially recover results of [17], [18] and [19]. The latter works used the method of metric entropy.

2 A Talagrand-Fernique tool

In this section we give further development of a tool which was introduced by Talagrand [28] and developed by Fernique [14].

Let $X \ge 0$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and define the function φ by

$$\varphi(\lambda) := \mathbf{E}[\frac{X^q}{\lambda^q} \wedge 1] = \frac{q}{\lambda^q} \int_0^{\lambda} t^{q-1} \mathbf{P}[X > t] dt = q \int_0^1 t^{q-1} \mathbf{P}[X > \lambda t] dt \qquad \forall \lambda > 0\,.$$

Clearly, $\varphi(\lambda)$ is continuous, non-increasing, and also satisfies $\lim_{\lambda \downarrow 0} \varphi(\lambda) = \mathbf{P}[X > 0]$ and $\lim_{\lambda \to \infty} \varphi(\lambda) = 0$.

Let $\{X_n\}$ be a sequence of complex random variables, for which $\sum_{n=1}^{\infty} \mathbf{P}[|X_n| > 0] = \infty$, and assume that there exist $1 < q \le 2$ and $\lambda_0 > 0$, such that

$$\sum_{n=1}^{\infty} \mathbf{E}[\frac{|X_n|^q}{\lambda_0^q} \wedge 1] < \infty\,. \tag{1}$$

Then, for every $l \ge 1$, the function, $\lambda \mapsto \sum_{n=l}^{\infty} \mathbf{E}[(\frac{|X_n|^q}{\lambda^q}) \wedge 1]$ is well defined on $(0, \infty)$, i.e., converges at any $\lambda > 0$. Moreover, it is continuous non-increasing, tends to infinity when λ decreases to 0 and to 0 when λ goes to infinity.

Hence for every unbounded non-decreasing sequence $\{A_n\}$ of positive numbers, we can define a *non-increasing* sequence $\{\lambda_m\} (= \{\lambda_m(q)\})$ of positive numbers enjoying the following property

$$2^m = \sum_{A_n \ge 2^m} \mathbf{E}[\frac{|X_n|^q}{\lambda_m^q(q)} \wedge 1] \qquad \forall m \ge 1. \tag{2}$$

When it will be clear from the context, we will omit the dependence in q.

Proposition 2.1. *Let $\{X_n\}$ be a sequence of complex random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, with $\sum_{n=1}^{\infty} \mathbf{P}[X_n \ne 0] = \infty$. Let $1 < q \le 2$ such that (1) is satisfied. Let $\{A_n\}$ be an unbounded non-decreasing sequence of positive numbers and $\{\lambda_n(q)\}$ be defined by (2). If for some $\alpha \ge 1$ we have $\sum_{m=1}^{\infty} 2^{m\alpha} \lambda_m(q) < \infty$, then*

$$\sum_{m=1}^{\infty} 2^{m(\alpha - 1/q)} \big(\sum_{A_n \ge 2^m} \mathbf{E}[|X_n|^q \mathbf{1}_{\{|X_n| \le \lambda_m\}}] \big)^{1/q} < \infty \tag{3}$$

and

$$\sum_{m=1}^{\infty} \sum_{2^m \le A_n < 2^{m+1}} \mathbf{E}[|X_n| \mathbf{1}_{\{\lambda_m < |X_n| \le \lambda_0\}}] < \infty \,. \tag{4}$$

In particular,

$$\sum_{m=1}^{\infty} 2^{m(\alpha - 1/q)} \Big(\sum_{A_n \ge 2^m} |X_n|^q \mathbf{1}_{\{|X_n| \le \lambda_m\}} \Big)^{1/q} < \infty \qquad \mathbf{P}\text{-}a.s. \tag{5}$$

and

$$\sum_{m=1}^{\infty} \sum_{2^m \le A_n < 2^{m+1}} |X_n| \mathbf{1}_{\{|X_n| > \lambda_m\}} < \infty. \qquad \mathbf{P}\text{-}a.s. \tag{6}$$

Proof. By definition of $\{\lambda_m\}$ we have

$$2^m = \sum_{A_n \ge 2^m} \frac{\mathbf{E}[|X_n|^q \mathbf{1}_{\{|X_n| \le \lambda_m\}}]}{\lambda_m^q} + \sum_{A_n \ge 2^m} \mathbf{P}[|X_n| > \lambda_m] \qquad \forall m \ge 1. \tag{7}$$

Hence

$$\Big(\sum_{A_n \ge 2^m} \mathbf{E}[|X_n|^q \mathbf{1}_{\{|X_n| \le \lambda_m\}}] \Big)^{1/q} \le \lambda_m 2^{m/q},$$

which proves the convergence of the series (3).

Since $\{\lambda_m\}$ is non-increasing, we have, by changing the order of summation and using (7) for the last inequality,

$$\begin{aligned}
\sum_{m=2}^{\infty} \sum_{2^m \le A_n < 2^{m+1}} \mathbf{E}[|X_n| \mathbf{1}_{\{\lambda_m < |X_n| \le \lambda_0\}} &\le \sum_{m=2}^{\infty} \sum_{2^m \le A_n < 2^{m+1}} \sum_{l=0}^{m-1} \lambda_l \mathbf{P}[\lambda_{l+1} < |X_n| \le \lambda_l] \\
\le \sum_{m=2}^{\infty} \sum_{2^m \le A_n < 2^{m+1}} \sum_{l=0}^{m-1} \lambda_l \mathbf{P}[|X_n| > \lambda_{l+1}] &= \sum_{l=0}^{\infty} \lambda_l \sum_{m=l+1}^{\infty} \sum_{2^m \le A_n < 2^{m+1}} \mathbf{P}[|X_n| > \lambda_{l+1}] \\
&= \sum_{l=0}^{\infty} \lambda_l \sum_{A_n \ge 2^{l+1}} \mathbf{P}[|X_n| > \lambda_{l+1}] \le 2 \sum_{l=0}^{\infty} 2^l \lambda_l < \infty.
\end{aligned}$$

This proves the convergence of the series (4).

Now, by definition of λ_0 (see also (7)), $\sum_{n=1}^{\infty} \mathbf{P}[|X_n| > \lambda_0] < \infty$. Hence, the events $\{|X_n| > \lambda_0\}$ take place a.s. a finite number of times. Then, the a.s. convergence of the series (5) and (6), follows from (3) and (4) by Beppo Levi's theorem (and Jensen's inequality for the series (5)). □

As already noticed in [14], conditions of the type $\sum_{m=1}^{\infty} 2^{m\alpha} \lambda_m < \infty$ may be simplified in the case where $X_n = a_n Y_n$ under some quite general conditions on the tails of the variables $\{Y_n\}$. In our application we will only consider the case where $\alpha = 1/2 + 1/q$. Hence, for the sake of clarity, the next proposition is stated in this case, while clearly, more general situations may be treated.

Proposition 2.2. *Let $\{Y_n\}$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that for some $1 \le p \le 2$ and some $K > 0$ we have $t^p \mathbf{P}[|Y_n| > t] \le K$, for every $n \ge 1$ and every $t \ge 1$. Let $\{a_n\}$ be a sequence of complex numbers, and assume that $\sum_{n=1}^{\infty} \mathbf{P}[a_n Y_n \ne 0] = \infty$. Let $\{A_n\}$ be an unbounded non-decreasing sequence of positive numbers, and let $1 < q \le 2$. Then, the sequence $\{\lambda_m(q)\}$ is well defined for $\{a_n Y_n\}$ and $\sum_{m=1}^{\infty} 2^{m(1/2+1/q)} \lambda_m(q) < \infty$, in either of the following cases:*

(i) $p=1$, $q=2$, *and* $\sum_{n=1}^{\infty}|a_n|\log A_n<\infty$,

(ii) $1<p<q\le 2$ *and* $\displaystyle\sum_{n=1}^{\infty}\frac{\big(\sum_{2^{A_k}\ge n}|a_k|^p\big)^{1/p}}{n(\log n)^{1/p+1/2-1/q}}<\infty$,

(iii) $p=q=2$ *and* $\displaystyle\sum_{n=1}^{\infty}\frac{\big(\sum_{2^{A_k}\ge n}|a_k|^2\cdot[1+\log^+(1/|a_k|)]\big)^{1/2}}{n(\log n)^{1/2}}<\infty$,

(iv) $p=q=2$, $\sup_{n\ge1}\mathbf{E}|Y_n|^2<\infty$, *and* $\displaystyle\sum_{n=1}^{\infty}\frac{\big(\sum_{2^{A_k}\ge n}|a_k|^2\big)^{1/2}}{n(\log n)^{1/2}}<\infty\,.$

Moreover, in the cases (ii), (iii) *or* (iv) *we have* $\sum_{n=1}^{\infty}\mathbf{E}[|a_nY_n|\mathbf{1}_{\{|a_nY_n|>\lambda_0\}}]<\infty$.

Proof. Let $n\ge1$ and let $\lambda>0$. If $|a_n|\ge\lambda$, we have

$$\mathbf{E}\big[\frac{|a_nY_n|^q}{\lambda^q}\wedge1\big]\le1\le\frac{|a_n|}{\lambda}\le\Big(\frac{|a_n|}{\lambda}\Big)^p.$$

Generally, we have

$$\mathbf{E}\big[\frac{|a_nY_n|^q}{\lambda^q}\wedge1\big]=\Big(\frac{|a_n|}{\lambda}\Big)^q\int_0^{\lambda/|a_n|}qt^{q-1}\mathbf{P}[|Y_n|>t]dt\,.$$

So, on the other hand, if $\lambda>|a_n|$ and $q>p$, we have

$$\mathbf{E}\big[\frac{|a_nY_n|^q}{\lambda^q}\wedge1\big]\le\Big(\frac{|a_n|}{\lambda}\Big)^q(1+Kq\int_1^{\lambda/|a_n|}t^{q-p-1}dt)\le\big(1+K\frac{q}{q-p}\big)\Big(\frac{|a_n|}{\lambda}\Big)^p\qquad(*)$$

Hence in any case, if $q>p$, we obtain

$$\mathbf{E}\big[\frac{|a_nY_n|^q}{\lambda^q}\wedge1\big]\le\big(1+K\frac{q}{q-p}\big)\Big(\frac{|a_n|}{\lambda}\Big)^p$$

Since, (i) or (ii) yield that $\sum_{n=1}^{\infty}|a_n|^p<\infty$, the sequence $\{\lambda_m\}$ is well defined for $\{a_nY_n\}$, and by definition satisfies $2^m=\sum_{A_n\ge2^m}\mathbf{E}[\frac{|a_nY_n|^q}{\lambda_m^q}\wedge1]$, for every $m\ge1$. Then

$$2^{m/p}\lambda_m\le\Big(1+K\frac{q}{q-p}\Big)^{1/p}\Big(\sum_{A_n\ge2^m}|a_n|^p\Big)^{1/p}.\qquad(**)$$

For $p=1$ and $q=2$, we have, using (**) and by changing the order of summation

$$\sum_{m=1}^{\infty}2^m\lambda_m\le(1+2K)\sum_{m=1}^{\infty}\sum_{A_n\ge2^m}|a_n|=\frac{1+2K}{\log2}\sum_{A_n\ge2}|a_n|\log A_n\,.$$

This proves (i).

For $1<p<q$, we have

$$\sum_{n=2}^{\infty}\frac{\big(\sum_{2^{A_k}\ge n}|a_k|^p\big)^{1/p}}{n(\log n)^{1/p+1/2-1/q}}=\sum_{l=0}^{\infty}\sum_{n=2^{2^l}}^{2^{2^{l+1}}-1}\frac{\big(\sum_{2^{A_k}\ge n}|a_k|^p\big)^{1/p}}{n(\log n)^{1/p+1/2-1/q}}\ge$$

$$\sum_{l=0}^{\infty}\Big(\sum_{2^{A_k}\ge2^{2^{l+1}}}|a_k|^p\Big)^{1/p}\sum_{n=2^{2^l}}^{2^{2^{l+1}}-1}\frac{1}{n(\log n)^{1/p+1/2-1/q}}\ge\qquad(***)$$

$$C\sum_{l=0}^{\infty}2^{(l+1)(1/2+1/q-1/p)}\Big(\sum_{A_k\ge2^{l+1}}|a_k|^p\Big)^{1/p},$$

for some positive constant C. Hence, using (**) the first part of (ii) follows.

For $p = q = 2$, we deduce using (*) that

$$\mathbf{E}\big[\frac{|a_nY_n|^2}{\lambda^2} \wedge 1\big] \leq \frac{|a_n|^2}{\lambda^2} \cdot \big(1 + 2K\log(\frac{\lambda}{|a_n|})\big) \leq \frac{|a_n|^2}{\lambda^2} \cdot \big(1 + 2K\log^+\lambda + 2K\log^+(\frac{1}{|a_n|})\big)\,.$$

Since (iii) yields $\sum_{n=1}^\infty |a_n|^2 \cdot [1 + \log^+(1/|a_n|)) < \infty$, the sequence $\{\lambda_m\}$ is well defined. By definition, $\lambda_m \to 0$, so we may and do assume that $0 < \lambda_m \leq 1$ for every $m \geq 1$. Hence, using the above computation we obtain

$$2^{m/2}\lambda_m \leq \Big(\sum_{A_n \geq 2^m} |a_n|^2 \cdot [1 + 2K\log^+(\frac{1}{|a_n|})]\Big)^{1/2}\,.$$

Hence, under (iii) the convergence of $\sum_{m=1}^\infty 2^m\lambda_m$ follows from a calculus similar to the one in (***).

If $\sup_{n\geq 1}\mathbf{E}|Y_n|^2 < \infty$, then $\mathbf{E}\big[\frac{|a_kY_k|^2}{\lambda^2} \wedge 1\big] \leq \frac{|a_k|^2}{\lambda^2}\sup_{n\geq 1}\mathbf{E}|Y_n|^2$. Since (iv) yields $\sum_{n=1}^\infty |a_n|^2 < \infty$, the sequence $\{\lambda_m\}$ is well defined. The convergence of $\sum_{m=1}^\infty 2^m\lambda_m$ follows from a calculus similar to the one in (***).

Now, assume that one of the cases (ii), (iii) or (iv) hold, so in particular $p > 1$. We have

$$\mathbf{E}[|a_nY_n|\mathbf{1}_{\{|a_nY_n|>\lambda_0\}}] = |a_n|\int_{\lambda_0/|a_n|}^\infty \mathbf{P}[|Y_n| > t]dt + \lambda_0\mathbf{P}(|a_nY_n| > \lambda_0) \leq$$
$$\frac{K}{p-1}\frac{|a_n|^p}{\lambda_0^{p-1}} + \lambda_0\mathbf{P}(|a_nY_n| > \lambda_0)\,.$$

Since in each of the cases (ii), (iii) or (iv) we have $\sum_{n=1}^\infty |a_n|^p < \infty$, and since we have, in any case, $\sum_{n=1}^\infty \mathbf{P}(|a_nY_n| > \lambda_0) < \infty$ (e.g., by (7)), the last assertion of the proposition follows. □

Corollary 2.3. *Let $\{X_n\}$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\sum_{n=1}^\infty \mathbf{P}(X_n \neq 0) = \infty$. Let $\{A_n\}$ be an unbounded non-decreasing sequence of positive numbers. If for some $1 < p < q \leq 2$ or for $p = q = 2$ the series*

$$\sum_{n=1}^\infty \frac{\big(\sum_{2^{A_k}\geq n}\mathbf{E}|X_k|^p\big)^{1/p}}{n(\log n)^{1/p+1/2-1/q}} \tag{8}$$

converges, then the sequence $\{\lambda_n(q)\}$ is well defined for $\{X_n\}$ and $\sum_{n=1}^\infty \lambda_n 2^{n(1/2+1/q)} < \infty$. Moreover, $\sum_{n=1}^\infty \mathbf{E}[|X_n|\mathbf{1}_{\{|X_n|>\lambda_0\}}] < \infty$.

Proof. For every $n \geq 1$ put $a_n := \|X_n\|_p$. If $a_n \neq 0$ put $Y_n = \frac{X_n}{a_n}$, otherwise put $Y_n = 0$. Then $t^p\mathbf{P}[|Y_n| > t] \leq 1$ for every $t \geq 1$, and by (8) we have $\sum_{n=1}^\infty \frac{\big(\sum_{2^{A_k}\geq n} a_k^p\big)^{1/p}}{n(\log n)^{1/p+1/2-1/q}} < \infty$. So, the corollary follows from Proposition 2.2. The case $1 < p < q \leq 2$ follows from point (ii) and the case $p = q = 2$ from point (iv). □

3 Convergence of random almost periodic series

In [14], Fernique used special cases of Proposition 2.1 and Proposition 2.2 to recover some results of Marcus and Pisier [18], [19], concerning uniform convergence of random Fourier series. Marcus and Pisier obtained their results in the more general setting of almost periodic series. Using estimates that we obtained recently [8], and using Proposition 2.1, we show how to deduce some of the results of [18] and [19]. Notice that in [18] and [19],

the results are proved for symmetric independent random variables, while we reach here the case of centered independent random variables (for symmetrization procedures see also [7]).

Let us recall a general convergence result that we obtained in [8]. We give its proof for the sake of completeness. In the following, $\mathbf{B}$ is a separable Banach space and $L_r(\Omega, \mathbf{P}; \mathbf{B})$ is the Banach space of all $\mathbf{B}$-valued random variables X, with $\int_\Omega \|X\|^r d\mathbf{P} < \infty$.

Theorem 3.1. *Let $\{X_n\} \subset L_r(\Omega, \mathbf{P}; \mathbf{B})$, with $1 \le r < \infty$. Let $\{\alpha_n\}$ be sequences of non-negative numbers, and let $\{A_n\}$ be an unbounded non-decreasing sequence with $A_1 \ge 1$. Assume there exist $s \ge 1$ such that, for every $m > n \ge 0$, we have*

$$\mathbf{E}\Big[\max_{n<l\le m}\Big\|\sum_{k=n+1}^{l} X_k\Big\|^r\Big] \le A_m\Big(\sum_{k=n+1}^{m}\alpha_k\Big)^s.$$

If the series

$$\sum_{n=1}^{\infty} 2^{n/r}\Big(\sum_{\{k:\, 2^n\le A_k<2^{n+1}\}}\alpha_k\Big)^{s/r} < \infty \tag{9}$$

converges, then the series $\sum_{n=1}^{\infty} X_n$ converges a.s. and in $L_r(\Omega, \mathbf{P}; \mathbf{B})$. Moreover

$$\Big\|\sup_{n\ge1}\|\sum_{k=1}^{n} X_k\|\Big\|_r \le 4\sum_{n=1}^{\infty} 2^{n/r}\Big(\sum_{\{k:\, 2^n\le A_k<2^{n+1}\}}\alpha_k\Big)^{s/r}.$$

Proof. We define by induction two sequences of natural numbers $\{\kappa_n\}$ and $\{l_n\}$. Put $\kappa_1 = l_1 = 0$, and now assume that κ_n and l_n are already defined. Let l_{n+1} be the integer such that $2^{l_{n+1}} \le A_{\kappa_n+1} < 2^{l_{n+1}+1}$. Now, by considering the unboundedness and the non-decreasingness of $\{A_n\}$, we define $\kappa_{n+1} = \max\{m \ge \kappa_n+1 :\, 2^{l_{n+1}} \le A_m < 2^{l_{n+1}+1}\}$. By construction, $\{\kappa_n\}$ and $\{l_n\}$ are strictly increasing. We have

$$\Big(\int \max_{\kappa_n<m\le\kappa_{n+1}}\Big\|\sum_{k=\kappa_n+1}^{m} X_k\Big\|^r d\mathbf{P}\Big)^{1/r} \le$$

$$A_{\kappa_{n+1}}^{1/r}\Big(\sum_{k=\kappa_n+1}^{\kappa_{n+1}}\alpha_k\Big)^{s/r} \le 2^{\frac{l_{n+1}+1}{r}}\Big(\sum_{\{k:\, 2^{l_{n+1}}\le A_k<2^{l_{n+1}+1}\}}\alpha_k\Big)^{s/r}.$$

Hence,

$$\sum_{n=1}^{\infty}\Big\|\max_{\kappa_n<m\le\kappa_{n+1}}\|\sum_{k=\kappa_n+1}^{m} X_k\|\Big\|_r \le 2\sum_{n=0}^{\infty} 2^{n/r}\Big(\sum_{\{k:\, 2^n\le A_k<2^{n+1}\}}\alpha_k\Big)^{s/r}.$$

This proves the a.s. convergence and the norm convergence.

Let $\kappa_n < m \le \kappa_{n+1}$. We have

$$\Big\|\sum_{k=1}^{m} X_k\Big\| \le \sum_{l=0}^{n-1}\Big\|\sum_{\kappa_l+1}^{\kappa_{l+1}} X_k\Big\| + \max_{\kappa_n<j\le\kappa_{n+1}}\Big\|\sum_{k=\kappa_n+1}^{j} X_k\Big\|$$

The maximal inequality then follows from the previous computations. □

Remark. Condition (9) is clearly satisfied if we have $\displaystyle\sum_{n=2}^{\infty}\frac{\big(\sum_{\{k:\, 2^{A_k}\ge n\}}\alpha_k\big)^{s/r}}{n(\log n)^{1-1/r}} < \infty$.

Now, we present an estimate for almost periodic polynomials. It is a simple consequence of Corollary 3.3 in [8]. In the one-dimensional case, i.e., $d = 1$, the following estimate can be deduced also from Theorem 7 of [13].

Notations. Let $d \geq 1$, and consider the Euclidian space $\mathbf{R}^d$. We denote by boldface, e.g., $\mathbf{t} = (t_1, \cdots, t_d)$, a vector in $\mathbf{R}^d$. For any $\mathbf{t}, \mathbf{u} \in \mathbf{R}^d$ we denote by $\langle \mathbf{t}, \mathbf{u} \rangle = t_1 u_1 + \cdots + t_d u_d$ the inner product in $\mathbf{R}^d$. Also we write $|\mathbf{t}| = \max\{|t_1|, \ldots, |t_d|\}$, and for any two real numbers a and b we write $a \vee b = \max\{a, b\}$. For a positive sequence $\{c_n\}$ we write $c_m^* = \max\limits_{1 \leq n \leq m} c_n$. If $\{\boldsymbol{\gamma}_n\} = \{(\gamma_n^{(1)}, \ldots, \gamma_n^{(d)})\}$ is a sequence of vectors in $\mathbf{R}^d$, then by our notations we have

$$|\boldsymbol{\gamma}_m|^* = \max_{1 \leq i \leq d} |\gamma_m^{(i)}|^* = \max_{1 \leq n \leq m} \max_{1 \leq i \leq d} |\gamma_n^{(i)}| \,.$$

Proposition 3.2. *Let $\{Y_n\} \subset L_2(\Omega, \mathbf{P})$ be a sequence of complex centered independent random variables. Let $\boldsymbol{\gamma}_n = (\gamma_n^{(1)}, \cdots, \gamma_n^{(d)})$ be a sequence of vectors in $\mathbf{R}^d$. Then, there exits a positive constant C, such that for every $T > 0$ and every $m > n \geq 1$, we have*

$$\mathbf{E}[\max_{n < l \leq m} \max_{\mathbf{t} \in [-T,T]^d} |\sum_{k=n+1}^{l} Y_k \mathrm{e}^{i\langle \boldsymbol{\gamma}_k, \mathbf{t} \rangle}|^2] \leq C \log(2+T) \log(m \vee |\boldsymbol{\gamma}_m|^*) \sum_{k=n+1}^{m} \mathbf{E}|Y_k|^2 \quad (10)$$

Remark. If $\{\boldsymbol{\gamma}_n\} \subset (\mathbf{R}^+)^d$, and each of the $\{\gamma_n^{(i)}\}$, $1 \leq i \leq d$, is an unbounded and non-decreasing sequence, then we may replace in (10) the term $\log(m \vee |\boldsymbol{\gamma}_m|^*)$ by $\log(|\boldsymbol{\gamma}_m|)$ (see Example 2.1 and Corollary 3.3 in [8]).

Theorem 3.3. *Let $\{X_n\}$ be a sequence of complex valued independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\sum_{n=1}^{\infty} \mathbf{P}[X_n \neq 0] = \infty$ and (1) is satisfied for $q = 2$ and some $\lambda_0 > 0$. Let $\{\boldsymbol{\gamma}_n\} \subset \mathbf{R}^d$, and let $\{\lambda_n\}$ be defined by (2) with $A_n := \log(n \vee |\boldsymbol{\gamma}_n|^*)$. Assume that $\sum_{m=1}^{\infty} 2^m \lambda_m < \infty$, and assume also that the series*

$$\sum_{n=1}^{\infty} \mathbf{E}[X_n \mathbf{1}_{\{|X_n| \leq \lambda_0\}}] e^{i\langle \boldsymbol{\gamma}_n, \mathbf{t} \rangle} \quad (11)$$

converges uniformly on $[-T, T]^d$, for some $T > 0$. Then, there exists a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, the series $\sum_{n=1}^{\infty} X_n(\omega) e^{i\langle \boldsymbol{\gamma}_n, \mathbf{t} \rangle}$ converges uniformly on $[-T, T]^d$.

Remark. If $\{X_n\}$ is a symmetric sequence or if $\sum_{n=1}^{\infty} |\mathbf{E}[X_n \mathbf{1}_{\{|X_n| \leq \lambda_0\}}]| < \infty$, then the series (11) converges uniformly on $\mathbf{R}^d$. In particular, it is uniformly convergent under each of the conditions (ii), (iii) or (iv) of Proposition 2.2.

Proof. Define a sequence of random variables $\{Y_k\}$ in the following way. If $A_k < 1$ put $Y_k = X_k$. If $2^n \leq A_k < 2^{n+1}$, for some $n \geq 0$, put $Y_k := X_k \mathbf{1}_{\{|X_k| \leq \lambda_n\}} - \mathbf{E}[X_k \mathbf{1}_{\{|X_k| \leq \lambda_n\}}]$.

By the assumption on $\{\lambda_n\}$ and Proposition 2.1, we have

$$\sum_{n=1}^{\infty} 2^{n/2} \Big(\sum_{A_k \geq 2^n} \mathbf{E}|Y_k|^2 \Big)^{1/2} < \infty \,.$$

Since the sequence $\{Y_n\}$ satisfies (10), we can apply Theorem 3.1, taking for $\mathbf{B}$ the Banach space of continuous functions on $[-T, T]^d$, with the *sup*-norm. Then, the series $\sum_{n=1}^{\infty} Y_n e^{i\langle \boldsymbol{\gamma}_n, \mathbf{t} \rangle}$ is a.s. uniformly convergent.

Let $k, n \geq 1$, such that $2^n \leq A_k < 2^{n+1}$. We have

$$X_k = Y_k \quad + \quad \mathbf{E}[X_k \mathbf{1}_{\{|X_k| \leq \lambda_0\}}] \quad - \quad \mathbf{E}[X_k \mathbf{1}_{\{\lambda_n < |X_k| \leq \lambda_0\}}] \quad + \quad X_k \mathbf{1}_{\{|X_k| > \lambda_n\}}.$$

So, for almost every $\omega \in \Omega$, the series $\sum_{n=1}^{\infty} X_n(\omega) \mathrm{e}^{i\langle \boldsymbol{\gamma}_n, \mathbf{t}\rangle}$ is the sum of four series converging uniformly on $[-T,T]^d$. Indeed, the first series, with the Y_k's, converges uniformly as shown above. The second series converges uniformly by (11). The third and the forth series are uniformly convergent by (4) and (6), respectively. □

Remarks. 1. The assumption $\sum_{n=1}^{\infty} \mathbf{P}[X_n \neq 0] = \infty$ is not restrictive since otherwise there is nothing to prove.

2. Theorem 3.3 was proved by Fernique [14] for symmetric $\{X_n\}$, when $d = 1$ and $\gamma_n = n$.

Corollary 3.4. *Let $\{Y_n\}$ be a sequence of complex valued symmetric independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that for some $K > 0$ we have $t\mathbf{P}[|Y_n| > t] \leq K$ for every $n \geq 1$ and every $t \geq 1$. Let $\{\boldsymbol{\gamma}_n\} \subset \mathbf{R}^d$, and let $\{a_n\}$ be a sequence of complex numbers such that*

$$\sum_{n=1}^{\infty} |a_n| \log\log(n \vee |\boldsymbol{\gamma}_n|^*) < \infty\,.$$

Then the series $\sum_{n=1}^{\infty} a_n Y_n \mathrm{e}^{i\langle \boldsymbol{\gamma}_n, \mathbf{t}\rangle}$ is a.s. uniformly convergent on any bounded set in $\mathbf{R}^d$.

Proof. For every $n \geq 1$ put $X_n = a_n Y_n$. We may and do assume that $\sum_{n=1}^{\infty} \mathbf{P}[X_n \neq 0] = \infty$, otherwise the result is trivial. Put $A_n = \log(n \vee |\boldsymbol{\gamma}_n|^*)$. Hence, under the condition above Proposition 2.2(i) yields that $\sum_{m=1}^{\infty} 2^m \lambda_m < \infty$. Since, $\{Y_n\}$ are symmetric, the series in (11) is null, so the result follows from Theorem 3.3. □

Corollary 3.5. *Let $1 < p \leq 2$, and let $\{\boldsymbol{\gamma}_n\} \subset \mathbf{R}^d$. Let $\{Y_n\}$ be a sequence of complex valued centered independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that for some $K > 0$ we have $t^p\mathbf{P}[|Y_n| > t] \leq K$ for every $t \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n Y_n \mathrm{e}^{i\langle \boldsymbol{\gamma}_n, \mathbf{t}\rangle}$ is a.s. uniformly convergent on any bounded set in $\mathbf{R}^d$ in either of the following situations:*

(i) $1 < p < 2$ and $\sum_{n=1}^{\infty} \frac{\left(\sum_{\{k \vee |\boldsymbol{\gamma}_k|^* \geq n\}} |a_k|^p\right)^{1/p}}{n(\log n)^{1/p}}$.

(ii) $p = 2$ and $\sum_{n=1}^{\infty} \frac{\left(\sum_{\{k \vee |\boldsymbol{\gamma}_k|^* \geq n\}} |a_k|^2 [1 + \log^+(1/|a_k|)]\right)^{1/2}}{n(\log n)^{1/2}}$

(iii) $p = 2$, $\sup_{n \geq 1} \mathbf{E}|Y_n|^2 < \infty$, and $\sum_{n=1}^{\infty} \frac{\left(\sum_{\{k \vee |\boldsymbol{\gamma}_k|^* \geq n\}} |a_k|^2\right)^{1/2}}{n(\log n)^{1/2}}$.

Proof. According to the above cases, Proposition 2.2(ii) − (iv), with $q = 2$ and $A_n = \log(n \vee |\boldsymbol{\gamma}_n|^*)$, yields that $\sum_{m=1}^{\infty} 2^m \lambda_m < \infty$. Hence, in order to apply Theorem 3.3, we just need to check the uniform convergence of (11). This follows from the facts that the $\{Y_n\}$ are centered, and from the last assertion of Proposition 2.2. □

Remarks. 1. If $\{\boldsymbol{\gamma}_n\} \subset (\mathbf{R}^+)^d$, and each of the sequences $\{\gamma_n^{(i)}\}$, $1 \leq i \leq d$, is unbounded and non-decreasing, then, using the Remark after Proposition 3.2, we may replace in Corollary 3.4 and Corollary 3.5 the expression $(n \vee |\boldsymbol{\gamma}_n|^*)$ by $(|\boldsymbol{\gamma}_n|)$.

2. Corollary 3.4 was proved in Fernique [14] for one dimensional Fourier series with *symmetric* independent coefficients, i.e., $d = 1$ and $\gamma_n = n$.

3. The a.e. uniform convergence of multidimensional almost periodic series with square integrable *symmetric* independent coefficients was considered in Marcus and Pisier [17, Ch. VII, §1]. In [18] and [19], they treated the case of p-stable variables or, more generally, of symmetric variables satisfying the tail condition of Corollary 3.5. The sufficient conditions obtained in [17], [18] and [19] are expressed in terms of the metric entropy. Then they (implicitly) derived the sufficiency of conditions (i) or (ii) of Corollary 3.5 (see [18, p. 294] or [19, p. 186]).

4. In general, extending results beyond the scope of symmetric random variables is possible, under various conditions, by a symmetrization procedure (see e.g., [7]).

Corollary 3.6. *Let $\{X_n\} \subset L_p(\Omega, \mathbf{P})$, $1 < p \leq 2$, be a sequence of centered independent complex valued random variables. Let $\{\boldsymbol{\gamma}_n\} \subset (\mathbf{R}^+)^d$, such that each of the sequences $\{\gamma_n^{(i)}\}_{1 \leq i \leq d}$ is unbounded and non-decreasing. If*

$$\sum_{n=1}^{\infty} \frac{\left(\sum_{|\boldsymbol{\gamma}_k| \geq n} \mathbf{E}|X_k|^p\right)^{1/p}}{n(\log n)^{1/p}} < \infty\,,$$

then the series $\sum_{n=1}^{\infty} X_n \mathrm{e}^{i\langle \boldsymbol{\gamma}_n, \mathbf{t}\rangle}$ is a.s. uniformly convergent on any bounded set in $\mathbf{R}^d$.

Proof. Apply Corollary 3.5 by considering Remark 1 above. □

Remark. Corollary 3.6 was obtained by Hunt [16] for $p = 2$ and $d = 1$, and under the stronger assumption $\sum_{n=1}^{\infty} \mathbf{E}|X_n|^2 (\log |\gamma_n|)^{1+\epsilon} < \infty$, for some $\epsilon > 0$.

4 Random ergodic theorems

We would like to prove random ergodic theorems for d-dimensional semi-flows or commuting family of measure preserving transformations (or isometries). We will state and prove our results in a unified setting. Then, we will deduce results in the specific case of flows or dynamical systems.

Let S_d be either the semigroup $\mathbf{N}^d$ or $(\mathbf{R}^+)^d$, G_d be either the group $\mathbf{Z}^d$ or $\mathbf{R}^d$, and H_d be either $[-\pi, \pi)^d$ or $\mathbf{R}^d$ (H_d is identified to the dual group of G_d).

Definition 4.1. A family $\{V_{\mathbf{s}}\}_{\mathbf{s} \in S_d}$ of isometries of a Hilbert space $\mathcal{H}$ is said to be a representation of S_d by isometries on $\mathcal{H}$ ($S_d = \mathbf{N}^d$ or $(\mathbf{R}^+)^d$), if $V_0 = I$, $V_{\mathbf{s}_1} V_{\mathbf{s}_2} = V_{\mathbf{s}_1 + \mathbf{s}_2}$ for every $\mathbf{s}_1, \mathbf{s}_2 \in S_d$, and if the function $S_d \ni \mathbf{s} \mapsto \langle V_{\mathbf{s}} h_1, h_2 \rangle_{\mathcal{H}}$ is continuous for every $h_1, h_2 \in \mathcal{H}$.

Definition 4.2. A family $\{U_{\mathbf{g}}\}_{\mathbf{g} \in G_d}$ of unitary operators of a Hilbert space $\mathcal{H}$ is said to be a representation of G_d by unitary operators on $\mathcal{H}$ ($G_d = \mathbf{Z}^d$ or $\mathbf{R}^d$), if $U_0 = I$, $U_{\mathbf{g}_1} U_{\mathbf{g}_2} = U_{\mathbf{g}_1 + \mathbf{g}_2}$ for every $\mathbf{g}_1, \mathbf{g}_2 \in G_d$, and if the function $G_d \ni \mathbf{g} \mapsto \langle U_{\mathbf{g}} h_1, h_2 \rangle_{\mathcal{H}}$ is continuous for every $h_1, h_2 \in \mathcal{H}$.

In particular, elements of such families are pairwise commuting. In the case where S_d or G_d are discrete, the continuity assumption is trivial.

Definition 4.3. Let $S_d = (\mathbf{R}^+)^d$ (or $\mathbf{N}^d$). A family $\{T_{\mathbf{s}}\}_{\mathbf{s} \in S_d}$ of commuting measure preserving transformations on a probability space (Y, Σ, π) is said to be an S_d-action, if it induces a representation of S_d by isometries on $L_2(\pi)$.

Usually, $(\mathbf{R}^+)^d$-actions are called continuous semi-flows. Note that any $\mathbf{N}^d$-action (representation) is generated by d pairwise commuting and measure preserving transformations (isometries).

For a representation of S_d by isometries, we recall a weak version of the unitary dilation, e.g., see Riesz and Nagy [25] (Main Theorem of the appendix) or Nagy and Foias [22].

Proposition 4.1. *Let $\{V_{\mathbf{s}}\}_{\mathbf{s} \in S_d}$ be a representation of S_d by isometries on $\mathcal{H}$. There exist a Hilbert space $\mathcal{H}' \supset \mathcal{H}$ and a representation $\{U_{\mathbf{g}}\}_{\mathbf{g} \in G_d}$ of G_d by unitary operators on $\mathcal{H}'$, such that, if P denotes the orthogonal projection from $\mathcal{H}'$ onto $\mathcal{H}$, then $V_{\mathbf{s}} h = P U_{\mathbf{s}} h$, for any $\mathbf{s} \in S_d$ and $h \in \mathcal{H}$.*

Remarks. 1. One can extend the equality $V_{\mathbf{g}}|_{\mathcal{H}} = P U_{\mathbf{g}}|_{\mathcal{H}}$ for all $\mathbf{g} \in G_d$ in order to obtain a *regular dilation* (see [22, Ch. I, Theorem 9.1] or a theorem of Brehmer [21, Ch. 6].

2. Let T_1 and T_2 be two commuting contractions on $\mathcal{H}$. The map $\mathbf{N}^2 \ni (n,m) \mapsto T_1^n T_2^m$ is a representation of $\mathbf{N}^2$ by contractions. One can consider the unitary dilation of the system $\{T_1, T_2\}$, i.e., a Hilbert space $\mathcal{H}' \supset \mathcal{H}$, two commuting unitary operators U_1 and U_2 on $\mathcal{H}'$, and an orthogonal projection P on $\mathcal{H}$, such that for every $f \in \mathcal{H}$ and every $n, m \geq 0$, we have $T_1^n T_2^m f = P U_1^n U_2^m f$ (see [22, Theorem 6.4]). By this unitary dilation theorem, all of our results later on, for $S_2 (= \mathbf{N}^2)$-representations by two commuting isometries, are also valid for $\mathbf{N}^2$-representations by two commuting contractions.

3. Parrott [24] gave an example (see also [22, Ch. I, §3]) which showed that the dilation theorem is no longer true in the case of more than two commuting contractions (for an analogue for commuting Markov operators see [11]).

Now, by a classical generalization of Stone's Theorem (see e.g. Riesz and Nagy [25, Ch. X, §140]), we know that a unitary representation of G_d admits a spectral representation, that is, for any $h \in \mathcal{H}'$, the exists a positive finite measure μ_h on H_d, such that, for every $\mathbf{g} = (g^{(1)}, \ldots, g^{(d)}) \in G_d$.

$$\langle U_{\mathbf{g}} h, h \rangle = \int_{H_d} \mathrm{e}^{i(g^{(1)} t_1 + \cdots + g^{(d)} t_d)} d\mu_h(t_1, \ldots, t_d).$$

From this and the dilation theorem, we deduce that for any $h \in \mathcal{H}$, any $\{\mathbf{s}_1, \ldots, \mathbf{s}_l\}$ in S_d, and any complex numbers $a_1, \ldots, a_l$,

$$\begin{aligned} \|a_1 V_{\mathbf{s}_1} h + \ldots + a_l V_{\mathbf{s}_l} h\|_{\mathcal{H}}^2 &\leq \|a_1 U_{\mathbf{s}_1} h + \ldots + a_l U_{\mathbf{s}_l} h\|_{\mathcal{H}'}^2 \\ &\leq \max_{\mathbf{t} \in H_d} |a_1 \mathrm{e}^{i\langle \mathbf{s}_1, \mathbf{t} \rangle} + \ldots + a_l \mathrm{e}^{i\langle \mathbf{s}_l, \mathbf{t} \rangle}|^2 \|h\|_{\mathcal{H}}^2 , \end{aligned} \tag{12}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbf{R}^d$.

We want to prove random ergodic theorems for S_d-representations or S_d-actions. We recall our notation $|\mathbf{t}| = \max\{|t_1|, \ldots, |t_d|\}$, for $\mathbf{t} \in \mathbf{R}^d$.

Definition 4.4. Let $\{\mathbf{s}_n\} \subset S_d$. A deterministic sequence $\{\beta_n\}$ is said to be a log-universally good weight for S_d-representations along $\{\mathbf{s}_n\}$ if the following conditions are satisfied: for any probability space (Y, Σ, π), any S_d-representation $\{V_{\mathbf{s}}\}_{\mathbf{s} \in S_d}$ on $L_2(Y, \Sigma, \pi)$, and any $f \in L_2(\pi)$, such that $\int_{H_d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t}) < \infty$, the series $\sum_{n=1}^{\infty} \beta_n V_{\mathbf{s}_n} f$ converges a.s. and $\| \sup_{n \geq 1} |\sum_{k=1}^{n} \beta_k V_{\mathbf{s}_n} f| \|_2 \leq C \big(\int_{H_d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t}) \big)^{1/2}$, for a constant $C > 0$ depending only on $\{\beta_n\}$ and $\{\mathbf{s}_n\}$.

Similarly we define,

Definition 4.5. Let $\{\mathbf{s}_n\} \subset S_d$. A deterministic sequence $\{\beta_n\}$ is said to be a log-universally good weight for S_d-actions along $\{\mathbf{s}_n\}$ if the following conditions are satisfied: for any probability space (Y, Σ, π), any S_d-action $\{T_{\mathbf{s}}\}_{\mathbf{s} \in S_d}$ on it, and any $f \in L_2(\pi)$, such that $\int_{H_d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t}) < \infty$, the series $\sum_{n=1}^{\infty} \beta_n f \circ T_{\mathbf{s}_n}$ converges a.s. and $\| \sup_{n \geq 1} |\sum_{k=1}^{n} \beta_k f \circ T_{\mathbf{s}_k}| \|_2 \leq C \big(\int_{H_d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t}) \big)^{1/2}$, for a constant $C > 0$ depending only on $\{\beta_n\}$ and $\{\mathbf{s}_n\}$.

Remark. The condition $\int_{H_d} \log(2 + |\mathbf{t}|)\ \mu_f(d\mathbf{t}) < \infty$ is clearly satisfied when $H_d = [-\pi, \pi)^d$. However, it is convenient to state these definitions in this unified setting. In this (discrete) situation the prefix log in the log-universality property is redundant.

Definition 4.6. Let $\{\mathbf{s}_n\} \subset S_d$. A sequence $\{X_n\}$ of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be a.s. a log-universally good weight for S_d-representations (actions) along $\{\mathbf{s}_n\}$ if there exists a measurable set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, the deterministic sequence $\{X_n(\omega)\}$ is a log-universally good weight for S_d-representations (actions) along $\{\mathbf{s}_n\}$.

We need a uniform estimate on random almost periodic polynomials that we obtained recently (see Theorem 1.1 of [8]). We recall our notation $c_m^* = \max_{1\le n\le m} c_n$, for any positive sequence $\{c_n\}$.

Proposition 4.2. *Let $\{Y_n\} \subset L_q(\Omega, \mathbf{P})$, $1 < q \le 2$, be a sequence of complex valued centered independent random variables, and let $\{\boldsymbol{\gamma}_n\} \subset \mathbf{R}^d$. Then there exist some positive constants ϵ and C, independent of $\{Y_n\}$, such that (with $0/0$ interpreted as 1)*

$$\Big\| \sup_{m\ge 2} \max_{1\le n<m} \sup_{T\ge 1} \exp\Big\{\epsilon\cdot \frac{\max_{\mathbf{t}\in[-T,T]^s} |\sum_{k=n+1}^m Y_k e^{i\langle \boldsymbol{\gamma}_k,\mathbf{t}\rangle}|^2}{\log(1+T)\log(m\vee|\boldsymbol{\gamma}_m|^*)\cdot(\sum_{k=n+1}^m |Y_k|^q + \|Y_k\|_q^q)^{2/q}}\Big\}\Big\|_1 \le C\,.$$

In addition, we need the following results of Móricz [20] (see also Proposition 2.2 and Proposition 2.3 in [6]), which extend some results of Stechkin (see Gaposhkin [15, §1.3]), and Menchoff (e.g., Doob [10, Ch. IV, Lemma 4.1, p. 156]).

Proposition 4.3. *Let $1 < r < \infty$, and let $\{h_k\} \subset L_r(\pi)$ be a sequence of complex valued random variables on some probability space. Let $m > n \ge 0$ be integers, and assume that there exist positive numbers $\{b_k\}_{n<k\le m}$ and constants $D > 0$ and $s \ge 1$, such that*

$$\int \Big|\sum_{k=j+1}^{l} h_k\Big|^r d\pi \le D\Big(\sum_{k=j+1}^{l} b_k\Big)^s \qquad \textit{for every} \qquad n \le j < l \le m.$$

Then, if $s > 1$, there exists a constant $C_{r,s} > 0$, such that

$$\int \max_{n<l\le m} \Big|\sum_{k=n+1}^{l} h_k\Big|^r d\pi \le DC_{r,s}\Big(\sum_{k=n+1}^{m} b_k\Big)^s.$$

If s=1, we have

$$\int \max_{n<l\le m} \Big|\sum_{k=n+1}^{l} h_k\Big|^r d\pi \le D(\log_2(4m))^r \sum_{k=n+1}^{m} b_k\,.$$

We want to prove the following

Theorem 4.4. *Let $\{X_n\}$ be a sequence of complex valued independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\sum_{n=1}^\infty \mathbf{P}[X_n \ne 0] = \infty$ and (1) is satisfied for some $1 < q \le 2$ and some $\lambda_0 > 0$. Let $\{\mathbf{s}_n\} \subset S_d$, and let $\{\lambda_n(q)\}$ be defined by (2) with $A_n := \log(n \vee |\mathbf{s}_n|^*)$ if $1 < q < 2$, or $A_n := (\log n)^2 \log(n \vee |\mathbf{s}_n|^*)$ if $q = 2$. Assume that*

$$\sum_{m=1}^\infty 2^{m(1/2+1/q)}\lambda_m(q) < \infty$$

and

$$\sum_{n=1}^\infty \sum_{2^n\le A_k<2^{n+1}} |\mathbf{E}[X_k \mathbf{1}_{\{|X_k|\le\lambda_n\}}]| < \infty\,. \tag{13}$$

Then $\{X_n\}$ is a.s. a log*-universally good weight for S_d-representations along $\{\mathbf{s}_n\}$.*

Remarks. 1. Notice that in this theorem, the variables are assumed neither to be centered nor to be integrable.

2. The assumption $\sum_{n=1}^\infty \mathbf{P}[X_n \ne 0] = \infty$ is technical. It is assumed in order to be able to define the sequence $\{\lambda_n(q)\}$. In any case, if it does not hold there is nothing to prove.

Proof. Since $\sum_{m=1}^{\infty} 2^{m(1/2+1/q)}\lambda_m < \infty$, by Proposition 2.1, (3) and (4) are satisfied with $\alpha = 1/2 + 1/q$ and there exists $\Omega_1 \subset \Omega$, with $\mathbf{P}(\Omega_1) = 1$, on which (5) and (6) are satisfied.

Define a sequence of centered independent random variables $\{Y_k\}$ in the following way. If $A_k < 1$ put $Y_k = X_k$. If $2^n \le A_k < 2^{n+1}$, for some $n \ge 0$, put $Y_k := X_k \mathbf{1}_{\{|X_k| \le \lambda_n\}} - \mathbf{E}[X_k \mathbf{1}_{\{|X_k| \le \lambda_n\}}]$.

Then, for every $\omega \in \Omega_1$, we have

$$\sum_{m=1}^{\infty} 2^{m/2} \Big(\sum_{A_k \ge 2^m} \big(|Y_k(\omega)|^q + \mathbf{E}[|Y_k|^q] \big) \Big)^{1/q} < \infty \,. \qquad (*)$$

By Proposition 4.2, there exists $\Omega_2 \subset \Omega$, with $\mathbf{P}(\Omega_2) = 1$, such that for every $\omega \in \Omega_2$, there exists $C_\omega > 0$, such that for every $\mathbf{t} \in \mathbf{R}^d$, for every $m > n \ge 1$, we have

$$\Big| \sum_{k=n+1}^{m} Y_k(\omega) e^{i\langle \mathbf{s}_k, \mathbf{t} \rangle} \Big|^2 \le C_\omega \log(2 + |\mathbf{t}|) \log(m \vee |\mathbf{s}_m^*|) \Big(\sum_{k=n+1}^{m} (|Y_k(\omega)|^q + \mathbf{E}|Y_k|^q) \Big)^{2/q}.$$

Let $\Omega^* := \Omega_1 \cap \Omega_2$ and $\omega \in \Omega^*$.

Let (Y, Σ, π) be any probability space, and let $\{V_{\mathbf{s}}\}_{\mathbf{s} \in S_d}$ be any representation of S_d by isometries on $L_2(\pi)$. Let $f \in L_2(\pi)$ such that $K_f := \int_{H_d} \log(2 + |\mathbf{t}|) \mu_f(d\mathbf{t}) < \infty$. Using dilation and the spectral theorem (see (12)), we have

$$\Big\| \sum_{k=n+1}^{m} Y_k(\omega) V_{\mathbf{s}_k} f \Big\|_2^2 \le K_f C_\omega \log(m \vee |\mathbf{s}_m^*|) \Big(\sum_{k=n+1}^{m} (|Y_k(\omega)|^q + \mathbf{E}|Y_k|^q) \Big)^{2/q}.$$

Since it is true for every $m > n \ge 1$, by Proposition 4.3 we obtain

$$\Big\| \max_{n < l \le m} \Big| \sum_{k=n+1}^{l} Y_k(\omega) V_{\mathbf{s}_k} f \Big| \Big\|_2^2 \le$$
$$K_f C_\omega C_{2,2/q} \log(m \vee |\mathbf{s}_m^*|) \Big(\sum_{k=n+1}^{m} (|Y_k(\omega)|^q + \mathbf{E}|Y_k|^q) \Big)^{2/q} \qquad (1 < q < 2) \quad (**)$$

and

$$\Big\| \max_{n < l \le m} \Big| \sum_{k=n+1}^{m} Y_k(\omega) V_{\mathbf{s}_k} f \Big| \Big\|_2^2 \le$$
$$K_f C_\omega (\log_2(4m))^2 \log(m \vee |\mathbf{s}_m^*|) \sum_{k=n+1}^{m} (|Y_k(\omega)|^2 + \mathbf{E}|Y_k|^2) \qquad (q = 2) \quad (***)$$

By (*), (**) or (***) (according to $1 < q < 2$ or $q = 2$), and using Theorem 3.1 with $r = 2$, $s = 2/q$, and the appropriate $\{A_n\}$, we obtain that, for every $\omega \in \Omega^*$, the series $\sum_{n=1}^{\infty} Y_n(\omega) V_{\mathbf{s}_k} f$ converges π-a.s and in $L_2(\pi)$. Moreover, for every $\omega \in \Omega^*$ there exists $C'_\omega > 0$, such that

$$\Big\| \sup_{n \ge 1} \Big| \sum_{k=1}^{n} Y_k(\omega) V_{\mathbf{s}_k} f \Big| \Big\|_2 \le \sqrt{C'_\omega K_f} \,.$$

Let $k, n \ge 1$, such that $2^n \le A_k < 2^{n+1}$. We have

$$X_k = Y_k \quad + \quad \mathbf{E}[X_k \mathbf{1}_{\{|X_k| \le \lambda_n\}}] \quad + \quad X_k \mathbf{1}_{\{|X_k| > \lambda_n\}}.$$

Recall that (13) is satisfied, and on Ω_1 (6) is satisfied. Hence, for every $\omega \in \Omega^*$, the series $\sum_{k=1}^{\infty} X_k V_{\mathbf{s}_k} f$ is π-a.s convergent since the series with the Y_k's is π-a.s convergent and we have

$$\sum_{k=1}^{\infty} |X_k - Y_k| \cdot \|V_{\mathbf{s}_k} f\|_2 = \|f\|_2 \sum_{k=1}^{\infty} |X_k - Y_k| < \infty \,.$$

The maximal inequality (as stated in the definition of log-universality) clearly follows. □

Corollary 4.5. *Let $\{Y_n\}$ be a sequence of complex valued symmetric independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that for some $K > 0$ we have $t\mathbf{P}[|Y_n| > t] \le K$ for every $n \ge 1$ and every $t \ge 1$. Let $\{\mathbf{s}_n\} \subset S_d$, and let $\{a_n\}$ be a sequence of complex numbers. If the series $\sum_{n=1}^{\infty} |a_n| \log\log(n \vee |\mathbf{s}_n|^*)$ converges, then the sequence $\{a_n Y_n\}$ is a.s. a* log*-universally good weight for S_d-representations along $\{\mathbf{s_n}\}$.*

Proof. We may and do assume that $\sum_{n=1}^{\infty} \mathbf{P}[a_n Y_n \neq 0] = \infty$, otherwise the result is trivial. By assumption the series $\sum_{n=1}^{\infty} |a_n| \log\log(n \vee |\mathbf{s}_n|^*)$ converges, so Proposition 2.2(i), applied with $A_n = (\log n)^2 \log(n \vee |\mathbf{s}_n|^*)$, yields that $\sum_{m=1}^{\infty} 2^m \lambda_m < \infty$. Since $\{Y_n\}$ is symmetric, condition (13) is trivially satisfied. Hence, the result follows from Theorem 4.4 with $q = 2$ and the above choice of $\{A_n\}$. □

Corollary 4.6. *Let $\{X_n\} \subset L_p(\Omega, \mathbf{P})$, $1 < p < 2$, be a sequence of complex valued centered independent random variables. Let $\{\mathbf{s}_n\} \subset S_d$, and assume that for some $\delta > 0$, the series $\sum_{n=1}^{\infty} \|X_n\|_p^p (\log(n \vee |\mathbf{s}_n|^*))^{p-1+\delta}$ converges. Then the sequence $\{X_n\}$ is a.s. a* log*-universally good weight for S_d-representations along $\{\mathbf{s}_n\}$.*

Proof. As usual, we may and do assume that $\sum_{n=1}^{\infty} \mathbf{P}[X_n \neq 0] = \infty$. For every $n \ge 1$, put $a_n = \|X_n\|_p$. If $a_n \neq 0$ put $Y_n = X_n/a_n$, otherwise put $Y_n = 0$. Let $1 < q < 2$ such that $q > \frac{2p}{p+2\delta}$. We use Proposition 2.2(ii) and Proposition 2.1 to show that the conditions of Theorem 4.4 are satisfied with this q. By assumption, we have

$$\sum_{\{k:\, \log(k \vee |\mathbf{s}_k|^*) \ge \log n/\log 2\}} |a_k|^p \le \frac{C}{(\log n)^{p-1+\delta}} \,,$$

for some positive constant C. Hence, with $A_k = \log(k \vee |\mathbf{s}_k|^*)$, we have

$$\sum_{n=1}^{\infty} \frac{\left(\sum_{2^{A_k} \ge n} |a_k|^p\right)^{1/p}}{n(\log n)^{1/p+1/2-1/q}} \le \sum_{n=1}^{\infty} \frac{C}{n(\log n)^{(1/p+1/2-1/q)+(p-1+\delta)/p}}$$
$$\le \sum_{n=1}^{\infty} \frac{C}{n(\log n)^{(3/2-1/q+\delta/p)}} < \infty \,,$$

by our choice of q. Hence, $\sum_{m=1}^{\infty} 2^{m(1/2+1/q)} \lambda_m(q) < \infty$ by Proposition 2.2(ii). In particular, we can apply Proposition 2.1. Since the $\{X_n\}$ are centered, by the last assertion in Proposition 2.2 and the convergence in (4), (13) is satisfied. We conclude by Theorem 4.4. □

Remark. It was proved in Theorem 4.2 of [5], that the convergence of the series $\sum_{n=1}^{\infty} \|X_n\|_p^p (\log n)^p (\log\log n)^{p/2+\epsilon}$, for some $\epsilon > 0$, is sufficient for $\{X_n\}$ to be a.s. universally good weight for $\mathbf{N}$-representations by *contractions* along $\{\mathbf{s}_n = n\}$, i.e., for power series of a single L_2-contraction. Hence our sufficient condition is weaker and our result more general.

One could apply Theorem 4.4, to extend the previous corollary to the case $p = 2$. It turns out that Theorem 4.4 is not efficient in this case. We need another method, using the following proposition (see e.g. [8])

Proposition 4.7. *Let $\{X_n\} \subset L_2(\Omega, \mathbf{P})$. Let $\{\alpha_n\}$ be a sequence of non-negative numbers. Let γ and C be positive constants, and assume there exists a positive non-decreasing (possibly constant) sequence $\{A_n\}$, with $A_n \le Cn^\gamma$, such that for every $m > n \ge 0$ we have*

$$\mathbf{E}\Big[\Big|\sum_{k=n+1}^{m} X_k\Big|^2\Big] \le A_m \sum_{k=n+1}^{m} \alpha_k\,. \tag{14}$$

If $\sum_{n=1}^{\infty} \alpha_n A_n (\log n)^2$ converges, then the series $\sum_{n=1}^{\infty} X_n$ converges almost everywhere and in $L_2(\mathbf{P})$. Furthermore, we have

$$\Big\|\sup_{n\ge1}|\sum_{k=1}^{n} X_k|\Big\|_2 \le 2\mathrm{e}^{\frac12}[1+2^{\frac12}(2+\gamma)]\Big(\sum_{n=1}^{\infty}\alpha_n A_n(\log n)^2\Big)^{1/2}.$$

Theorem 4.8. *Let $\{X_n\} \subset L_2(\Omega, \mathbf{P})$ be a sequence of complex valued centered independent random variables, and let $\{\mathbf{s}_n\} \subset S_d$. If*

$$\sum_{n=2}^{\infty} \|X_n\|^2 (\log n)^2 \log(n \vee |\mathbf{s}_n|^*) < \infty\,,$$

then $\{X_n\}$ is a.s. a log*-universally good weight for S_d-representations along $\{\mathbf{s}_n\}$.*

Proof. We need to split our sequence of random variables $\{X_n\}$ into two parts, in order to apply the previous proposition. Let $\{Y_n\}$ and $\{Z_n\}$ be defined as follows: $Y_n = X_n$ if $|\mathbf{s}_n|^* \le \mathrm{e}^n$, otherwise $Y_n \equiv 0$; $Z_n = X_n - Y_n$.

By assumption and by Beppo Levi theorem, there exists $\Omega_1 \subset \Omega$, with $\mathbf{P}(\Omega_1) = 1$, such that for every $\omega \in \Omega_1$ the series $\sum_{n=1}^{\infty} |X_n(\omega)|^2 (\log n)^2 \log(n \vee |s_n|^*)$ converges.

We first deal with $\{Z_n\}$, the easy part. Let $\{n_k\}$ be the sequence of integers for which $|\mathbf{s}_{n_k}|^* > \mathrm{e}^{n_k}$. Hence $Z_n = 0$ if $n \notin \{n_k\}$. We have, for every $\omega \in \Omega_1$,

$$\begin{aligned}\sum_{n=2}^{\infty} |Z_n(\omega)| &\le \Big(\sum_{k=1}^{\infty} |Z_{n_k}(\omega)|^2 (\log n_k)^2 \log(n_k \vee |\mathbf{s}_{n_k}|^*)\Big)^{\frac12} \Big(\sum_{k=1}^{\infty} \frac{1}{\log(n_k \vee |\mathbf{s}_{n_k}|^*)(\log n_k)^2}\Big)^{\frac12} \\ &\le \Big(\sum_{n=2}^{\infty} |X_n(\omega)|^2 (\log n)^2 \log(n \vee |s_n|^*)\Big)^{\frac12} \Big(\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}\Big)^{\frac12} < \infty\,.\end{aligned}$$

So, we just need to prove that $\{Y_n(\omega)\}$ is a.s. a log-universally good weight for S_d-representations along $\{\mathbf{s}_n\}$.

Since Y_n is null when $|\mathbf{s}_n|^* > \mathrm{e}^n$, modifying $\{\mathbf{s}_n\}$ when necessary we can assume that $|\mathbf{s}_n|^* \le \mathrm{e}^n$, for every $n \ge 2$.

By Proposition 4.2, there exists $\Omega_2 \subset \Omega$, with $\mathbf{P}(\Omega_2) = 1$, such that for every $\omega \in \Omega_2$, there exists $C_\omega > 0$, such that for every $\mathbf{t} \in H_d$ and for every $m > n \ge 1$, we have

$$\Big|\sum_{k=n+1}^{m} Y_k(\omega)\mathrm{e}^{i\langle \mathbf{s}_k, \mathbf{t}\rangle}\Big|^2 \le C_\omega \log(1+|\mathbf{t}|)\log(m \vee |\mathbf{s}_m|^*) \sum_{k=n+1}^{m} (|Y_k(\omega)|^2 + \mathbf{E}|Y_k|^2).$$

Let $\Omega^* := \Omega_1 \cap \Omega_2$ and $\omega \in \Omega^*$.

Let (Y, Σ, π) be any probability space and $\{V_{\mathbf{s}}\}_{\mathbf{s}\in S_d}$ any S_d-representation on $L_2(\pi)$. Let $f \in L_2(\pi)$ such that $K_f := \int_{H_d} \log(1+|\mathbf{t}|)\mu_f(d\mathbf{t}) < \infty$. By the spectral theorem (see (12)), we have

$$\Big\|\sum_{k=n+1}^{m} Y_k(\omega) V_{\mathbf{s}_k} f\Big\|_2^2 \le K_f C_\omega \log(m \vee |\mathbf{s}_m|^*) \sum_{k=n+1}^{m} (|Y_k(\omega)|^2 + \mathbf{E}|Y_k|^2).$$

Since $|Y_n| \le |X_n|$ for every $n \ge 2$, (14) holds with $A_n = K_f C_\omega \log(n \vee |\mathbf{s}_n|^*)$ and $\alpha_n = |X_n(\omega)|^2 + \mathbf{E}|X_n|^2$. Hence, by Proposition 4.7, the series $\sum_{n=1}^\infty Y_n(\omega) V_{\mathbf{s}_n} f$ converges π-a.s. Moreover, for every $\omega \in \Omega^*$, there exists $C'_\omega > 0$ such that

$$\Big\| \sup_{n\ge 1} \Big| \sum_{k=1}^n Y_k(\omega) V_{\mathbf{s}_k} f \Big| \Big\|_2 \le \sqrt{C'_\omega K_f}.$$

Combining the two results for $\{Y_n\}$ and $\{Z_n\}$ yield the result. □

Remarks. 1. Theorem 4.8 was proved in [8] for $\mathbf{N}^d$-representations.

2. Theorem 4.8 was proved in [5] in the setting of a single L_2-contraction on some probability space, along the sequence of natural numbers. In this case, i.e., $d = 1$, $\mathbf{s}_n = n$, and a single contraction, our sufficient condition is exactly the same as the one given in [5, Theorem 4.2].

3. In the case of a single contraction, Boukhari and Weber [4] obtained a different sufficient conditions, in term of the convergence of two series. While one of these series is close to the one we consider, we do not know how to compare the second one to our sufficient condition. The result in [4] is stated for a monotonic sequence of positive integers s_n, but, as noticed by Weber (personal communication) it extends easily to the setting of a $\mathbf{N}^d$-valued sequence.

As we already mentioned, the condition on μ_f, in Definition 4.4 or Definition 4.5, is trivially satisfied when $H_d = [-\pi, \pi)^d$ (i.e. when $S_d = \mathbf{N}^d$). Hence, the conclusion of Corollary 4.6 and Theorem 4.8 holds for any $f \in L_2(\pi)$ (without the "log-integrability"). So, in the case where $S_d = \mathbf{N}^d$, we can use Stein's interpolation (as in [8], see also [26], [5] or [4]) to extend Corollary 4.6 and Theorem 4.8. We obtain

Theorem 4.9. *Let $\{X_n\} \subset L_p(\Omega, \mathbf{P})$, $1 < p \le 2$, be a sequence of complex centered independent random variables. Let $\mathbf{s}_n = (s_n^{(1)}, \dots, s_n^{(d)})$ be a sequence of vectors in $\mathbf{N}^d$. Assume either of the series $\sum_{n=1}^\infty \|X_n\|_p^p (\log(n \vee |\mathbf{s}_n|^*))^{p-1+\delta}$, for some $\delta > 0$ ($1 < p < 2$) or $\sum_{n=1}^\infty \|X_n\|_2^2 (\log n)^2 \log(n \vee |\mathbf{s}_n|^*)$ ($p = 2$) converges. Then there exist a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any probability space (Y, Σ, π), any pairwise commuting and measure preserving transformations $\tau_1, \dots, \tau_d$ on Y, and any $f \in L_r(Y, \pi)$, $1 < r \le 2$, the series*

$$\sum_{n=1}^\infty \frac{X_n(\omega) f \circ \tau_1^{s_n^{(1)}} \circ \cdots \circ \tau_d^{s_n^{(d)}}}{n^{(2-r)(p-1)/pr}} \tag{15}$$

converges π-a.s. Moreover, for every $\omega \in \Omega^*$, there exists $K_\omega > 0$ such that

$$\Big\| \sup_{n\ge 1} \Big| \sum_{k=1}^n \frac{X_n(\omega) f \circ \tau_1^{s_n^{(1)}} \circ \cdots \circ \tau_d^{s_n^{(d)}}}{n^{(2-r)(p-1)/pr}} \Big| \Big\|_r \le K_\omega. \tag{16}$$

The case $p = 2$ was proved in Theorem 5.3 of [8]. The proof for $1 < p < 2$ is quite similar. We give the main steps in the Appendix, for the sake of completeness.

Remark. Of course, a centered i.i.d. sequence $\{Y_n\} \subset L_p(\Omega, \mathbf{P})$ never satisfies either of the conditions of the theorem. However, in the case of polynomial growth of $\{|s_n|^*\}$ the theorem can be applied to $X_n = Y_n/n^\alpha$ for some $\alpha > 1/p$, to obtain π-a.s. convergence of

$$\sum_{n=1}^\infty \frac{Y_n(\omega) f \circ \tau_1^{s_n^{(1)}} \circ \cdots \circ \tau_d^{s_n^{(d)}}}{n^\beta}$$

for $\beta = \alpha + (2-r)(p-1)/pr$. Applying Kronecker's lemma we obtain a randomly weighted ergodic theorem with rate $o(1/n^{1-\beta})$. A better rate can be obtained by applying the theorem to $X_n := Y_n/n^{1/p}(\log n)^\gamma$ with an appropriate γ.

Theorem 4.10. *Let $\{\boldsymbol{\theta}_n\}$ be a sequence of independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, with values in $\mathbf{R}^d$. Let Φ be some positive non-decreasing function on $\mathbf{R}^+$, such that there exists $\eta > 0$, for which $\Phi(x) \geq x^\eta$ for every $x \geq 0$. Assume that there exists $\delta > 0$, such that $\sum_{n=1}^\infty \mathbf{P}[|\boldsymbol{\theta}_n| > \Phi(n)^\delta] < \infty$. Let $\{a_n\}$ be a sequence of complex numbers. Then there exist universal contants $\epsilon > 0$ and $C > 0$, such that*

$$\mathbf{E}\left[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\left\{\epsilon\cdot\frac{\max_{|\mathbf{t}|\leq T}\left|\sum_{k=n+1}^m a_k(\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}-\mathbf{E}[\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}])\right|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^m|a_k|^2}\right\}\right]\leq C \tag{17}$$

The proof, being of a technical nature, is given in the Appendix.

Remarks. 1. In the proof of Theorem 4.10, we will use Proposition 4.2 for the sequence of powers $\{\boldsymbol{\theta}_n(\omega)\}$. The constant C, that comes from Proposition 4.2, depends on the ℓ_4-norm of $\{1/(n \vee |\boldsymbol{\theta}_n(\omega)|^*)\}$, hence may be chosen independently of $\{\boldsymbol{\theta}_n(\omega)\}$ (see Theorem 3.5 in [8] and the remarks after it).

2. Theorem 4.10 is a generalization of Theorem 9 of [29] and Theorem 1.5 of [12] (see also Lemma 2.3 of [4] and Lemma 1 of [23]).

3. Theorem 4.10 gives integrability in the Orlicz space associated with the function $\exp x^2 - 1$, while the above cited results are in L_1, i.e., without the exponential of the square.

4. The assumption on $\{\boldsymbol{\theta}_n\}$ is weaker than the ones previously assumed in [29], [4], [23], and [12]. Furthermore, only $\{\boldsymbol{\theta}_n\}$ with $\mathbf{N}^d$-values were considered in the above cited papers, except in [12], where also $\mathbf{R}^d$-values were considered.

5. The supremum over T in (17) is crucial for the case of $(\mathbf{R}^+)^d$-representations in the next theorem.

We deduce the following

Theorem 4.11. *Let $\{\boldsymbol{\theta}_n\}$ be a sequence of independent random variables, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, with values in $(\mathbf{R}^+)^d$. Let Φ be some positive non-decreasing function on $\mathbf{R}^+$, such that there exists $\eta > 0$, for which $\Phi(x) \geq x^\eta$ for every $x \geq 0$. Assume that there exists $\delta > 0$, such that $\sum_{n=1}^\infty \mathbf{P}[|\boldsymbol{\theta}_n| > \Phi(n)^\delta] < \infty$. Let $\{a_n\}$ be a sequence of complex numbers satisfying*

$$\sum_{n=1}^\infty |a_n|^2(\log n)^2\log\Phi(n) < \infty\,. \tag{18}$$

Then there exists a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any probability space (Y, Σ, π), any S_d-representation $\{V_\mathbf{s}\}$ on $L_2(\pi)$, and any $f \in L_2(\pi)$ with $\int_{H_d}\log(2+|\mathbf{t}|)d\mu_f(\mathbf{t}) < \infty$, the series

$$\sum_{n=1}^\infty a_n(V_{\boldsymbol{\theta}_n}(f) - \mathbf{E}[V_{\boldsymbol{\theta}_n}](f))$$

converges π-a.s.

Proof. The proof is exactly the same as the one of Theorem 4.8, using Theorem 4.10 instead of Proposition 4.2. □

We now give some applications of Theorem 4.11 concerning semi-flows. We obtain

Theorem 4.12. *Let $\{\boldsymbol{\theta}_n\} \subset (\mathbf{R}^+)^d$ be i.i.d. random variables, with $\mathbf{E}[|\boldsymbol{\theta}_1|^\sigma] < \infty$, for some $\sigma > 0$. Let $\{\mathbf{s}_n\} \subset (\mathbf{R}^+)^d$, with $|\mathbf{s}_n|^* = O(2^{n^\gamma})$, for some $0 < \gamma < 1$. Then, there exits a set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any probability space (Y, Σ, π), any continuous semi-flow $\{T_\mathbf{t}\}_{\mathbf{t}\in(\mathbf{R}^+)^d}$, and any $f \in L_2(\pi)$ with $\int_{\mathbf{R}^d}\log(2+|\mathbf{t}|)\mu_f(d\mathbf{t}) < \infty$, the series*

$$\sum_{n=1}^\infty \frac{T_{\mathbf{s}_n+\boldsymbol{\theta}_n(\omega)}f - T_{\mathbf{s}_n}(\mathbf{E}[T_{\boldsymbol{\theta}_1}]f)}{n}$$

converges π-a.s. In particular, the sequence

$$\frac{1}{n}\sum_{k=1}^{n}(T_{\mathbf{s}_k+\boldsymbol{\theta}_k(\omega)}f - T_{\mathbf{s}_k}(\mathbf{E}[T_{\boldsymbol{\theta}_1}]f)$$

converges to zero π-a.s.

Proof. Apply Theorem 4.11 to the sequence of independent random variables $\{\boldsymbol{\theta}_n + \mathbf{s}_n\}$, with $a_n = 1/n$, $\Phi(n) = (n^{1/\sigma} + |\mathbf{s}_n|^*)^\sigma$, and $\delta = 1/\sigma$. □

Remark. It follows that under the conditions of Theorem 4.12, if the ergodic theorem holds along the sequence of "times" $\{\mathbf{s}_n\}$, then it holds also along $\{\mathbf{s}_n + \theta_n(\omega)\}$ for $\mathbf{P}$-a.e. $\omega \in \Omega$ and the limit is independent of ω.

For the next result we consider the particular case of the flow induced by all rotations of the unit circle. Let $(Y, \Sigma, \pi) = ([0,1), \mathcal{B}, m)$, where $\mathcal{B}$ is the σ-algebra of Borel sets on $[0,1)$ and m is the Lebesgue measure. For every $t \in \mathbf{R}$ define the rotation $\tau_t : [0,1) \mapsto [0,1)$ by $\tau_t(x) = x+t$ modulo 1. For any $f \in L_2([0,1), m)$, let μ_f be the spectral measure induced by f and the flow $\{\tau_t\}$, i.e., that measure for which $\int_0^1 f(x+t)\overline{f}(x)dm(x) = \int_{\mathbf{R}} \mathrm{e}^{ity} d\mu_f(y)$, for every $t \in \mathbf{R}$.

Corollary 4.13. *Let $\{\theta_n\}$ be a sequence of i.i.d. random variables, with values in $[0,1)$. Then there exists a measurable set $\Omega^* \subset \Omega$, with $\mathbf{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$, for any $f \in L_2([0,1), m)$, with $\int_{\mathbf{R}} \log(2+|y|)d\mu_f(y) < \infty$, the sequence*

$$\frac{1}{n}\sum_{k=1}^{n} f \circ \tau_{\theta_k(\omega)} \qquad \textit{converges } m\textit{-a.s.} \tag{19}$$

Proof. By applying Theorem 4.12 for $\mathbf{s}_n = n$ and the special flow induced by the rotations of the circle, and by considering the previous remark the result follows. □

Remark. This result is to be contrasted with Remark 2 of [3] (see Theorem A of [3]). Indeed, it was proved in [3], that in the case where θ_1 admits the Lebesgue measure m as distribution, for almost every realization $\{\theta_n(\omega)\}$, there exists a bounded f on $[0,1)$, such that the convergence in (19) fails. Hence, Corollary 4.13 shows that under the condition $\int_{\mathbf{R}} \log(2+|y|)d\mu_f(y) < \infty$ the question of A. Bellow, mentioned in [3] (i.e., the a.s. convergence in (19)), holds.

The spectral condition in Corollary 4.13, may be simplified, in this particular case, to obtain Theorem D. Indeed, let $f \in L_2([0,1), m)$ and write $f(t) = \sum_{n\in\mathbf{Z}} \hat{f}(n)e^{2i\pi nt}$. Then

$$\int_0^1 f(x+t)\overline{f}(x)dm(x) = \sum_{n\in\mathbf{Z}} |\hat{f}(n)|^2 e^{2i\pi nt}.$$

Hence, $\mu_f = \sum_{n\in\mathbf{Z}} |\hat{f}(n)|^2 \delta_{2n\pi}$, and Theorem D follows.

A Appendix: proofs of Theorem 4.9 and Theorem 4.10

Proof of Theorem 4.9. As we mentioned, we only need to prove the case $1 < p < 2$. Let $1 < q < 2$, such that $q > \frac{2p}{p+2\delta}$, and let $\{A_n\}$ be defined by $A_n = \log(n \vee |\mathbf{s}_n|^*)$, for every $n \geq 1$. We saw in the proof of Corollary 4.6, that under our conditions the sequence $\{\lambda_m(q)\}$, defined in (2), is well defined for $\{X_n\}$, for these choices of q and $\{A_n\}$. Hence, the conclusions of Proposition 2.1 and Proposition 2.2 hold. In particular, we saw that (13) holds, i.e., $\sum_{n=1}^{\infty}\sum_{2^n \leq A_k < 2^{n+1}} |\mathbf{E}[X_k \mathbf{1}_{\{|X_k| \leq \lambda_n\}}]| < \infty$.

Let $\{Y_n\}$ be the sequence of random variables defined by: $Y_k = X_k$ if $A_k < 1$, and $Y_k = X_k \mathbf{1}_{\{|X_k|\le\lambda_n\}} - \mathbf{E}[X_k \mathbf{1}_{\{|X_k|\le\lambda_n\}}]$ if $2^n \le A_k < 2^{n+1}$, for some $k, n \ge 1$. So, by the above absolute convergence, we need to prove the theorem only for the sequence $\{Y_n\}$.

As in the proof of Theorem 4.4, since Proposition 2.1 holds, for $\mathbf{P}$-almost every $\omega \in \Omega$ we have

$$\sum_{m=1}^{\infty} 2^{m/2} \Big(\sum_{A_k \ge 2^m} \big(|Y_k(\omega)|^q + \mathbf{E}[|Y_k|^q] \big) \Big)^{1/q} < \infty . \qquad (*)$$

On the other hand, we can deduce from Theorem 4.2, that for $\mathbf{P}$-almost every $\omega \in \Omega$, there exists $K_\omega > 0$ such that, for every $\eta \in \mathbf{R}$, every $\mathbf{t} \in \mathbf{R}^d$, and every $m > n \ge 1$, we have

$$\sup_{\mathbf{t}\in[-\pi,\pi)^d} \Big| \sum_{k=n+1}^{m} Y_k(\omega) \mathrm{e}^{-i\frac{p-1}{p}\eta \log k} \mathrm{e}^{i\langle \mathbf{s}_k, \mathbf{t}\rangle} \Big|^2 \le$$
$$K_\omega \log(2+|\eta|)(\log(m \vee |\mathbf{s}_m|^* + 1) \Big(\sum_{k=n+1}^{m} |Y_k(\omega)|^q + \|Y_k\|_q^q \Big)^{2/q} .$$

Let $\{T_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{N}^d}$ be the family of operators (on each $L_r(Y)$, $1 < r \le 2$) defined by $T_{\mathbf{s}}(f) = f \circ \tau_1^{s^{(1)}} \circ \cdots \circ \tau_d^{s^{(d)}}$, where $\mathbf{s} = (s^{(1)}, \ldots, s^{(d)})$. So, by the spectral theorem

$$\Big\| \sum_{k=n+1}^{m} Y_k(\omega) \mathrm{e}^{-i\frac{p-1}{p}\eta \log k} T_{\mathbf{s}_k}(f) \Big\|_2^2 \le$$
$$K_\omega \|f\|_2^2 \log(2+|\eta|)(\log(m \vee |\mathbf{s}_m|^* + 1) \Big(\sum_{k=n+1}^{m} |Y_k(\omega)|^q + \|Y_k\|_q^q \Big)^{2/q} .$$

Using Proposition 4.3 we have

$$\Big\| \max_{n<l\le m} \Big| \sum_{k=n+1}^{l} Y_k(\omega) \mathrm{e}^{-i\frac{p-1}{p}\eta \log k} T_{\mathbf{s}_k}(f) \Big| \Big\|_2^2 \le$$
$$K_\omega \|f\|_2^2 C_{2,2/q} \log(2+|\eta|)(\log(m \vee |\mathbf{s}_m|^* + 1) \Big(\sum_{k=n+1}^{m} |Y_k(\omega)|^q + \|Y_k\|_q^q \Big)^{2/q} . \qquad (**)$$

Finally, by our assumption for $\mathbf{P}$-almost every $\omega \in \Omega$, we have

$$\sum_{n=1}^{\infty} |Y_n(\omega)|_p^p (\log(n \vee |\mathbf{s}_n|^*))^{p-1+\delta} . \qquad (***)$$

Let Ω^* be the set on which (*), (**), and (***) are satisfied, and fix $\omega \in \Omega^*$.

For any complex numbers $\zeta = \xi + i\eta$, with $0 \le \xi \le 1$, and any $n \ge 1$, define the operator $\Psi_{n,\zeta}(\omega) = \sum_{k=1}^{n} Y_k(\omega) k^{-\frac{p-1}{p}\zeta} T_{\mathbf{s}_k}$. By (*) and (**) we may and do apply Theorem 3.1, with $A_n = K_\omega \|f\|_2^2 C_{2,2/q} \log(2+|\eta|) \log(n \vee |\mathbf{s}_n|^* + 1)$ and $\alpha_n = |Y_n(\omega)|^q + \|Y_n\|_q^q$, in order to obtain,

$$\Big\| \sup_{n\ge1} |\Psi_{n,i\eta}(\omega) f| \Big\|_2 \le C_1 \sqrt{\log(2+|\eta|)} \|f\|_2 \qquad \text{for any} \qquad f \in L_2(\pi) \qquad (20)$$

for some $C_1 > 0$, which does not depend either on η or on f. On the other hand, by (***) we have

$$\Big\| \sup_{n\geq 1} |\Psi_{n,1+i\eta}(\omega) f| \Big\|_1 \leq$$

$$\|f\|_1 \Big(\sum_{n=1}^{\infty} |Y_n(\omega)|_p^p (\log(n \vee |\mathbf{s}_n|^*))^{p-1+\delta} \Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} \frac{1}{n(\log(n \vee |\mathbf{s}_n|^*))^{1+\delta p/(p-1)}} \Big)^{\frac{p-1}{p}} < \infty \tag{21}$$

As in [8], (20) and (21) allows one to use Stein's complex interpolation theorem [30, Theorem XII.1.39] in order to evaluate $\Psi_{n,t}(\omega)$, with $\frac{1}{r} = \frac{1-t}{2} + t$, and to conclude that

$$\Big\| \sup_{n\geq 1} \Big| \sum_{k=1}^{n} \frac{Y_n(\omega) T_{\mathbf{s}_n}(f)}{n^{(2-r)(p-1)/pr}} \Big| \Big\|_r < \infty \qquad \text{for any} \quad f \in L_r(\pi)\,.$$

So, we deduce that (16) is satisfied. Now, since the series (15) converges π-a.s. for every $f \in L_2(\pi)$, by Corollary 4.6, we conclude by the Banach principle.

□

Proof of Theorem 4.10. We want to truncate the variables $\{\boldsymbol{\theta}_n\}$ by means of the assumption made on them. For any $k \geq 1$ put $\Omega_k = \{|\boldsymbol{\theta}_k| \leq \Phi(k)^\delta\}$, and by $\overline{\Omega}_k$ denote its complementary.

Let $m > n \geq 1$ and $\mathbf{t} \in \mathbf{R}^d$. We have

$$\begin{aligned}
\Big| \sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle}]) \Big| &\leq \Big| \sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k}]) \Big| \\
&\quad + \sum_{k=n+1}^{m} |a_n| \mathbf{1}_{\overline{\Omega}_k} + \sum_{k=n+1}^{m} \mathbf{E}[|a_n| \mathbf{1}_{\overline{\Omega}_k}] \\
&\leq \Big| \sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k}]) \Big| \\
&\quad + \Big(\sum_{k=n+1}^{m} |a_k|^2 \Big)^{1/2} \Big(\sum_{l=1}^{\infty} \mathbf{1}_{\overline{\Omega}_l} \Big)^{1/2} + \Big(\sum_{k=n+1}^{m} |a_k|^2 \Big)^{1/2} \Big(\sum_{l=1}^{\infty} (\mathbf{P}[\overline{\Omega}_l])^2 \Big)^{1/2}.
\end{aligned}$$

Using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we deduce that

$$\frac{\big| \sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle}]) \big|^2}{\log(1+T)\log(1+\Phi(m)) \sum_{k=n+1}^{m} |a_k|^2} \leq$$

$$3 \cdot \frac{|\sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k}])|^2}{\log(1+T)\log(1+\Phi(m)) \sum_{k=n+1}^{m} |a_k|^2} + \frac{3}{(\log 2)^2} \sum_{l=1}^{\infty} \mathbf{1}_{\overline{\Omega}_l} + \frac{3}{(\log 2)^2} \sum_{l=1}^{\infty} \mathbf{P}[\overline{\Omega}_l].$$

Hence, for every $\alpha > 0$ we have

$$\mathbf{E}\Big[\sup_{T\geq 1} \sup_{m>n\geq 1} \exp\Big\{ \alpha \cdot \frac{\max_{|\mathbf{t}|\leq T} \big| \sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle}]) \big|^2}{\log(1+T)\log(1+\Phi(m)) \sum_{k=n+1}^{m} |a_k|^2} \Big\} \Big] \leq$$

$$\mathbf{E}\Big[\sup_{m>n\geq 1} \sup_{T\geq 1} \exp\Big\{ 3\alpha \cdot \frac{\max_{|\mathbf{t}|\leq T} \big| \sum_{k=n+1}^{m} a_k (\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k} - \mathbf{E}[\mathrm{e}^{i\langle \boldsymbol{\theta}_k, \mathbf{t}\rangle} \mathbf{1}_{\Omega_k}]) \big|^2}{\log(1+T)\log(1+\Phi(m)) \sum_{k=n+1}^{m} |a_k|^2} \Big\} \times$$

$$\exp\Big\{ \frac{3\alpha}{(\log 2)^2} \sum_{l=1}^{\infty} \mathbf{1}_{\overline{\Omega}_l} \Big\} \Big] \times \exp\Big\{ \frac{3\alpha}{(\log 2)^2} \sum_{l=1}^{\infty} \mathbf{P}[\overline{\Omega}_l] \Big\}.$$

By Cauchy-Schwartz inequality (in $L_2(\mathbf{P})$) for the first two factors above, and by putting $C_1 = \exp\big\{\frac{3\alpha}{(\log 2)^2}\sum_{l=1}^{\infty}\mathbf{P}[\overline{\Omega}_l]\big\}$, we have

$$\mathbf{E}\Big[\sup_{T\geq 1}\sup_{m>n\geq 1}\exp\Big\{\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k(\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}-\mathbf{E}[\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}])\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k|^2}\Big\}\Big]\leq$$

$$C_1\Big(\mathbf{E}\Big[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\Big\{6\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k(\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k}-\mathbf{E}[\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k}])\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k|^2}\Big\}\Big]\Big)^{1/2}$$
$$\times\Big(\mathbf{E}\Big[\exp\Big\{\frac{6\alpha}{(\log 2)^2}\sum_{l=1}^{\infty}\mathbf{1}_{\overline{\Omega}_l}\Big\}\Big]\Big)^{1/2}.$$

Put $L = 6\alpha/(\log 2)^2$. Now, by independence and since $\sum_{n=1}^{\infty}\mathbf{P}[\overline{\Omega}_n]<\infty$, we have

$$\mathbf{E}\big[\exp(L\sum_{l=1}^{\infty}\mathbf{1}_{\overline{\Omega}_l})\big]=\prod_{l=1}^{\infty}\mathbf{E}[\exp(L\mathbf{1}_{\overline{\Omega}_l})]\leq\prod_{l=1}^{\infty}\Big(1+(e^L-1)\mathbf{P}[\overline{\Omega}_l]\Big)<\infty\,.$$

Hence it is enough to find sufficiently small $\alpha>0$, such that the quantity below

$$\mathbf{E}\Big[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\Big\{\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k(\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k}-\mathbf{E}[\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k}])\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k|^2}\Big\}\Big],$$

which we denoted by A, is finite.

Let $\{\boldsymbol{\theta}'_n\}$ be an independent copy of $\{\boldsymbol{\theta}_n\}$, living on a probability space $(\Omega',\mathcal{F}',\mathbf{P}')$. We denote by $\mathbf{E}'$ the corresponding expectation.

Using the positivity of the expectation and the convexity of $x\mapsto\exp(\alpha x^2)$, we have

$$A\leq\mathbf{E}\mathbf{E}'\Big[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\Big\{\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k(\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k}-\mathrm{e}^{i\langle\boldsymbol{\theta}'_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k})\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k|^2}\Big\}\Big]$$

Let $\{\varepsilon_n\}$ be a sequence of Rademacher random variables, living on $(\Omega_\varepsilon,\mathcal{F}_\varepsilon,\mathbf{P}_\varepsilon)$. We denote by $\mathbf{E}_\varepsilon$ the corresponding expectation. Put $\Omega'_k=\{|\boldsymbol{\theta}'_n|\leq\Phi(n)^\delta\}$ and by $\overline{\Omega}'_k$ denote its complementary.

Notice that the sequence of functions $\{\mathrm{e}^{i\langle\boldsymbol{\theta}_n,\mathbf{t}\rangle}\mathbf{1}_{\Omega_n}-\mathrm{e}^{i\langle\boldsymbol{\theta}'_n,\mathbf{t}\rangle}\mathbf{1}_{\Omega'_n}\}$, with values in $(C([-T,T]^d),\|.\|_\infty)$, is symmetric independent, so it is stochastically equivalent to the sequence $\{\epsilon_n(\mathrm{e}^{i\langle\boldsymbol{\theta}_n,\mathbf{t}\rangle}\mathbf{1}_{\Omega_n}-\mathrm{e}^{i\langle\boldsymbol{\theta}'_n,\mathbf{t}\rangle}\mathbf{1}_{\Omega'_n})\}$. Hence,

$$A\leq\mathbf{E}\mathbf{E}'\mathbf{E}_\epsilon\Big[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\Big\{\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k\epsilon_k(\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega_k}-\mathrm{e}^{i\langle\boldsymbol{\theta}'_k,\mathbf{t}\rangle}\mathbf{1}_{\Omega'_k})\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k|^2}\Big\}\Big].$$

Using $(a+b)^2\leq 2(a^2+b^2)$; Cauchy-Schwartz inequality in $L_2(\Omega_\epsilon)$; independence of $\{\boldsymbol{\theta}_n\}$ and $\{\boldsymbol{\theta}'_n\}$; Jensen's inequality for the concave function $x\mapsto\sqrt{x}$; the fact that $\{\boldsymbol{\theta}_n\}$ and $\{\boldsymbol{\theta}'_n\}$ are independent copy, we obtain

$$\begin{aligned}A&\leq\mathbf{E}\mathbf{E}_\epsilon\Big[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\Big\{4\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k\mathbf{1}_{\Omega_k}\epsilon_k\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k|^2}\Big\}\Big]\\&\leq\mathbf{E}\mathbf{E}_\epsilon\Big[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\Big\{4\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\big|\sum_{k=n+1}^{m}a_k\mathbf{1}_{\Omega_k}\epsilon_k\mathrm{e}^{i\langle\boldsymbol{\theta}_k,\mathbf{t}\rangle}\big|^2}{\log(1+T)\log(1+\Phi(m))\sum_{k=n+1}^{m}|a_k\epsilon_k\mathbf{1}_{\Omega_k}|^2}\Big\}\Big].\end{aligned}$$

For every $\omega\in\Omega$, we consider the sequence of independent and bounded random variables $\{Y_n\}$ and the sequence $\{\boldsymbol{\gamma}_n\}\subset\mathbf{R}^d$, given by: $Y_n:=a_n\epsilon_n\mathbf{1}_{\Omega_n}$ and $\boldsymbol{\gamma}_n(\omega):=\boldsymbol{\theta}_n(\omega)$, if $\omega\in\Omega_n$, and $\boldsymbol{\gamma}_n(\omega):=(0,\ldots,0)$ otherwise.

By Proposition 4.2, there exist constants $C, \alpha > 0$, independent of ω (see Remark 1 after the formulation of Theorem 4.10), such that

$$\mathbf{E}_\epsilon\left[\sup_{m>n\geq 1}\sup_{T\geq 1}\exp\left\{4\alpha\cdot\frac{\max_{|\mathbf{t}|\leq T}\left|\sum_{k=n+1}^m a_k\mathbf{1}_{\Omega_k}\epsilon_k e^{i\langle\boldsymbol{\gamma}_k(\omega),\mathbf{t}\rangle}\right|^2}{\log(1+T)\log(m\vee|\boldsymbol{\gamma}_m(\omega)|^*)\sum_{k=n+1}^m|a_k\epsilon_k\mathbf{1}_{\Omega_k}|^2}\right\}\right]\leq C.$$

Now, by the assumption on Φ and by the construction of $\{\boldsymbol{\gamma}_n(\omega)\}$, there exists $D > 0$, such that $\log(m\vee|\boldsymbol{\gamma}_m(\omega)|^*)\leq D\log(1+\Phi(m))$, for every $\omega\in\Omega$ and for every $m\geq 1$. Hence, by taking the expectation $\mathbf{E}$ from both sides the result follows. □

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