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THE CLT FOR ROTATED ERGODIC SUMS AND RELATED PROCESSES

ABSTRACT. Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be an ergodic dynamical system. The rotated ergodic sums of a function f on Ω for $\theta \in \mathbb{R}$ are $S_n^\theta f := \sum_{k=0}^{n-1} e^{2\pi i k \theta} f \circ \tau^k$, $n \geq 1$. Using Carleson's theorem on Fourier series, Peligrad and Wu proved in [14] that $(S_n^\theta f)_{n \geq 1}$ satisfies the CLT for a.e. θ when $(f \circ \tau^n)$ is a regular process.

Our aim is to extend this result and give a simple proof based on the Fejér-Lebesgue theorem. The results are expressed in the framework of processes generated by K -systems. We also consider the invariance principle for modified rotated sums. In a last section, we extend the method to \mathbb{Z}^d -dynamical systems.

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Introduction. Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a dynamical system, i.e., a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a measurable transformation $\tau : \Omega \rightarrow \Omega$ which preserves \mathbb{P} . The rotated ergodic sums of a function f on Ω are defined for $\theta \in \mathbb{R}$ by

$$S_n^\theta f := \sum_{k=0}^{n-1} e^{2\pi i k \theta} f \circ \tau^k, \quad n \geq 1. \quad (1)$$

Using Carleson's theorem on Fourier series, Peligrad and Wu proved in [14] that $(S_n^\theta f)_{n \geq 1}$ satisfies the CLT for a.e. θ when $(f \circ \tau^n)$ is a regular process.

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It is remarkable that this result does not require a rate of decorrelation or regularity of the function f generating the stationary process $(f \circ \tau^n)$. Our aim is to give a simple proof based on the Fejér-Lebesgue theorem of this result and to extend it. The results are expressed in the framework of processes generated by K -systems. We also consider the invariance principle for modified rotated sums like in [13].

As an application, we prove a recurrence property for a class of rotated stationary processes. In the last section, the method is extended to \mathbb{Z}^d -dynamical systems. Other extensions will be presented in a further paper.

The proofs are based on the approximation of the ergodic sums by a martingale and lead to two separate questions: validity of a mean square approximation by a martingale, limiting behavior of the approximating martingale.

1. Preliminaries.

1.1. Rotated ergodic sums, T -filtrations. Let T be a unitary operator on a Hilbert space \mathcal{H} . The *spectral measure* with respect to T of a vector $f \in \mathcal{H}$ is the positive measure ν_f on the unit circle \mathbb{T}^1 with Fourier coefficients $\hat{\nu}_f(n) = \langle T^n f, f \rangle, n \in \mathbb{Z}$. The results in this section are purely spectral, although later T will be the Koopman operator induced by an automorphism.

We denote by φ_f the density of the absolutely continuous part of the spectral measure ν_f with respect to the Lebesgue measure λ on \mathbb{T}^1 . Let K_n be the Fejér kernel. For the rotated ergodic sums $S_n^\theta f$ defined by (1), we have

$$\frac{1}{n} \|S_n^\theta f\|_2^2 = \int_0^1 \frac{1}{n} \left| \frac{\sin \pi n(t - \theta)}{\sin \pi(t - \theta)} \right|^2 \nu_f(dt) = (K_n * \nu_f)(\theta). \quad (2)$$

This formula implies by the Fejér-Lebesgue theorem (cf. [19, Ch. III, Th. 8.1]):

Lemma 1.1. *For every $f \in L_0^2(\mathbb{P})$, we have, for a.e. θ , $\lim_{n \rightarrow \infty} \frac{1}{n} \|S_n^\theta f\|_2^2 = \varphi_f(\theta)$.*

It follows that the asymptotic variance of the rotated sums $\lim_{n \rightarrow \infty} \frac{1}{n} \|S_n^\theta f\|_2^2$ exists for a.e. θ . Therefore a question is the behavior of the normalized ergodic sums in distribution, for a.e. θ .

Remark that $\|S_n^\theta f\|_2 = o(\sqrt{n})$ for a.e. θ , if the spectral measure is singular.

T -filtrations

Definitions, notations 1.2. Let T be a unitary operator on a Hilbert space \mathcal{H} . A closed subspace \mathcal{H}_0 of \mathcal{H} such that $T^{-1}\mathcal{H}_0 \subset \mathcal{H}_0$ defines a Hilbertian T -filtration \mathcal{F} , i.e., an increasing family of closed subspaces \mathcal{H}_n of \mathcal{H} with $\mathcal{H}_n = T^n \mathcal{H}_0, \forall n \in \mathbb{Z}$.

The closed subspace generated by $\bigcup_{n \in \mathbb{Z}} \mathcal{H}_n$ is denoted by \mathcal{H}_∞ and the intersection $\bigcap_{n \in \mathbb{Z}} \mathcal{H}_n$ by $\mathcal{H}_{-\infty}$. We say that an element $f \in \mathcal{H}$ is \mathcal{F} -regular (or simply regular), if f belongs to \mathcal{H}_∞ and is orthogonal to $\mathcal{H}_{-\infty}$.

Orthogonal decomposition Let Π_n be the orthogonal projection on the subspace $\mathcal{K}_n := \mathcal{H}_n \ominus \mathcal{H}_{n-1}$ of vectors in \mathcal{H}_n which are orthogonal to \mathcal{H}_{n-1} . We have the orthogonal decomposition $\mathcal{H}_\infty = \mathcal{H}_{-\infty} \oplus_{n \in \mathbb{Z}} \mathcal{K}_n$ and for $f \in \mathcal{H}_\infty$:

$$f = \Pi_{-\infty} f + \sum_{n \in \mathbb{Z}} \Pi_n f, \quad \|f\|_2^2 = \|\Pi_{-\infty} f\|_2^2 + \sum_{n \in \mathbb{Z}} \|\Pi_n f\|_2^2. \quad (3)$$

Note that $\mathcal{K}_\ell = T^\ell \mathcal{K}_0$ and, for every g in \mathcal{K}_0 , $\Pi_\ell T^k g = \delta_{k,\ell} T^k g, \forall k, \ell \in \mathbb{Z}$, where $\delta_{k,\ell}$ is the Kronecker symbol.

Let $(\psi_j)_{j \in J}$ be an orthonormal basis of $\mathcal{K}_0 = \mathcal{H}_0 \ominus \mathcal{H}_{-1}$. It yields an orthonormal basis $(T^k \psi_j)_{j \in J, k \in \mathbb{Z}}$ of the subspace $\bigoplus_{n \in \mathbb{Z}} \mathcal{K}_n$ generated by the filtration. Let f be a \mathcal{F} -regular function in \mathcal{H}_∞ . By restricting to the closed subspace generated by $(T^n f)_{n \in \mathbb{Z}}$, we can assume that J is countable. Setting

$$a_{j,n} := \langle f, T^n \psi_j \rangle, \quad (4)$$

we have $\Pi_n f = \sum_{j \in J} a_{j,n} T^n \psi_j$ and $\|f\|_2^2 = \sum_{n \in \mathbb{Z}} \|\Pi_n f\|_2^2 = \sum_{n \in \mathbb{Z}} \sum_{j \in J} |a_{j,n}|^2$.

Notations 1.3. For each $j \in J$, let γ_j be an everywhere finite square integrable function with Fourier coefficient $a_{j,n}$, $n \in \mathbb{Z}$, defined by (4), that is,

$$\gamma_j(t) = \sum_{n \in \mathbb{Z}} a_{j,n} e^{2\pi i n t}. \quad (5)$$

A simple computation shows that $|\gamma_j|^2$ is the spectral density of the orthogonal projection f_j of f on the subspace generated by $(T^k \psi_j)_{k \in \mathbb{Z}}$; hence

$$\varphi_f = \sum_{j \in J} |\gamma_j|^2 \text{ and } \int \sum_{j \in J} |\gamma_j(\theta)|^2 d\theta = \sum_{j \in J} \sum_{k=-\infty}^{\infty} |a_{j,k}|^2 = \int \varphi_f(\theta) d\theta = \|f\|_2^2. \quad (6)$$

By (6), the set $\Lambda_0 := \{\theta \in \mathbb{T} : \sum_{j \in J} |\gamma_j(\theta)|^2 < \infty\}$ has full measure. For $\theta \in \Lambda_0$, we define $M_\theta f \in \mathcal{K}_0$ by:

$$M_\theta f := \sum_j \gamma_j(\theta) \psi_j.$$

The function $\theta \rightarrow \|M_\theta f\|_{2, \mathbb{P}}^2 = \sum_{j \in J} |\gamma_j(\theta)|^2$ is a version of the spectral density φ_f of f .

1.2. Approximation of the rotated ergodic sums. Here we show that, for almost every θ , the process with orthogonal increments $(\sum_0^{n-1} e^{2\pi i k \theta} T^k M_\theta)_{n \geq 1}$ approximates the rotated process in mean square.

Proposition 1.4. *If $f \in \mathcal{H}_\infty$ is \mathcal{F} -regular, the set $\Lambda(f) \subset \Lambda_0$ of $\theta \in \mathbb{T}^1$, such that*

$$\frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi i k \theta} T^k (f - M_\theta f) \right\|_2^2 \rightarrow 0,$$

has full Lebesgue measure in \mathbb{T}^1 .

Proof. The spectral density of $f - M_\theta f$ is

$$\varphi_{f - M_\theta f}(t) = \sum_{j \in J} |\gamma_j(t) - \gamma_j(\theta)|^2 = \sum_j |\gamma_j(t)|^2 + \sum_j |\gamma_j(\theta)|^2 - \sum_j \gamma_j(t) \bar{\gamma}_j(\theta) - \sum_j \bar{\gamma}_j(t) \gamma_j(\theta).$$

We have (cf. Lemma 1.1):

$$\begin{aligned} \frac{1}{n} \|S_n^\theta(f - M_\theta f)\|_2^2 &= \int_0^1 \frac{1}{n} \left| \sum_{k=0}^{n-1} e^{2\pi i k(t-\theta)} \right|^2 \varphi_{f - M_\theta f}(t) dt \\ &= \int_0^1 K_n(t - \theta) \varphi_{f - M_\theta f}(t) dt. \end{aligned} \quad (7)$$

Let Λ'_0 be the set of full measure of θ 's such that the convergence given by the Fejér-Lebesgue theorem holds at θ for the functions $|\gamma_j|^2$, γ_j , $\forall j \in J$, and $\sum_{j \in J} |\gamma_j|^2$.

Take $\theta \in \Lambda_0 \cap \Lambda'_0$. Let $\varepsilon > 0$ and let $J_0 = J_0(\varepsilon, \theta)$ be a finite subset of J such that $\sum_{j \notin J_0} |\gamma_j(\theta)|^2 < \varepsilon$. Since

$$\begin{aligned} & \lim_n \int_0^1 K_n(t - \theta) \sum_{j \notin J_0} |\gamma_j(t)|^2 dt \\ = & \lim_n \int_0^1 K_n(t - \theta) \sum_{j \in J} |\gamma_j(t)|^2 dt - \lim_n \int_0^1 K_n(t - \theta) \sum_{j \in J_0} |\gamma_j(t)|^2 dt = \sum_{j \notin J_0} |\gamma_j(\theta)|^2, \end{aligned}$$

we have

$$\begin{aligned} & \limsup_n \int_0^1 K_n(t - \theta) \varphi_{f - M_\theta f}(t) dt \\ & \leq \lim_n \int_0^1 K_n(t - \theta) \sum_{j \in J_0} [|\gamma_j(t)|^2 + |\gamma_j(\theta)|^2 - \gamma_j(t)\bar{\gamma}_j(\theta) - \bar{\gamma}_j(t)\gamma_j(\theta)] dt \\ & \quad + 2 \lim_n \int_0^1 K_n(t - \theta) \sum_{j \notin J_0} [|\gamma_j(t)|^2 + |\gamma_j(\theta)|^2] dt \\ & = \sum_{j \in J_0} [|\gamma_j(\theta)|^2 + |\gamma_j(\theta)|^2 - \gamma_j(\theta)\bar{\gamma}_j(\theta) - \bar{\gamma}_j(\theta)\gamma_j(\theta)] + 4 \sum_{j \notin J_0} |\gamma_j(\theta)|^2 \leq 0 + 4\varepsilon. \end{aligned}$$

This shows that $\Lambda_0 \cap \Lambda'_0 \subset \Lambda(f)$; hence $\Lambda(f)$ has full measure. \square

Corollary 1.5. *Let $f_{-\infty}$ be the orthogonal projection on $\mathcal{H}_{-\infty}$ of a vector $f \in \mathcal{H}_\infty$. If the spectral measure of $f_{-\infty}$ is singular, then for a.e. θ*

$$\frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi i k \theta} T^k (f - M_\theta(f - f_{-\infty})) \right\|_2^2 \rightarrow 0.$$

Proof. By Lemma 1.1, $\lim_n \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi i k \theta} T^k f_{-\infty} \right\|_2^2 = 0$ for a.e. θ . As $f - f_{-\infty}$ is regular, Proposition 1.4 implies the result. \square

Remark 1.6. We may also deal with T isometry. For example, when τ is an endomorphism (i.e., non-invertible measure preserving), the induced Koopman operator T is only an isometry. We may transfer some results which hold for automorphisms to endomorphisms in the following way (cf. Nagy-Foias [12, Proposition 6.2]). If U is the unitary dilation of T and h is a polynomial, by the construction of the dilation we obtain $\|h(T)\| \leq \|h(U)\|$. In particular, Proposition 1.4 holds for an endomorphism and the associated decreasing filtration. The spectral measure that is used in the proof is replaced by the spectral measure of f with respect to the unitary dilation. Analogous results hold for any commuting finite family of endomorphisms.

The following result will be used later for the computation of the variance for rotated processes.

Lemma 1.7. *Let μ be an atomless finite complex measure on $[-\frac{1}{2}, \frac{1}{2}]$ and denote by $(\hat{\mu}(n))$ its Fourier-Stieltjes coefficients. Let $\theta_1, \theta_2 \in [-\frac{1}{2}, \frac{1}{2}]$ with $\theta_1 \neq \theta_2$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{p=0}^{n-1} e^{2\pi i (k\theta_1 - p\theta_2)} \hat{\mu}(k-p) = 0.$$

Proof. It is enough to prove that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \hat{\mu}(\ell) e^{2\pi i \ell \theta_2} \sum_{k=0}^{\ell} e^{2\pi i k(\theta_1 - \theta_2)} = 0$.

From the inequality $|\sum_{k=1}^{\ell} e^{2\pi i k(\theta_1 - \theta_2)}| \leq \frac{C}{|\theta_1 - \theta_2|}$ and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \left| \frac{1}{n} \sum_{\ell=0}^{n-1} \hat{\mu}(\ell) e^{2\pi i \ell \theta_2} \sum_{k=0}^{\ell} e^{2\pi i k(\theta_1 - \theta_2)} \right|^2 &\leq \left(\frac{1}{n} \sum_{\ell=0}^{n-1} |\hat{\mu}(\ell)|^2 \right) \left(\frac{1}{n} \sum_{\ell=0}^{n-1} \left| \sum_{k=0}^{\ell} e^{2\pi i k(\theta_1 - \theta_2)} \right|^2 \right) \\ &\leq \frac{C_{\theta_1, \theta_2}}{n} \sum_{\ell=0}^{n-1} |\hat{\mu}(\ell)|^2 \rightarrow C_{\theta_1, \theta_2} \sum_x |\mu\{x\}|^2 = 0. \end{aligned}$$

□

For a unitary operator T and for $f_1, f_2 \in L^2(\Omega)$, there is a complex spectral measure $\mu_{1,2}$ on the torus (*the cross spectral measure* of (f_1, f_2) with respect to T) such that $\hat{\mu}(k) := \langle T^k f_1, f_2 \rangle = \int e^{-2\pi i k t} d\mu_{1,2}(t)$. The measure $\mu_{1,2}$ is a linear combination of the spectral measures of $f_1 \pm f_2$ and $f_1 \pm i f_2$ with respect to T . If T is weakly mixing, then $\mu_{1,2}$ is continuous for every f_1 and f_2 .

Proposition 1.8. *Let T be a unitary operator on \mathcal{H} and let $f_1, f_2 \in \mathcal{H}$. Denote by φ_{f_i} , $i = 1, 2$ the densities of the absolutely continuous part of the spectral measures of f_i with respect to T . Assume that (f_1, f_2) has an atomless cross spectral measure with respect to T . Then for a.e. $(\theta_1, \theta_2) \in \mathbb{T}^2$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} (e^{2\pi i k \theta_1} T^k f_1 + e^{2\pi i k \theta_2} T^k f_2) \right\|^2 = \varphi_{f_1}(\theta_1) + \varphi_{f_2}(\theta_2).$$

Proof. We expand

$$\begin{aligned} \frac{1}{n} \left\| \sum_{k=0}^{n-1} (e^{2\pi i k \theta_1} T^k f_1 + e^{2\pi i k \theta_2} T^k f_2) \right\|^2 &= \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi i k \theta_1} T^k f_1 \right\|^2 + \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi i k \theta_2} T^k f_2 \right\|^2 + \\ &\quad \frac{2}{n} \Re \left\langle \sum_{k=0}^{n-1} e^{2\pi i k \theta_1} T^k f_1, \sum_{k=0}^{n-1} e^{2\pi i k \theta_2} T^k f_2 \right\rangle. \end{aligned}$$

By the Fejér-Lebesgue theorem, for a.e. θ_1, θ_2 the first two terms tend respectively to $\varphi_{f_1}(\theta_1)$ and $\varphi_{f_2}(\theta_2)$. The inner product term tends to 0 by the previous lemma. □

The results of this section were of spectral nature and valid for general stationary processes of second order. Now we apply them to regular processes, with the terminology of dynamical systems and specifically of K -systems.

2. CLT for rotated processes. Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a dynamical system. If $\mathcal{B} \subset \mathcal{A}$ is a sub σ -algebra, then $L_0^2(\mathcal{B})$ is the subspace of centered \mathcal{B} -measurable functions in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Recall (cf. Rohlin [15]) that any dynamical system has a largest zero entropy factor $\mathcal{P}(\tau)$ (Pinsker factor) and a sub σ -algebra \mathcal{A}_0 of \mathcal{A} with the following properties: (i) (*increasing*) $\mathcal{A}_0 \subset \tau^{-1} \mathcal{A}_0$; (ii) (*generating*) $\mathcal{A}_\infty := \sigma(\cup_{n=-\infty}^{\infty} \tau^{-n} \mathcal{A}_0) = \mathcal{A}$; (iii) $\mathcal{A}_{-\infty} := \cap_{n=-\infty}^{\infty} \tau^{-n} \mathcal{A}_0 = \mathcal{P}(\tau)$.

The increasing sequence $\mathcal{A}_n := \tau^{-n} \mathcal{A}_0$ defines a *filtration* with the corresponding Hilbertian T -filtration:

$$L_0^2(\mathcal{P}(\tau)) = L_0^2(\mathcal{A}_{-\infty}) \subset \dots \subset L_0^2(\mathcal{A}_{-1}) \subset L_0^2(\mathcal{A}_0) \subset L_0^2(\mathcal{A}_1) \subset \dots \subset L_0^2(\mathcal{A}_\infty) = L_0^2(\mathcal{A}).$$

Every function in $L^2(\mathcal{A})$ which is orthogonal to $L^2(\mathcal{P}(\tau))$ is regular for the associated Hilbertian T -filtration.

In particular, when the system is a K -system (equivalently has completely positive entropy, i.e., with a trivial Pinsker factor), there exists an increasing generating sub σ -algebra \mathcal{A}_0 of \mathcal{A} such that $\mathcal{A}_{-\infty}$ is trivial. Every function in $L_0^2(\mathbb{P})$ is then regular for the associated Hilbertian T -filtration. Therefore Proposition 1.4 applies to K -systems.

If the Pinsker factor is not trivial but has a singular spectral type, i.e., if the spectral measure ν_f is singular for every $f \in L_0^2(\mathcal{P}(\tau))$, then Corollary 1.5 applies.

When τ is an endomorphism we have a *decreasing* filtration $\mathcal{A}_n = \tau^{-n}\mathcal{A}$, $n \geq 0$. Here $\mathcal{A}_\infty = \mathcal{A}$ and $\mathcal{A}_{-\infty}$ is replaced by $\bigcap_{n=0}^\infty \mathcal{A}_n$. This induces a *one-sided* Hilbertian T -filtration. Every function in $L^2(\mathcal{A})$ orthogonal to $L^2(\mathcal{A}_{-\infty})$ is regular. If the system is *exact*, that is, $\mathcal{A}_{-\infty}$ is trivial, every function in $L_0^2(\mathcal{A})$ is regular. Using the unitary dilation (see Remark 1.6) and reversing the filtration order, Proposition 1.4 holds for endomorphisms.

2.1. CLT. In this section we show a Central Limit Theorem (CLT) for “rotated” processes.

Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a K -system. Let \mathcal{A}_0 be an increasing and generating sub- σ algebra of \mathcal{A} as in the beginning of Section 2 and $\mathcal{A}_n = \tau^{-n}\mathcal{A}_0$. With the notations of Subsection 1.1, we put $\mathcal{K}_0 := L_0^2(\mathcal{A}_0) \ominus L_0^2(\mathcal{A}_{-1})$.

For $\theta \in \mathbb{R}$, we consider the product system $(\mathbb{T}^1 \times \Omega, \lambda \otimes \mathbb{P}, \tau_\theta)$, where

$$\tau_\theta : (x, \omega) \rightarrow (x + \theta \bmod 1, \tau\omega).$$

The Pinsker factor of τ_θ is isomorphic to the rotation by θ on the circle. Let $(\psi_j)_{j \in J}$ be an orthonormal family of real functions which is a basis of \mathcal{K}_0 .

Let us consider a function $F \in L^2(\mathbb{T}^1 \times \Omega)$ orthogonal to the functions depending only on ω ,

$$F(x, \omega) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} e^{2\pi i \ell x} f_\ell(\omega). \quad (8)$$

We investigate the question of the CLT for $\sum_{k=0}^{n-1} F(x+k\theta, \tau^k\omega)$ for a.e. θ . When $F(x, \omega) = e^{2\pi i x} f(\omega)$, we will obtain the CLT for “rotated ergodic sums” of Peligrad and Wu mentioned in the introduction.

We always assume $\int f_\ell d\mathbb{P} = 0$. Let φ_{f_ℓ} be the density of the spectral measure of f_ℓ for the action of τ .

Like in Subsection 1.1, (cf. Formula (4)), the orthonormal basis $(T^k \psi_j)_{k \in \mathbb{Z}, j \in J}$ of $L_0^2(\mathcal{A})$ is used to decompose the components f_ℓ of F . The L^2 -norm of F is

$$\|F\|_2^2 = \sum_{\ell \neq 0} \|f_\ell\|_2^2 = \sum_j \sum_{\ell \neq 0} \sum_k |\langle f_\ell, T^k \psi_j \rangle|^2 < \infty.$$

For each $j \in J$, let $\gamma_{j,\ell}$ be an everywhere finite square integrable function with Fourier series $\gamma_{j,\ell}(t) = \sum_n \langle f_\ell, T^n \psi_j \rangle e^{2\pi i n t}$ (cf. (5)). Let $\tilde{f}_\ell(x, \omega) := e^{2\pi i \ell x} f_\ell(\omega)$. The spectral density of \tilde{f}_ℓ for τ_θ is $\varphi_{\tilde{f}_\ell}(t + \ell\theta) = \sum_j |\gamma_{j,\ell}(t + \ell\theta)|^2$ and we have

$$\sum_j \int |\gamma_{j,\ell}(t + \ell\theta)|^2 dt = \sum_j \int |\gamma_{j,\ell}(t)|^2 dt = \sum_j \sum_k |\langle f_\ell, T^k \psi_j \rangle|^2.$$

The spectral density of F in the dynamical system $(\mathbb{T}^1 \times \Omega, \tau_\theta)$ is

$$\varphi_F(t) = \sum_{\ell \neq 0} \sum_j |\gamma_{j,\ell}(t + \ell\theta)|^2.$$

The above formula defines for every θ a nonnegative function in $L^1(\mathbb{T}^1)$ with integral $\int_{\mathbb{T}^1} \varphi_F dt = \|F\|_2^2$. The *variance* at time n is

$$\begin{aligned} & \frac{1}{n} \int_{\mathbb{T}^1 \times \Omega} \left| \sum_{k=0}^{n-1} F(x + k\theta, \tau^k \omega) \right|^2 dx \mathbb{P}(d\omega) = \\ & \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \int_{\Omega} \left| \sum_{k=0}^{n-1} e^{2\pi i \ell k \theta} f_\ell(\tau^k \omega) \right|^2 \mathbb{P}(d\omega) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} (K_n * \varphi_{f_\ell})(\ell\theta). \end{aligned}$$

The asymptotic variance $\sigma_\theta^2(F)$ for the action of τ_θ exists whenever the limit $\lim_n \sum_{\ell \in \mathbb{Z} \setminus \{0\}} (K_n * \varphi_{f_\ell})(\ell\theta)$ exists and is finite.

Definition of the approximating martingale The functions $\sum_j \gamma_{\ell,j}(\ell\theta) \psi_j$ are in \mathcal{K}_0 . Let $M_\theta f$ (or simply M_θ) be defined by:

$$M_\theta(x, \omega) := \sum_{\ell \neq 0} e^{2\pi i \ell x} \sum_j \gamma_{\ell,j}(\ell\theta) \psi_j. \quad (9)$$

We have

$$\int_{\mathbb{T}^1 \times \Omega} |M_\theta(x, \omega)|^2 dx \mathbb{P}(d\omega) = \sum_{\ell \neq 0} \sum_{j \in J} |\gamma_{j,\ell}(\ell\theta)|^2 = \sum_{\ell \neq 0} \varphi_{f_\ell}(\ell\theta). \quad (10)$$

Since $\sum_{\ell \neq 0} \sum_{j \in J} \int |\gamma_{j,\ell}(\ell\theta)|^2 d\theta = \sum_{\ell \neq 0} \sum_{j \in J} \int |\gamma_{j,\ell}(\theta)|^2 d\theta = \|F\|_2^2 < \infty$, M_θ is well defined for θ in the set of *full measure*

$$\Lambda_0(F) := \left\{ \theta \in \mathbb{T} : \sum_{\ell \neq 0} \sum_{j \in J} |\gamma_{j,\ell}(\ell\theta)|^2 < \infty \right\}.$$

Denote by \mathcal{B} the Borel σ -algebra of \mathbb{T} . With respect to the filtration $(\mathcal{B} \times \mathcal{A}_n)$, $(M_\theta(x + n\theta, \tau^n \omega))_{n \in \mathbb{Z}}$ is a τ_θ -stationary ergodic sequence of differences of martingale, with variance given by (10). Notice that M_θ is real valued if F is a real valued function. The following two propositions are the main steps in the proof of the CLT for rotated sums.

Proposition 2.1. *Let F be a real function in $L^2(\mathbb{T}^1 \times \Omega)$ represented as in (8). For $\theta \in \Lambda_0(F)$, the asymptotic distribution with respect to the measure $\lambda \otimes \mathbb{P}$ on $\mathbb{T}^1 \times \Omega$ of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} M_\theta(x + k\theta, \tau^k \omega)$ is the normal law with variance $\sum_{\ell \neq 0} \varphi_{f_\ell}(\ell\theta)$.*

Proof. The result follows from Billingsley-Ibragimov theorem on stationary ergodic martingale differences (cf. [9]). \square

Proposition 2.2. *Suppose $\sum_\ell \|f_\ell\|_2 < \infty$.*

1) *Then we have*

$$\sum_\ell \sup_n (K_n * \varphi_{f_\ell})(\ell\theta) < \infty, \text{ for a.e. } \theta. \quad (11)$$

2) *The asymptotic variance $\sigma_\theta^2(F) = \sum_\ell \varphi_{f_\ell}(\ell\theta)$ exists for a.e. θ . The set $\Lambda(F)$ of elements θ in Λ_0 such that*

$$\delta_n^\theta(F) := \frac{1}{n} \int_{\mathbb{T}^1 \times \Omega} \left| \sum_{k=0}^{n-1} [F(x + k\theta, \tau^k \omega) - M_\theta(x + k\theta, \tau^k \omega)] \right|^2 dx \mathbb{P}(d\omega) \rightarrow 0 \quad (12)$$

has full measure.

Proof. 1) By the maximal inequality for the convolution by the Fejér kernels K_n (see Zygmund [19, Th. 8.1] and Garsia [7, p. 7]), there is C such that, for every $\varphi \in L^1(\mathbb{T}^1)$, $\lambda(\sup_n |(K_n * \varphi)| > \alpha) \leq \frac{C}{\alpha} \|\varphi\|_1$.

Recall that $\|\varphi_{f_\ell}\|_{L^1(dt)} = \|f_\ell\|_2^2$. Using the invariance of the Lebesgue measure on \mathbb{T}^1 by the map $\theta \rightarrow \ell\theta \bmod 1$, $\ell \neq 0$, and the maximal inequality, we obtain (implicitly we restrict the sum to indices ℓ such that $\|f_\ell\|_2 \neq 0$) for $s > 0$ the following maximal inequality which implies (11):

$$\begin{aligned} \lambda(\theta : \sum_{\ell} \sup_n (K_n * \varphi_{f_\ell})(\ell\theta) \geq s \sum_{\ell} \|f_\ell\|_2) &\leq \sum_{\ell} \lambda(\theta : \sup_n (K_n * \varphi_{f_\ell})(\ell\theta) \geq s \|f_\ell\|_2) \\ &\leq \sum_{\ell} \lambda(\theta : \sup_n (K_n * \varphi_{f_\ell})(\theta) \geq s \|f_\ell\|_2) \leq \frac{C}{s} \sum_{\ell} \|f_\ell\|_2^{-1} \|\varphi_{f_\ell}\|_1 \leq \frac{C}{s} \sum_{\ell} \|f_\ell\|_2. \end{aligned}$$

2) The existence of the asymptotic variance $\sigma_\theta^2(F)$ for a.e. θ follows from (11). We have

$$\begin{aligned} &\int_{\mathbb{T}^1 \times \Omega} \left| \sum_{k=0}^{n-1} [F(x + k\theta, \tau^k \omega) - M_\theta(x + k\theta, \tau^k \omega)] \right|^2 dx \mathbb{P}(d\omega) \\ &= \int_{\mathbb{T}^1 \times \Omega} \left| \sum_{k=0}^{n-1} \sum_{\ell \neq 0} e^{2\pi i \ell (x + k\theta)} (f_\ell(\tau^k \omega) - \sum_j \gamma_{j,\ell}(\ell\theta) \psi_j(\tau^k \omega)) \right|^2 dx \mathbb{P}(d\omega) \\ &= \int_{\mathbb{T}^1 \times \Omega} \left| \sum_{\ell \neq 0} e^{2\pi i \ell x} \sum_{k=0}^{n-1} e^{2\pi i k \ell \theta} (f_\ell(\tau^k \omega) - \sum_j \gamma_{j,\ell}(\ell\theta) \psi_j(\tau^k \omega)) \right|^2 dx \mathbb{P}(d\omega) \\ &= \sum_{\ell \neq 0} \int_{\Omega} \left| \sum_{k=0}^{n-1} e^{2\pi i k \ell \theta} (f_\ell(\tau^k \omega) - \sum_j \gamma_{j,\ell}(\ell\theta) \psi_j(\tau^k \omega)) \right|^2 \mathbb{P}(d\omega), \end{aligned}$$

hence (cf. (7)):

$$\begin{aligned} \delta_n^\theta(F) &= \sum_{\ell \neq 0} \frac{1}{n} \|S_n^{\ell\theta}(f_\ell - M_{\ell\theta,\ell} f_\ell)\|_2^2 = \sum_{\ell \neq 0} \int_0^1 \frac{1}{n} \left| \sum_{k=0}^{n-1} e^{2\pi i k(t-\ell\theta)} \right|^2 \varphi_{f_\ell - M_{\ell\theta,\ell} f_\ell}(t) dt \\ &= \sum_{\ell \neq 0} \int_0^1 K_n(t - \ell\theta) \varphi_{f_\ell - M_{\ell\theta,\ell} f_\ell}(t) dt, \end{aligned} \tag{13}$$

where the spectral density (for τ) of $f_\ell - M_{\ell\theta,\ell} f_\ell$ is $\varphi_{f_\ell - M_{\ell\theta,\ell} f_\ell}(t) = \sum_{j \in J} |\gamma_{j,\ell}(t) - \gamma_{j,\ell}(\ell\theta)|^2$.

From Proposition 1.4, for a.e. θ and for each ℓ , $\lim_n \int_0^1 K_n(t - \ell\theta) \varphi_{f_\ell - M_{\ell\theta,\ell} f_\ell}(t) dt = 0$. By (11), for a.e. θ we can permute the limit and the sum in (13):

$$\lim_n \sum_{\ell \neq 0} \int_0^1 K_n(t - \ell\theta) \varphi_{f_\ell - M_{\ell\theta,\ell} f_\ell}(t) dt = \sum_{\ell \neq 0} \lim_n \int_0^1 K_n(t - \ell\theta) \varphi_{f_\ell - M_{\ell\theta,\ell} f_\ell}(t) dt = 0.$$

□

Theorem 2.3. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a K -system. Let F be a real function in $L^2(\mathbb{T}^1 \times \Omega)$ orthogonal to the functions depending only on ω with Fourier expansion (8) such*

that $\sum \|f_\ell\|_2 < +\infty$. Then for a.e. θ we have with respect to the measure $dx \otimes \mathbb{P}(d\omega)$

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} F(x + k\theta, \tau^k \omega) \xrightarrow[n \rightarrow \infty]{distr} \mathcal{N}(0, \sigma_\theta^2(F)),$$

with asymptotic variance $\sigma_\theta^2(F) = \sum_\ell \varphi_{f_\ell}(\ell\theta)$.

Proof. The result follows from the CLT for the martingale defined by (9) (Proposition 2.1) and the approximation (12) valid for a.e. θ , as shown in Proposition 2.2). \square

Corollary 2.4. *Let f be a real function in $L_0^2(\mathbb{P})$. For a.e. θ , the CLT holds for the rotated sums:*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \xrightarrow[n \rightarrow \infty]{distr} \mathcal{N}(0, \Gamma_\theta)$$

with respect to the measure $\mathbb{P}(d\omega)$, with covariance matrix

$$\Gamma_\theta = \begin{pmatrix} \frac{1}{2} \varphi_f(\theta) & 0 \\ 0 & \frac{1}{2} \varphi_f(\theta) \end{pmatrix}.$$

Proof. Theorem 2.3 implies the CLT for the sums $e^{2\pi i x} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega)$ with respect to $dx \otimes \mathbb{P}(d\omega)$. An easy computation gives the covariance matrix for the (complex valued) martingale M_θ associated to $e^{2\pi i x} f(\omega)$. Using the lemma below, this implies the result. \square

Lemma 2.5. *For a.e. $\theta \in \mathbb{T}^1$, the asymptotic distribution of $\frac{1}{\sqrt{n}} e^{2\pi i x} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega)$ with respect to $\lambda \otimes \mathbb{P}$ is the same as that of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega)$ with respect to \mathbb{P} .*

Proof. Let $Z_n^\theta(\omega) := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega)$. By Theorem 2.3 for a.e. θ , $e^{2\pi i x} Z_n^\theta(\omega)$ satisfies the CLT with respect to $\lambda \otimes \mathbb{P}$. Fix θ in this set of full measure. By Theorem 4 of Eagleson [5] (see also [18]), if we replace $\lambda \otimes \mathbb{P}$ by an absolutely continuous probability measure with respect to $\lambda \otimes \mathbb{P}$, the asymptotic distribution of $e^{2\pi i x} Z_n^\theta(\omega)$ is unchanged (i.e., the limit is *mixing*).

For $\varepsilon > 0$ let J_ε be an interval neighborhood of 0 of length ε . Put $\zeta_\varepsilon := \varepsilon^{-1} 1_{J_\varepsilon}$. We consider the probability measure $\zeta_\varepsilon(x) \lambda(dx) \otimes \mathbb{P}(d\omega)$. If Φ is a Lipschitz function defined on \mathbb{C} with Lipschitz constant C , then for $x \in J_\varepsilon$ we have $|\Phi(e^{2\pi i x} Y) - \Phi(Y)| \leq C\varepsilon|Y|$, for every $Y \in \mathbb{C}$. Hence using the Cauchy-Schwarz inequality, we have

$$\left| \int (\Phi(e^{2\pi i x} Z_n^\theta) - \Phi(Z_n^\theta)) \zeta_\varepsilon(x) dx d\mathbb{P}(d\omega) \right| \leq C\varepsilon \left(\int |Z_n^\theta|^2 d\mathbb{P} \right)^{\frac{1}{2}}.$$

Since $\sup_n \int |Z_n^\theta|^2 d\mathbb{P} < +\infty$, it shows that:

$$\limsup_{\varepsilon \rightarrow 0} \sup_n \left| \int (\Phi(e^{2\pi i x} Z_n^\theta) - \Phi(Z_n^\theta)) \zeta_\varepsilon(x) dx \mathbb{P}(d\omega) \right| = 0.$$

We obtain the result by applying what precedes to the family $\Phi_t(Y) = e^{i \Re e(tY)}$, for t and $Y \in \mathbb{C}$. \square

Remarks 2.6. 1) The CLT for the sums $\sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega)$ with respect to (θ, ω) under the measure $d\theta \otimes \mathbb{P}(d\omega)$ follows from Corollary 2.4.

2) A variant of the proof of Corollary 2.4 consists in using a CLT for martingales with the weights $\cos(2\pi k \theta)$. For this, we use the ergodic theorem to prove the

convergence of the conditional variance and apply Brown's Central Limit Theorem for martingale which extends Billingsley-Ibragimov theorem or Corollary 3.4 in [9].

Extensions In the definition (1) of the rotated sums, it is natural to replace $e^{2\pi i k \theta}$ for θ in the circle by $e^{2\pi i \langle \ell, \theta \rangle}$, where θ belongs to the d -dimensional torus. This leads to the following construction. For $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, let us consider the system defined on $\mathbb{T}^d \times \Omega$, by $\tau_\theta : (x, \omega) \rightarrow (x + \theta \bmod 1, \tau\omega)$. We denote by λ the Lebesgue measure on \mathbb{T}^d .

Let F be a function in $L^2(\mathbb{T}^d \times \Omega)$ orthogonal to the functions depending only on ω , i.e.,

$$F(x, \omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, x \rangle} f_\ell(\omega). \quad (14)$$

Its L^2 -norm is

$$\|F\|_2^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \|f_\ell\|_2^2 = \sum_j \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_k |\langle f_\ell, T^k \psi_j \rangle|^2 < \infty.$$

Observe that the push forward of the Lebesgue measure on \mathbb{T}^d by the map $\theta \in \mathbb{T}^d \rightarrow \langle \ell, \theta \rangle \in \mathbb{T}^1$ is the Lebesgue measure on \mathbb{T}^1 if $\ell \in \mathbb{Z}^d \setminus \{0\}$. Therefore the previous method applies. Now the martingale M_θ is defined by

$$M_\theta(x, \omega) := \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, x \rangle} M_{\langle \ell, \theta \rangle, \ell}(\omega), \quad \text{where } M_{\langle \ell, \theta \rangle, \ell} := \sum_j \gamma_{\ell, j}(\langle \ell, \theta \rangle) \psi_j.$$

Its variance is

$$\int_{\mathbb{T}^d \times \Omega} |M_\theta(x, \omega)|^2 dx \mathbb{P}(d\omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in J} |\gamma_{\ell, j}(\langle \ell, \theta \rangle)|^2 < +\infty, \quad \text{for a.e. } \theta. \quad (15)$$

The same proof as in Theorem 2.3 shows:

Theorem 2.7. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a K -system. Let F be a real function in $L^2(\mathbb{T}^d \times \Omega)$ orthogonal to the functions depending only on ω with Fourier expansion (14) satisfying $\sum \|f_\ell\|_2 < +\infty$. Then for a.e. $\theta \in \mathbb{T}^d$ the CLT holds for $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} F(x + k\theta, \tau^k \omega)$ with respect to the measure $\lambda \otimes \mathbb{P}$.*

Remark that the proof applies if we replace \mathbb{T}^d by any compact abelian connected group G , since for any character on G the map $g \in G \rightarrow \chi(g) \in \mathbb{T}^1$ is a surjective homomorphism and the push forward of the Haar measure on G is the Lebesgue measure on \mathbb{T}^1 .

Now, as in Corollary 2.4, we consider the distribution with respect to the measure \mathbb{P} . Note that, with the condition $\sum \|f_\ell\|_2 < +\infty$, the function $\omega \rightarrow F(\theta, \omega) := \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, \theta \rangle} f_\ell(\omega)$ is in $L^2(\Omega, \mathbb{P})$ for every θ .

Theorem 2.8. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a K -system. Let $F(\theta, \omega)$ with Fourier expansion (14). If $\sum \|f_\ell\|_2^{\frac{2}{3}} < +\infty$, then for a.e. $\theta \in \mathbb{T}^d$ the CLT holds for $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} F(k\theta, \tau^k \omega)$ with respect to the measure \mathbb{P} and $\lim_n \frac{1}{n} \|\sum_{k=0}^{n-1} F(k\theta, \tau^k \cdot)\|^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \varphi_{f_\ell}(\langle \ell, \theta \rangle)$.*

Proof. It follows from the hypothesis (cf. proof of Proposition 2.2):

$$\begin{aligned} \lambda(\theta : \sum_{\ell} \sup_n [K_n * \varphi_{f_{\ell}}(\langle \ell, \theta \rangle)]^{\frac{1}{2}} \geq s \sum_{\ell} \|f_{\ell}\|_2^{\frac{2}{3}}) &\leq \sum_{\ell} \lambda(t : \sup_n [K_n * \varphi_{f_{\ell}}(t)]^{\frac{1}{2}} \geq s \|f_{\ell}\|_2^{\frac{2}{3}}) \\ &\leq \frac{C}{s^2} \sum_{\ell} \|f_{\ell}\|_2^2 / \|f_{\ell}\|_2^{\frac{4}{3}} = \frac{C}{s^2} \sum_{\ell} \|f_{\ell}\|_2^{\frac{2}{3}} < +\infty. \end{aligned}$$

With the notation $S_{\ell,n}^{\theta}(\omega) := \sum_{k=0}^{n-1} e^{2\pi i \langle \ell, k\theta \rangle} f_{\ell}(\tau^k \omega)$, using the triangle inequality this implies:

$$\begin{aligned} \lambda(\theta : \sup_n \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} \sum_{\ell \in \mathbb{Z}^d} e^{2\pi i \langle \ell, k\theta \rangle} f_{\ell} \circ \tau^k \right\|_2 \geq s \sum_{\ell \in \mathbb{Z}^d} \|f_{\ell}\|_2^{\frac{2}{3}}) \\ \leq \lambda\left(\theta : \sum_{\ell} \sup_n \frac{1}{\sqrt{n}} \|S_{\ell,n}^{\theta}\|_2 \geq s \sum_{\ell} \|f_{\ell}\|_2^{\frac{2}{3}}\right) \\ \leq \sum_{\ell} \lambda\left(\theta : \sup_n (K_n * \varphi_{f_{\ell}})(\langle \ell, \theta \rangle) \geq s^2 \|f_{\ell}\|_2^{\frac{4}{3}}\right) \leq \frac{C}{s^2} \sum_{\ell} \|f_{\ell}\|_2^{\frac{2}{3}} < +\infty. \end{aligned} \quad (16)$$

If L is a finite set of indices in \mathbb{Z}^d , we have by the preceding inequalities:

$$\lambda(\theta : \sup_n \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} \sum_{\ell \in \mathbb{Z}^d \setminus L} e^{2\pi i \langle \ell, k\theta \rangle} f_{\ell} \circ \tau^k \right\|_2 \geq s \sum_{\ell \in \mathbb{Z}^d \setminus L} \|f_{\ell}\|_2^{\frac{2}{3}}) \leq \frac{C}{s^2} \sum_{\ell \in \mathbb{Z}^d \setminus L} \|f_{\ell}\|_2^{\frac{2}{3}}.$$

Therefore, for an increasing sequence of finite sets (L_p) in \mathbb{Z}^d with union $\mathbb{Z}^d \setminus \{0\}$ and with

$$Z_n^{\theta,p} := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \sum_{\ell \in L_p} e^{2\pi i \langle \ell, k\theta \rangle} f_{\ell} \circ \tau^k, \quad Z_n^{\theta} := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, k\theta \rangle} f_{\ell} \circ \tau^k,$$

we have $\lim_p \sup_n \|Z_n^{\theta} - Z_n^{\theta,p}\|_2 = 0$ for a.e. θ . Now, if we prove that $Z_n^{\theta,p} \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma_p(\theta))$ for every p , then we may use Theorem 3.2 in Billingsley [1] to conclude that $Z_n^{\theta} \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma(\theta))$, with $\sigma(\theta) := \lim_p \sigma_p(\theta) \in [0, +\infty[$.

Let $Y_{\ell} = S_{\ell,n}^{\theta}(\omega)$ and let $\gamma_{\ell} \in \mathbb{C}$ be any family of complex numbers. We have

$$|e^{i\Re e(t \sum_{\ell \in L_p} \gamma_{\ell} Y_{\ell})} - e^{i\Re e(t \sum_{\ell \in L_p} Y_{\ell})}| \leq \sum_{\ell \in L_p} |e^{i\Re e(t\gamma_{\ell} Y_{\ell})} - e^{i\Re e(tY_{\ell})}| \leq |t| \sum_{\ell \in L_p} |\gamma_{\ell} - 1| |Y_{\ell}|.$$

Therefore we can use the argument of Lemma 2.5. We obtain the CLT for $Z_n^{\theta,p}$ for a fixed p with respect to \mathbb{P} with the same variance for the limit law as for the process $\frac{1}{\sqrt{n}} \sum_{\ell \in L_p} e^{2\pi i \langle \ell, x \rangle} \sum_{k=0}^{n-1} e^{2\pi i \langle \ell, k\theta \rangle} f_{\ell} \circ \tau^k$ with respect to $\lambda \times \mathbb{P}$.

Hence we obtain $Z_n^{\theta,p} \Rightarrow_n \mathcal{N}(0, \sigma_p(\theta))$, with for a.e. θ (cf. (15))

$$\sigma_p^2(\theta) = \sum_{\ell \in L_p} \sum_{j \in J} |\gamma_{j,\ell}(\langle \ell, \theta \rangle)|^2.$$

Since $\sigma^2(\theta) = \lim_p \sigma_p^2(\theta) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in J} |\gamma_{j,\ell}(\langle \ell, \theta \rangle)|^2$, we obtain the result. The convergence of the variance will be proved below. \square

Corollary 2.9. *Let h be a continuous function on \mathbb{T}^d with 0 integral such that $\sum_{\ell \in \mathbb{Z}^d} |\hat{h}(\ell)|^{\frac{2}{3}} < \infty$. Let f be in $L^2(\Omega)$. Then, for a.e. $\theta \in \mathbb{T}^d$, the CLT holds for $\sum_{k=0}^{n-1} h(k\theta) f \circ \tau^k$ with respect to $\mathbb{P}(d\omega)$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} h(k\theta) T^k f \right\|^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} |\hat{h}(\ell)|^2 \varphi_f(\langle \ell, \theta \rangle)$.*

Proof. We apply Theorem 2.8 to $F(\theta, \omega) = h(\theta)f(\omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, \theta \rangle} \hat{h}(\ell) f(\omega)$. \square

Remark 2.10. 1) The previous results extend to vector valued functions.

2) Let $\mathcal{P}(\tau)$ be the Pinsker factor of a dynamical systems $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$. The previous results extend to functions F in $L^2(\mathbb{T}^d \times \Omega)$ with f_ℓ orthogonal to $L^2(\mathcal{P}(\tau))$ for every $\ell \in \mathbb{Z}^d \setminus \{0\}$, under the condition $\sum \|f_\ell\|_2^{\frac{2}{3}} < +\infty$.

They are valid for dynamical systems such that $\mathcal{P}(\tau)$ has a singular spectrum (with a possibly degenerated limit law, see Corollary 1.5).

Computation of the variance Here we give a proof of the convergence of the variance as stated in Theorem 2.8. For this result we need only to assume that τ is weakly mixing in order that the cross spectral measures with respect to the unitary operator T induced by τ are without atoms.

Theorem 2.11. *If F with Fourier series $F(x, \omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, x \rangle} f_\ell(\omega)$ is such that $\sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \|f_\ell\|_2^{\frac{2}{3}} < +\infty$, then for a.e. $\theta \in \mathbb{T}^d$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} F(k\theta, \tau^k \omega) \right\|_{\mathbb{P}(d\omega)}^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \varphi_{f_\ell}(\langle \ell, \theta \rangle).$$

Proof. From Proposition 1.8 applied with distinct $\theta_1 = \ell\theta$ and $\theta_2 = \tilde{\ell}\theta$, we obtain the result when F is trigonometric polynomial. We consider now the general case.

We put as above $S_{\ell, n}^\theta(\omega) := \sum_{k=0}^{n-1} e^{2\pi i k \langle \ell, \theta \rangle} f_\ell(\tau^k \omega)$. The assumption of the theorem and Inequality (16) in Theorem 2.8 imply

$$\sum_{\ell \neq 0} \sup_n \left\| \frac{1}{\sqrt{n}} S_{\ell, n}^\theta \right\| < +\infty, \text{ for a.e. } \theta. \quad (17)$$

It follows

$$\frac{1}{n} \left\| \sum_{k=0}^{n-1} F(k\theta, \tau^k \cdot) \right\|^2 = \lim_L \left\| \sum_{0 \neq \ell \in [-L, L]^d} \frac{1}{\sqrt{n}} S_{\ell, n}^\theta \right\|^2 = \lim_L \left\| \sum_{0 \neq \ell \in [-L, L]^d} \frac{1}{\sqrt{n}} S_{\ell, n}^\theta \right\|^2.$$

So, actually we need to compute $\lim_n \lim_L \left\| \sum_{0 \neq \ell \in [-L, L]^d} \frac{1}{\sqrt{n}} S_{\ell, n}^\theta f \right\|^2$. After expanding the square of the norm, we have to compute the following two limits:

$$\lim_n \sum_{\ell \neq 0} \frac{1}{n} \|S_{\ell, n}^\theta\|^2 \text{ and } \Re e \left\{ \lim_n \sum_{\ell > \tilde{\ell}} \left\langle \frac{1}{\sqrt{n}} S_{\ell, n}^\theta, \frac{1}{\sqrt{n}} S_{\tilde{\ell}, n}^\theta \right\rangle \right\}. \quad (18)$$

By (17) $\sum_{\ell \neq 0} \sup_n \frac{1}{n} \|S_{\ell, n}^\theta\|^2 < +\infty$ and we may use Lebesgue dominated convergence theorem to permute \lim_n with the sum in the first expression of (18). This expression converges to the desired limit of the theorem. For the second term, by Lemma 1.7 we have that $\lim_n \langle \frac{1}{\sqrt{n}} S_{\ell, n}^\theta, \frac{1}{\sqrt{n}} S_{\tilde{\ell}, n}^\theta \rangle = 0$ for $\ell \neq \tilde{\ell}$ and for every $\theta \in \mathbb{T}^d$ with rationally independent components. Since

$$\sum_{\ell \neq \tilde{\ell}} \sup_n \left| \left\langle \frac{1}{\sqrt{n}} S_{\ell, n}^\theta, \frac{1}{\sqrt{n}} S_{\tilde{\ell}, n}^\theta \right\rangle \right| \leq \left(\sum_{\ell} \sup_n \left\| \frac{1}{\sqrt{n}} S_{\ell, n}^\theta \right\| \right)^2,$$

we may use (17) and Lebesgue's dominated convergence theorem to invert \lim_n with the sum in the second expression of (18) to conclude that the limit for n tending to ∞ of the mixed terms is 0. \square

2.2. Invariance principle for a.e. θ for modified sums. Let $(\Omega, \mathbb{P}, \tau)$ be a K -system and $(T^n f)$ a process generated by $f \in L_0^2(\Omega, \mathbb{P})$ where $Tf = f \circ \tau$. Let us consider for a parameter α in $]0, 1]$ the modified rotated sums

$$S_{n,\alpha}^\theta f = \sum_{k=0}^{n-1} e^{2\pi i k \theta} \left(1 - \left(\frac{k}{n}\right)^\alpha\right) T^k f$$

and the interpolated normalized continuous time process defined for $t \in [0, 1]$ (with the convention $\sum_{k=0}^{-1} = 0$) by

$$W_{n,t,\alpha}^\theta f = \frac{t^\alpha}{\sqrt{n}} \sum_{k=0}^{[nt]-1} e^{2\pi i k \theta} \left(1 - \left(\frac{k}{[nt]}\right)^\alpha\right) T^k f.$$

For $\alpha \in]0, 1]$, we have $\lim_n \frac{1}{n} \sum_{k=1}^{n-1} \left(1 - \left(\frac{k}{n}\right)^\alpha\right)^2 = \frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}$. A computation analogous to that in [13] gives the following result:

Proposition 2.12. *Let α be in $]0, 1]$. We have for a.e. θ*

$$\| \sup_{0 \leq t \leq 1} |W_{n,t,\alpha}^\theta f - W_{n,t,\alpha}^\theta M_\theta f| \|_2 \rightarrow 0. \quad (19)$$

Proof. The inequalities $\sum_{j=k}^n \int_j^{j+1} \frac{dt}{t^{1-\alpha}} \leq \sum_{j=k}^n \frac{1}{j^{1-\alpha}} \leq \sum_{j=k}^n \int_{j-1}^j \frac{dt}{t^{1-\alpha}}$ imply

$$\sum_{j=k}^n \frac{1}{j^{1-\alpha}} = \frac{1}{\alpha} (n^\alpha - k^\alpha) + \rho(k, n),$$

$$\text{with } 0 \leq \rho(k, n) \leq \int_{k-1}^n \frac{dt}{t^{1-\alpha}} - \int_k^n \frac{dt}{t^{1-\alpha}} = \frac{k^\alpha - (k-1)^\alpha}{\alpha}. \quad (20)$$

We have the following relation,

$$\begin{aligned} \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} \frac{S_k^\theta f}{k^{1-\alpha}} &= \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i (k-1)\theta} \left(\frac{1}{k^{1-\alpha}} + \dots + \frac{1}{[nt]^{1-\alpha}}\right) T^{k-1} f \\ &= \frac{1}{\alpha} \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i (k-1)\theta} ([nt]^\alpha - k^\alpha + \rho(k, [nt])) T^{k-1} f \\ &= \frac{1}{\alpha} \left(\frac{[nt]}{nt}\right)^\alpha \frac{t^\alpha}{n^{\frac{1}{2}}} \sum_{k=1}^{[nt]} e^{2\pi i (k-1)\theta} \left(1 - \left(\frac{k}{[nt]}\right)^\alpha\right) T^{k-1} f + \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i (k-1)\theta} \rho(k, [nt]) T^{k-1} f; \end{aligned}$$

hence:

$$W_{n,t,\alpha}^\theta f = \left(\frac{nt}{[nt]}\right)^\alpha \frac{\alpha}{n^{\frac{1}{2}+\alpha}} \left[\sum_{k=1}^{[nt]} \frac{S_k^\theta f}{k^{1-\alpha}} - \left(\frac{nt}{[nt]}\right)^\alpha \frac{\alpha}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i (k-1)\theta} \rho(k, [nt]) T^{k-1} f \right].$$

Since $\frac{nt}{[nt]} \leq 2$ for $\frac{1}{n} \leq t \leq 1$, putting $\theta(n, t) = (\frac{nt}{[nt]})^\alpha$, we have from the previous relation:

$$\begin{aligned} \sup_{\frac{1}{n} \leq t \leq 1} |W_{n,t,\alpha}^\theta f| &\leq \sup_{\frac{1}{n} \leq t \leq 1} \left[\frac{\theta(n, t)}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} \frac{|S_k^\theta f|}{k^{1-\alpha}} \right] + \sup_{\frac{1}{n} \leq t \leq 1} \left[\frac{\theta(n, t)}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} |\rho(k, [nt])| T^{k-1} |f| \right] \\ &\leq \sup_{\frac{1}{n} \leq t \leq 1} \left[\frac{2}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} \frac{|S_k^\theta f|}{k^{1-\alpha}} \right] + \sup_{\frac{1}{n} \leq t \leq 1} \left[\frac{2}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^{[nt]} [k^\alpha - (k-1)^\alpha] T^{k-1} |f| \right] \\ &\leq \frac{2}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^n \frac{|S_k^\theta f|}{k^{1-\alpha}} + \frac{2}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^n [k^\alpha - (k-1)^\alpha] T^{k-1} |f|. \end{aligned}$$

We use the previous inequality with $f - M_\theta f$ instead of f . By the triangle inequality, Cauchy-Schwarz inequality and (20) it follows:

$$\begin{aligned} \left\| \sup_{0 \leq t \leq 1} |W_{n,t,\alpha}^\theta f - W_{n,t,\alpha}^\theta M_\theta f| \right\|_2 &\leq \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^n \frac{\|S_k^\theta f - S_k^\theta M_\theta f\|_2}{k^{1-\alpha}} \\ &\quad + \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{k=1}^n [k^\alpha - (k-1)^\alpha] (\|T^{k-1} f\|_2 + \|T^{k-1} M_\theta f\|_2) \\ &\leq \frac{1}{n^{\frac{1}{2}+\alpha}} \left(n \sum_{k=1}^n \frac{\|S_k^\theta f - S_k^\theta M_\theta f\|_2^2}{k^{2(1-\alpha)}} \right)^{\frac{1}{2}} + \frac{n^\alpha}{n^{\frac{1}{2}+\alpha}} (\|f\|_2 + \|M_\theta f\|_2) \\ &= \left(\frac{1}{n^{2\alpha}} \sum_{k=1}^n \frac{\|S_k^\theta f - S_k^\theta M_\theta f\|_2^2}{k^{2(1-\alpha)}} \right)^{\frac{1}{2}} + \frac{1}{n^{\frac{1}{2}}} (\|f\|_2 + \|M_\theta f\|_2). \end{aligned}$$

As $\|S_k^\theta f - S_k^\theta M_\theta f\|_2^2 = o(k)$ for a.e. θ , this implies (19). \square

Theorem 2.13. *For a.e. $\theta \in \mathbb{T}$, the real and imaginary parts of the process $(W_{n,t,\alpha}^\theta f)$ are asymptotically independent and satisfy the invariance principle with convergence to $\frac{1}{2}\varphi_f(\theta)Z_\alpha(t)$, where $Z_\alpha(t)$ is a Gaussian continuous process with covariance for $s \leq t$,*

$$\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} \frac{s^{2-\alpha}}{(1-\alpha)(2-\alpha)} \left[t^{1-\alpha} - \frac{1}{3-2\alpha} s^{1-\alpha} \right].$$

Proof. By Proposition 2.12 it is enough to prove the statement for the process $(W_{n,t,\alpha}^\theta M_\theta f)$. We use the Cramér-Wold device [3] to identify the 2-d limit. Then the proof goes along the same lines as in Peligrad-Peligrad [13]. Since the system is weakly mixing the factor $\frac{1}{2}\varphi_f(\theta)$ comes from the following limit, for a.e. θ

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \cos^2(2\pi k\theta) T^k |M_\theta f|^2 = \frac{1}{2} \varphi_f(\theta), \text{ a.e. and in } L^1(\mathbb{P})$$

and the same with $\sin^2(2\pi k\theta)$. The asymptotic independence follows from the fact that for a.e. θ the asymptotic correlation between the real and imaginary parts of the process is zero a.e. and in $L^1(\mathbb{P})$ -norm. \square

The above result is an a.e. result, but for modified sums. In [14] an invariance principle for the usual rotated sums is shown for the product measure. We present below their result (and include a proof for the sake of completeness) which is valid also in our setting. The key step is the following lemma, where Carleson's result on Fourier series is used:

Lemma 2.14. *For every regular function f , the family $(\frac{1}{n} \sup_{1 \leq k \leq n} |S_k^\theta f|^2)_{n \geq 1}$ is uniformly integrable for the product measure $d\theta \times \mathbb{P}(d\omega)$.*

Proof. For every $m \geq 1$, we have:

$$\begin{aligned} S_k^\theta f &= \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell f = \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \sum_{j=-\infty}^{\infty} \Pi_j f \\ &= \sum_{j=-m}^m \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \Pi_j f + \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \sum_{j \notin [-m, m]} \Pi_j f. \end{aligned} \quad (21)$$

First, let us consider the second term in (21). By Carleson-Hunt's inequality, there is a constant C such that:

$$\int_{\mathbb{T}^1} \max_{1 \leq k \leq n} \left| \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \sum_{j \notin [-m, m]} \Pi_j f(\omega) \right|^2 d\theta \leq C \sum_{\ell=0}^{n-1} |T^\ell \sum_{j \notin [-m, m]} \Pi_j f(\omega)|^2;$$

hence:

$$\begin{aligned} &\int_{\Omega} \int_{\mathbb{T}^1} \frac{1}{n} \max_{1 \leq k \leq n} \left| \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \sum_{j \notin [-m, m]} \Pi_j f(\omega) \right|^2 d\theta \mathbb{P}(d\omega) \\ &\leq C \frac{1}{n} \int_{\Omega} \sum_{\ell=0}^{n-1} \left| \sum_{j \notin [-m, m]} T^\ell \Pi_j f(\omega) \right|^2 \mathbb{P}(d\omega) = C \sum_{j \notin [-m, m]} \|\Pi_j f\|^2 \xrightarrow{m \rightarrow \infty} 0. \end{aligned} \quad (22)$$

In particular, we obtain that $\frac{1}{n} \max_{1 \leq k \leq n} \left| \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \sum_{j \notin [-m, m]} \Pi_j f(\omega) \right|^2$ is uniformly integrable for the product measure $d\theta \times \mathbb{P}(d\omega)$. Now we consider the first term in (21). For each $j \in [-m, m]$ and $\theta \in \mathbb{T}$, the sequence $M_j^\theta(k, \omega) := \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \Pi_j f(\omega)$ is a square integrable martingale with respect to $(\mathcal{A}_{j+k-1})_{k \geq 1}$. We conclude that $(M_j^\theta(k, \omega))_{k \geq 1}$ is a square integrable martingale with respect to $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{A}_{j+k-1})_{k \geq 1}$. Proposition 1(a) in Dedecker and Rio [4] yields that $(\frac{1}{n} \max_{1 \leq k \leq n} |M_j^\theta(k, \omega)|^2)_{n \geq 1}$ is uniformly integrable for $d\theta \times \mathbb{P}(d\omega)$.

As a finite sum of uniformly integrable sequences is a uniformly integrable sequence, the sequence $\sum_{j \in [-m, m]} \frac{1}{n} \max_{1 \leq k \leq n} |M_j^\theta(k, \omega)|^2$ is uniformly integrable. This observation and (22) imply the assertion for $\frac{1}{n} \max_{1 \leq k \leq n} |S_k^\theta|^2$. \square

As a consequence of standard results from Billingley [1] now it follows:

Theorem 2.15 ([14]). *Under the product measure $d\theta \times \mathbb{P}(d\omega)$, the process $(\frac{1}{\sqrt{n}} S_{[nt]}^\theta f)$ is tight in $D(0, 1)$ and*

$$\left(\Re e \left\{ \frac{1}{\sqrt{n}} S_{[nt]}^\theta f \right\}, \Im e \left\{ \frac{1}{\sqrt{n}} S_{[nt]}^\theta f \right\} \right) \Rightarrow \sqrt{\frac{1}{2}} \varphi_f(U) (W'(t), W''(t)),$$

where U is a random variable uniformly distributed on $[0, 2\pi]$ and $W'(t)$ and $W''(t)$ are two independent standard Brownian motions independent of U .

2.3. Application to recurrence of rotated stationary walks in \mathbb{C} . Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be an ergodic dynamical system and φ be measurable function on Ω with values in \mathbb{R}^d , $d \geq 1$. The skew product $\tau_\varphi : (\omega, y) \rightarrow (\tau\omega, y + \varphi(\omega))$ leaves invariant the σ -finite measure $\mathbb{P} \times dy$. It is known that it is either dissipative or conservative with respect to this measure. The latter case occurs if and only if the ‘‘stationary walk’’ (or ‘‘cocycle’’) $S_n\varphi(\omega) := \sum_{k=0}^{n-1} \varphi(\tau^k\omega)$ is *recurrent*, i.e., for a.e. ω , $S_{n_k(\omega)}\varphi(\omega)$ belongs to an arbitrary neighborhood of 0 for an infinite sequence of times $n_k(\omega)$.

For $d = 1$, when φ is integrable, the condition $\int \varphi d\mathbb{P} = 0$ is necessary and sufficient for recurrence. The construction of 2-dimensional stationary walks deserves attention, since, as shown by the i.i.d. case, the dimension $d = 2$ is critical for the recurrence property. We use the following sufficient recurrence criterion:

Recurrence criterion (K. Schmidt [17]) *Let $B(\eta)$ be the ball of radius $\eta > 0$ centered at the origin in \mathbb{R}^2 . If there exists $\delta > 0$ such that, for every $\eta > 0$, we have $\lim_n \mathbb{P}(n^{-1/2}S_n\varphi \in B(\eta)) \geq \delta\eta^2$, then $(S_n\varphi)$ is recurrent in \mathbb{R}^2 .*

We apply the preceding results to rotated 2-dimensional walks, by modifying the increments of the cocycle at each step by a rotation.

Theorem 2.16. *Let $(\Omega, \mathbb{P}, \tau)$ be a K -system or an exact system. Let $\varphi \in L^2(\mathbb{P})$ with values in \mathbb{C} . For a.e. $\theta \in \mathbb{R}$, the process $\sum_{k=0}^{n-1} e^{2\pi ik\theta} \varphi(\tau^k\omega)$ is recurrent and the map $\tau_{\theta, \varphi} : (\omega, z) \rightarrow (\tau\omega, e^{2\pi i\theta}z + \varphi(\omega))$ from $\Omega \times \mathbb{C}$ into itself is conservative.*

Proof. We consider the system $(\mathbb{T}^1 \times \Omega, \lambda \otimes \mathbb{P}, \tau_\theta)$, where $\tau_\theta : (x, \omega) \rightarrow (x + \theta \bmod 1, \tau\omega)$. Let $F(x, \omega) := e^{2\pi ix} \varphi(\omega)$.

Since F satisfies the CLT by Theorem 2.3 (see Remark 2.10), the recurrence criterion implies that for a.e. θ , for a.e. (x, ω) , the process $\sum_{k=0}^{n-1} F(x + k\theta, \tau^k\omega)$ is recurrent in any neighborhood of 0. Hence for a.e. θ the process $\sum_{k=0}^{n-1} e^{2\pi ik\theta} \varphi(\tau^k\omega)$ is also recurrent.

The map $\tau_{\theta, \varphi}$ defines a measure preserving transformation on $(\Omega \times \mathbb{C}, \mathbb{P} \times m)$, where m is the Lebesgue measure on \mathbb{R}^2 . By iteration of this map, we get: $\tau_{\theta, \varphi}^n = (\tau^n\omega, e^{2\pi in\theta}z + \varphi_n(\theta, \omega))$, with $\varphi_n(\theta, \omega) = \sum_{k=0}^{n-1} e^{2\pi i(n-1-k)\theta} \varphi(\tau^k\omega)$ which behaves in modulus like the rotated ergodic sums $\sum_{k=0}^{n-1} e^{-2\pi ik\theta} \varphi(\tau^k\omega)$.

This implies the conservativity of $\tau_{\theta, \varphi}$. Indeed, suppose that $\tau_{\theta, \varphi}$ is not conservative on $\Omega \times \mathbb{C}$ for the measure $\mu := d\mathbb{P}(\omega) \otimes dm(z)$. Then there is a measurable subset D of positive measure in $\Omega \times \mathbb{C}$ such that the sets $\tau_{\theta, \varphi}^n D$, $n \in \mathbb{Z}$, are pairwise disjoint. Let K be a compact set in \mathbb{C} such that $(\Omega \times K) \cap D$ has a positive measure and let V be a compact neighborhood of zero in \mathbb{C} . We set $B_K = \Omega \times (K + V)$. We have $\sum_{n \geq 0} \mu(\tau_{\theta, \varphi}^{-n} B_K \cap D) = \sum_{n \geq 0} \mu(B_K \cap \tau_{\theta, \varphi}^n D) \leq \mu(B_K) < +\infty$. Therefore, by the Borel-Cantelli Lemma, for a.e. (ω, z) there is $N(\omega, z)$ such that $(\omega, z) \notin \tau_{\theta, \varphi}^{-n} B_K \cap D$ for $n > N(\omega, z)$. This contradicts that for a.e. point (ω, z) in $(\Omega \times K)$, in particular for a.e. points of $(\Omega \times K) \cap D$, $\tau_{\theta, \varphi}^n(\omega, z)$ belong to B_K infinitely often. \square

Another situation is provided by isometric extensions.

Theorem 2.17. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a K -system or an exact system. Let φ be a measurable real function on Ω and $\varphi_k(\omega) := \varphi(\omega) + \varphi(\tau\omega) + \dots + \varphi(\tau^{k-1}\omega)$. For a.e. θ the asymptotic distribution of $n^{-1/2} \sum_0^{n-1} e^{2\pi ik\theta} e^{2\pi i\varphi_k}$ with respect to \mathbb{P} is a normal law. For a.e. θ , for \mathbb{P} -a.e. ω the process $\sum_0^{n-1} e^{2\pi ik\theta} e^{2\pi i\varphi_k(\omega)}$ visits infinitely often every neighborhood of 0.*

Proof. The isometric extension $\tau_\varphi : (\omega, y) \rightarrow (\tau\omega, y + \varphi(\omega) \bmod 1)$ defines a measure preserving dynamical system on $\Omega \times \mathbb{T}^1$. Its Pinsker factor $\mathcal{P}(\tau_\varphi)$ is invariant for

the circle action on $\Omega \times \mathbb{T}^1$ on the second coordinate. The sets which are invariant by this action belong to the Pinsker factor of τ , hence are trivial. Therefore the circle action on $\mathcal{P}(\tau_\varphi)$ is ergodic. Since it has a discrete spectrum, the map τ_φ , which commutes with this action, has also a discrete (hence singular) spectrum in restriction to $\mathcal{P}(\tau_\varphi)$.

Let $F \in L^2(\mathbb{P} \times dy)$. By Corollary 2.4 and Remark 1.6, the CLT (with a limiting law which can be degenerated) is satisfied for a.e. θ for the rotated ergodic sums $S_n^\theta F := \sum_{k=0}^{n-1} e^{2\pi i k \theta} F \circ \tau_\varphi^k$, $n \geq 1$.

In particular consider the function $F(\omega, y) := e^{2\pi i y}$ on $\Omega \times \mathbb{T}^1$. Its ergodic sums for the action of τ_φ read $e^{2\pi i y} \sum_{k=0}^{n-1} e^{2\pi i k \theta} e^{2\pi i \varphi_k(\omega)}$. The result follows from the previous theorem. \square

Example: The map $\tau : x \rightarrow 2x \bmod 1$ is exact for the Lebesgue measure on \mathbb{T}^1 . For φ on the circle, let $\varphi_k(x) := \varphi(x) + \varphi(2x) + \dots + \varphi(2^{k-1}x)$. It is known that, if φ is Hölderian, the process $(\sum_0^{n-1} e^{2\pi i \varphi_k})$ is recurrent in \mathbb{C} (cf. [8]). The previous theorem shows that for any measurable real function φ the process $(\sum_0^{n-1} e^{2\pi i k \theta} e^{2\pi i \varphi_k})_{n \geq 1}$ is recurrent in \mathbb{C} for a.e. θ .

3. CLT for rotated \mathbb{Z}^d -actions. The above results for rotated ergodic sums rely on the duality between \mathbb{Z} , the group action corresponding to 1-dimensional discrete time dynamical systems, and the group of the circle. In higher dimension this suggests that the same method applies to multidimensional abelian actions. In this section, we show how to obtain a CLT for rotated ergodic sums of L^2 -functions on \mathbb{Z}^d dynamical systems ($d > 1$) with K -property which allows to use martingale methods as in the previous section. For simplicity we take $d = 2$.

We write $\mathbf{n} = (n_1, n_2)$ for an element of \mathbb{Z}^2 and we put on \mathbb{Z}^2 the lexicographic order $\mathbf{n} \leq \mathbf{p} \Leftrightarrow (n_1 < p_1) \text{ or } (n_1 = p_1 \text{ and } n_2 \leq p_2)$.

Spectral properties Part of the spectral considerations of section 1.1 extends to a unitary action of a discrete abelian group on a Hilbert space. For an action of \mathbb{Z}^d , the Fourier analysis on \mathbb{T}^1 , used for the spectral properties of a single map, is replaced by the Fourier analysis on the d -dimensional torus. First let us recall some classical facts about Fourier series in several variables. For simplicity, we consider two variables.

Using the usual one-dimensional Fejér kernel, the two-dimensional Fejér kernel on \mathbb{T}^2 corresponding to rectangles is defined as $K_{N,P}(\theta_1, \theta_2) = K_N(\theta_1) \cdot K_P(\theta_2)$. For an integrable function φ on \mathbb{T}^2 with Fourier coefficients $(\hat{\varphi}_{k,\ell})_{k,\ell \in \mathbb{Z}}$, the Fejér kernel applied to φ gives

$$K_{N,P}(\varphi)(\theta_1, \theta_2) := (K_{N,P} * \varphi)(\theta_1, \theta_2) = \frac{1}{NP} \sum_{n=1}^N \sum_{p=1}^P \left(\sum_{k=-n+1}^{n-1} \sum_{\ell=-p+1}^{p-1} \hat{\varphi}_{k,\ell} e^{2\pi i(k\theta_1 + \ell\theta_2)} \right).$$

We have the following results (cf. Zygmund [19, Ch. XVII], Th. 2.14, Th. 3.1):

Proposition 3.1. (i) If φ is integrable, then $\lim_{N \rightarrow \infty} K_{N,N}(\varphi) = \varphi$ a.e.

(ii) If $\varphi \log^+ |\varphi|$ is integrable, then $\lim_{\min\{N,P\} \rightarrow \infty} K_{N,P}(\varphi) = \varphi$ a.e.

Rotated ergodic sums and approximation Let \mathcal{H} be a Hilbert space. If T_1, T_2 are two commuting unitary operators on \mathcal{H} , they define a \mathbb{Z}^2 -unitary action on \mathcal{H} by $f \in \mathcal{H} \rightarrow T_1^{k_1} T_2^{k_2} f$, $\mathbf{k} \in \mathbb{Z}^2$.

The rotated ergodic sum corresponding to a rectangle $R_{N,L} = [0, N[\times]0, L[$ is, for f in \mathcal{H} ,

$$S_{N,L}^{\theta_1, \theta_2} f = \sum_{\mathbf{k} \in R_{N,L}} e^{2\pi i(k_1 \theta_1 + k_2 \theta_2)} T_1^{k_1} T_2^{k_2} f.$$

For $L = N$ we simply write R_N for $R_{N,N}$ and $S_N^{\theta_1, \theta_2} f$. The variance of the rotated 2-dimensional process (for squares), when it exists, is $\lim_N \frac{1}{N^2} \|S_N^{\theta_1, \theta_2} f\|_2^2$.

Suppose that \mathcal{K}_0 is a closed subspace of \mathcal{H} such that the subspaces $T_1^{k_1} T_2^{k_2} \mathcal{K}_0$ are pairwise orthogonal. If $(\psi_j)_{j \in J}$ is an orthonormal basis of \mathcal{K}_0 , the family $(T_1^{l_1} T_2^{l_2} \psi_j, j \in J, \mathbf{l} \in \mathbb{Z}^2)$ is an orthonormal basis of the closed subspace \mathcal{H}_∞ generated by these subspaces.

Notations 3.2. For $f \in \mathcal{H}_\infty$, we set $a_{j, \mathbf{n}} := \langle f, T_1^{n_1} T_2^{n_2} \psi_j \rangle$. For $j \in J$, let γ_j be an everywhere finite square integrable function on \mathbb{T}^2 with Fourier coefficients $a_{j, \mathbf{n}}$. The density of the spectral measure for the \mathbb{Z}^2 action is $\varphi_f = \sum_{j \in J} |\gamma_j|^2$.

Since $\int_{\mathbb{T}^2} \sum_{j \in J} |\gamma_j(\theta_1, \theta_2)|^2 d\theta_1 d\theta_2 = \sum_{j \in J} \sum_{\mathbf{n} \in \mathbb{Z}^2} |a_{j, \mathbf{n}}|^2 = \int \varphi_f(\theta_1, \theta_2) d\theta_1 d\theta_2 = \|f\|^2 < \infty$, the set $\Lambda_0 := \{\theta \in \mathbb{T}^2 : \sum_{j \in J} |\gamma_j(\theta)|^2 < \infty\}$ has full measure. For $\theta \in \Lambda_0$, let

$$M_\theta f := \sum_j \gamma_j(\theta) \psi_j \in \mathcal{K}_0. \quad (23)$$

With a proof analogous to that of Proposition 1.4, Proposition 3.1 implies:

Proposition 3.3. *For $f \in \mathcal{H}_\infty$, the set $\Lambda(f)$ of elements $\theta = (\theta_1, \theta_2)$ in Λ_0 such that*

$$\lim_N \frac{1}{N^2} \left\| \sum_{\mathbf{k} \in R_N} e^{2\pi i(k_1 \theta_1 + k_2 \theta_2)} T_1^{k_1} T_2^{k_2} (f - M_\theta f) \right\|_2^2 = 0.$$

has full measure in \mathbb{T}^2 .

K-systems Now we consider two commuting automorphisms τ_1 and τ_2 of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. They define a measure preserving action of \mathbb{Z}^2 . We write $T_1 f$ (resp. $T_2 f$) for $f \circ \tau_1$ (resp. $f \circ \tau_2$). Recall that the entropy of such an action can be defined, as well as the Pinsker factor which is the largest factor with zero entropy. The action has a completely positive entropy if every non trivial factor has positive entropy (equivalently its Pinsker factor is trivial). The notion of K -system can be extended to a \mathbb{Z}^2 -action in the following way:

For a sub σ -algebra \mathcal{B} of \mathcal{A} , we denote by \mathcal{B}^{τ_i} , $i = 1, 2$, the σ -algebra generated by $(\bigcup_{n=-N}^N \tau_i^n \mathcal{B}, N \geq 1)$.

A sub- σ -algebra \mathcal{A}_0 is *increasing* if $(n', p') \leq (n, p) \Rightarrow \tau_1^{n'} \tau_2^{p'} \mathcal{A}_0 \subset \tau_1^n \tau_2^p \mathcal{A}_0$. An increasing sub- σ -algebra \mathcal{A}_0 of \mathcal{A} has the property K if

- (i) it is generating: the sub σ -algebra is generated by $\bigcup_{n,p} \tau_1^n \tau_2^p \mathcal{A}_0$ is \mathcal{A} ;
- (ii) its remote past is trivial, i.e., $\bigcap_\ell \tau_2^\ell \mathcal{A}_0 = \tau_1^{-1} \mathcal{A}_0^{\tau_2}$ and $\bigcap_\ell \tau_1^\ell \mathcal{A}_0^{\tau_2}$ is trivial.

The system is a K -system if there exists an increasing sub σ -algebra with the property K . In that case, the system has completely positive entropy. Conversely B. Kamiński ([11]) proved the existence of a sub σ -algebra with the K -property, when the action has completely positive entropy.

Examples 1) The simplest examples of K -systems for a \mathbb{Z}^2 -action are the \mathbb{Z}^2 -Bernoulli schemes (full 2-shift) defined as the shift action on the coordinates on the product $\Omega = I^{\mathbb{Z}^2}$ endowed with a product probability measure $p^{\otimes \mathbb{Z}^2}$, where p is a probability vector on a finite space I .

2) For K -systems of algebraic origin see the book of K. Schmidt ([16]).

3) Examples are also provided by statistical mechanics (for instance, see [10] for examples of \mathbb{Z}^2 -systems which are K but not Bernoulli).

Rotated ergodic sums and martingale approximation Let \mathcal{A}_0 be a sub σ -algebra and \mathcal{K}_0 be a subspace in $L^2(\mathcal{A}_0)$ of functions which satisfy

$$\mathbb{E}(M_\theta f \circ \tau_1^{k_1} \tau_2^{k_2} | \tau_1^{-k'_1} \tau_2^{-k'_2} \mathcal{A}_0) = 0, \quad \forall \mathbf{k} > \mathbf{k}', \quad (24)$$

Let f be in $L^2_0(\mathbb{P})$. Let $M_{(\theta_1, \theta_2)} \in \mathcal{K}_0$ be associated to f for a.e. (θ_1, θ_2) by (23).

Theorem 3.4.

$$\frac{1}{n} \sum_{\mathbf{k} \in R_n} M_\theta(\tau_1^{k_1} \tau_2^{k_2} \omega) \xrightarrow{distr} \mathcal{N}(0, \Gamma(\theta))$$

with respect to \mathbb{P} , where $\Gamma(\theta)$ is the covariance matrix

$$\Gamma(\theta) = \begin{pmatrix} \frac{1}{2} \varphi_f(\theta_1, \theta_2) & 0 \\ 0 & \frac{1}{2} \varphi_f(\theta_1, \theta_2) \end{pmatrix}.$$

Proof. Although we consider the 2-dimensional case, as long as we deal with finite sums, by ordering, we may consider them as 1-dimensional sums. The ergodic theorem for the ergodic means on squares implies the convergence of the conditional variance. The theorem then follows from Theorem 3.2 in [9] for martingales (cf. Remark 2.6). \square

Now we can apply the martingale approximation to the 2-dimensional rotated ergodic sums.

Theorem 3.5. *Let $(\Omega, \mathcal{A}, \mathbb{P}, (\tau_1, \tau_2))$ be a 2-d K -system. Let f be in $L^2_0(\mathbb{P})$ with spectral density φ_f . Then for Lebesgue-a.e. θ the asymptotic distribution (with respect to \mathbb{P}) of*

$$\left(\frac{1}{n} \sum_{\mathbf{k} \in R_n} \cos(2\pi(k_1\theta_1 + k_2\theta_2)) f(\tau_1^{k_1} \tau_2^{k_2} \cdot), \frac{1}{n} \sum_{\mathbf{k} \in R_n} \sin(2\pi(k_1\theta_1 + k_2\theta_2)) f(\tau_1^{k_1} \tau_2^{k_2} \cdot) \right)$$

is the centered normal law in \mathbb{R}^2 with covariance matrix $\frac{1}{2} \begin{pmatrix} \varphi_f(\theta_1, \theta_2) & 0 \\ 0 & \varphi_f(\theta_1, \theta_2) \end{pmatrix}$.

If the spectral density φ_f is in $L_1 \log L_1$, then the result is valid for ergodic sums on rectangles.

Proof. By Proposition 3.3, the mean square approximation of the bi-dimensional process by a bi-dimensional array satisfying a martingale property holds. The result then follows from the previous theorem. The asymptotic covariance computation follows from the same argument as in the proof of Theorem 2.13.

If φ_f is in $L^1 \log L^1$, ii) of Proposition 3.1 allows to extend the result to ergodic sums on sequences of rectangles $R_{N,L}$ and take the limit when $\min\{N, L\} \rightarrow \infty$. \square

Examples of commuting endomorphisms

Let us consider the commuting (non invertible) endomorphisms on \mathbb{T}^1 given by $\tau_1 x = 2x \bmod 1$ and $\tau_2 x = 3x \bmod 1$ (2 and 3 can be replaced by any pair of

coprime integers > 1). They generate an action of the semi-group \mathbb{N}^2 . The invertible extension of this action has a Lebesgue spectrum.

Let J be the subset of all non-zero integers which are not divisible by 2 or 3. Every non-zero integer $n \in \mathbb{N}$ can be written in a unique way as $n = 2^{k_1} 3^{k_2} j$, for some non-negative integers k_1, k_2 and $j \in J$.

We define \mathcal{K}_0 as the closed subspace of $L_0^2(\mathbb{T}^1)$ generated by $e^{2\pi i j x}$ for $j \in J$. The subspaces $T_1^{k_1} T_2^{k_2} \mathcal{K}_0$ are pairwise orthogonal and with the previous notations $\mathcal{H}_\infty = L_0^2(\mathbb{T}^1)$.

Let f be a function in $L_0^2(\mathbb{T}^1)$ with Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}_n e^{2\pi i n x}$. Let

$$\gamma_j(\theta_1, \theta_2) = \sum_{k_1, k_2 \geq 0} \hat{f}_{2^{k_1} 3^{k_2} j} e^{2\pi i (k_1 \theta_1 + k_2 \theta_2)}.$$

As $j \in J$ is not divisible by 2 and by 3, for any f in \mathcal{K}_0 and any g in L_0^2 , $T_1^{k_1} T_2^{k_2} f$ is orthogonal to $T_1^{k'_1} T_2^{k'_2} g$ if $\mathbf{k} < \mathbf{k}'$. Let $M_\theta f(x) := \sum_{j \in J} \gamma_j(\theta) e^{2\pi i j x}$.

Let us mention that a Central Limit Theorem like Theorem 3.5 can be shown ([2]). The main step in the proof is a result analogous to Theorem 3.4 for the sums $\sum_{\mathbf{k} \in R_n} M_\theta(\tau_1^{k_1} \tau_2^{k_2} x)$. This can be done by using the properties of the set of integers $(2^{k_1} 3^{k_2}, (k_1, k_2) \in \mathbb{N}^2)$ as in [6].

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