ITERATES OF A PRODUCT OF CONDITIONAL EXPECTATION OPERATORS

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Abstract. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T = P_1 P_2 \cdots P_d$ be a finite product of conditional expectations with respect to the sub σ-algebras $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_d$. We show that for every $f \in L_p(\mu)$, $1 < p \leq 2$, the sequence $\{T^n f\}$ converges μ-a.e., with

$$\lim_{n\to\infty} T^n f = \mathbb{E}[f|\mathcal{F}_1 \cap \mathcal{F}_2 \cap \cdots \cap \mathcal{F}_d] \quad \mu \text{ - a.e.}$$

1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T = P_1 P_2 \cdots P_d$ be a finite product of conditional expectations with respect to the sub σ-algebras $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_d$. Since conditional expectations are contractions of all $L_p(\mu)$ spaces, $p \in [1, \infty]$, so is T.

When $d = 2$, Burkholder and Chow [2] proved that for every $f \in L_2(\mu)$ the iterates $T^n f$ converge a.s. (and thus also in L_2-norm) to the conditional expectation with respect to $\mathcal{F}_1 \cap \mathcal{F}_2$. The L_2-norm convergence had been proved by von-Neumann [5, Lemma 22]. The main property of T when $d = 2$ is that $T^n = (P_1 P_2 P_1)^{n-1} P_2$ with $P_1 P_2 P_1$ self-adjoint in L_2, so from the work of Stein [9] it follows that the a.e. convergence of $\{T^n f\}$ holds also for any $f \in L_p(\mu)$, $p > 1$ (one needs to show only for $p < 2$). Rota's work [7] yields a different proof, which in fact proves the a.e. convergence of $\{T^n f\}$ when $f \in L \log^+ L$ (see [1]). Ornstein [6] showed that for $f \in L_1(\mu)$ almost everywhere convergence need not hold (although L_1-norm convergence does).

For arbitrary d, the L_2-norm convergence of $T^n f$, $f \in L_2(\mu)$, was proved by Halperin [4] (and the limit is the conditional expectation with respect to $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \cdots \cap \mathcal{F}_d$). Zaharopol [12] proved that the iterates $T^n f$ converge in L_p-norm for $f \in L_p(\mu)$, $p \geq 1$ (for $p \leq 2$ this follows from [4]). Delyon and Delyon [3] proved that $T^n f$ converges a.e. for any $f \in L_2(\mu)$.

In this note we show that for every $f \in L_p(\mu)$, $p > 1$, the sequence $\{T^n f\}$ converges μ-a.e., with

$$\lim_{n\to\infty} T^n f = \mathbb{E}[f|\mathcal{F}_1 \cap \mathcal{F}_2 \cap \cdots \cap \mathcal{F}_d] \quad \mu \text{ - a.e.}$$

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2. Pointwise convergence

Since [3] gives a.e. convergence of $T^n f$ for $f \in L_2(\mu)$, we have the convergence for $f \in L_p$, $p > 2$. For $1 < p < 2$, a maximal inequality in L_p will prove our result. We will combine techniques from [9] and [3].

Theorem 2.1 (Delyon-Delyon [3]). *Let V be an operator on a Hilbert space and let σ be a closed bounded convex subset of $\mathbb{C}$ containing the numerical range of V, i.e., containing $\Theta(V) = \{\langle f, Vf\rangle : \|f\| = 1\}$. Then there exists a constant K_σ, which depends only on σ, such that for any finite sequence of rational functions $u_1, \ldots, u_l$ with poles outside σ we have*

$$\Big\| \sum_{i=1}^{l} u_i(V)^* u_i(V) \Big\| \le K_\sigma^2 \sup_{z\in\sigma} \sum_{i=1}^{l} |u_i(z)|^2.$$

Remark. For any $0 \le \alpha \le 1$, denote by D_α the closed disk of radius $1-\alpha$ centered at $(\alpha, 0)$. For any real ϵ, denote by H_ϵ the closed half-plane, containing $(0,0)$ and having $(1,0)$ on its boundary, defined by

$$H_\epsilon = \big\{ z : \Re\{(1+i\epsilon)(1-z)\} \ge 0 \big\}.$$

As was noted in [3, §6], when we consider our specific operator T in Theorem 2.1, there exist $\alpha, \epsilon > 0$, such that the set $\sigma = D_\alpha \cap H_\epsilon \cap H_{-\epsilon}$ satisfies the conditions of Theorem 2.1. It is then possible to check that $\sup_{z\in\sigma} \frac{|1-z|}{1-|z|} < \infty$.

Notations.

Put $M_n = \frac{1}{n+1}\sum_{k=0}^{n} T^k$ and $M(f) = \sup_{n\ge 0} |M_n(f)|$. Also put

$$\Delta^0 T^n = T^n, \ \Delta T^n = T^n - T^{n-1}, \ \Delta^2 T^n = T^n - 2T^{n-1} + T^{n-2}, \ldots$$
$$\Delta^r T^n = \Delta(\Delta^{r-1} T^n) = T^{n-r}(T-I)^r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} T^{n-j}.$$

We agree that $\Delta^r T^n = 0$ for $n < r$.

The next proposition refines and extends the inequalities of [3], and is crucial to the use of Stein's method [9] in the non-symmetric case $d > 2$ (for $d = 2$ it follows from [9, Lemma 2]).

Proposition 2.2. *Let T be the product of d conditional expectations. For every integer $r = 0, 1, 2, \cdots$, there exists a positive constant B_r, such that for every $f \in L_2(\mu)$ we have*

$$\| \sup_{n\ge r} n^r |\Delta^r T^n f| \,\|_2 \le B_r \|f\|_2.$$

Proof. By Delyon-Delyon [3] (see the proof in §6), for some absolute constant $B_0 > 0$, we have $\| \sup_{n\ge 0} |T^n f| \,\|_2 \le B_0 \|f\|_2$ – this is the case $r = 0$.

By two successive applications of Abel's summation by parts we obtain

$$T^n - M_n = \frac{1}{n+1}\sum_{k=1}^{n} k\Delta T^k = \frac{1}{n+1}\Big[\frac{n(n+1)\Delta T^n}{2} - \sum_{k=2}^{n} \frac{k(k-1)}{2}\Delta^2 T^k\Big].$$

Hence, in order to estimate the norm $\| \sup_{n\geq 0} n|\Delta T^n f| \|_2$ it is enough to estimate $\| \sup_{n\geq 0} |T^n f| \|_2$, $\|M(f)\|_2$, and $\Big\| \sup_{n\geq 2} |\frac{1}{n+1}\sum_{k=2}^n \frac{k(k-1)}{2}\Delta^2 T^k f|\Big\|_2$, so only the last quantity should be estimated.

Using the Cauchy-Schwarz inequality we have

$$\Big|\frac{1}{n+1}\sum_{k=2}^n \frac{k(k-1)}{2}\Delta^2 T^k f\Big|^2 \leq \frac{1}{8}\sum_{k=2}^\infty k^3|\Delta^2 T^k f|^2 = \frac{1}{8}\sum_{k=2}^\infty k^3|T^{k-2}(T-I)^2 f|^2.$$

Hence, using Beppo Levi's theorem and Theorem 2.1,

$$\int \Big|\sup_{n\geq 2}|\frac{1}{n+1}\sum_{k=2}^n \frac{k(k-1)}{2}\Delta^2 T^k f|\Big|^2 d\mu \leq \int \sum_{k=2}^\infty k^3 |T^{k-2}(T-I)^2 f|^2 d\mu =$$

$$\lim_{n\to\infty}\sum_{k=2}^n k^3\langle [T^{k-2}(T-I)^2]^* T^{k-2}(T-I)^2 f, f\rangle \leq$$

$$K_\sigma \|f\|_2^2 \sup_{z\in\sigma}\sum_{k=2}^\infty k^3 |z^{k-2}|^2|(z-1)^2|^2 \leq$$

$$CK_\sigma\|f\|_2^2 \sup_{z\in\sigma}\frac{|1-z|^4}{(1-|z|^2)^4} \leq CK_\sigma\|f\|_2^2\sup_{z\in\sigma}\Big(\frac{|1-z|}{1-|z|}\Big)^4 < \infty.$$

So, combining all facts we obtain that $\| \sup_{n\geq 0} n|\Delta T^n f| \|_2 \leq B_1\|f\|_2$ for some absolute constant B_1.

By successive applications of Abel's summation by parts, it is possible to show that in order to estimate the norm $\| \sup_{n\geq 0} n^r|\Delta^r T^n f| \|_2$, one needs to estimate all $\| \sup_{n\geq 0} n^j|\Delta^j T^n f| \|_2$, $j = 0,\dots,r-1$, and $\|M(f)\|_2$, and also $\| \sum_{k=r+1}^\infty k^{2r+1}|\Delta^{r+1}T^k f|^2\|_2$. Hence, we use Theorem 2.1 to estimate

$$\int \sum_{k=r+1}^\infty k^{2r+1}|\Delta^{r+1}T^k f|^2 d\mu \leq K_\sigma\|f\|_2^2\sup_{z\in\sigma}\sum_{k=r+1}^\infty k^{2r+1}|z^{k-r-1}|^2|(z-1)^{r+1}|^2 \leq$$

$$CK_\sigma\|f\|_2^2\sup_{z\in\sigma}\frac{|1-z|^{2r+2}}{(1-|z|^2)^{2r+2}} \leq CK_\sigma\|f\|_2^2\sup_{z\in\sigma}\Big(\frac{|1-z|}{1-|z|}\Big)^{2r+2} < \infty.$$

By combining all the above estimates the result follows. $\square$

In order to use Stein's complex interpolation [8] as in [9], we need to define $C(\lambda)$-Cesàro sums of a *complex order* λ, (See [13, §III.1] for the standard notations and Stein and Weiss [11, §3] for extensibility to complex orders). Denote $A_0^\lambda = 1$ and

$$A_k^\lambda = \frac{(\lambda+1)\cdot(\lambda+2)\cdots(\lambda+k)}{k!} \quad \text{for an integer } k>0.$$

Here A_k^λ is the k^{th}-coefficient of the Taylor expansion of $\frac{1}{(1-x)^{1+\lambda}}$, $-1<x<1$. $\{A_k^\lambda\}$ are also called *generalized binomial coefficients*.

The following estimate is known (e.g., see Zygmund [13, §III.1]):

Lemma 2.3. *If $\alpha \in \mathbb{R}\backslash\{-1,-2,\dots\}$, then $A_n^\alpha \cong \frac{n^\alpha}{\Gamma(\alpha+1)}$. Hence there exists a positive constant b_α, which depends only on α, such that for every $n \ge 0$ we have*

$$(n+1)^\alpha/b_\alpha \le A_n^\alpha \le b_\alpha(n+1)^\alpha.$$

The next lemma extends [11, Lemma 6], with similar computations.

Lemma 2.4. *If $\alpha \in \mathbb{R}\backslash\{-1,-2,\dots\}$ and $\beta \in \mathbb{R}$, then there exist positive constants c_α and C_α, which depend only on α, such that for every $n \ge 0$ we have*

$$1 \le |A_n^{\alpha+i\beta}/A_n^\alpha| \le c_\alpha \mathrm{e}^{2\beta^2} \qquad \textit{and} \qquad |A_n^{\alpha+i\beta}| \le C_\alpha \mathrm{e}^{2\beta^2}(n+1)^\alpha.$$

Proof. For $\alpha > -1$ this is Lemma 6 in [11] and application of Lemma 2.3. Let $\alpha < -1$ be non-integer, and put $r = [|\alpha|]+1$, so $-r < \alpha < -r+1$. For $n > r$ by definition

$$\Big|\frac{A_n^{\alpha+i\beta}}{A_n^\alpha}\Big|^2 = \prod_{k=1}^n \Big|1+\frac{i\beta}{k+\alpha}\Big|^2 = \prod_{k=1}^n \Big(1+\frac{\beta^2}{(k+\alpha)^2}\Big) =$$

$$\prod_{k=1}^r \Big(1+\frac{\beta^2}{(k+\alpha)^2}\Big)\cdot \prod_{k=r+1}^n \Big(1+\frac{\beta^2}{(k+\alpha)^2}\Big)\ .$$

Using $\beta^{2j} \le \beta^{2r}+1$ for $j<r$ in majorizing the polynomial given by left hand product (which dominates $|A_n^{\alpha+i\beta}/A_n^\alpha|^2$ when $n \le r$), we obtain

$$\Big|\frac{A_n^{\alpha+i\beta}}{A_n^\alpha}\Big|^2 \le c_\alpha^2\Big(1+\frac{\beta^{2r}}{r!}\Big)\prod_{k=1}^\infty\Big(1+\frac{\beta^2}{k^2}\Big).$$

Using the estimates $1+\frac{x^{2r}}{r!} \le \mathrm{e}^{x^2}$ and $1+x^2 \le \mathrm{e}^{x^2}$, we obtain

$$\Big|\frac{A_n^{\alpha+i\beta}}{A_n^\alpha}\Big|^2 \le c_\alpha^2 \mathrm{e}^{\beta^2}\cdot \mathrm{e}^{\left(\sum_{k=1}^\infty \frac{\beta^2}{n^2}\right)} \le c_\alpha^2\mathrm{e}^{\beta^2}\mathrm{e}^{\frac{\pi^2}{6}\beta^2} \le c_\alpha^2\mathrm{e}^{4\beta^2}.$$

The second inequality follows from Lemma 2.3, with $C_\alpha = b_\alpha c_\alpha$. □

For a (formal) series of numbers $\sum_{j=0}^\infty a_j$, the Cesàro sums of order λ are defined by

$$S_n^\lambda(\Sigma a_j) = \sum_{k=0}^n A_{n-k}^\lambda a_k$$

It is known [11] that for every two complex numbers λ and δ one has

$$S_n^{\lambda+\delta}(\Sigma a_j) = \sum_{k=0}^n A_{n-k}^{\lambda-1}S_k^\delta(\Sigma a_j).$$

Notations. For an integer $n \ge 0$ and a complex number λ we define the Cesàro sums operators $S_n^\lambda := \sum_{k=0}^n A_{n-k}^\lambda T^k$, so $S_n^\lambda(f)(x) = S_n^\lambda(\Sigma T^j f(x))$; for $n<0$ put $S_n^\lambda = 0$.

For $f \in L_1(\mu)$ put $S_*^\lambda(f) = \sup_{n\ge 0}|(n+1)^{-(\lambda+1)}S_n^\lambda(f)|$, and $f_r^* = S_*^{-(r+1)}(f)$ for non-negative integers r.

Note that
(i): $S_n^{-(r+1)} = T^{n-r}(T-I)^r = \Delta^r T^n$, for any integers $r \geq 0$ and $n \geq r$.
(ii): $S_n^{-1} = T^n$ for $n \geq 0$, and $S_*^{-1} f = \sup_{n\geq 0} |T^n f| = f_0^*$.
(iii): $S_n^0 = \sum_{k=0}^n T^k$, and $S_*^0(f) = \sup_{n\geq 0} |M_n(f)| = M(f)$.
For any non-negative integer r we have $S_n^\lambda = \sum_{k=0}^n A_{n-k}^{\lambda+r} S_k^{-(r+1)}$.

Proposition 2.5. *Let $\lambda = \alpha + i\beta$ be a complex number with $\alpha > 0$. Then there exists a positive constant C'_α, which depends only on α, such that for every $f \in L_1(\mu)$,*

$$S_*^\lambda(f) \leq C'_\alpha \mathrm{e}^{2\beta^2} M(f).$$

Consequently, for every $f \in L_p(\mu)$, $1 < p \leq \infty$, we have

$$\|S_*^\lambda(f)\|_p \leq \frac{p}{p-1} C'_\alpha\, \mathrm{e}^{2\beta^2} \|f\|_p\,.$$

Proof. By the above properties of Cesàro sums we have $S_n^\lambda = \sum_{k=0}^n A_{n-k}^{\lambda-1} S_k^0$. By the maximal ergodic theorem, $|S_k^0(f)| \leq (k+1)M(f) < \infty$ μ-a.e. Using Lemma 2.4 with $\alpha - 1 > -1$ we obtain

$$|S_n^\lambda(f)| \leq \sum_{k=0}^n |A_{n-k}^{\lambda-1}||S_k^0(f)| \leq C_{\alpha-1}\mathrm{e}^{2\beta^2} M(f) \sum_{k=0}^n (n+1-k)^{\alpha-1}(k+1) \leq$$

$$C_{\alpha-1}\mathrm{e}^{2\beta^2} M(f)(n+1)^{\alpha+1}.$$

So,

$$|S_*^\lambda(f)| = \sup_{n\geq 0} |(n+1)^{-(\lambda+1)} S_n^\lambda(f)| = \sup_{n\geq 0} |(n+1)^{-(\alpha+1)} S_n^\lambda(f)| \leq C'_\alpha \mathrm{e}^{2\beta^2} M(f)\,,$$

with $C'_\alpha = C_{\alpha-1}$. The second part follows from the first by the maximal ergodic theorem, since $\|M(f)\|_p \leq p/(p-1)\|f\|_p$. □

Proposition 2.6. *Let $\lambda = \alpha + i\beta$ be a complex number with $\alpha \leq 0$ and $\alpha \neq -1, -2, \ldots$. Then there exist positive constants D_α and D'_α, which depend only on α, such that for every $f \in L_2(\mu)$*

$$S_*^\lambda(f) \leq D_\alpha \mathrm{e}^{2\beta^2}(f_0^* + f_1^* + \cdots + f_{[|\alpha|]}^*).$$

Consequently,

$$\|S_*^\lambda(f)\|_2 \leq D'_\alpha \mathrm{e}^{2\beta^2} \|f\|_2\,.$$

Proof. By Proposition 2.2, for every integer $r \geq 0$ we have $\|f_r^*\|_2 < \infty$, since, by property (i),

$$f_r^* = \Big(\sup_{n\geq r} |(n+1)^r \Delta^r T^n f|\Big) \vee \Big(\max_{0\leq n<r} |(n+1)^r S_n^{-(r+1)} f|\Big).$$

Hence $f_r^* < \infty$ a.e.; by the definitions, $|S_k^{-(r+1)}(f)| \leq (k+1)^{-r} f_r^*$ for $k \geq 0$.

In the case $-1 < \alpha \leq 0$, using Lemma 2.4 we have

$$|S_n^\lambda(f)| \leq \sum_{k=0}^n |A_{n-k}^\lambda||S_k^{-1}(f)| \leq C_\alpha \mathrm{e}^{2\beta^2} f_0^* \sum_{k=0}^n (n+1-k)^\alpha \leq C_\alpha \mathrm{e}^{2\beta^2} f_0^* (n+1)^{\alpha+1}$$

Hence we have,

$$S_*^\lambda(f) = \sup_{n\geq 0}\left|\frac{S_n^\lambda(f)}{(n+1)^{(\lambda+1)}}\right| = \sup_{n\geq 0}\left|\frac{S_n^\lambda(f)}{(n+1)^{(\alpha+1)}}\right| \leq C_\alpha \mathrm{e}^{2\beta^2} f_0^*.$$

This proves the first part of the proposition for $-1 < \alpha \leq 0$, with $D_\alpha = C_\alpha$. The second part follows by taking the L_2-norm and application of Proposition 2.2, where D'_α depends on C_α and B_0.

Now, let $-2 < \alpha < -1$. This time we use the identity $S_n^\lambda = \sum_{k=0}^n A_{n-k}^{\lambda+1} S_k^{-2}$. First we assume that n is even. We have

$$S_n^\lambda(f) \leq \Big|\sum_{k=0}^{n/2} A_{n-k}^{\lambda+1} S_k^{-2}(f)\Big| + \sum_{k=n/2+1}^{n} |A_{n-k}^{\lambda+1}||S_k^{-2}(f)| = \Sigma_I + \Sigma_{II}.$$

First we estimate Σ_{II}. Using Lemma 2.4 for $\alpha+1 > -1$, we obtain

$$\Sigma_{II} \leq C_{\alpha+1}\mathrm{e}^{2\beta^2} f_1^* \sum_{k=n/2+1}^{n} (n+1-k)^{\alpha+1}\frac{1}{k+1} \leq$$

$$C_{\alpha+1}\mathrm{e}^{2\beta^2} f_1^* \sum_{k=n/2+1}^{n} (n+1-k)^{\alpha+1}\frac{2}{n} \leq \frac{2C_{\alpha+1}\mathrm{e}^{2\beta^2} f_1^* n^{\alpha+2}}{n(\alpha+2)} \frac{2}{\alpha+2} C_{\alpha+1}\mathrm{e}^{2\beta^2} f_1^* n^{\alpha+1}.$$

In order to estimate Σ_I we apply Abel's summation by parts. Note that $\Delta S_k^{-1} = S_k^{-2}$ and $A_n^\lambda = A_n^{\lambda+1} - A_{n-1}^{\lambda+1} = \Delta A_n^{\lambda+1}$. Also for two sequences $\{a_n\}$ and $\{b_n\}$, with $b_{-1} = 0$, we use the identity

$$\sum_{k=0}^{n} a_k \Delta b_k = a_n b_n - \sum_{k=0}^{n-1} b_k \Delta a_{k+1}.$$

Hence using Lemma 2.4 we obtain,

$$\Sigma_I = \Big|\sum_{k=0}^{n/2} A_{n-k}^{\lambda+1}\Delta S_k^{-1}(f)\Big| = \Big|A_{n/2}^{\lambda+1} S_{n/2}^{-1}(f) + \sum_{k=0}^{n/2-1} A_{n-k}^{\lambda} S_k^{-1}(f)\Big| \leq$$

$$|A_{n/2}^{\lambda+1} S_{n/2}^{-1}(f)| + \sum_{k=0}^{n/2-1} |A_{n-k}^{\lambda}||S_k^{-1}(f)| \leq$$

$$C_{\alpha+1}\mathrm{e}^{2\beta^2} f_0^* n^{\alpha+1} + C_\alpha \mathrm{e}^{2\beta^2} f_0^* \sum_{k=0}^{n/2-1} (n+1-k)^\alpha \leq$$

$$(C_{\alpha+1} + \frac{C_\alpha}{\alpha+1})\mathrm{e}^{2\beta^2} f_0^* n^{\alpha+1}.$$

Combining Σ_I and Σ_{II} we obtain

$$|S_n^\lambda(f)| \leq D_\alpha \mathrm{e}^{2\beta^2}(f_0^* + f_1^*) n^{\alpha+1},$$

where D_α depends on C_α and $C_{\alpha+1}$. This inequality holds for n even. For odd n, we split $\sum_{k=0}^{n} = \sum_{k=0}^{\frac{n+1}{2}} + \sum_{\frac{n+1}{2}+1}^{n}$ and make the same computations as above. Hence we have,

$$S_*^\lambda(f) = \sup_{n\geq 0}\left|\frac{S_n^\lambda(f)}{(n+1)^{(\lambda+1)}}\right| = \sup_{n\geq 0}\left|\frac{S_n^\lambda(f)}{(n+1)^{(\alpha+1)}}\right| \leq D_\alpha e^{2\beta^2}(f_0^* + f_1^*).$$

This proves the first inequality of the proposition in the case $-2 < \alpha < -1$. The second inequality follows by taking the L_2-norm and using Proposition 2.2, where D'_α depends on C_α, $C_{\alpha+1}$, B_0, and B_1.

Similarly, one can prove the case $-3 < \alpha < -2$. We first assume that $n \geq 4$ is even and we start with

$$S_n^\lambda(f) \leq \Big|\sum_{k=0}^{n/2} A_{n-k}^{\lambda+2}\Delta S_k^{-2}(f)\Big| + \sum_{k=n/2+1}^{n} |A_{n-k}^{\lambda+2}||S_k^{-3}(f)| = \Sigma_I + \Sigma_{II}.$$

Using Lemma 2.4 for $\alpha+2 > -1$, we obtain

$$\Sigma_{II} \leq C_{\alpha+2}e^{2\beta^2} f_2^* \sum_{k=n/2+1}^{n} (n+1-k)^{\alpha+2}\frac{1}{k+1^2} \leq$$

$$C_{\alpha+2}e^{2\beta^2} f_2^* \sum_{k=n/2+1}^{n} (n+1-k)^{\alpha+2}\frac{4}{n^2} \leq \frac{4C_{\alpha+2}e^{2\beta^2} f_2^* n^{\alpha+3}}{n^2(\alpha+3)} = \frac{4}{\alpha+3}C_{\alpha+2}e^{2\beta^2} f_2^* n^{\alpha+1}.$$

In order to estimate Σ_I we apply Abel's summation by parts twice successively. Hence we obtain

$$\Sigma_I \leq |A_{n/2}^{\lambda+2}S_{n/2}^{-2}(f)| + \Big|\sum_{k=0}^{n/2-1} A_{n-k}^{\lambda+1}\Delta S_k^{-1}(f)\Big| \leq$$

$$A_{n/2}^{\lambda+2}|S_{n/2}^{-2}(f)| + |A_{n/2+1}^{\lambda+1}||S_{n/2-1}^{-1}(f)| + \sum_{k=0}^{n/2-2} |A_{n-k}^{\lambda}||S_k^{-1}(f)| \leq$$

$$2C_{\alpha+2}e^{2\beta^2} n^{\alpha+2}\frac{f_1^*}{n} + C_{\alpha+1}e^{2\beta^2} n^{\alpha+1} f_0^* + C_\alpha e^{2\beta^2} f_0^* n^{\alpha+1}.$$

Combining Σ_I and Σ_{II} we obtain

$$|S_n^\lambda(f)| \leq D_\alpha e^{2\beta^2}(f_0^* + f_1^* + f_2^*)n^{\alpha+1},$$

where now D_α depends on C_α, $C_{\alpha+1}$, and $C_{\alpha+2}$. Similar considerations yield the same for n odd. Hence we have,

$$S_*^\lambda(f) \leq D_\alpha e^{2\beta^2}(f_0^* + f_1^* + f_2^*).$$

By taking the L_2-norm in the above inequality and using Proposition 2.2, we obtain the second inequality (the second assertion of the proposition) for the case $-3 < \alpha < -2$, where D'_α depends on $C_{\alpha+j}$ and B_j, $j = 0, 1, 2$.

Consequently, if $-(r+1) < \alpha < -r$, for some non-negative integer r, then after r-successive applications of Abel's summation by parts, we obtain

$$S_*^\lambda(f) \leq D_\alpha e^{2\beta^2}(f_0^* + f_1^* + \cdots + f_r^*),$$

for D_α depends on $C_{\alpha+j}$, $j = 0, 1, \ldots, r$. By taking the L_2-norm in the above inequality and using Proposition 2.2, we obtain the second inequality for the case $-(r+1) < \alpha < -r$, where D'_α depends on $C_{\alpha+j}$ and B_j, $j = 0, 1, \ldots, r$. □

Remark. In the general context of T a self-adjoint Dunford-Schwartz contraction (i.e., T is a contraction of each L_p, $1 \le p \le \infty$), Proposition 2.5 and Proposition 2.6 are Lemma 4 and Lemma 3 in [9], respectively. Only short indications of proofs were given in [9]. Also a continuous version of these propositions (for the analogous problem of a.e. convergence for a semigroup $\{T_t : t \ge 0\}$) was addressed in [9]. In Stein's book [10] proofs were given for the continuous version. In this case, Cesàro summability of complex order is replaced by fractional integration and fractional derivation. Since the proofs of Proposition 2.5 and Proposition 2.6 are not immediate consequences of their continuous analogues, the proofs are given here for the sake of completeness. While Proposition 2.5 holds for *any* Dunford-Schwartz contraction, the more complicated Proposition 2.6 relies on specific estimates and inequalities in L_2, provided in our case by [3].

For the reader's convenience we now describe Stein's complex interpolation method [8] (see also [13, Theorem XII.1.39]).

Let (X, ν) and (Y, η) be two measure spaces and let $\{T_z : z \in \mathbb{C}\}$ be a family of linear transformations from the simple functions on (X, ν) to measurable functions on (Y, η). Such a family is called an *analytic family of operators* if for any simple functions f and g on X and Y, respectively, $\Phi(z) := \int T_z(f) g \, d\eta$ is analytic in the strip $0 < \Re(z) < 1$ and continuous in $0 \le \Re(z) \le 1$.

The analytic family $\{T_z\}$ is said to have an *admissible growth* if for f and g as above there exist two positive constants A and $a < \pi$, which depend only on f and g, such that for every $z = \alpha + i\beta$, with $0 \le \alpha \le 1$, we have $\log |\Phi(\alpha + i\beta)| \le A e^{a|\beta|}$.

Stein's complex interpolation theorem. *Let $\{T_z\}$ be an analytic family of operators with admissible growth. Suppose that $1 \le p_1, p_2, q_1, q_2 \le \infty$, and that $1/p = (1-t)/p_1 + t/p_2$ and $1/q = (1-t)/q_1 + t/q_2$, where $0 \le t \le 1$. Also suppose that for every simple function f on X,*

$$\|T_{iy}(f)\|_{q_1} \le A_0(y) \|f\|_{p_1} \quad \text{and} \quad \|T_{1+iy}(f)\|_{q_2} \le A_1(y) \|f\|_{p_2}.$$

We also assume that for some absolute positive constants A and $a < \pi$

$$\log |A_i(y)| \le A e^{a|y|}, \qquad i = 0, 1.$$

Then $\|T_t(f)\|_q \le A_t \|f\|_p$ for some positive constant A_t depending only on t and the functions $A_0(y)$ and $A_1(y)$. Consequently, T_t may be extended to a bounded linear operator from all of $L_p(X, \nu)$ into $L_q(Y, \eta)$.

Theorem 2.7. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, let $\mathcal{F}_j$, $1 \le j \le d$, be sub σ-algebras of $\mathcal{F}$ with corresponding conditional expectations P_j, and put $T = P_1 P_2 \cdots P_d$. Then for every $1 < p \le \infty$ there exists a positive constant*

A_p, such that for every $f \in L_p(\mu)$ we have

$$\| \sup_{n\geq 0} |T^n f| \, \|_p \leq A_p \|f\|_p \, .$$

Consequently, the sequence $\{T_n f\}$ converges μ-a.e. with

$$\lim_{n\to\infty} T^n f = \mathbb{E}[f|\mathcal{F}_1 \cap \mathcal{F}_2 \cap \cdots \cap \mathcal{F}_d].$$

Proof. For the pointwise convergence we have to prove only when $1 < p < 2$ (the case $p = 2$ was proved by [3]). The maximal inequality when $2 < p < \infty$ can be proved similarly.

Take $1 < p < 2$ and fix $1 < p_0 < p < 2$. Find $0 < t^* < 1$, such that $1/p = (1-t^*)/2 + t^*/p_0$. For $K > 1/(1-t^*) > 0$ define $\alpha_0 := -1 - Kt^* < -1$ and $\alpha_1 := -1 + K(1 - t^*) > 0$. We may and do choose K such that $\alpha_0 \neq -2, -3, \ldots$.

Let $N(\omega)$ be any bounded $\mathbb{N}$-valued $\mathcal{F}$-measurable function. For $\lambda \in \mathbb{C}$ and any simple function f define

$$R_{\lambda,N}(f)(\omega) = (N(\omega)+1)^{-(\alpha_0+\lambda(\alpha_1-\alpha_0)+1)} \cdot S_{N(\omega)}^{\alpha_0+\lambda(\alpha_1-\alpha_0)}(f)(\omega).$$

Now fix the bounded function $N(\omega)$. Since for $z \in \mathbb{C}$,

$$\int g(\omega) \frac{S^z_{N(\omega)}(f)(\omega)}{(N(\omega)+1)^{z+1}} d\mu = \sum_{n=0}^{\max N} \sum_{k=0}^{n} \frac{A^z_{n-k}}{(n+1)^{z+1}} \int_{\{N(\omega)=n\}} g(\omega) T^k f(\omega) d\mu,$$

it follows that for any simple function g, the function $\int g R_{\lambda,N}(f)\, d\mu$ is continuous in $0 \leq \Re(\lambda) \leq 1$ and analytic in $0 < \Re(\lambda) < 1$. Using Proposition 2.5 or Proposition 2.6 and Hölder's inequality we conclude that $\{R_{\lambda,N}\}$ is an analytic family with admissible growth in the strip $0 \leq \Re(\lambda) \leq 1$. Furthermore, using both propositions and $|R_{\lambda,N}(f)| \leq |S_*^{\alpha_0+\lambda(\alpha_1-\alpha_0)}(f)|$ we conclude that

$$\|R_{i\beta,N}(f)\|_2 \leq \|S_*^{\alpha_0+i\beta(\alpha_1-\alpha_0)(f)}\|_2 \leq D'_{\alpha_0} \mathrm{e}^{2[\beta(\alpha_1-\alpha_0)]^2} \|f\|_2$$

$$\|R_{1+i\beta,N}(f)\|_{p_0} \leq \|S_*^{\alpha_1+i\beta(\alpha_1-\alpha_0)}(f)\|_{p_0} \leq \frac{p_0}{p_0-1} C'_{\alpha_1} \mathrm{e}^{2[\beta(\alpha_1-\alpha_0)]^2} \|f\|_{p_0},$$

where D'_{α_0} and C'_{α_1} are absolute constants, which are independent of f or of the choice of $N(\omega)$. By the interpolation theorem we obtain that $\|R_{t^*,N}(f)\|_p \leq A_p \|f\|_p$ for $f \in L_p(\mu)$, with A_p a positive constant which is independent of $N(\omega)$ and f (but may depend on p, p_0, α_0, α_1, and d).

Given $f \in L_p(\mu)$, let $N_k(\omega)$ be the first integer where $\max\limits_{1\leq n\leq k} |T^n f(\omega)|$ is attained. Then $|R_{t^*,N_k}(f)| = |S^{-1}_{N_k}(f)| = |T^{N_k} f| = \max\limits_{1\leq n\leq k} |T^n f| \,$.

Now, by Lebesgue's monotone convergence theorem, we obtain the asserted maximal inequality.

The μ-a.e. convergence of $\{T^n f\}$ follows from the already known convergence for functions in $L_2(\mu)$ (by [3]) and by the Banach principle. The identification of the limit follows from [4] or [12].

The maximal inequality for the case $2 < p \leq \infty$ is achieved by a similar interpolation procedure, now between 2 to ∞. ∎

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