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On the convergence of the rotated one-sided ergodic Hilbert transform

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Abstract Sufficient conditions have been given for the convergence in norm and a.e. of the ergodic Hilbert transform ([13], [6], [7]). Here we apply these conditions to the rotated ergodic Hilbert transform $\sum_{n=1}^{\infty} \frac{\lambda^n}{n} T^n f$, where λ is a complex number of modulus 1. When T is a contraction in a Hilbert space, we show that the logarithmic Hausdorff dimension of the set of λ's for which this series does not converge is at most 2 and give examples where this bound is attained.

Keywords Contractions · spectral measure · one-sided rotated ergodic Hilbert transform · Hausdorff dimension

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1 Introduction

Let T be a normal contraction on a Hilbert space $\mathcal{H}$. Let D be the unit disk and, for $f \in \mathcal{H}$, denote the spectral measure of f with respect to T by σ_f. In [6] (see also [3] for earlier results) it was proved that the one-sided ergodic Hilbert transform (EHT) $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ converges in $\mathcal{H}$ if and only if $\int_D \log^2 |1-z|\, d\sigma_f < \infty$. It is proved in [6] that when T is a contraction (even not normal) on $\mathcal{H} = L_2(m)$ of a σ-finite measure, then for $f \in L_2(m)$ the convergence of $\sum_{n=3}^{\infty} \frac{\langle T^n f, f\rangle \log n (\log\log\log n)^2}{n}$ ensures m-a.e. and norm convergence of $\sum_{n=1}^{\infty} \frac{T^n f}{n}$. Convergence of the EHT is a strengthening of the convergence of the ergodic means given by the ergodic theorems.

Denote by Γ the unit circle and take $\lambda \in \Gamma$. By the mean ergodic theorem we know that for every contraction T on $\mathcal{H}$ the averages $\frac{1}{n}\sum_{k=1}^{n} \lambda^k T^k f$ converge in norm. When T is induced by a measure-preserving transformation on a probability space (Ω, m), the Wiener-Wintner theorem [20] says that for $f \in L_1(\Omega, m)$ and for m-a.e. $\omega \in \Omega$, the averages $\frac{1}{n}\sum_{k=1}^{n} \lambda^k T^k f(\omega)$ converge for *every* $\lambda \in \Gamma$.

It is then natural to consider the convergence for $\lambda \in \Gamma$ of the *rotated* EHT

$$\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n}. \tag{1}$$

For the *two-sided* rotated ergodic Hilbert transform $\sum_{n=1}^{\infty} \frac{\lambda^n T^n f - \overline{\lambda}^n T^{*n} f}{n}$, for every contraction T (not necessarily normal) on $\mathcal{H}$ and for *every* $f \in \mathcal{H}$, convergence in norm holds for every $\lambda \in \Gamma$ (Campbell [4]). Lacey and Terwilleger proved recently that if T is induced by an invertible measure preserving transformation on a probability space, then for every $f \in L_p(m)$, $p > 1$, m-a.e. the two-sided rotated ergodic Hilbert transform converges for every $\lambda \in \Gamma$ ([17], Corollary 7.2).

For the one-sided rotated ergodic Hilbert transform, by Theorem 4.2 in [6] applied to λT for a contraction T and Carleson's theorem, convergence in norm holds in (1) for Lebesgue a.e. λ. The aim of this paper is to answer the following question: For a given $f \in \mathcal{H}$, *what is the size of the set of λ's such that the series in* (1) *does not converge in norm* ? We will show that the logarithmic Hausdorff dimension of this set is at most 2, and construct examples where it can be 2. We consider also a.e. convergence when $\mathcal{H}$ is the space $L_2(m)$ of a σ-finite measure m.

In some cases there is norm convergence in (1) for every $\lambda \in \Gamma$. If $\{T^n f\}$ are centered i.i.d. with finite variance the convergence holds by orthogonality. Furthermore, by Theorem 1(iv) of Cuzick and Lai [9] if $\int |f| \log^+ |f| dm < \infty$, then m-a.e. the series (1) converges uniformly in λ.

Remark that, unlike the two-sided rotated ergodic Hilbert transform, there are examples where there is norm-convergence in (1) for every $\lambda \in \Gamma$, but the set of points where convergence holds for every λ is empty. As an example consider the map : $(x, y) \to (x, y+x)$ on the 2-torus. The spectrum is continuous on the subspace orthogonal to the functions depending only on x. For the function $f(x, y) := e^{2\pi i y}$ there is convergence in norm in (1) for every $\lambda \in \Gamma$, but the set of points (x, y) such that pointwise convergence holds for every λ is empty. By elementary Fourier analysis, for every fixed λ the series (1) converges a.e., (note that this is a particular case of the general result of [7]), but the null-set of non-convergence depends on λ.

2 Ergodic Hilbert transform and rotated ergodic Hilbert transform

2.1 A lemma on Fourier series

We begin with some preliminaries which slightly extend results of [6].

Let $b : u \to b(u)$ be a positive *slowly varying* function defined for $u \geq 1$ (i.e. for every $\delta > 0$, $u^\delta b(u)$ is increasing and $u^{-\delta} b(u)$ is decreasing for u large enough). Write $B(t) := \int_1^t \frac{b(u)}{u} du$.

It is known that, if $\sum_{n=1}^\infty \frac{b(n)}{n} = \infty$, then, as $t \to \infty$, $B(t) \cong \sum_{n \leq t} \frac{b(n)}{n}$ and $b(t) = o(B(t))$ (see [21], Ch. V, p. 188).

Lemma 2.1.1 *Let ν be a finite measure on the interval $[-\pi, \pi[$. For a positive slowly varying function b such that $\sum_{n=1}^\infty \frac{b(n)}{n} = \infty$, the following conditions are equivalent:*

$$\int_{-\pi}^{\pi} B(\frac{1}{|t|})\, \nu(dt) < \infty; \tag{2}$$

$$\sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{ikt}\, b(k)}{k}\ \nu(dt) \ \textit{converges}; \tag{3}$$

$$\liminf_{n \to +\infty} \sum_{k=1}^{n} \int_{-\pi}^{\pi} \frac{\cos(kt)\, b(k)}{k}\ \nu(dt) < +\infty. \tag{4}$$

Proof Clearly (3) $\Rightarrow$ (4). The proof of (2) $\Rightarrow$ (3) is similar to that of $(i) \Rightarrow (ii)$ of Theorem 3.3 given in [6] for the special case $b(k) = \log k$. We prove the general result for the sake of completeness.

Assume that (2) holds.

For every $n \geq 1$ and $t \in [-\pi, \pi] \setminus \{0\}$, we have $|\sum_{k=1}^n \mathrm{e}^{ikt}| \leq \frac{\pi}{|t|}$. Since $b(u)$ is slowly varying, the sequence $\{b(n)/n\}_{n\geq 1}$ decreases to zero and Abel's summation by parts yields that the series $\sum_{n=1}^\infty \frac{\mathrm{e}^{int} b(n)}{n}$ converges for every $t \in [-\pi, \pi] \setminus \{0\}$, and that the partial sums are uniformly bounded on the set $\{t \in [-\pi, \pi] : |t| \geq \varepsilon > 0\}$ for every $\varepsilon > 0$.

As $\nu(\{0\}) = 0$ by (2), $\sum_{k=1}^\infty \frac{\mathrm{e}^{ikt} b(k)}{k}$ converges ν-a.e. To prove (3), by the Lebesgue dominated convergence theorem and by (2), it suffices to prove that, for t in a neighborhood of 0, $\sup_{n\geq 1} |\sum_{k=1}^n \frac{\mathrm{e}^{ikt} b(k)}{k}|$ is dominated by $B(|t|^{-1})$.

We will bound $\sup_{n\geq 1} |\sum_{k=1}^n \frac{\mathrm{e}^{ikt} b(k)}{k}|$ for $0 < |t| \leq \frac{1}{3}$. Let $n_t := [|t|^{-1}]$.

For $n \leq n_t$, we have:

$$\Big|\sum_{k=1}^n \frac{\mathrm{e}^{ikt} b(k)}{k}\Big| \leq \sum_{k=1}^n \frac{b(k)}{k} \leq CB(n) \leq CB(|t|^{-1}).$$

For $n > n_t$, we use the decomposition

$$\sum_{k=1}^n \frac{\mathrm{e}^{ikt} b(k)}{k} = \sum_{k=1}^{n_t} \frac{\mathrm{e}^{ikt} b(k)}{k} + \sum_{k=n_t+1}^{n} \frac{\mathrm{e}^{ikt} b(k)}{k} = P_1 + P_2,$$

with P_1 estimated above. Let $S_j := \sum_{k=1}^j \mathrm{e}^{ikt}$, for $j \geq 1$. Since $n \geq n_t + 1 > |t|^{-1}$ and $\{b(n)/n\}$ is decreasing, using Abel's summation, we obtain

$$\begin{aligned} |P_2| &\leq \frac{b(n)}{n}|S_n| + \sum_{k=n_t+1}^{n-1} \Big(\frac{b(k)}{k} - \frac{b(k+1)}{k+1}\Big)|S_k| + \frac{b(n_t+1)}{n_t+1}|S_{n_t}| \\ &\leq \frac{b(n)}{n}\frac{\pi}{|t|} + 2\frac{b(n_t+1)}{n_t+1}\frac{\pi}{|t|} \;\leq\; 3\pi \frac{b(|t|^{-1})}{|t|^{-1}}\frac{1}{|t|} \;=\; 3\pi b(|t|^{-1}). \end{aligned}$$

The two cases together give

$$\sup_{n\geq 1}\Big|\sum_{k=1}^n \frac{\mathrm{e}^{ikt} b(k)}{k}\Big| \leq C' b(|t|^{-1}) + CB(|t|^{-1}),\ \forall t \in [-\pi, \pi] \setminus \{0\}.$$

This prove our claim, since $b(u) = o(B(u))$ as $u \to \infty$.

Now we prove (4) $\Rightarrow$ (2). For $\alpha \in]0, 1[$, the partial sums $\sum_{k=1}^n \frac{\cos(kt)}{k^\alpha}$ are uniformly bounded from below (see Zygmund ([21], Ch. V, Th. 2.29)). Hence, by Abel's summation by parts (using the fact that $\{b(n)/n^{1-\alpha}\}$ decreases) the partial sums $\sum_{k=1}^n \frac{\cos(kt)\, b(k)}{k}$ are uniformly bounded from below, say by $-C$.

We have $\nu(\{0\}) = 0$, since (4) implies

$$0 \leq \liminf_{n\to\infty} \int_{-\pi}^{\pi} \Big(C + \sum_{k=1}^n \frac{\cos(kt) b(k)}{k}\Big)\, \nu(dt) < \infty.$$

Using again that the sequence $\{b(k)/k\}$ decreases to zero, and Abel's summation by parts, we have the convergence of the series $\sum_{k=1}^{\infty} \frac{\cos(kt)\, b(k)}{k}$ for every $0 \neq t \in [-\pi, \pi]$, hence its convergence ν-a.e., and by Fatou's lemma

$$\int_{-\pi}^{\pi} \lim_{n\to\infty} \Big(C + \sum_{k=1}^{n} \frac{\cos(kt)b(k)}{k} \Big) \nu(dt) \leq \liminf_{n\to\infty} \int_{-\pi}^{\pi} \Big(C + \sum_{k=1}^{n} \frac{\cos(kt)b(k)}{k} \Big) \nu(dt)\,.$$

The integrand in (2) is bounded for $|t| \geq \varepsilon > 0$. Since $\sum_{k=1}^{\infty} \frac{\cos(kt)\, b(k)}{k}$ behaves like $B(|t|^{-1})$ as $t \to 0$ (see Zygmund ([21], Ch. V, Th. 2.15)), condition (2) is satisfied.

Note that although we only assume that the lim inf in (4) is not $+\infty$, the proof shows that it can not be $-\infty$, and that in fact the series converges. $\square$

2.2 The one-sided ergodic Hilbert transform

Let T be a contraction of a Hilbert space $\mathcal{H}$. Define $T_n := T^n$ for $n \geq 0$ and $T_n := (T^*)^{|n|}$ for $n < 0$. Then $\{\langle T_n f, f\rangle\}$ is a positive semi-definite sequence ([19], Appendix, §9) and therefore by Herglotz's theorem it is the sequence of the Fourier coefficients of a positive finite measure ν_f on the unit circle Γ. We will still denote by ν_f the representation of the measure ν_f as a measure on the interval $I = [-\pi, \pi[$ and use both representations.

By the unitary dilation theorem of B. Sz. Nagy ([19], Theorem III, p. 469), there exist a larger Hilbert space $\mathcal{H}'$, an orthogonal projection $P_{\mathcal{H}}$ from $\mathcal{H}'$ onto $\mathcal{H}$, and an unitary operator U on $\mathcal{H}'$ such that

$$T_n P_{\mathcal{H}} g = P_{\mathcal{H}} U^n g, \forall g \in \mathcal{H}', \forall n \in \mathbb{Z}.$$

For $f \in \mathcal{H}$, the above identity yields

$$\langle T_n f, f\rangle = \langle P_{\mathcal{H}} U^n f, f\rangle = \langle U^n f, P_{\mathcal{H}}^* f\rangle = \langle U^n f, P_{\mathcal{H}} f\rangle = \langle U^n f, f\rangle.$$

By the spectral representation theorem for unitary operators, ν_f is the spectral measure of f with respect to U, with Fourier coefficients $\{\hat{\nu}_f(n) = \langle T_{-n} f, f\rangle\}$.

Definition 2.2.1 For a contraction T on $\mathcal{H}$ and $f \in \mathcal{H}$, ν_f is called the *unitary spectral measure* of f (with respect to T). When ν_f is absolutely continuous with respect to the Lebesgue measure, we say that f has a *spectral density.*

Let $b(u)$ be a positive slowly varying function such that $\sum_{n=1}^{\infty} \frac{b(n)}{n} = \infty$. With the previous notations, the equivalence given by Lemma 2.1.1 yields immediately the equivalence between the following conditions:

$$\int_{-\pi}^{\pi} B(\frac{1}{|t|})\, \nu_f(dt) < \infty; \tag{5}$$

$$\sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{ikt}\, b(k)}{k}\ \nu_f(dt) = \sum_{k=1}^{\infty} \frac{b(k)}{k}\ \hat{\nu}_f(-k) \textit{ converges.} \tag{6}$$

If T is a normal contraction, then the previous conditions are equivalent to

$$\sum_{n=1}^{\infty} \frac{\| \sum_{k=1}^{n} T^k f\|^2 b(n)}{n^3} < \infty. \tag{7}$$

Indeed the proof of the equivalence between (5), (6) and (7) for the special case $b(n) = \log n$ (and hence $B(u) = \log^2 u$) was given in [6], Theorem 3.3, for T a normal contraction and can be adapted for a more general $b(n)$.

Let us also mention Cuny ([8], Lemma 2.1) for the equivalence (5) $\Leftrightarrow$ (7) and, for the case $b(n) = \log n (\log\log\log n)^2$, Gaposhkin ([13], conditions (33) and (34)) who has indicated that in the unitary case, with this choice of $b(n)$, (5) and (7) are equivalent and both are implied by (6). Here we see that these three conditions are equivalent.

Theorem 2.2.1 *1) Let T be a contraction on a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$ with unitary spectral measure ν_f. Then the following conditions are equivalent:*

$$\int_{-\pi}^{\pi} \log^2 |t| \, \nu_f(dt) < \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{\langle T^n f, f\rangle \log n}{n} \text{ converges.} \tag{9}$$

These conditions imply

$$\sum_{n=1}^{\infty} \frac{T^n f}{n} \text{ converges in norm.} \tag{10}$$

2) If T is a normal contraction, then (8), (9), (10) and (11) below (where σ_f is the spectral measure of f) are equivalent

$$\int_D \log^2 |1 - z| \, \sigma_f(dz) < \infty. \tag{11}$$

Proof The theorem is essentially in [6], except that it is shown here that all the information about the convergence of the one-sided EHT is contained in the unitary spectral measure, since the equivalence (8) $\Leftrightarrow$ (9) is a particular case of (5) $\Leftrightarrow$ (6).

The implication (9) $\Rightarrow$ (10) is Theorem 4.2 in [6], where also the equivalence (11) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10) is shown for a normal contraction. □

Remarks 1) If T is a normal contraction and U its unitary dilation, then $\sum_{n=1}^{\infty} \frac{U^n f}{n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ converges. Indeed, the "only if" follows by the continuity of the projection $P_{\mathcal{H}}$. For the "if" condition we apply the theorem to the unitary operator U, since (9) holds for U.

2) As mentioned in [6], if T is an isometry and U its unitary dilation, then $T^n f = U^n f$ for every $f \in \mathcal{H}$. Hence, when T is an isometry, (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10).

3) If T is a contraction of $L_2(m)$ of a σ-finite measure space and f is in $L_2(m)$, the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\langle T^n f, f\rangle \log n (\log\log\log n)^2}{n}$ implies the convergence in norm and m-a.e. of $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n}$ by Theorem 4.3 in [6].

4) Gaposhkin [13] has shown that in the family of *all* unitary operators T, the condition $\int_{-\pi}^{\pi} b(t)\, \nu_f(dt) < \infty$ with $b(t) := \big(\log(|t|^{-1}) \log\log|\log(|t|^{-1})|\big)^2$ is sharp. As well, by the equivalence (2) $\Leftrightarrow$ (3) in Lemma 2.1.1, the factor $\log n(\log\log\log n)^2$ in claim 3) of Theorem 2.2.1 can not be replaced by any slowly varying function $b(u)$ with $\int_1^\infty \frac{b(u)}{u} du = \infty$ and $b(n) = o\big(\log n(\log\log\log n)^2\big)$ for the class of unitary operators.

2.3 The rotated one-sided ergodic Hilbert transform $\sum_{n=1}^{\infty} \frac{\lambda^n}{n} T^n f$

Let T be a contraction on $\mathcal{H}$ and let $f \in \mathcal{H}$. Let ν_f be the unitary spectral measure of f with respect to T. For $\lambda \in \Gamma$ we have $(\lambda T)_n = \lambda^n T_n$, for every $n \in \mathbb{Z}$. Hence the unitary spectral measure of f with respect to λT is $\nu_f(\lambda^{-1}\cdot)$ (denoted by $\delta_\lambda * \nu_f$).

Similarly, if T is a normal contraction on $\mathcal{H}$ and if σ_f is the spectral measure of f with respect to T, then the spectral measure of f with respect to λT is $\sigma_f(\lambda^{-1}\cdot)$. Therefore the following proposition results immediately from Theorem 2.2.1 and Theorem 4.3 in [6].

Proposition 2.3.1 *Let T be a contraction on a Hilbert space $\mathcal{H}$, let f be in $\mathcal{H}$ with unitary spectral measure ν_f, and let $\lambda \in \Gamma$.*

1) Then the following conditions are equivalent:

$$\int_\Gamma \log^2|1-z|\ \delta_\lambda * \nu_f(dz) < \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{\lambda^n \langle T^n f, f\rangle \log n}{n} \ \textit{converges}. \tag{13}$$

These conditions imply

$$\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n} \ \textit{converges in norm}. \tag{14}$$

2) If T is a normal contraction, then (12), (13), (14) and (15) below are equivalent

$$\int_D \log^2|1-z|\ \delta_\lambda * \sigma_f(dz) < \infty. \tag{15}$$

3) Assume that T is contraction of the space $L_2(m)$ of a σ-finite measure space and $f \in L_2(m)$.

Then the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n \langle T^n f, f\rangle \log n\, (\log\log\log n)^2}{n}$ *implies that*

$$\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n} \text{ converges in norm and } m-a.e. \tag{16}$$

Remarks 1) If T be a positive contraction on $L_2(m)$ of a σ-finite measure m, or if T is a Dunford-Schwartz operator, then $\displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n}$ converges m-a.e. for $\lambda \in \Gamma$ such that (14) holds. Since the modulus of λT is T in the first case, the linear modulus of T in the second case, this results from Theorem 2.1 in Cuny [7] applied to λT.

2) Let the measure ν_f be an absolutely continuous with $d\nu_f/dt \in L_p(dt)$, for some $p > 1$. Put $B(1/|t|) := (\log(1/|t|))^2(\log\log|\log(1/|t|)|)^2$. Since $B(1/|t|)$ is in $L_q([-\pi,\pi[,dt)$ for every $1 \le q < \infty$, Hölder's inequality implies that $\int_{-\pi}^{\pi} B(1/(t-s))\frac{d\nu_f}{dt}(s)ds < \infty$. So, a rotated version of the equivalence (5) $\Leftrightarrow$ (6) yields (16) for *every* $\mathrm{e}^{is} \in \Gamma$. Another approach for proving this result is as follow. We may assume $1 < p \le 2$. Since $d\nu_f/dt \in L_p(dt)$, by the Hausdorff-Young theorem $\{\hat{\nu}_f(k)\} \in \ell_q$, so $\{\lambda^k\langle T^k f, f\rangle\} \in \ell_q$ and Hölder's inequality yields (16).

3) Let T be induced by an ergodic dynamical system defined on a probability space (Ω, m). By a result of Halmos [14], (for non-atomic spaces) there is always $f \in L_2^0(m)$ such that $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ does not converge in norm. Assume that T has Lebesgue spectrum. Then there is a dense set of functions in the space $L_2^0(m)$ of functions in $L_2(m)$ with zero integral such that (16) holds for every $\lambda \in \Gamma$ (for K-automorphisms, see Theorems 5 and 10 of Assani [1]).

Indeed, when T has Lebesgue spectrum, there is an orthogonal decomposition $\bigoplus_{j\in J} H_j$ of the space $L_2^0(m)$, where J is the spectral multiplicity, and H_j, $j \in J$, is the closed subspace of $L_2^0(m)$ spanned by $\{T^k f_j, k \in \mathbb{Z}\}$ for some function $f_j \in L_2^0(m)$ such that $\langle f_j, T^k f_j\rangle = 0$, for every $k \neq 0$. The finite linear combinations of $\{T^k f_j, j \in J, k \in \mathbb{Z}\}$ are dense in $L_2^0(m)$ and these functions have a polynomial spectral density. The result then follows from Remark 2).

4) If (14) holds for some λ, then it holds for the orthogonal projection of f on any T-invariant subspace, hence f is orthogonal to the eigenspace corresponding to the eigenvalue $\overline{\lambda}$ (if there is a $\overline{\lambda}$-eigenfunction) and $\sigma_f(\{\overline{\lambda}\}) = 0$.

Proposition 2.3.1 shows that, for a normal contraction, if for $f \in \mathcal{H}$ we have norm convergence of $\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n}$ for *every* $\lambda \in \Gamma$, then it is not only that σ_f is a continuous measure but it has a rate in its modulus of continuity. For every subset $B \subset D$ containing $\overline{\lambda}$, with $0 < \delta = \sup_{z \in B} |z - \overline{\lambda}| \le 1$, we have

$$\sigma_f(B) = \int_B d\sigma_f \le \frac{1}{\log^2 \delta} \int_D \log^2 |1 - \lambda z| \, d\sigma_f(z) \le \frac{C_\lambda}{\log^2 \delta}.$$

5) For any aperiodic dynamical system and any $\lambda \in \Gamma$, there is a dense G_δ set of functions $f \in L_2^0(m)$ such that (16) does not hold (del Junco and Rosenblatt, see Remark following Corollary 3.3 in [15]).

We construct in Section 3 a stationary process such that the set of λ's for which (16) does not hold is "big" in some sense. The same construction can be performed for any dynamical system with Lebesgue spectrum and provides functions f such that the set of $\lambda \in \Gamma$ for which (16) does not hold has a logarithmic Hausdorff dimension 2.

In the opposite direction, for every contraction, 2 is always a bound for logarithmic Hausdorff dimension of the set of such λ's:

Theorem 2.3.1 *Let T be a contraction on a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$. The set of $\lambda \in \Gamma$ such that $\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n}$ does not converge in norm has a logarithmic Hausdorff dimension at most 2.*

Proof By Proposition 2.3.1 the set of non-convergence is included in the set $\{\lambda : \int_\Gamma \log^2 |1 - z| \; \delta_\lambda * \nu_f(dz) = +\infty\}$. The result then follows from Theorem 4.2.1 on h-Hausdorff dimensions (see Appendix), with h defined by $h(0) = 0$, $h(x) = 1/|\log x|$ for $0 < x \le 1/2$, and $h(x) = 1/\log 2$, for $x > 1/2$. $\square$

3 Examples

3.1 Moving averages

Let $\{\xi_k\}$ be a sequence of centered i.i.d. complex random variables on a probability space (Ω, m) with $\mathbb{E}|\xi_1|^2 dm = 1$. Let $\{c_k\}_{k \in \mathbb{Z}}$ be a sequence in $\ell_2(\mathbb{Z})$ and put $c(t) := \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{ikt}$.

For $n \in \mathbb{Z}$, we define the *moving averages* $f_n := \sum_{k=-\infty}^{\infty} c_k \xi_{n+k}$. This series converges in $L_2(m)$ by the Riesz-Fischer theorem, and almost everywhere by the Khintchine-Kolmogorov theorem. Clearly $f_n = T^n f_0$, where T is induced by the two-sided shift that generates $\{\xi_k\}$. The spectral measure ν of f_0 with respect to T is absolutely continuous and $\frac{d\nu}{dt}(t) = |c(t)|^2$. Conversely, for any function $c \in L_2([-\pi, \pi[, dt)$, $|c(t)|^2$ is the spectral density of a moving average (see Doob [10, Ch. X, §8]). Therefore, for any nonnegative function g with $\int_{-\pi}^{\pi} g \, dt = 1$, there is a stationary moving average model with g as spectral density. If we choose $\{\xi_k\}$ to be Gaussian, then the resulting stationary process $\{f_n\}$ is also Gaussian.

In terms of the function c which generates the moving averages, the condition for the convergence (14) (condition (12) of Proposition 2.3.1) reads, for $\lambda = e^{is}$, $\int_{-\pi}^{\pi} |c(t)|^2 \log^2 |s-t|\, dt < \infty$. Since T in the examples of this section is unitary, (12) is equivalent to the convergence in norm of the rotated EHT.

The next proposition shows that, for this class of examples, the set of λ's where $\sum_{n=1}^{\infty} \frac{\lambda^n f_n}{n}$ does not converge is the same for the convergence in probability, in norm, and a.e. That is, in general, we can not reduce the size of this set by weakening the mode of convergence.

Proposition 3.1.1 *Assume that the random variables $\{\xi_k\}$ are in $L_4(\Omega, m)$. Then, for any sequence of complex numbers $\{a_n\}$, the convergence in probability and the convergence in L_2-norm of $\sum_{n=1}^{\infty} a_n f_n$ are equivalent.*

When $a_n = \lambda^n/n$, convergence in probability, convergence in norm, and a.e. convergence are equivalent.

Proof For $k \in \mathbb{Z}$ and $N \geq 1$, let $b_k^N := \sum_{n=1}^{N} a_n c_{k-n}$. Since $\{c_k\}$ is in $\ell_2(\mathbb{Z})$, for every $N \geq 1$ we have $b^N := \{b_k^N\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, and

$$\sum_{n=1}^{N} a_n f_n = \sum_{k=-\infty}^{\infty} \xi_k \sum_{n=1}^{N} a_n c_{k-n} = \sum_{k=-\infty}^{\infty} \xi_k \, b_k^N . \tag{17}$$

Assume convergence in probability of the sequence $(\sum_{n=1}^{N} a_n f_n)_{N\geq 1}$. By (17) we conclude that for every $\varepsilon > 0$,

$$\lim_{N,M\to\infty} m\Big(|\sum_{k=-\infty}^{\infty} \xi_k (b_k^N - b_k^M)| > \varepsilon \Big) = 0 \,.$$

Now we prove that $\{b^N\}_{N\geq 1}$ is a Cauchy sequence in $\ell_2(\mathbb{Z})$. Otherwise, there would exist $\varepsilon_0 > 0$ and a sequence of integers $N_j \uparrow \infty$, such that $\|b^{N_{j+1}} - b^{N_j}\|_2 \geq \varepsilon_0$. Since $\xi_k \in L_4$, by the Paley-Zygmund inequality (cf. [16], p. 31, Theorem 3), for a fixed $\kappa \in]0,1[$ there exists $\eta > 0$, such that

$$\begin{aligned} & m\Big(\big| \sum_{k=-\infty}^{\infty} \xi_k (b_k^{N_{j+1}} - b_k^{N_j}) \big| > \kappa \varepsilon_0 \|\xi_0\|_2 \Big) \geq \\ & m\Big(\big| \sum_{k=-\infty}^{\infty} \xi_k (b_k^{N_{j+1}} - b_k^{N_j}) \big| > \kappa \|\xi_0\|_2 \, \big\| b^{N_{j+1}} - b^{N_j} \big\|_2 \Big) > \eta. \end{aligned}$$

Since the left hand side of the above inequality tends to zero as $j \to \infty$, we have a contradiction.

Hence $\{b^N\}_{N\geq 1}$ is a Cauchy sequence and therefore converges to some sequence $b = \{b_k\} \in \ell_2(\mathbb{Z})$. Using (17) we obtain

$$\Big\| \sum_{n=N+1}^{M} a_n f_n \Big\|^2 = \Big\| \sum_{k=-\infty}^{\infty} \xi_k \, (b_k^M - b_k^N) \Big\|^2 = \|b^M - b^N\|^2 \|\xi_0\|^2 \longrightarrow 0.$$

This implies that $\sum_{n=1}^{N} a_n f_n$ converges in norm to $\sum_{k=-\infty}^{\infty} b_k \xi_k$, with $b_k = \sum_{n=1}^{\infty} a_n c_{k-n}$.

When $a_n = \lambda^n/n$, the equivalence with a.e. convergence follows from the result of Cuny [7, Theorem 2.1] mentioned in the remarks of Section 2.3. $\square$

3.2 Examples with an uncountable set of λ's of non-convergence

We construct now by a different method stationary Gaussian processes, first with a countable set of non-convergence, then with an uncountable one.

Proposition 3.2.1 *There is a Gaussian stationary process $\{X_n\}$ with a spectral density such that the series $\sum_{n=1}^{\infty} \frac{\lambda^n X_n}{n}$ does not converge in norm only for λ in an infinite countable subset of Γ.*

Proof The computations are done on the interval $[-\pi, \pi[$. Let $\{s_k\}$ be a sequence in $[0, \mathrm{e}^{-1})$ converging to a limit s_∞. Let $\{c_k\} \in \ell_1$ be a positive sequence. On $[-\pi, \pi]$ define $g_k(t) := \frac{\mathbf{1}_{[s_k, \mathrm{e}^{-1}]}(t)}{(t-s_k)\log^2(t-s_k)}$. Since the integral on $[-\pi, \pi]$ of g_k is less than $\int_0^{\mathrm{e}^{-1}} \frac{dt}{t \log^2(t)} = 1$, the series $g(t) := \sum_{k=1}^{\infty} c_k g_k(t)$ is a.e. convergent and defines an integrable function g such that $\|g\|_1 \le \sum_{k=1}^{\infty} c_k$.

By Doob [10, Th. 3.1, p. 72], there is a stationary Gaussian process $\{X_n\}$ with spectral density g. Let T be the transformation such that $X_n = T^n X_0$. The spectral density of X_0 with respect to T is g. As the series is positive:

$$\int_{-\pi}^{\pi} \log^2 |t-s|\, g(t)\, dt = \sum_{k=1}^{\infty} \int_{s_k}^{\mathrm{e}^{-1}} \log^2 |t-s| \frac{c_k}{(t-s_k)\log^2(t-s_k)}\, dt.$$

Hence, the integral on the left hand side is infinite for every $s \in \{s_k\}$. If $s \notin \{s_k\} \cup s_\infty$, then, in a neighborhood of s, $\frac{1}{(t-s_k)\log^2(t-s_k)}$ is bounded uniformly with respect to k and $\log^2 |t-s|$ is integrable; so the integral converges. $\square$

Now we modify this example to get a set of nonconvergence of positive logarithmic Hausdorff dimension. (See Assani [2] for a related question).

Proposition 3.2.2 *For every $\alpha < 2$, there exists a non-empty perfect nowhere dense subset $P \subset \Gamma$, with logarithmic Hausdorff dimension $\ge \alpha$, and a Gaussian stationary process $\{X_n\}$, with a spectral density, such that the series $\sum_{n=1}^{\infty} \frac{\lambda^n X_n}{n}$ does not converge in norm for every $\lambda \in P$.*

Proof The construction of $\{X_n\}$ is like in the previous example, but we change the definition of $\{s_k\}$ and g_k. We build a sequence $\{s_k\} \subset [0, \mathrm{e}^{-\mathrm{e}}]$ whose elements are the endpoints of intervals in the construction of a Cantor type set of a non-constant ratio of dissection. The closure of $\{s_k\}$ will be a uncountable perfect nowhere dense set. Observe that each of the end points appears infinitely often in the sequence $\{s_k\}$.

For any interval $[x, x+l]$ and $\eta \in]0, 1/2[$, let us consider the closed disjoint intervals $[x, x+l\eta]$ and $[x+l(1-\eta), x+l]$. These intervals are called 'end intervals' and the complementary open interval with respect to $[x, x+l]$, called 'middle interval', is removed. This dissection of $[x, x+l]$ will be said of type $[2; \eta]$.

Let $\varepsilon > 0$. Let $\eta_1 := 1/3^{(2+\varepsilon)^{\frac{1}{2}}}$, and for $k \geq 2$

$$\eta_k = [\eta_{k-1} \cdot \eta_{k-2} \cdots \eta_1]^{-1} 3^{-(2+\varepsilon)^{\frac{k}{2}}} = \frac{3^{(2+\varepsilon)^{\frac{k-1}{2}}}}{3^{(2+\varepsilon)^{\frac{k}{2}}}}. \tag{18}$$

Starting from $[0, \mathrm{e}^{-\mathrm{e}}]$, we perform a dissection of type $[2; \eta_1]$ and remove the middle interval. On each remaining end interval we perform a dissection of type $[2; \eta_2]$, and remove the middle intervals – and so on. After n operations we have 2^n end intervals, each of length $\mathrm{e}^{-\mathrm{e}} \eta_n \eta_{n-1} \cdots \eta_1$. As $n \to \infty$ we obtain a non-empty perfect nowhere dense set P (necessarily uncountable) of Lebesgue measure zero.

The sequence $\{s_k\}$ is defined as follows. At the first dissection we have two end intervals $[s_1, s_2]$ and $[s_3, s_4]$. At the second dissection we have 4 end intervals. Let $s_5, \ldots, s_{12}$ be their endpoints (in increasing order) – and so on. One easily sees that the closure of $\{s_k\}$ is the same Cantor type set P obtained in the dissection process above.

Let B_n be the set of indices k such that s_k belongs to the n-th operation. Let $c_k := \frac{1}{(2+\varepsilon)^n}$, for $k \in B_n$. Since B_n contains 2^{n+1} elements, clearly $\{c_k\} \in \ell_1$.

The functions g_k are defined by:

$$g_k(t) := \frac{\mathbf{1}_{[s_k, \mathrm{e}^{-\mathrm{e}}]}(t)}{(t-s_k)|\log(t-s_k)|(\log|\log(t-s_k)|)^2}.$$

The function $g(t) := \sum_{k=1}^{\infty} c_k g_k(t)$ is integrable with $\|g\|_1 = \sum_{k=1}^{\infty} c_k \int_{-\pi}^{\pi} g_k(t)dt \leq \sum_{k=1}^{\infty} c_k$, and we have the following lower bound for every $s \in [0, \mathrm{e}^{-\mathrm{e}}]$:

$$\begin{aligned}
\int_{-\pi}^{\pi} \log^2 |t-s| \, g(t)dt &\geq \sum_{k:\, s_k < s} c_k \int_{s_k}^{\frac{1}{2}(s+s_k)} \frac{\log^2 |t-s| \, dt}{(t-s_k)|\log(t-s_k)|(\log|\log(t-s_k)|)^2} \\
&\geq \sum_{k:\, s_k < s} c_k \log^2(s-s_k) \int_{s_k}^{\frac{1}{2}(s+s_k)} \frac{dt}{(t-s_k)|\log(t-s_k)|(\log|\log(t-s_k)|)^2} \\
&= \sum_{k:\, s_k < s} c_k \frac{\log^2(s-s_k)}{|\log|\log(\frac{1}{2}(s-s_k))||}.
\end{aligned}$$

From the construction it follows that, for any $s \in P$ and any integer $n \geq 1$, there exists an interval endpoint $s(n)$, such that $s(n) < s$ with $s - s(n) \leq$

$\mathrm{e}^{-\mathrm{e}}\eta_n \cdots \eta_1 < 3^{-(2+\varepsilon)^{\frac{n}{2}}}$, and $s(n)$ belongs to the n-th operation. Hence, for $s \in P$:

$$\int_{-\pi}^{\pi} \log^2 |t-s|\, g(t)dt \geq \sum_{k:\, s_k<s} c_k \frac{\log^2(s-s_k)}{|\log|\log(\frac{1}{2}(s-s_k))||}$$
$$= \sum_{n=1}^{\infty} \sum_{k\in B_n, s_k<s} c_k \frac{\log^2(s-s_k)}{|\log|\log(\frac{1}{2}(s-s_k))||} \geq \sum_{n=1}^{\infty} \frac{1}{(2+\varepsilon)^n} \frac{\log^2(s-s(n))}{|\log|\log(\frac{1}{2}(s-s(n)))||}$$
$$\geq \sum_{n=1}^{\infty} \frac{1}{(2+\varepsilon)^n} \frac{[(2+\varepsilon)^{\frac{n}{2}}\log 3]^2}{|\log[(2+\varepsilon)^{\frac{n}{2}}\log 3 - \log 2]|} = +\infty.$$

Define the function $K(x) = (\log(1/x)^+)^\alpha$, $\alpha > 0$. Using Carleson [5, §IV, Theorem 3], we conclude that P has a positive K-capacity if and only if the series $\sum_{k=1}^{\infty} 2^{-k} K(\eta_k)$ converges. According to (18), we have $\sum_{k=2}^{\infty} 2^{-k} K(\eta_k) = \log^\alpha(3) \sum_{k=2}^{\infty} 2^{-k}[(2+\varepsilon)^{\frac{k}{2}} - (2+\varepsilon)^{\frac{k-1}{2}}]^\alpha$ and the series converges if and only if $\alpha < 2\log(2)/\log(2+\varepsilon)$. Theorem 1 in [5, §IV] implies that the logarithmic Hausdorff dimension is $\geq \alpha$. As ε is arbitrary the assertion follows. □

The bound 2 of the logarithmic Hausdorff dimension of the set of $\lambda \in \Gamma$ such that $\sum_{n=1}^{\infty} \frac{\lambda^n T^n f}{n}$ does not converge in norm can be attained:

Theorem 3.2.1 *There exist an uncountable subset $P \subset \Gamma$ with logarithmic Hausdorff dimension 2 and a Gaussian stationary process $\{X_n\}$ with a spectral density, such that the series $\sum_{n=1}^{\infty} \frac{\lambda^n X_n}{n}$ does not converge in norm for every $\lambda \in P$. The closure of P is a perfect set.*

Proof Take $\alpha_j \uparrow 2$ and for every $j \geq 1$ build the associated function g_j and and set P_j as in Proposition 3.2.2. We can assume that $\|g_j\|_1 = 1$ for every $j \geq 1$. Now, define $P = \cup_{j=1}^{\infty} P_j$ and put $g = \sum_{j=1}^{\infty} \beta_j g_j$, where $\{\beta_j\}$ is a summable sequence of positive numbers. Clearly, g is an integrable function and there is a Gaussian stationary process $\{X_n\}$ with spectral density g. For $j \geq 1$ we have $\int_{-\pi}^{\pi} \log^2 |t-s|\, g(t)dt \geq \beta_j \int_{-\pi}^{\pi} \log^2 |t-s|\, g_j(t)dt$. Hence, for every $j \geq 1$ and for every $s \in P_j$, we have $\int_{-\pi}^{\pi} \log^2 |t-s|\, g(t)dt = \infty$. We conclude that for every $s \in P$ the series $\sum_{n=1}^{\infty} \frac{\mathrm{e}^{ins} X_n}{n}$ does not converge in norm.

Since each set P_j is perfect, so is the closure of P. Furthermore, the logarithmic Hausdorff dimension of P is not less than the dimension of any P_j. So, the logarithmic dimension of P is ≥ 2. Using Theorem 2.3.1 we obtain that the logarithmic dimension of P is 2. □

4 Appendix: Hausdorff dimension of a set of divergence

4.1 h-Hausdorff dimension

We prove in this appendix the result used in Theorem 2.3.1 (Theorem 4.2.1). It is likely to belong to the "folklore" of Hausdorff dimension theory, but for the sake of completeness we prove the result which fits exactly to our need. First we recall general notions and results on the construction of Hausdorff measures on a metric space (X,d). For our purpose it suffices to take $X=\mathbb{R}$.

Let $\mathcal{F}$ be a family of subsets of X and ζ be a map from $\mathcal{F}$ to $[0,\infty]$. A generalized Hausdorff measure ϕ on (X,d) is associated to $(\mathcal{F},\zeta)$ in a standard way: For every $\delta>0$, we define the set function ϕ_δ by

$$\phi_\delta(A)=\inf\{\sum_{i\in\mathbb{N}}\zeta(U_i)\},\ A\subset X,$$

where the infimum is taken over all countable families $(U_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{F}$ such that $A\subset\cup_{i\in\mathbb{N}}U_i$ and $\operatorname{diam}U_i\le\delta$ for all i. The *Hausdorff measure* of a subset A is $\phi(A)=\lim_{\delta\to0}\phi_\delta(A)=\sup_{\delta>0}\phi_\delta(A)$.

We will use the following result *on approximation by compact sets* (which holds for any Suslin subset, cf. [Fe], p. 186, Corollary 2.10.23).

Theorem 4.1.1 *Let X be a metric space such that all bounded closed subsets are compact and $\mathcal{F}$ be the family of all compact subsets of X. Suppose that the map ζ is continuous for the Hausdorff distance and that $\zeta(C)>0$ whenever* $\operatorname{diam}C>0$. *Then every Borel subset S of X satisfies*

$$\phi(S)=\sup\{\phi(C):C\in\mathcal{F},\ C\subset S\}.$$

Let $G:X\times X\to[0,\infty]$ be a LSC (lower semi-continuous) kernel. For a positive measure μ on X, for $x\in X$ the *potential* associated with μ and G, and the potential associated with μ and the dual kernel $G^*(x,y)=G(y,x)$ are defined by

$$G\mu(x):=\int G(x,y)\,\mu(dy),\ G^*\mu(x):=\int G(y,x)\,\mu(dy).$$

Since G is LSC, Fatou's lemma implies that $G\mu$ and $G^*\mu$ are also LSC. The *energy* of μ for the kernel G is

$$I_G(\mu):=\int\int G(x,y)\,\mu(dx)\mu(dy)=\int G\mu(x)\,\mu(dx)=\int G^*\mu(y)\,\mu(dy).$$

We will consider functions ζ of the following form: let $h:[0,\infty[\to[0,\infty[$ be an increasing continuous function with $h(0)=0$ and let $\mathcal{F}$ be the family of all bounded subsets of $\mathbb{R}$. For each $s>0$, we can take as ζ the map

$$\zeta_s:\mathcal{F}\to[0,\infty[,\ A\to h^s(\operatorname{diam}A).$$

The corresponding set functions are denoted by ϕ^s_δ and ϕ^s.

Using the continuity at 0 and the monotonicity of h, it is easy to see that if $\phi^s(A) < \infty$ for some $s > 0$, then $\phi^t(A) = 0$ for all $t > s$. The generalized *h-Hausdorff dimension* of A is defined by

$$\dim_h(A) := \inf\{s > 0 : \phi^s(A) = 0\}.$$

4.2 h-Hausdorff dimension of the set of divergence of the potential

We are interested by the size of the set where the potential of a measure μ is infinite for the kernels G_s, $s > 0$, defined (with the convention $\frac{1}{0} = +\infty$) by

$$G_s : (x, y) \in \mathbb{R}\times\mathbb{R} \to \frac{1}{h^s(|x-y|)}. \tag{19}$$

For a positive measure μ on $\mathbb{R}$ with finite mass and a parameter $s_0 > 0$, we consider the following set

$$F_{\mu,s_0} := \{x \in \mathbb{R}\colon G_{s_0}\mu(x) = \int_{\mathbb{R}} \frac{1}{h^{s_0}(|x-y|)}\,\mu(dy) = +\infty\}. \tag{20}$$

Theorem 4.2.1 *Let $h : [0, +\infty[\to [0, +\infty[$ be a continuous increasing function with $h(0) = 0$, continuously differentiable outside a discrete subset of $[0, +\infty[$. Assume that there exists a constant C such that $h(2x) \leq Ch(x), \forall x \geq 0$. Then for any μ with finite mass and $s_0 > 0$, we have $\dim_h(F_{\mu,s_0}) \leq s_0$.*

Theorem 4.2.1 is a kind of generalization of the relation of Hausdorff dimension to capacity, e.g., see [18, Ch. 8] or [11, Ch. 4]. The proof is based on the proposition and the lemma below. Some details are straightforward extensions of proofs which can be found in standard books on Hausdorff measures ([5], [11], [12], [18]) and are omitted.

Proposition 4.2.1 *Let G be a LSC kernel such that, for some positive constant C_1,*

$$\forall x, y, z \in X, d(y, z) \leq 3d(x, z) \Rightarrow G(z, x) \leq C_1 G(z, y).$$

Let μ be a finite positive measure on X and let $F = \{x \in X : G\mu(x) = +\infty\}$. Then the energy $I_G(\nu)$ of any non zero positive measure ν with support in F is infinite.

Proof Let λ be a positive measure and $E = \operatorname{supp}\lambda$. For all $x \in X$, there exists $y \in E$ such that $d(x, y) \leq 2d(x, E)$. For all $z \in E$, we have $d(y, x) \leq 2d(x, E) \leq 2d(x, z)$, therefore $d(y, z) \leq d(y, x) + d(x, z) \leq 3d(x, z)$. Hence $G(z, x) \leq C_1 G(z, y)$ and

$$G^*\lambda(x) = \int G(z, x)\,\lambda(dz) \leq \int C_1 G(z, y)\,\lambda(dz) = C_1 G^*\lambda(y).$$

Therefore
$$\sup_{x\in X} G^*\lambda(x) \leq C_1 \sup_{y\in E} G^*\lambda(y).$$
Moreover suppose that $E = \operatorname{supp}\lambda \subset \mathrm{F}$, then
$$+\infty = \int G\mu(y)\,\lambda(dy) = \int G^*\lambda(x)\,\mu(dx).$$
Since μ is finite, it follows that $\sup_{x\in X} G^*\lambda(x) = +\infty$. Hence $\sup_{y\in E} G^*\lambda(y) = +\infty$.

Now let ν be a non zero positive measure positive with $\operatorname{supp}\nu \subset \mathrm{F}$, and assume that the energy of ν is finite, $\int G^*\nu(x)\,\nu(dx) = I_G(\nu) < +\infty$. We will apply the first part of the proof to a measure λ deduced from ν and get a contradiction.

For all $a > 0$, we have $\nu(\{x \in X : G^*\nu(x) > a\}) \leq I_G(\nu)/a$. Hence, if a is large enough, the set $A = \{x \in X : G^*\nu(x) \leq a\}$ has a positive measure. Choose such a number a. The set A is closed, since $G^*\nu$ is LSC.

Consider the measure λ defined by $\lambda(B) = \nu(A \cap B)$ for all Borel subsets B of X. On the one hand, the choice of a ensures that λ is not zero. On the other hand, $\lambda \leq \nu$ and therefore $G^*\lambda \leq G^*\nu \leq a$ on the set A since A contains $E = \operatorname{supp}\lambda$. This contradicts $\sup_{y\in E} G^*\lambda(y) = +\infty$, and therefore the energy of ν cannot be finite. □

Lemma 4.2.1 *If $\phi^s(F_{\mu,s_0}) > 0$ for some $s > 0$, there exists a finite positive measure ν_0 with support in F_{μ,s_0} such that, for all $0 < t < s$,*
$$I_{G_t}(\nu_0) = \int\int \frac{1}{h^t(|x-y|)}\,\nu_0(dx)\,\nu_0(dy) < +\infty.$$

Proof The measure ν_0 is constructed by restricting ϕ^s to a suitable compact subset K_0 of F_{μ,s_0}.

1) The continuity of the kernel $G_s : (x,y) \in \mathbb{R}\times\mathbb{R} \to \frac{1}{h^s(|x-y|)} \in [0,+\infty]$ implies that F_{μ,s_0} is a G_δ and hence a Borel subset of $\mathbb{R}$. Theorem 4.1.1 with $S = F_{\mu,s_0}$ and $\phi = \phi^s$ provides a compact set $K_1 \subset F_{\mu,s_0}$ such that $\phi^s(K_1) > 0$.

Now there is a compact subset $K_2 \subset K_1$ such that $0 < \phi^s(K_2) < +\infty$. The proof of this assertion can be easily adapted from [11] p. 62, Theorem 4.10, where the same result is proved for the usual Hausdorff measure.

2) Using the inequality $h(2x) \leq Ch(x)$ and standard arguments (covering lemma and Egoroff's theorem (cf. [11] Proposition 4.9 p. 61 and Proposition 4.11 p. 63)) one can show that there exists a compact subset $K_0 \subset K_2$ and a finite constant b such that $\phi^s(K_0) > 0$ and
$$\phi^s(K_0 \cap B(x,r)) \leq bh^s(r), \forall x \in \mathbb{R}, r > 0. \tag{21}$$

Let ν_0 be the measure defined by $\nu_0(A) = \phi^s(K_0 \cap A)$. For $0 < t < s$, let $G_t\nu_0(x) := \int_{\mathbb{R}} \frac{\nu_0(dy)}{h^t(|x-y|)}$. For all $x \in \mathbb{R}$ and $r \geq 0$, set $m_x(r) = \nu_0(B(x,r))$. We have $m_x(0) = 0$ since $h^s(r)$ tends to 0 as $r \to 0$, and $m_x(r) \leq bh^s(r)$ by (21). By the same computation as in [11, p. 65-66], we obtain:

$$G_t\nu_0(x) = \int_{B(x,1)} \frac{\nu_0(dy)}{h^t(|x-y|)} + \int_{B(x,1)^C} \frac{\nu_0(dy)}{h^t(|x-y|)} \leq$$

$$\int_0^1 h^{-t}(r)\, dm_x(r) + \nu_0(\mathbb{R})h^{-t}(1)$$

$$= [h^{-t}(r)\, m_x(r)]_0^1 + \int_0^1 th^{-t-1}(r)h'(r)\, m_x(r)dr + \nu_0(\mathbb{R})h^{-t}(1)$$

$$\leq h^{-t}(1)\, b\, h^s(1) + \int_0^1 th^{-t-1}(r)\, h'(r)\, bh^s(r)\, dr + \nu_0(\mathbb{R})h^{-t}(1)$$

$$\leq b + \nu_0(\mathbb{R})h^{-t}(1) + b\,t \left[\frac{h^{s-t}(r)}{s-t} \right]_0^1 .$$

Therefore, $G_t\nu_0$ is bounded on $\mathbb{R}$ and $I_{G_t}(\nu_0) < +\infty$. □

Proof of Theorem 4.2.1 Let μ be a positive finite measure on $\mathbb{R}$ and let $s_0 > 0$. Suppose that there exists $s > s_0$ such that $\phi^s(F_{\mu,s_0}) > 0$, where F_{μ,s_0} is defined by (20). It follows from Proposition 4.2.1 that for all positive measures ν with support in F_{μ,s_0}, $I_{G_{s_0}}(\nu) = +\infty$. But this contradicts the existence of a measure ν_0 with support in F_{μ,s_0}, such that $I_{G_{s_0}}(\nu) < +\infty$, as asserted by Lemma 4.2.1. □

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