Almost sure convergence of weighted sums of independent random variables

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Dedicated to Arkady Tempelman

Abstract. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\{X_n\}$ be a sequence of integrable centered i.i.d. random variables. In this paper we consider what conditions should be imposed on a complex sequence $\{b_n\}$ with $|b_n| \to \infty$, in order to obtain a.s. convergence of $\sum_n \frac{X_n}{b_n}$, whenever X_1 is in a certain class of integrability. In particular, our condition allows us to generalize the rate obtained by Marcinkiewicz and Zygmund when $\mathbf{E}[|X_1|^p] < \infty$ for some $1 < p < 2$. When applied to weighted averages, our result strengthens the SLLN of Jamison, Orey, and Pruitt in the case X_1 is symmetric. An analogous question is studied for $\{X_n\}$ an L_p-bounded martingale difference sequence. An extension of Azuma's SLLN for weighted averages of uniformly bounded martingale difference sequences is also presented. Applications are made also to modulated averages and to strong consistency of least squares estimators in a linear regression. The main tool for the general approach is (a generalization of) the counting function introduced by Jamison *et al.* for the SLLN for weighted averages.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\{X_n\}$ be a sequence of integrable centered independent random variables. In this paper, which is largely expository, we consider what conditions should be imposed on a complex sequence $\{b_n\}$ with $|b_n| \to \infty$, in order to obtain a.s. convergence of the series $\sum_n \frac{X_n}{b_n}$, whenever $\{X_n\}$ is in a certain class of integrability. Of particular interest is the case of weighted averages, when $\{w_n\}$ is a sequence of positive numbers (weights) with divergent sum and $b_n = \sum_{k=1}^n w_k/w_n$; a.s. convergence of the above series implies the strong law of large numbers (SLLN) for the weighted averages. Another case of interest is when $\{c_n\}$ is a sequence with $\sum |c_k|^2 = \infty$ and we take $b_n = \sum_{k=1}^n |c_k|^2/c_n$; a.s. convergence of the series implies strong consistency of the least square estimator (LSE) in a linear regression model.

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Hill [**Hi**], in his 1945 study of almost everywhere convergence of regular summability methods applied to sequences of zeros and ones, observed that if $\{w_n\}$ is a sequence of positive numbers with partial sums $W_n := \sum_{k=1}^n w_k$ tending to infinity, such that $\sum_{k=1}^\infty \left(\frac{w_k}{W_k}\right)^2 < \infty$, then any Rademacher sequence $\{\epsilon_n\}$ (i.e., $\{\epsilon_n\}$ i.i.d. with $\mathbf{P}(\epsilon_1 = \pm 1) = \frac{1}{2}$) satisfies

$$\frac{1}{W_n}\sum_{k=1}^n w_k\epsilon_k \to 0 \quad a.s. \tag{1}$$

This result, which is a consequence of the Khintchine-Kolmogorov theorem, raised the question about conditions on a positive sequence $\{w_n\}$ which are sufficient for (1). Since (1) implies $\frac{w_n\epsilon_n}{W_n} \to 0$ a.s., a necessary condition is $\frac{w_n}{W_n} \to 0$. The insufficiency of this condition was noted by Maruyama and by Tsuchikura (see references in [**Ts**]). Tsuchikura [**Ts**] proved that (1) holds if $\{w_n\}$ is increasing and satisfies

$$\lim_{n\to\infty} \frac{w_n \log\log W_n}{W_n} = 0 \tag{2}$$

Salem and Zygmund [**SZ**] proved Tsuchikura's result differently, assuming $W_n \to \infty$ instead of monotonicity of $\{w_n\}$. Note that every bounded $\{w_n\}$ with divergent sum satisfies (2).

By the Khintchine-Kolmogorov theorem, Hill's assumptions imply that

$$\frac{1}{W_n}\sum_{k=1}^n w_k X_k \to 0 \quad a.s. \tag{3}$$

whenever $\{X_n\}$ are centered independent random variables with $\sup_n \mathbf{E}|X_n|^2 < \infty$, so Tsuchikura's work raises several questions, lumped together in the following:

Find conditions on a sequence of positive weights $\{w_n\}$ *(with divergent sum) which ensure that the weighted SLLN* (3) *holds for every sequence of centered independent random variables in a given class*, e.g., for all uniformly bounded $\{X_n\}$, for all centered i.i.d. sequences with finite variance, etc.

Jamison, Orey, and Pruitt [**JOP**] gave a necessary and sufficient condition on a weight sequence $\{w_n\}$ with divergent sum for (3) to hold for *every* i.i.d. sequence $\{X_n\}$ with $\mathbf{E}|X_1| < \infty$ and $\mathbf{E}X_1 = 0$. They introduced the *counting function* $N(t) = \#\{n \ge 1 : W_n/w_n \le t\}$, which is finite when $w_n/W_n \to 0$, and (assuming also $W_n \to \infty$) they proved [**JOP**, Theorem 2] that if $\{X_n\}$ is a sequence of integrable centered i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$\mathbf{E}\Big[|X_1|^2 \int_{t\ge|X_1|} \frac{N(t)dt}{t^3}\Big] < \infty\,, \tag{4}$$

then the weighted averages $\frac{1}{W_n}\sum_{k=1}^n w_k X_k$ converge to zero a.s.

This result was used there to prove (see [**JOP**, Theorem 3]) that the condition $\limsup_{t\to\infty} N(t)/t < \infty$ is necessary and sufficient for a.s. convergence to zero of the weighted averages $\frac{1}{W_n}\sum_{k=1}^n w_k X_k$, for every sequence of integrable centered i.i.d. random variables $\{X_n\}$.

Azuma [**Az**] proved that Tsuchikura's condition (2) implies that the weighted SLLN (3) holds for every uniformly bounded martingale difference sequence $\{X_n\}$, in particular for uniformly bounded centered independent random variables.

The problem of finding sufficient conditions for the weighted SLLN (3) to hold for every centered i.i.d. with $\mathbf{E}|X_1|^p < \infty$ ($p > 1$ fixed) was recently treated by Lin and Weber [**LW**]; see additional references there.

The above strong laws for weighted averages raise the question of a.s. convergence of $\sum_{n=1}^\infty \frac{w_n X_n}{W_n}$. Marcinkiewicz and Zygmund [**MZ-1**, Theorem 6] extended Kolmogorov's SLLN, by proving that if $\{X_n\}$ is a sequence of integrable centered i.i.d. random variables, such that $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$ or X_1 is symmetric, then the series $\sum_{n=1}^\infty \frac{X_n}{n}$ converges a.s. When neither additional condition holds, the generalization of [**JOP**] by Heyde [**He**] yields a rate of growth of the partial sums. A natural question is, under the same assumptions on $\{X_n\}$, what conditions should be imposed on a sequence $\{b_n\}$ of complex numbers (with $|b_n| \to \infty$) in order to obtain a.s. convergence of the series $\sum_{n=1}^\infty \frac{X_n}{b_n}$.

Although Kolmogorov's three series theorem gives a necessary and sufficient condition for a.s. convergence of a series of independent random variables, in the context described above we are interested in finding conditions on $\{b_n\}$ which imply a.s. convergence of the series $\sum_{n=1}^\infty \frac{X_n}{b_n}$ for *every* sequence of centered i.i.d. $\{X_n\}$ in a given class of integrability.

We obtain some new results and improvements of old ones. We will show in §2 that an analogue of the Marcinkiewicz-Zygmund result quoted above is valid for the SLLN for weighted averages of centered i.i.d. random variables. Our approach, in the generality of the question about sequences $\{b_n\}$, allows us to generalize at the same time also the rate obtained in [**MZ-1**, Theorem 9] when $\mathbf{E}[|X_1|^p] < \infty$ for some $1 < p < 2$. For fixed $1 < p < 2$, we also obtain (Corollary 2.6) a necessary and sufficient condition on the weights $\{w_n\}$, in terms of the counting function, for the SLLN for weighted averages of all i.i.d. $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$.

In §3 we obtain (Theorem 3.2), for fixed $1 < p \le 2$, a necessary and sufficient condition on the weights, in terms of the counting function, for a.s. convergence of $\sum_{n=1}^\infty \frac{w_n Y_n}{W_n}$ for every L_p-norm bounded martingale difference sequence $\{Y_n\}$, and we extend some results of Lin and Weber [**LW**].

In §4 we obtain another proof of Azuma's result, as a corollary of a more general result; in fact, the weighted SLLN for uniformly bounded martingale differences is obtained under a condition slightly weaker than (2).

In §5 we study the a.s. convergence of the series $\sum_{n=1}^\infty \frac{c_n X_n}{n}$ (which implies a SLLN for modulated averages) when $\{X_n\}$ are centered i.i.d. with $\mathbf{E}[|X_1|\log^+|X_1|]$ finite.

§6 treats more specifically the problem of strong consistency of the LSE in a linear regression model with i.i.d. noise: for $1 < p \le 2$ we obtain conditions on the regressors $\{c_n\}$ which ensure a.s. convergence of the series $\displaystyle\sum_{n=1}^\infty \frac{c_n X_n}{\sum_{k=1}^n |c_k|^2}$ for all i.i.d. noises $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$.

In some sections we will discuss the applicability of the results to strictly stationary random weights ('typical' realizations of dynamical systems). Let $(S, \mathcal{A}, \mu)$

be a probability space and let $\theta : S \mapsto S$ be a μ-measure preserving map. Assani [**A1**] introduced a variant of the counting function when $w_n = w_n(x) = g(\theta^n x)$, for some non-negative $g \in L_p(\mu)$, $p > 1$. Among other results, it was shown there that

$$\limsup_{t\to\infty} \#\{n \geq 1 : g(\theta^n x)/n \geq 1/t\}/t < \infty \quad \mu - \text{a.e.} \tag{5}$$

This result was used (in the i.i.d. case) to obtain an extension of Bourgain's return times theorem beyond its duality assumption.

When θ is ergodic, (5) yields that $\limsup_{t\to\infty} N(t)/t < \infty$ for μ-a.e. realization $\{w_n(x)\}$, which leads to a SLLN for weighted averages with random stationary weights: for x in a subset $S_0 \subset S$ of full μ-measure, the SLLN (3) holds for all weighted averages, with weights $\{w_n(x)\}$, of i.i.d. random variables having finite expectation.

Later, Assani [**A2**] proved that for g with $\int g \log^+ g \, d\mu$ finite, the left hand side of (5) is integrable (see also Baxter *et al.* [**BJLO**]). Using an entropy inequality of Stein and Weiss [**SW**] and techniques from [**A1**], Demeter and Quas [**DQ**] proved that (5) holds even for g with $\int g \log^+ \log^+ g \, d\mu < \infty$. Immediate consequences about a.s. convergence of the averages were also noted there. Recently, Assani, Buczolich, and Mauldin [**ABM**] proved that (5) may fail to hold for $g \in L_1(\mu)$.

Similar properties of counting functions were investigated in other contexts too: by Marcus and Pisier [**MP**, Theorem 3.3] in connection with uniform convergence of random Fourier series, by Jin and Chen [**JC**] in connection with least squares estimates (LSE) in regression models, and by Chen *et al.* [**CZF**] in connection with general SLLN. In the context of the last two references see Theorem 2.9.

In this paper we deal with sequences of random variables, like independent sequences $\{X_n\}$ or martingale differences sequences $\{Y_n\}$, which are assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with expectation denoted by $\mathbf{E}$.

When treating strictly stationary random weights, we use a second probability space $(S, \mathcal{A}, \mu)$, with a μ-measure preserving map θ. Properties which hold with probability one are referred to as *almost sure* (a.s.) if they are related to $\mathbf{P}$, and as *almost everywhere* (a.e.) if they are related to μ.

2. Weighted sums of i.i.d. random variables

In this section we study the SLLN for weighted averages of centered i.i.d. random variables $\{X_n\}$ via the a.s. convergence of series of the form $\sum_{n=1}^{\infty} \frac{X_n}{b_n}$.

Note that for any non-negative sequence $\{\alpha_n\}$, the (non-decreasing) counting function $N_{\{\alpha_n\}}(t) := \#\{n : \alpha_n \leq t\}$ is finite for every $t > 0$ if and only if $\alpha_n \to \infty$ (since existence of a bounded subsequence $\alpha_{n_k} \leq M$ implies $N_{\{\alpha_n\}}(t) = \infty$ for $t > M$).

The properties of this counting function will be used in our study.

Lemma 2.1. *Let $\{\alpha_n\}$ be a positive sequence, tending to infinity. Let $\varphi(t)$ be a differentiable, positive, and non-increasing function on $[1, \infty)$, such that $\varphi(t) \underset{t\to\infty}{\longrightarrow} 0$. Put $N(t) = N_{\{\alpha_n\}}(t) = \#\{n : \alpha_n \leq t\}$. Then the series $\sum_{n=1}^{\infty} \varphi(\alpha_n)$ converges if and only if $\int_1^{\infty} \varphi'(t) N(t) dt$ converges.*

Proof. Let $\{\alpha_{n_j}\}$ be the non-decreasing rearrangement of $\{\alpha_n\}$. Clearly, $N(t)$ is a non-decreasing, right continuous, step function with jumps appearing along a *subsequence* of points $\{\alpha_{n_j}\}$.

Now, by the above considerations, and using the definition of Riemann-Stieltjes integral, for any $z > 1$ we have

$$\sum_{\{k\colon 1<\alpha_k\le z\}} \varphi(\alpha_k) = \int_1^z \varphi(t)dN(t) = \varphi(z)N(z) - \varphi(1)N(1) - \int_1^z \varphi'(t)N(t)dt\,.$$

Note that the last term on the right hand side has, in fact, positive sign since φ is non-increasing.

If $\sum_{k=1}^\infty \varphi(\alpha_k) < \infty$, then clearly the integral of the statement converges. If the integral converges, by monotonicity of $N(t)$ and the assumption $\varphi(t) \underset{t\to\infty}{\to} 0$, we have

$$\varphi(z)N(z) \le -\int_z^\infty \varphi'(t)N(t)dt \to 0 \quad \text{as} \quad z\to\infty$$

hence, the series $\sum_{k=1}^\infty \varphi(\alpha_k)$ converges. □

Remark. Only asymptotic properties of φ are important, i.e., all the requirements could be satisfied starting from $t \ge t_0$. In this case it is enough to check whether $\int_{t_0}^\infty \varphi'(t)N(t)dt$ converges.

From the usual SLLN it follows that if $\sup_n n/|b_n|$ is finite, then $\frac{1}{b_n}\sum_{k=1}^n X_k$ converges a.s. to 0 for every integrable centered i.i.d. sequence $\{X_n\}$. Clearly, if $\sup_n n/|b_n|$ is finite, then $\limsup_{t\to\infty} \#\{n \ge 1 : |b_n| \le t\}/t < \infty$. The following theorem gives more precise information.

Theorem 2.2. *Let $\{b_n\}$ be a non-zero sequence of complex numbers. Put $N(t) = \#\{n \ge 1 : |b_n| \le t\}$ and assume that $\limsup_{t\to\infty} N(t)/t^p < \infty$, for some $1 \le p < 2$. Then for a sequence of integrable centered i.i.d. random variables $\{X_n\}$, the series $\displaystyle\sum_{n=1}^\infty \frac{X_n}{b_n}$ converges a.s. in the following cases:*

(i) $1 < p < 2$ and $\mathbf{E}[|X_1|^p] < \infty$.

(ii) $p = 1$ and $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$.

(iii) $p = 1$ and X_1 is symmetric.

Proof. First we make two remarks: (a) recall that since $N(t)$ is finite valued, $|b_n| \to \infty$; (b) we may and do assume that $|b_n| \ge 1$ for every $n \ge 1$.

Now we prove the theorem. Since $\limsup_{t\to\infty} N(t)/t^p < \infty$, for $X_1 \in L_p(\mathbf{P})$ we have $\mathbf{E}[N(|X_1|)] < \infty$. Hence,

$$\sum_{n=1}^\infty \mathbf{P}\big(|X_n| \ge |b_n|\big) = \mathbf{E}\big[\sum_{n=1}^\infty \mathbf{1}_{\{|b_n|\le|X_1|\}}\big] = \mathbf{E}[N(|X_1|)] < \infty \qquad (*)$$

So, it is enough to prove that $\sum_{n=1}^\infty \frac{X_n\mathbf{1}_{\{|X_n|\le|b_n|\}}}{b_n}$ converges a.s.

Using Lemma 2.1 (see its proof) we compute

$$\sum_{n=1}^\infty \frac{\mathbf{E}[|X_n|^2\mathbf{1}_{\{|X_n|\le|b_n|\}}]}{|b_n|^2} = \mathbf{E}\Big[|X_1|^2 \sum_{|b_n|\ge|X_1|} \frac{1}{|b_n|^2}\Big] \le \mathbf{E}\Big[|X_1|^2 \int_{|X_1|}^\infty \frac{N(t)dt}{t^3}\Big]\,.$$

Since $p < 2$, the last expectation above is finite thanks to $\limsup_{t\to\infty} N(t)/t^p < \infty$. This proves that

$$\sum_{n=1}^{\infty} \frac{X_n \mathbf{1}_{\{|X_n|\le |b_n|\}} - \mathbf{E}[X_n \mathbf{1}_{\{|X_n|\le |b_n|\}}]}{b_n}$$

converges a.s. for every $1 \le p < 2$ (even for $p = 1$).

Now we prove that $\sum_{n=1}^{\infty} \frac{|\mathbf{E}[X_n \mathbf{1}_{\{|X_n|\le |b_n|\}}]|}{|b_n|}$ converges. In case (iii) the terms are zero, by symmetry of X_n and symmetry of the truncation. In the other cases, by Lemma 2.1 (see its proof),

$$\sum_{n=1}^{\infty} \frac{|\mathbf{E}[X_n \mathbf{1}_{\{|X_n|\le |b_n|\}}]|}{|b_n|} \le \mathbf{E}\Big[|X_1| \sum_{n=1}^{\infty} \frac{\mathbf{1}_{\{|X_1|>|b_n|\}}}{|b_n|}\Big] \le$$

$$\mathbf{E}\Big[|X_1| \sum_{\{n:\, 1\le |b_n|\le |X_1|\}} \frac{1}{|b_n|}\Big] \le \mathbf{E}\Big[|X_1|\Big(\frac{N(|X_1|)}{|X_1|} + \int_1^{|X_1|} \frac{N(t)dt}{t^2}\Big)\Big] \quad (**)$$

□

Remarks. 1. When $p = 1$ and we put $b_n = n$, parts (ii) and (iii) yield a strengthening of the SLLN due to Marcinkiewicz-Zygmund [**MZ-1**, Theorem 6]. An example there shows that in general, when $p = 1$ the condition $\mathbf{E}[|X_1|] < \infty$ alone is not is not sufficient for the desired convergence of the series.

2. When we take $1 < p < 2$ and $b_n = n^{1/p}$, part (i) yields the case $1 < p < 2$ of [**MZ-1**, Theorem 9]. Part (i) of the theorem is false for $p = 2$ – take $b_n = \sqrt{n}$ and use the central limit theorem.

3. Note that in the proof of the theorem we obtain that if $\limsup_{t\to\infty} N(t)/t^p$ is finite, for $1 < p \le 2$, and $\mathbf{E}[|X_1|^p] < \infty$, then the series $\sum_{n=1}^{\infty} \frac{|\mathbf{E}[X_n \mathbf{1}_{\{|X_n|\le |b_n|\}}]|}{|b_n|}$ converges. The same holds if $p = 1$ and $\mathbf{E}[|X_1| \log^+ |X_1|] < \infty$.

From Theorem 2.2 we obtain the following new additional information on the SLLN (3) for weighted averages of centered i.i.d. random variables under the condition of [**JOP**]; for equal weights this strengthening of the SLLN is in [**MZ-1**, Theorem 6].

Corollary 2.3. *Let $\{w_n\}$ be a weight sequence with $W_n = \sum_{k=1}^{n} w_k \to \infty$, such that* $\limsup N(t)/t < \infty$. *Then* $\sum_{n=1}^{\infty} \frac{w_n X_n}{W_n}$ *converges a.s. whenever $\{X_n\}$ are centered integrable i.i.d. random variables with X_1 symmetric or* $\mathbf{E}[|X_1| \log^+ |X_1|]$ *finite.*

It is not hard to follow the computations in the proof of Theorem 2.2 in order to see that if we assume that $\limsup_{t\to\infty} N(t)/t(\log t)^{\gamma} < \infty$, for some non-negative γ, then the series $\sum_{n=1}^{\infty} \frac{X_n}{b_n}$ converges a.s. for any centered i.i.d. $\{X_n\}$ with $\mathbf{E}[|X_1|(\log^+ |X_1|)^{\gamma+1}] < \infty$. This yields the following.

PROPOSITION 2.4. *Let $\{w_n\}$ be a bounded weight sequence with $W_n \to \infty$. Then $\sum_{n=1}^{\infty} \frac{w_n X_n}{W_n}$ converges a.s. whenever $\{X_n\}$ are centered i.i.d. random variables with $\mathbf{E}[|X_1|(\log^+ |X_1|)^2] < \infty$, or with X_1 symmetric and $\mathbf{E}[X_1 \log^+ |X_1|] < \infty$.*

PROOF. For $b_n = W_n/w_n$ we have $\limsup N(t)/t \log t < \infty$, by Lemma 2 of [**JOP**]. The first assertion follows from the preceding discussion. The second one follows from the proof of part (iii) of Theorem 2.2. □

Remarks. 1. For bounded weights with divergent sum, the weighted SLLN holds for i.i.d. random variables with $\mathbf{E}[|X_1| \log^+ |X_1|] < \infty$, by [**JOP**].

2. Lin and Weber [**LW**, Theorem 4.14] proved for unbouded weights that if

$$\sup_{n\geq 1} \frac{1}{W_n} \sum_{k=1}^{n} w_k (\log(1+w_k))^{\beta} < \infty \qquad (\beta > 1)$$

then the weighted SLLN holds for $\{X_n\}$ centered i.i.d. with $\mathbf{E}[|X_1|(\log^+ |X_1|)^{\gamma}] < \infty$ for some $\gamma > 1$.

3. See Corollary 3.5 for additional properties of bounded weights with divergent sum.

PROPOSITION 2.5. *Let $\{b_n\}$ be a non-zero sequence of complex numbers, and put $N(t) = \#\{n \geq 1 : |b_n| \leq t\}$. For $p \geq 1$ the following are equivalent:*

(i) $\limsup_{t\to\infty} N(t)/t^p < \infty$.

(ii) The sequence $\frac{X_n}{b_n}$ converges a.s. to 0 for every symmetric *i.i.d. random variables $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$.*

PROOF. We first observe that the identity (*) in the proof of Theorem 2.2 is valid for every $p \geq 1$.

$(ii) \Rightarrow (i)$: In particular, $\frac{\epsilon_n}{b_n} \to 0$ a.s. for a Rademacher sequence $\{\epsilon_n\}$ (i.e., an i.i.d. sequence with $\mathbf{P}(\epsilon_1 = \pm 1) = 1/2$), which yields $|b_n| \to \infty$; hence $N(t)$ is a finite valued function.

Also, by the Borel-Cantelli lemma for independent sets, the identity in (*) in the proof of Theorem 2.2 and $\frac{X_n}{b_n} \to 0$ a.s. yield that $\mathbf{E}[N(|X_1|)] < \infty$ for every symmetric i.i.d. $\{X_n\}$ with $X_1 \in L_p(\mathbf{P})$.

One can show (e.g., by the uniform boundedness principle) that for any non-negative unbounded sequence $\{\beta_n\}$, there exists a non-negative sequence $\{\alpha_n\}$, such that $\sum_{n=1}^{\infty} \alpha_n$ converges but $\sum_{n=1}^{\infty} \alpha_n \beta_n$ diverges.

If, on the contrary, $\limsup_{t\to\infty} N(t)/t^p = \infty$, then there exists a sequence of positive numbers $\{t_n\}$, such that $N(t_n)/t_n^p \to \infty$. By the observation above, there exists a non-negative sequence of numbers $\{p_n\}$, with $2\sum_{n=1}^{\infty} p_n = 1$, such that $\sum_{n=1}^{\infty} p_n t_n^p$ converges, but $\sum_{n=1}^{\infty} p_n N(t_n)$ diverges. Let $\{X_n\}$ be a symmetric sequence of i.i.d. random variables, which are defined by the law: $\mathbf{P}(X_1 = \pm t_n) = p_n$. So, we have $\mathbf{E}[|X_1|^p] = 2\sum_{n=1}^{\infty} p_n t_n^p < \infty$ and $\mathbf{E}[N(|X_1|)] = 2\sum_{n=1}^{\infty} p_n N(t_n) = \infty$. Hence we obtain a contradiction, so (i) must hold.

$(i) \Rightarrow (ii)$: For every $\epsilon > 0$ we have $\mathbf{E}[N(\frac{|X_1|}{\epsilon})] < \infty$ when $\mathbf{E}[|X_1|^p] < \infty$. Using (*) and the Borel-Cantelli lemma, we obtain that $\frac{X_n}{b_n} \to 0$ a.s. □

Remarks. 1. Note that, in fact, condition (i) above yields that $X_n/b_n \to 0$ a.s. for every identically distributed sequence $\{X_n\}$ (not necessarily independent) with $\mathbf{E}[|X_1|^p] < \infty$. On the other hand, if $X_n/b_n \to 0$ a.s. for every identically distributed sequence $\{X_n\}$, then condition (i) holds. This equivalence should be compared with Assani [**A1**, Theorem 8].

2. Let $\varphi(t)$ be a positive non-decreasing function on $[0,\infty)$, e.g., Orlicz's type functions like $t \mapsto t^p$, $t \mapsto t\log^+ t$, etc... It is not hard to see that the equivalence above holds true in the following sense: $\limsup_{t\to\infty} N(t)/\varphi(t) < \infty$ if and only if $X_n/b_n \to 0$ a.s. for every symmetric i.i.d. sequence $\{X_n\}$ with $\mathbf{E}[\varphi(|X_1|)] < \infty$.

Corollary 2.6. *Let $\{b_n\}$ be a non-zero sequence of complex numbers, and put $N(t) = \#\{n \ge 1 : |b_n| \le t\}$. For $1 < p < 2$ the following are equivalent:*

(i) $\limsup_{t\to\infty} N(t)/t^p < \infty$.

(ii) *The sequence $\frac{X_n}{b_n}$ converges a.s. to* 0 *for every* symmetric *i.i.d. random variables $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$.*

(iii) *The series $\sum_{n=1}^{\infty} \frac{X_n}{b_n}$ converges a.s. for every sequence of centered i.i.d. random variables $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$.*

Remarks. 1. The sequence $b_n = \sqrt{n}$ shows that when $p = 2$, (i) of the corollary does not imply (iii). On the other hand, by the previous remark, $X_n/\sqrt{n}$ converges to 0 a.s. even for $\{X_n\}$ identically distributed (not necessarily independent) with $\mathbf{E}[|X_1|^2] < \infty$.

2. Let $\{a_n\}$ be complex numbers and let $\{A_n\}$ be a non-decreasing sequence of positive numbers, tending to infinity. When we put in Corollary 2.6 $b_n = A_n/a_n$ (with $A_n/0$ interpreted as ∞), the function $N(t)$ is well defined (finite if $|b_n| \to \infty$). By Kronecker's lemma, condition (iii) of the corollary implies

(iv) $\frac{1}{A_n}\sum_{k=1}^{n} a_k X_k$ *converges to* 0 *a.s. for every $\{X_n\}$ centered i.i.d. random variables with $\mathbf{E}[|X_1|^p] < \infty$.*

Condition (iv) implies (ii) of the corollary, since $A_{n-1} \le A_n$ and

$$\frac{a_n X_n}{A_n} = \frac{1}{A_n}\sum_{k=1}^{n} a_k X_k - \frac{A_{n-1}}{A_n}\frac{1}{A_{n-1}}\sum_{k=1}^{n-1} a_k X_k.$$

Thus, (iv) is equivalent to the three conditions of the corollary, and we obtain a stronger result than [**CZF**, Theorem 2], where only $(i) \iff (iv)$ is proved; here we obtain from (i) the a.s. convergence of the series $\sum_{n=1}^{\infty} \frac{a_n X_n}{A_n}$, and show the equivalence of all four conditions. This discussion applies, in particular, to weighted averages when the weights $\{w_n\}$ have a divergent sum, and for $1 < p < 2$ it yields a complete characterization of the weighted SLLN for centered i.i.d. random variables $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$.

3. For $1 < p < 2$, (iii) may hold with $\sum_n |a_n/A_n|^p = \infty$ ([**LW**, p. 528]).

4. In the context of Remark 2, in general for $p = 2$ we have $(iii) \Rightarrow (iv) \Rightarrow (ii) \Leftrightarrow (i)$; however, condition (i) of Corollary 2.6 does not imply (iv) – take $a_n = 1$ and $A_n = \sqrt{n}$. Note that if the series $\sum_{n=1}^{\infty} \frac{a_n X_n}{A_n}$ converges a.s. for *one* centered i.i.d. sequence $\{X_n\}$ with finite variance, then by [**MZ-1**, Théorème 4] we have

$\sum_{n=1}^{\infty}|a_n/A_n|^2<\infty$ and (iii) holds. Now let $a_n=1$ and $A_n=\sqrt{n\log(n+1)}$; then $\sum_{n=1}^{\infty}(a_n/A_n)^2=\infty$ so (iii) fails, but from the Hartman-Wintner law of iterated logarithm (LIL) we obtain that (iv) holds. It is not clear if for weighted averages ($a_n=w_n$ and $A_n=\sum_{k=1}^{n}w_k$) (iv) implies (iii).

COROLLARY 2.7. *Let $\{b_n\}$ be a non-zero sequence of complex numbers, and put $N(t)=\#\{n\geq 1:|b_n|\leq t\}$. Then the following are equivalent:*

(i) $\limsup_{t\to\infty} N(t)/t\log t<\infty$.

(ii) *The sequence* $\dfrac{X_n}{b_n}$ *converges a.s. to* 0 *for every* symmetric *i.i.d. random variables $\{X_n\}$ with $\mathbf{E}[|X_1|\log^+|X_1|]<\infty$.*

(iii) *The series* $\displaystyle\sum_{n=1}^{\infty}\frac{X_n}{b_n}$ *converges a.s. for every sequence of symmetric i.i.d. random variables $\{X_n\}$ with $\mathbf{E}[|X_1|\log^+|X_1|]<\infty$.*

PROOF. $(i)\Rightarrow(iii)$ by the proof of part (iii) of Theorem 2.2, and obviously $(iii)\Rightarrow(ii)$. The proof of $(ii)\Rightarrow(i)$ is similar to the proof of $(ii)\Rightarrow(i)$ in Proposition 2.5 (see Remark 2 following it). □

The following corollary of Theorem 2.2(iii) and Proposition 2.5 deals with the case $p=1$ with no additional moment assumptions.

COROLLARY 2.8. *Let $\{b_n\}$ be a non-zero sequence of complex numbers, and put $N(t)=\#\{n\geq 1:|b_n|\leq t\}$. The following are equivalent:*

(i) $\limsup_{t\to\infty} N(t)/t<\infty$.

(ii) *The sequence* $\dfrac{X_n}{b_n}$ *converges a.s. to* 0 *for every* symmetric *integrable i.i.d. random variables $\{X_n\}$.*

(iii) *The series* $\displaystyle\sum_{n=1}^{\infty}\frac{X_n}{b_n}$ *converges a.s. for every sequence of* symmetric *integrable i.i.d. random variables $\{X_n\}$.*

Remark. Corollary 2.8 yields that for any sequence $\{c_n\}$ the series $\sum_{n=1}^{\infty}c_nX_n$ converges a.s. for every integrable symmetric i.i.d. $\{X_n\}$ if (and only if) $c_nX_n\to 0$ a.s. for every such sequence $\{X_n\}$ (we may assume $c_n\neq 0$, and put $b_n=1/c_n$).

The following theorem is partly inspired by [**JOP**] (see also [**CZF**]).

THEOREM 2.9. *Let $\{a_n\}$ be a sequence of complex numbers and let $\{A_n\}$ be a non-decreasing sequence of positive numbers, tending to infinity. The following conditions $(i)-(iv)$ are equivalent:*

(i) *The function $N(t)=\#\{n\geq 1: A_n/|a_n|\leq t\}$ is finite valued for every $t\geq 0$, and* $\limsup_{t\to\infty} N(t)/t<\infty$

(ii) *The series* $\displaystyle\sum_{n=1}^{\infty}\frac{a_nX_n}{A_n}$ *converges a.s. for every* symmetric *sequence of integrable i.i.d. random variables.*

(iii) *The averages* $\displaystyle\frac{1}{A_n}\sum_{k=1}^{n}a_kX_n$ *converge a.s. to* 0 *for every symmetric sequence of integrable i.i.d. random variables.*

(iv) The sequence $\frac{a_nX_n}{A_n}$ *converges a.s. to* 0 *for every symmetric sequence of integrable i.i.d. random variables.*

If in addition we assume that

$$\sup_{n\ge 1} \frac{\sum_{k=1}^n |a_k|}{A_n} < \infty\,, \tag{6}$$

then (i) implies that $\frac{1}{A_n}\sum_{k=1}^n a_kX_k$ *converges a.s. to* 0 *for every* $\{X_n\}$ *integrable, centered sequence of i.i.d. random variables (which are not necessarily symmetric).*

Proof. The equivalence of conditions (i), (ii) and (iv) follows by putting $b_n = A_n/a_n$ in the previous corollary. For their equivalence to (iii) see Remark 2 following Corollary 2.6.

Now we assume (6) and prove the last part. For integrable centered i.i.d., the computations in the proof of Theorem 2.2 yield that

$$\sum_{n=1}^\infty \frac{a_nX_n\mathbf{1}_{\{|X_n|\le A_n/|a_n|\}} - \mathbf{E}[a_nX_n\mathbf{1}_{\{|X_n|\le A_n/|a_n|\}}]}{A_n}$$

converges a.s. Hence

$$\frac{1}{A_n}\sum_{k=1}^n \Big\{a_kX_k\mathbf{1}_{\{|X_k|\le A_k/|a_k|\}} - \mathbf{E}\big[a_kX_k\mathbf{1}_{\{|X_k|\le A_k/|a_k|\}}\big]\Big\} \to 0 \qquad \text{a.s.}$$

Clearly, since $A_n/|a_n| \to \infty$ we have $\mathbf{E}[X_n\mathbf{1}_{\{|X_n|\le A_n/|a_n|\}}] \to 0$.

For every $n \ge 1$, put $\alpha_{n,k} = a_k/A_n$ for $1 \le k \le n$ and $\alpha_{n,k} = 0$ for $k > n$. By our assumption we have $\sup_{n\ge1} \frac{\sum_{k=1}^n |a_k|}{A_n} < \infty$, so we obtain that $\sup_{n\ge1}\sum_{k=1}^\infty |\alpha_{n,k}| < \infty$. Now by usual summability arguments we obtain that $\frac{1}{A_n}\sum_{k=1}^n \mathbf{E}[a_kX_k\mathbf{1}_{\{|X_k|\le A_k/|a_k|\}}] \to 0$, which in turn implies a.s. convergence to zero of $\frac{1}{A_n}\sum_{k=1}^n a_kX_k$. □

Remarks. 1. The main difference between the "averages" considered in Theorem 2.9 and the weighted averages considered in [**JOP**] is in the natural summability property that was present in [**JOP**]. More precisely, for the weighted averages $\frac{1}{W_n}\sum_{k=1}^n w_kX_k$ we have automatically $\sup_{n\ge1}\sum_{k=1}^\infty \alpha_{n,k} \equiv 1$.

2. The first part of Theorem 2.9 is new, even in the context of [**JOP**]. Condition (6) implies condition (1.3) of Chen, Zhu, and Fang [**CZF**, Theorem 1] (and for $\{a_n\}$ positive is equivalent to it). Therefore the second part of Theorem 2.9, which deals with not necessarily symmetric centered i.i.d. random variables, is already implied by Theorem 1 in [**CZF**]. Combined with the first part, it shows that under (6), a.s. convergence of the "averages" for every integrable *symmetric* i.i.d. $\{X_n\}$ implies the same also for all non-symmetric centered i.i.d. random variables.

3. Condition (6) was introduced by Tempelman [**T**, Theorem 5.6] for problems of L_2-consistency of the least square estimates in multivariate linear regression models (in which $A_n = \sum_{k=1}^n |a_k|^2$). This condition was re-investigated and extended in [**CLT**].

4. It was noted in [**CZF**], that if $\{a_n\}$ is a positive sequence, then (6) is necessary for (iii) to hold for every centered i.i.d., not necessarily symmetric, random variables. It follows that if the a_n's are positive, and condition (i) in Theorem 2.9

holds while (6) fails (for an example see Remark 4 in [**CZF**, §2]), then there exists a sequence of centered i.i.d. random variables $\{X_n\}$, which are necessarily not symmetric by Theorem 2.9, such that the weighted averages in (iii) fail to converge a.s. to zero. Hence the symmetry assumption cannot be dropped in the first part of Theorem 2.9. On the other hand, the theorem shows that in the symmetric case, (i) yields the stronger result of a.s. convergence of the series $\sum_{n=1}^{\infty} \frac{a_n X_n}{A_n}$, even without assuming condition (6).

5. By Theorem 2.2(ii), condition (i) implies that for every centered i.i.d. with $\mathbf{E}[|X_1| \log^+ |X_1|] < \infty$ the series in part (ii) of the theorem converges a.s., and therefore $\frac{1}{A_n} \sum_{k=1}^{n} a_k X_k \to 0$ a.s., even without (6).

Corollary 2.10. *Let $(S, \mathcal{A}, \mu)$ be a probability space, and let θ be a μ-measure preserving transformation on it. Let $g \geq 0$, with $\int g \log^+ \log^+ g \, d\mu < \infty$. Then there exists a subset $S_0 \subset S$ of full μ-measure, such that for every $x \in S_0$, we have the following: for any integrable centered sequence of i.i.d. random variables the following assertions hold:*

(i) *The averages* $\displaystyle\frac{1}{n} \sum_{k=1}^{n} g(\theta^k x) X_k$ *converge to zero a.s.*

(ii) *If X_1 is symmetric or $\mathbf{E}[|X_1| \log^+ |X_1|] < \infty$, then the series* $\displaystyle\sum_{n=1}^{\infty} \frac{g(\theta^n x) X_n}{n}$ *converges a.s.*

(iii) *If θ is ergodic, then the weighted averages* $\displaystyle\frac{\sum_{k=1}^{n} g(\theta^k x) X_k}{\sum_{k=1}^{n} g(\theta^k x)}$ *converge to zero a.s*

(iv) *If X_1 is symmetric or $\mathbf{E}[|X_1| \log^+ |X_1|] < \infty$, and θ is ergodic, then the series* $\displaystyle\sum_{n=1}^{\infty} \frac{g(\theta^n x) X_n}{\sum_{k=1}^{n} g(\theta^k x)}$ *converges a.s.*

Proof. (i) is Corollary 6 of Demeter and Quas [**DQ**].

(ii): For $x \in S$ put $A_n = n$ and $a_k(x) = g(\theta^k x)$. By [**DQ**, Theorem 5], for almost every x we have $\limsup_{t\to\infty} N(t)/t < \infty$, so part (iii) or part (ii) of Theorem 2.2 applies.

(iii): When θ is ergodic, $\frac{1}{n} \sum_{k=1}^{n} g(\theta^k x) \to \int g \, d\mu > 0$ a.e., by the ergodic theorem, so (i) yields (iii).

(iv): For $x \in S$, we now put $A_n(x) = \sum_{k=1}^{n} g(\theta^k x)$ and $a_n(x) = g(\theta^k x)$. For θ ergodic, the pointwise ergodic theorem and [**DQ**, Theorem 5] yield that $\limsup_{t\to\infty} N(t)/t < \infty$ for a.e. x, so part (iii) or part (ii) of Theorem 2.2 applies. $\square$

Remarks. 1. If $\{g_n\} \subset L_1(\mu)$ is a sequence of non-negative identically distributed random variables, by Sawyer [**S**, Lemma 3] the series $\sum_{n=1}^{\infty} \left(\frac{g_n}{n}\right)^p$ converges a.e. for every $p > 1$. Using Marcinkiewicz and Zygmund [**MZ-1**, Theorem 5'], we obtain that there exists a set of full μ-measure $S_0 \subset S$, such that for each $x \in S_0$, the series $\sum_{n=1}^{\infty} \frac{g_n(x) X_n}{n}$ converges a.s., for every centered *independent* sequence $\{X_n\} \subset L_p(\mathbf{P})$ (not necessarily identically distributed) with $\sup_{n\geq 1} \mathbf{E}[|X_n|^p] < \infty$. This result is [**BJLO**, Theorem 3.7], and in particular, if $\{X_n\}$ are identically distributed, it is [**A3**, Theorem 5(2)]. Corollary 2.10 assumes more on g, but requires a weaker integrability condition on X_1 for general centered i.i.d. $\{X_n\}$, and no additional integrability condition if X_1 is symmetric.

2. As a consequence of Proposition 2.5 and the previous remark, we obtain that for *any* identically distributed non-negative $\{g_n\} \subset L_1(\mu)$ we have the following: there exists a full μ-measure set $S_0 \subset S$, such that for any $x \in S_0$ and for any $p > 1$, we have $\limsup_{t\to\infty} \#\{n \geq 1 : g_n(x)/n \geq 1/t\}/t^p < \infty$. As we mention later in a remark before Theorem 5.2, this finiteness does not hold for $p = 1$. On the other hand, if g_n is induced by a dynamical system with $\int g_1 \log^+ \log^+ g_1 \, d\mu < \infty$, we already know by Demeter and Quas [**DQ**, Theorem 5] (see also Assani [**A2**, Theorem 5] for an earlier result) that

$$\limsup_{t\to\infty} \#\{n \geq 1 : g_n(x)/n \geq 1/t\}/t < \infty \qquad \text{a.e.}$$

3. Weighted sums of L_p-bounded martingale differences

In this section we relax the identical distribution assumption of the previous section. We observed in the remarks following Corollary 2.6 that a.s. convergence of the series $\sum_{k=1}^\infty \frac{w_k X_k}{W_k}$ for *one* centered i.i.d. sequence with finite variance is equivalent to $\sum_{k=1}^\infty (w_k/W_k)^2 < \infty$, which yields a.s. convergence of the series

$$\sum_{k=1}^\infty \frac{w_k Y_k}{W_k} \tag{7}$$

for *every* $\{Y_n\}$ centered independent random variables with $\sup_n \mathbf{E}|Y_n|^2 < \infty$.

For $p \geq 1$, we say that a sequence of random variables $\{Y_n\} \subset L_p(\mathbf{P})$ is $L_p(\mathbf{P})$-*norm bounded*, abbreviated L_p-*bounded*, if $\sup_n \mathbf{E}[|Y_n|^p] < \infty$.

For fixed $1 < p \leq 2$ we investigate in this section the a.s. convergence of (7) for every L_p-bounded sequence $\{Y_n\}$ of centered independent random variables. Theorem 3.2 below, which is a refinement of [**LW**, Proposition 4.3] (see the remark after the theorem), also adds the connection with the counting function. It turns out that independence can be relaxed, and we deal with martingale differences.

Lemma 3.1. *Let $\{b_n\}$ be a sequence of complex numbers, and define $N(t) = \#\{n \geq 1 : |b_n| \leq t\}$. Then for each $1 < p < \infty$, the series $\sum_{n=1}^\infty \frac{1}{|b_n|^p}$ converges if and only if $\int_1^\infty \frac{N(t)}{t^{p+1}} dt$ converges.*

Proof. Either condition implies that $|b_n| \to \infty$. Now apply Lemma 2.1 with $\alpha_n = |b_n|$ and $\varphi(t) = 1/t^p$. □

Theorem 3.2. *Let $\{a_n\}$ be a sequence of complex numbers, let $\{A_n\}$ be a non-decreasing sequence of positive numbers tending to infinity, and (with $A/0 = \infty$) put $N(t) = \#\{n \geq 1 : A_n/|a_n| \leq t\}$. For each $1 < p \leq 2$, the following assertions are equivalent:*

(i) The function $N(t)$ is finite valued and $\displaystyle\int_1^\infty \frac{N(t)}{t^{p+1}} dt$ converges.

(ii) The series $\displaystyle\sum_{n=1}^\infty \Big(\frac{|a_n|}{A_n}\Big)^p$ converges.

(iii) The series $\displaystyle\sum_{n=1}^\infty \frac{a_n Y_n}{A_n}$ converges a.s. for every L_p-bounded martingale difference sequence $\{Y_n\}$.

(iv) The series $\displaystyle\sum_{n=1}^{\infty}\frac{a_nX_n}{A_n}$ *converges a.s. for every* L_p*-bounded, centered independent sequence* $\{X_n\}$.

(v) The averages $\displaystyle\frac{1}{A_n}\sum_{k=1}^{n}a_kX_k$ *converge a.s. to* 0 *for every* L_p*-bounded, centered independent sequence* $\{X_n\}$.

(vi) The sequence $\displaystyle\frac{a_nX_n}{A_n}$ *converges a.s. to* 0 *for every* L_p*-bounded,* symmetric *independent sequence* $\{X_n\}$.

Proof. $(i) \Leftrightarrow (ii)$ by the previous lemma. The implication $(ii) \Rightarrow (iii)$ can be proved using Chow's extension of the Marcinkiewicz-Zygmund result [**Ch**, Corollary 5]. $(iii) \Rightarrow (iv)$ is trivial. $(iv) \Rightarrow (v)$ by Kronecker's lemma.

$(v) \Rightarrow (vi)$ since $A_{n-1}/A_n \le 1$ and

$$\frac{a_nX_n}{A_n} = \frac{1}{A_n}\sum_{k=1}^{n}a_kX_k - \frac{A_{n-1}}{A_n}\frac{1}{A_{n-1}}\sum_{k=1}^{n-1}a_kX_k.$$

$(vi) \Rightarrow (ii)$: validity of (vi) yields that $\frac{a_n\epsilon_n}{A_n} \to 0$ a.s. for a Rademacher sequence $\{\epsilon_n\}$ (i.e., $\{\epsilon_n\}$ i.i.d. with $\mathbf{P}(\epsilon_n = \pm 1) = 1/2$). So, $a_n/A_n \to 0$, and we may assume that $|a_n|/A_n \le 1$ for every $n \ge n_0$. We define a sequence of symmetric independent random variables $\{X_n : n \ge n_0\}$ according to the law: $\mathbf{P}(X_n = \pm\frac{A_n}{|a_n|}) = \frac{1}{2}(\frac{|a_n|}{A_n})^p$ and $\mathbf{P}(X_n = 0) = 1 - (\frac{|a_n|}{A_n})^p$. Clearly, $\mathbf{E}[|X_n|^p] \equiv 1$ and $\mathbf{P}(|\frac{a_nX_n}{A_n}| \ge 1) = (\frac{|a_n|}{A_n})^p$. So, if we assume $\sum_{n=1}^{\infty}(\frac{|a_n|}{A_n})^p = \infty$, we obtain by independence and the Borel-Cantelli lemma that (vi) fails. □

Remarks. 1. In the special case $a_n = w_n$ and $A_n = W_n$, the equivalences $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ were proved in [**LW**, Proposition 4.3] and Remark 5 following it.

2. The counter-example in the proof above is basically the counter-example constructed in Theorem 5 of [**MZ-1**] (see also the proof of [**Chu**, Theorem 2]). The same idea was used in [**LW**].

3. Note that $A_n \to \infty$ is only required for the implication $(iv) \Rightarrow (v)$. Hence, all parts of Theorem 3.2, except (v), could be formulated with b_n instead of A_n/a_n.

4. Note that if $\int_1^\infty \frac{N(t)}{t^{p+1}}dt < \infty$, then $\lim\limits_{t\to\infty} N(t)/t^p = 0$. On the other hand, $\limsup\limits_{t\to\infty} N(t)/t^p < \infty$ does not imply $\int_1^\infty \frac{N(t)}{t^{p+1}}dt < \infty$ (for examples when $1 < p < 2$ and $a_n = w_n$, $A_n = W_n$, see Remark 3 following [**LW**, Proposition 4.3]).

5. As noted earlier, for $p = 2$ a.s. convergence of $\sum_{k=1}^{\infty}\frac{a_kX_k}{A_k}$ for *one* centered i.i.d. sequence with finite variance is equivalent to condition (ii) of the theorem.

Corollary 3.3. *Let* $\{a_n\}$ *be a sequence of complex numbers and let* $\{A_n\}$ *be a non-decreasing sequence of positive numbers, tending to infinity. Assume that for every integrable symmetric i.i.d. sequence* $\{X_n\}$*, the sequence* $\frac{a_nX_n}{A_n}$ *converges a.s. to* 0*. Then for any* $1 < p \le 2$*, all the conditions of Theorem 3.2 hold.*

Proof. Theorem 2.9 yields that that $\limsup\limits_{t\to\infty} N(t)/t < \infty$. Hence condition (i) in Theorem 3.2 holds for every $p > 1$. □

The following lemma is Lemma 2 in Jamison, Orey, and Pruitt [**JOP**]:

Lemma 3.4. *Let $0 < w_n \le 1$ with $\sum_{n=1}^\infty w_n = \infty$. Then the counting function $N(t) = \#\{n \ge 1 : W_n/w_n \le t\}$ satisfies $\limsup_{t\to\infty} N(t)/(t\log t) \le 2$.*

By the previous lemma, the following corollary applies to bounded sequences with divergent sum.

Corollary 3.5. *Let $\{w_n\}$ be a sequence of positive numbers with divergent sum, such that $\limsup_{t\to\infty} N(t)/(t\log t) < \infty$. Then for each $1 < p \le 2$ we have $\sum_{n=1}^\infty \left(\frac{w_n}{W_n}\right)^p < \infty$, and for every L_p-bounded martingale difference sequence $\{Y_n\}$ the series $\sum_{n=1}^\infty \frac{w_n Y_n}{W_n}$ converges a.s.*

Proof. The assumption $\limsup_{t\to\infty} N(t)/(t\log t) < \infty$ yields condition (i) of Theorem 3.2, so Theorem 3.2 yields the result. $\square$

Remark. If $\{b_n\}$ is a non-zero sequence such that $N_{\{|b_n|\}}$ satisfies $\limsup_{t\to\infty} N(t)(\log t)^\gamma/t^2 < \infty$ for some $\gamma > 1$, then by Theorem 3.2 we have that $\sum_{n=1}^\infty \frac{Y_n}{b_n}$ converges a.s. for any L_2-bounded martingale difference sequence $\{Y_n\}$. In particular, this completes, in some sense, Theorem 2.2 for the case $p = 2$.

Example 3.1. For every $n \ge 1$, put $W_n = \mathrm{e}^{\log^2 n}$. By Lagrange's formula we have

$$\frac{2}{n}\log n\, \mathrm{e}^{\log^2 n} \le W_{n+1} - W_n \le \frac{2}{n+1}\log(n+1)\, \mathrm{e}^{\log^2(n+1)}$$

Hence, the finite limit $\lim_{t\to\infty} N(t)/(t\log t)$ exists (but $\lim_{t\to\infty} N(t)/t = \infty$).

Remark. Under the assumption $\limsup_{t\to\infty} N(t)/(t\log t) < \infty$, we obtain by (4) (as in [**JOP**, Theorem 4]), that the weighted averages $\frac{1}{W_n}\sum_{k=1}^n w_k X_k$ converge a.s. to zero for every centered i.i.d. $\{X_n\}$ with $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$ (even without $\{w_n\}$ being bounded!). Since the assumption on $N(t)$ is weaker than the case $p = 1$ in Theorem 2.2, the a.s. convergence of the series $\sum_{n=1}^\infty \frac{w_n X_n}{W_n}$ requires a stronger assumption on X_1 (e.g., $\mathbf{E}[|X_1|(\log^+|X_1|)^2(\log^+\log^+|X_1|)^{1+\epsilon}] < \infty$) – see the next theorem.

The following theorem improves the result of Corollary 3.5 in the case of independent random variables, and applies to bounded weights with divergent sum:

Theorem 3.6. *Let $\{b_n\}$ be a non-zero sequence of complex numbers. Put $N(t) = \#\{n \ge 1 : |b_n| \le t\}$, and assume that $\limsup_{t\to\infty} N(t)/(t\log t)$ is finite. Let $\psi(t)$ be a positive, non-decreasing, differentiable function for $t \ge 0$, such that $\psi(t)\log^2 t/t$ is non-increasing and $\int_{t_0}^\infty \frac{dt}{t\log t\,\psi(t)}$ converges, for some $t_0 > 0$. Also assume that $\sup_{t\ge t_0} t\,\psi'(t)/\psi(t) < \infty$. Then for any centered independent sequence $\{X_n\}$ with $\sup_{n\ge1}\mathbf{E}[|X_n|(\log^+|X_n|)^2\psi(|X_n|)]$ finite, the series $\sum_{n=1}^\infty \frac{X_n}{b_n}$ converges a.s.*

Furthermore, given a sequence of non-zero complex numbers $\{b_n\}$, satisfying $|b_n| \to \infty$ and $\liminf_{t\to\infty} N(t)/(t\log t) > 0$, the condition $\sup_{n\ge1}\mathbf{E}[|X_n|(\log^+|X_n|)^2] < \infty$ for centered independent $\{X_n\}$ does not ensure a.s. convergence to zero of $\{\frac{X_n}{b_n}\}$.

Proof. Assume the logarithm is to base 2. By the assumption $N(t)$ is finite, hence $|b_n| \to \infty$, so we may and do assume that $|b_n| \geq 2$ for $n \geq n_0$. Using $\limsup_{t\to\infty} N(t)/(t\log t) < \infty$ we apply Lemma 2.1 with $\varphi(t) = \frac{1}{t\log^2 t\psi(t)}$, by noting that our assumptions yield that $\varphi'(t) \approx \frac{1}{t^2 \log^2 t\,\psi(t)}$, to obtain that $\sum_{n=n_0}^{\infty} \frac{1}{|b_n|(\log(|b_n|))^2\psi(|b_n|)}$ converges. By applying Theorem 2(ii) in Chung [**Chu**] (inspection of the proof shows that the sequence $\{a_n\}$ there could be taken to be complex and monotonicity of $\{|a_n|\}$ is not needed), with the function $t \mapsto t\log^2 t\,\psi(t)$ and with the sequence $\{b_n\}$, we conclude the a.s. convergence result.

If $\sum_{n=1}^{\infty} \frac{X_n}{b_n}$ converges a.s., then necessarily $\frac{X_n}{b_n} \to 0$ a.s. We will construct a sequence of centered independent random variables $\{X_n\}$ with $\mathbf{E}[|X_n|(\log^+|X_n|)^2] \equiv 1$, while $\limsup_{n\to\infty}|\frac{X_n}{b_n}| \geq 1$ a.s. By assumptions, we have $\frac{1}{|b_n|(\log(|b_n|))^2} \leq \frac{1}{2}$ for $n \geq n_0$. Define the independent sequence $\{X_n : n \geq n_0\}$, according to the following law: $X_n = \pm|b_n|$ with probability $\frac{1}{2|b_n|(\log(|b_n|))^2}$, and $X_n = 0$ with probability $1 - \frac{1}{|b_n|(\log(|b_n|))^2}$. Clearly, $\mathbf{E}[X_n] = 0$ and $\mathbf{E}[|X_n|(\log^+|X_n|)^2] = 1$, for every $n \geq n_0$. But we have $\mathbf{P}(|\frac{X_n}{b_n}| \geq 1) = \frac{1}{|b_n|(\log(|b_n|))^2}$, and by Lemma 2.1 (this time with $\varphi(t) = 1/t(\log t)^2$) and by the condition $\liminf_{t\to\infty} N(t)/(t\log t) > 0$, the series $\sum_{n=n_0}^{\infty} \mathbf{P}(|\frac{X_n}{b_n}| \geq 1)$ diverges and the result follows. $\square$

Remarks. 1. For example, we can take $\psi(t) = (\log^+\log^+ t)^\gamma$, for some $\gamma > 1$.

2. As a consequence of Lemma 3.4, it was proved in [**JOP**, Theorem 4] that if $\{w_n\}$ is a bounded sequence of positive numbers with $\sum_{n=1}^{\infty} w_n = \infty$, then $\frac{1}{W_n}\sum_{k=1}^{n} w_k X_k$ converges a.s. to zero for every centered i.i.d. sequence $\{X_n\}$, with $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$. In Proposition 2.4 we obtain a.s. convergence of the series $\sum_{n=1}^{\infty} \frac{w_n X_n}{W_n}$ when $\mathbf{E}[|X_1|(\log^+|X_1|)^2] < \infty$. In Theorem 3.6 (with $b_n = W_n/w_n$) we assume a slightly stronger moment condition, and benefit by relaxing the identical distribution assumption. Note that in order to deduce the convergence of the weighted averages to zero, we also must assume $\sum_{n=1}^{\infty} w_n = \infty$. This will allow us to apply Kronecker's lemma.

Let $\{w_n\}$ be a weight sequence and fix $2 < p \leq \infty$. When the series $\sum_{k=1}^{\infty} \frac{w_k X_k}{W_k}$ converges a.s. for one i.i.d. sequence with $\mathbf{E}[|X_1|^p] < \infty$, we must have $\sum_n (\frac{w_n}{W_n})^2 < \infty$, so all the conditions of Theorem 3.2 with $p = 2$ are satisfied. Hence for $p > 2$ the problem is to characterize (or give sufficient conditions on) weight sequences such that the *weighted averages* converge a.s. for every L_p-bounded centered random variables. We mention the following result of Lin and Weber [**LW**, Theorem 4.12].

Theorem 3.7. *Let $\{w_n\}$ be a weight sequence with $M_n := \sum_{k=1}^{n} w_k^2 \to \infty$, and let $2 < p < \infty$. If for some $\alpha > 1$ we have*

$$\limsup_{n\to\infty} \frac{\sqrt{M_n}[\log M_n(\log\log M_n)^\alpha]^{1/p}}{W_n} < \infty, \tag{8}$$

then $\frac{1}{W_n}\sum_{k=1}^{n} w_k X_k \to 0$ a.s. for all L_p-bounded centered independent $\{X_n\}$.

4. Weighted sums of uniformly bounded martingale differences

In this section we look for conditions on a weight sequence $\{w_n\}$ which ensure the a.s. convergence of the weighted averages of every uniformly bounded centered independent random variables. It turns out, as in part of the results of the previous section, that we can even deal with martingale differences. The main result is due to Azuma [**Az**], but our proof is different, and allows some more general results.

We first present a consequence of a result of F. Móricz [**M**, Theorem 1] (see also Remark 2 after [**CC**, Proposition 2.6] or [**CL**, Proposition 2.3])

Theorem 4.1. *Let $1 < p < \infty$, and let $\{f_k\}_{k=m}^n \subset L_p(\Omega, \mathbf{P})$ be a sequence of random variables. Assume there exist non-negative numbers $\{\alpha_k\}_{k=m}^n$, and some constants $C > 0$ and $q > 1$, such that*

$$\mathbf{E}\Big[\Big|\sum_{k=j}^{l} f_k\Big|^p\Big] \le C\Big(\sum_{k=j}^{l} \alpha_k\Big)^q \qquad \text{for every} \qquad n \ge l \ge j \ge m.$$

Then

$$\mathbf{E}\Big[\Big|\max_{m\le l\le n} |\sum_{k=m}^{l} f_k|\Big|^p\Big] \le C_{p,q}\Big(\sum_{k=m}^{n} \alpha_k\Big)^q,$$

where $C_{p,q} = C\big(1 - \frac{1}{2^{(q-1)/p}}\big)^{-p}$.

Now we present a consequence of a result of E. Rio [**R**, Théorème 2.4]. For random variables $\{f_k\}$ we denote by $\mathcal{F}_n = \sigma(f_1, \dots, f_n)$ the σ-algebra generated by $\{f_1, \dots, f_n\}$.

Proposition 4.2. *Let $\{f_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of centered random variables and let $\mathcal{F}_n = \sigma(f_1, \dots, f_n)$. Then for any $l \ge j \ge 1$ and for every natural $p = 1, 2, \dots$, we have*

$$\mathbf{E}\Big[\Big|\sum_{k=j}^{l} f_k\Big|^{2p}\Big] \le \frac{(2p)!}{p!2^p}\Big(\sum_{k=j}^{l} \|f_k\|_\infty^2 + \sum_{k=j}^{l} \max_{k\le s\le l} \|2f_k \sum_{i=k+1}^{s} \mathbf{E}[f_i|\mathcal{F}_k]\|_\infty\Big)^p$$

(with the convention that $\sum_{i=k+1}^{k}$ is defined as 0).

Corollary 4.3. *Let $\{Y_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of martingale differences. Then for any $l \ge j \ge 1$ and for every natural $p = 1, 2, \dots$, we have*

$$\mathbf{E}\Big[\Big|\sum_{k=j}^{l} Y_k\Big|^{2p}\Big] \le \frac{(2p)!}{p!2^p}\big(\sum_{k=j}^{l} \|Y_k\|_\infty^2\big)^p$$

Remark. For $Y_n = a_n\epsilon_n$ (where $\{\epsilon_n\}$ is a Rademacher sequence), the corollary yields the classical Khintchine's inequality (e.g., [**Z**, Theorem V.8.4], [**LT**, Lemma 4.1]).

Notation. For $\{f_k\} \subset L_\infty$, put $S_n = \sum_{k=1}^n f_k$, $S_n^* = \max_{1\le l\le n} |\sum_{k=1}^l f_k|$, $B_n = \sum_{k=1}^n \|f_k\|_\infty^2$, and

$$R_n = \sum_{k=1}^{n} \|f_k\|_\infty^2 + \sum_{k=1}^{n} \max_{k\le s\le n} \|2f_k \sum_{i=k+1}^{s} \mathbf{E}[f_i|\mathcal{F}_k]\|_\infty. \tag{9}$$

When $\{f_n\}$ is a martingale difference, $R_n = B_n$.

COROLLARY 4.4. *Let $\{f_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of centered random variables. Then for every natural $p = 1, 2, \ldots$, we have*

$$\mathbf{E}\Big[\max_{1\le l\le n}\Big|\sum_{k=1}^{l} f_k\Big|^{2p}\Big] \le C\frac{(2p)!}{p!c_0^p}R_n^p, \tag{10}$$

for some absolute *positive constants C and c_0.*

PROOF. By Proposition 4.2, we can use Theorem 4.1 with $m = 1$ and

$$\alpha_k = \|f_k\|_\infty^2 + \max_{k\le s\le n}\|2f_k\sum_{i=k+1}^{s}\mathbf{E}[f_i|\mathcal{F}_k]\|_\infty, \qquad k = 1, \ldots, n,$$

so for every $p = 2, 3, \ldots$, inequality (10) holds with $c_0 = c_0(p) = 2\big(1 - \frac{1}{2^{(p-1)/2p}}\big)^2$ and $C = 1$. As we can see, $c_0(p)$ is an increasing function of p, hence inequality (10) holds, for every $p = 2, 3, \ldots$, with $c_0 = c_0(2)$ and $C = 1$. Now, for $p = 1$ we have

$$\mathbf{E}\Big[\max_{1\le l\le n}\Big|\sum_{k=1}^{l} f_k\Big|^{2}\Big] \le \Big(\mathbf{E}\Big[\max_{1\le l\le n}\Big|\sum_{k=1}^{l} f_k\Big|^{4}\Big]\Big)^{1/2} \le \Big(\frac{4!}{2c_0^2}R_n^2\Big)^{1/2} = \frac{\sqrt{12}}{c_0}R_n.$$

So, the inequality holds for $p = 1, 2, \ldots$ with, e.g., $C = 2 > \sqrt{12}/2$. □

PROPOSITION 4.5. *Let $\{f_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of centered random variables. Then for every $a > 0$, we have*

$$\mathbf{E}[\exp(aS_n^*)] \le C\exp(\frac{1}{c_0}a^2R_n), \tag{11}$$

for some absolute *positive constants C and c_0.*

PROOF. By inequality (10), with the constants C and c_0 defined there, we have

$$\mathbf{E}[\exp(aS_n^*)] \le 2\mathbf{E}[\cosh(aS_n^*)] = 2\sum_{p=0}^{\infty}\mathbf{E}[\frac{1}{(2p)!}a^{2p}|S_n^*|^{2p}] \le$$

$$2C\sum_{p=0}^{\infty}\frac{1}{(2p)!}\frac{a^{2p}(2p)!}{p!c_0^p}R_n^p = 2C\sum_{p=0}^{\infty}\frac{1}{p!}(\frac{1}{c_0}a^2R_n)^p = 2C\exp(\frac{1}{c_0}a^2R_n).$$

□

Remarks. 1. For $\{f_n = Y_n\}$ centered independent with finite moments of all orders, Lemma 2 of [**MZ-2**] yields $\mathbf{E}\exp(aS_n^*) \le 16\mathbf{E}\exp(a|S_n|)$. From this inequality, which does not depend on Theorem 4.1, Tsuchikura [**Ts**] obtained (11) when $Y_n = w_n\epsilon_n$, by applying Khintchine's inequality. In our general context Rio's Corollary 4.3 replaces Khintchine's inequality, and Theorem 4.1 gives a maximal inequality without independence.

2. For a martingale difference sequence $\{Y_n\} \subset L_p$, Doob's maximal inequality [**Do**, Ch. VII, Theorem 3.4] yields $\mathbf{E}[(S_n^*)^p] \le \big(\frac{p}{p-1}\big)^p\mathbf{E}[|S_n|^p]$ for $p > 1$. In this case, one does not need to use Theorem 4.1 (but only some arguments of [**Ts**]), in order to conclude Corollary 4.4, in particular to conclude Proposition 4.5.

The following theorem is our main result in this section; its proof is a generalization of the method of Tsuchikura [**Ts**].

THEOREM 4.6. *Let $\{A_n\}$ be a non-decreasing sequence of positive numbers, tending to infinity, such that $\limsup_n A_{n+1}/A_n < \infty$. Let $\{f_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of centered random variables, with R_n defined by (9). If*

$$\lim_{n\to\infty} \frac{R_n \log\log A_n}{A_n^2} = 0 \tag{12}$$

holds, then $\displaystyle\frac{1}{A_n}\sum_{k=1}^n f_k$ converges a.s. to 0.

Analogously, $\displaystyle\limsup_{n\to\infty} \Big|\frac{1}{A_n}\sum_{k=1}^n f_k\Big| < \infty$ a.s. if

$$\limsup_{n\to\infty} \frac{R_n \log\log A_n}{A_n^2} < \infty\ . \tag{13}$$

PROOF. Take $\alpha > \limsup_n A_{n+1}/A_n \geq 1$. We are going to construct a subsequence of natural numbers $\{n_j\}$, such that

$$A_{n_{j-1}} \leq A_{n_j} \leq 2\alpha A_{n_{j-1}} \qquad \text{and} \qquad A_{n_j+1} > 2\alpha A_{n_{j-1}}\ .$$

We start the construction process with n_1 large enough, such that $A_{n_1} > 1$ and $A_{n+1} < \alpha A_n$ for every $n \geq n_1$. Assume that $n_1, \dots, n_{j-1}$ are defined. By monotonicity of $\{A_n\}$ and by the assumption $\limsup_n A_{n+1}/A_n < \alpha$, we have $A_{n_{j-1}} \leq A_{n_{j-1}+1} < 2\alpha A_{n_{j-1}}$. Define n_j as the maximal n for which $A_n \leq 2\alpha A_{n_{j-1}}$, so $n_j \geq n_{j-1} + 1$.

For $n_{j-1} < k \leq n_j$ we have

$$\frac{|S_k|}{A_k} \leq \frac{\max_{n_{j-1}<l\leq n_j} |S_l|}{A_{n_{j-1}}} \leq 2\alpha \frac{\max_{n_{j-1}<l\leq n_j} |S_l|}{A_{n_j}}\ . \tag{14}$$

The a.s. convergence to 0 of $\frac{1}{A_n}\sum_{k=1}^n f_k$ will therefore follow if we prove that

$$\max_{n_{j-1}<l\leq n_j} |S_l|/A_{n_j} \to 0 \ \text{ a.s.}$$

Hence, it is enough to prove that the series $\sum_{j=1}^\infty \mathbf{P}(S^*_{n_j} > \delta A_{n_j})$ converges for every $\delta > 0$.

Fix $\delta > 0$. By Proposition 4.5, for fixed j and $a > 0$ we have

$$\mathbf{P}(S^*_{n_j} > \delta A_{n_j}) \exp(a\delta A_{n_j}) \leq \int \exp(a S^*_{n_j})\, d\mathbf{P} \leq C \exp(\frac{1}{c_0} a^2 R_{n_j})\,.$$

For fixed j, putting $a = \frac{c_0 \delta A_{n_j}}{2R_{n_j}}$ we conclude that

$$\mathbf{P}(S^*_{n_j} > \delta A_{n_j}) \leq C \exp(-\frac{c_0 \delta^2 A_{n_j}^2}{4R_{n_j}})\,.$$

Now, by condition (12) for large enough j we have

$$\frac{R_{n_j}}{A_{n_j}^2} \leq \frac{c_0\delta^2}{8} \cdot \frac{1}{\log\log A_{n_j}}\,, \tag{15}$$

and by construction we also have

$$A_{n_j} \geq \frac{1}{\alpha} A_{n_j+1} > \frac{2\alpha}{\alpha} A_{n_{j-1}} > 4A_{n_{j-2}} > \cdots > 2^{j-1}$$

Hence, combining everything together, we obtain

$$\mathbf{P}(S^*_{n_j} > \delta A_{n_j}) \le C \exp\{-2\log\log(2^{j-1})\} = \frac{C}{(\log 2)^2 (j-1)^2}.$$

Thus $\sum_{j=1}^{\infty} \mathbf{P}(S^*_{n_j} > \delta A_{n_j})$ converges for every $\delta > 0$ and thus $\frac{1}{A_n}\sum_{k=1}^n f_k$ converges to 0 a.s.

The proof of the second part, under assumption (13), is a modification of the previous proof: by (13), inequality (15) holds for *some* $\delta > 0$, and for this δ the series $\sum_{j=1}^{\infty} \mathbf{P}(S^*_{n_j} > \delta A_{n_j})$ converges; by (14), this implies $\limsup_n \left|\frac{1}{A_n}\sum_{k=1}^n f_k\right| < 2\alpha\delta$ a.s. □

Corollary 4.7. *Let $\{A_n\}$ be a non-decreasing sequence of positive numbers, tending to infinity, such that $\limsup_n A_{n+1}/A_n < \infty$. Let $\{Y_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of martingale differences. If*

$$\lim_{n\to\infty} \frac{(\|Y_1\|_\infty^2 + \cdots + \|Y_n\|_\infty^2)\log\log A_n}{A_n^2} = 0 \tag{16}$$

holds, then $\dfrac{1}{A_n}\displaystyle\sum_{k=1}^n Y_k$ converges a.s. to 0.

Analogously, $\displaystyle\limsup_{n\to\infty} \Big|\frac{1}{A_n}\sum_{k=1}^n Y_k\Big| < \infty$ a.s. if

$$\limsup_{n\to\infty} \frac{(\|Y_1\|_\infty^2 + \cdots + \|Y_n\|_\infty^2)\log\log A_n}{A_n^2} < \infty\ . \tag{17}$$

Corollary 4.8. *Let $\{f_n\} \subset L_\infty(\Omega, \mathbf{P})$ be a sequence of centered random variables, with $\{R_n\}$ tending to infinity. If $\limsup_n \frac{R_{n+1}}{R_n} < \infty$, then we have*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{R_n \log\log R_n}} \Big|\sum_{k=1}^n f_k\Big| < \infty \quad \textit{almost surely.}$$

Proof. Let $A_n := \sqrt{R_n \log\log R_n}$. It is easy to check that (13) holds, so the second part of Theorem 4.6 applies. □

Remarks. 1. When $\{Y_n\} \subset L_\infty(\Omega, \mathbf{P})$ are centered independent, V. Egorov's LIL [**E**] is proved under a condition which relates the size of $\|Y_n\|_\infty$ to $\sum_{k=1}^n \|Y_k\|_2^2$, the variance of $S_n := \sum_{k=1}^n Y_k$. In our result the size of $|S_n|$ is measured in terms of B_n. The lim sup is almost surely constant since it is a tail random variable.

2. The assumption $\limsup \frac{B_{n+1}}{B_n} < \infty$ is equivalent to $\limsup \frac{\|Y_n\|_\infty^2}{B_n} < 1$.

Let $\{a_n\}$ be a sequence numbers and let $\{g_n\}$ be a sequence of centered random variables, with $\sup_{n\ge 1}|g_n| \le 1$. Define $\mathcal{F}_n = \sigma(g_1, \ldots, g_n)$ and $\mathcal{F}'_n = \sigma(a_1 g_1, \ldots, a_n g_n)$. Clearly, $\mathcal{F}'_k \subset \mathcal{F}_k$. Since, for every bounded random variable g we have $\|E[g|\mathcal{F}'_k]\|_\infty \le \|E[g|\mathcal{F}_k]\|_\infty$, we obtain the following estimation for R_n (of the sequence $\{f_n = a_n g_n\}$)

$$R_n = \sum_{k=1}^n \|a_k g_k\|_\infty^2 + \sum_{k=1}^n \max_{k\le s\le n} \|2 a_k g_k \sum_{v=k+1}^s \mathbf{E}[a_v g_v|\mathcal{F}'_k]\|_\infty \le$$

$$\sum_{k=1}^n |a_k|^2 + 2\sum_{k=1}^n |a_k| \sum_{v=k+1}^n |a_v| \|\mathbf{E}[g_v|\mathcal{F}_k]\|_\infty.$$

We can use this estimate in order to obtain the following two corollaries:

COROLLARY 4.9. *Let $\{a_n\}$ be a sequence of real numbers. Let $\{A_n\}$ be a non-decreasing sequence of positive numbers tending to infinity with* $\limsup_n \frac{A_{n+1}}{A_n} < \infty$. *If*

$$\lim_{n\to\infty} \frac{R_n \log\log A_n}{A_n^2} = 0\,,$$

then for every uniformly bounded *sequence of centered random variables $\{g_n\}$ we have* $\frac{1}{A_n}\sum_{k=1}^{n} a_n g_k \to 0$ *a.s.*

COROLLARY 4.10. *Let $\{a_n\}$ be a sequence of complex numbers. Let $\{A_n\}$ be a non-decreasing sequence of positive numbers, tending to infinity, such that* $\limsup_n A_{n+1}/A_n < \infty$. *If*

$$\lim_{n\to\infty} \frac{(|a_1|^2 + \cdots + |a_n|^2)\log\log A_n}{A_n^2} = 0\,,$$

then for every uniformly bounded *sequence of martingale differences $\{X_n\}$ we have* $\frac{1}{A_n}\sum_{k=1}^{n} a_k X_k \to 0$ *a.s.*

LEMMA 4.11. *Let $\{w_n\}$ be a sequence of non-negative numbers, put $W_n = \sum_{k=1}^n w_k$, and assume that $\sum_{k=1}^n w_k$ diverges. Then*

$$\lim_{n\to\infty} \frac{w_n \log\log W_n}{W_n} = 0 \implies \lim_{n\to\infty} \frac{(w_1^2 + \cdots + w_n^2)\log\log W_n}{W_n^2} = 0 \tag{18}$$

If $\{w_n\}$ is non-decreasing, also the converse implication holds.

PROOF. If $\{w_n\}$ is bounded, say by K, then both sides of (18) are bounded by $K\log\log W_n/W_n$, so both tend to 0 since $W_n \to \infty$.

To prove the implication $\Rightarrow$ when $\{w_n\}$ is unbounded, we follow the steps in the proof of Theorem 1.2.1 of Salem and Zygmund [**SZ**]. Define $w_n^* = \max_{1\le l\le n}\{w_l\}$, and let $1 \le k(n) \le n$ be an integer with $w_{k(n)} = w_n^*$. Since the function $t \mapsto t/\log\log t$ increases for $t > \mathrm{e}^2$, and since $\{W_n\}$ is non-decreasing and unbounded, the sequence $\{W_n/\log\log W_n\}$ is non-decreasing for n large enough. Hence for large n we have

$$\frac{w_n^* \log\log W_n}{W_n} =$$

$$\frac{w_{k(n)}\log\log W_{k(n)}}{W_{k(n)}} \cdot \frac{W_{k(n)}}{\log\log W_{k(n)}} \cdot \frac{\log\log W_n}{W_n} \le \frac{w_{k(n)}\log\log W_{k(n)}}{W_{k(n)}} \to 0,$$

as $n \to \infty$ (since unboundedness of $\{w_n\}$ implies $k(n) \to \infty$). Now,

$$\frac{\sum_{k=1}^n w_k^2}{W_n^2} \le \frac{w_n^* \sum_{k=1}^n w_k}{W_n^2} = \frac{w_n^*}{W_n},$$

and we obtain the implication $\Rightarrow$.

Now we assume that $\{w_n\}$ is non-decreasing. We first note that the assumed convergence to zero implies $w_n/W_n \to 0$. Since $W_n/W_{n+1} = (W_{n+1} - w_{n+1})/W_{n+1}$, we obtain that $W_{n+1}/W_n \to 1$. As in the proof of Theorem 4.6, for any n large enough, there exists $m = m(n) > n$, such that $W_n \le W_m \le 2W_n$ and $W_{m+1} > 2W_n$. Hence, by monotonicity of $\{w_n\}$, we have

$$\frac{w_n}{W_n} = \frac{w_n W_m}{W_n W_m} \le \frac{w_n W_n}{W_n W_m} + \frac{\sum_{k=n+1}^m w_k^2}{W_n W_m} \le \frac{w_n}{W_n} \cdot \frac{W_{m+1}}{2W_m} + \frac{2(\sum_{k=1}^m w_k^2)}{W_m^2}.$$

For large n we have $\frac{W_{n+1}}{W_n} < \frac{4}{3}$, so the above and $m > n$ imply

$$\frac{1}{3}\frac{w_n}{W_n} \log\log W_n \le \frac{2(\sum_{k=1}^m w_k^2)}{W_m^2} \log\log W_n \le \frac{2(\sum_{k=1}^m w_k^2)}{W_m^2} \log\log W_m.$$

Since $m = m(n) \to \infty$ as $n \to \infty$, we obtain the implication $\Leftarrow$. □

Remark. Either of the conditions in the above lemma yields $w_n/W_n \to 0$, which (as mentioned in the proof) is equivalent to $W_{n+1}/W_n \to 1$.

Corollary 4.12. *Let $\{w_n\}$ be a sequence of non-negative numbers, and put $W_n = \sum_{k=1}^n w_k$. Assume that $\sum_{k=1}^n w_k$ diverges. If*

$$\lim_{n\to\infty} \frac{(w_1^2 + \cdots + w_n^2)\log\log W_n}{W_n^2} = 0, \tag{19}$$

in particular if

$$\lim_{n\to\infty} \frac{w_n \log\log W_n}{W_n} = 0, \tag{20}$$

then for every sequence $\{X_n\}$ of uniformly bounded martingale differences, the weighted averages $\dfrac{1}{W_n}\displaystyle\sum_{k=1}^n w_k X_k$ converge a.s. to 0.

Remarks. 1. The convergence under the assumption (20) (Tsuchikura's condition) is Azuma's result [**Az**].

2. For a martingale differences sequence $\{X_n\}$ which is uniformly bounded, say by 1, and $Y_n = w_n X_n$, Corollary 4.3 yields

$$\mathbf{E}\Big[\Big|\sum_{k=j}^l w_k X_k\Big|^{2p}\Big] \le \frac{(2p)!}{p!2^p}\Big(\sum_{k=j}^l w_k^2\Big)^p.$$

With this inequality our corollary can be obtained from the proof of [**SZ**, Theorem 1.4.1] (see the remark in [**SZ**]) whenever (20) holds. However, it seems that the method of [**SZ**] does not yield the more general condition (19) (or in general Theorem 4.6).

3. Conditions (19) and (20) are optimal: if the left hand side of (20) is only bounded, then the weighted averages need not converge – see [**SZ**, Theorem 1.5.1] or [**Ts**]. Since $\{w_n\}$ of the example in [**Ts**] (due to Maruyama) is increasing, it satisfies $\sum_{k=1}^n w_k^2 \le w_n W_n$, and therefore the left hand side of (19) is also bounded.

4. When $\{w_n\}$ is bounded with divergent sum, both (19) and (20) hold, so the corollary applies. However, in this case more is known: Corollary 3.5 yields that the series $\sum_{n=1}^\infty \frac{w_n Y_n}{W_n}$ converges a.s. for every sequence of martingale differences $\{Y_n\}$ with $\sup_{n\ge1} \mathbf{E}[|Y_n|^p] < \infty$, $1 < p \le 2$; by Theorem 4 in [**JOP**], the averages $\frac{1}{W_n}\sum_{k=1}^n w_k X_k$ converge a.s. to 0 for every integrable centered i.i.d. $\{X_n\}$ with $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$.

5. Tsuchikura [**Ts**] proved that if $\{\frac{w_n}{W_n}\}$ is non-increasing and $\sum_n \left(\frac{w_n}{W_n}\right)^2 < \infty$, then (20) holds.

Example 4.1. *A sequence $\{w_n\}$ satisfying (20) to which our L_p-results do not apply.*

Let $w_1 = e$ and $w_n = e^{\sqrt{n}} - e^{\sqrt{n-1}} \sim \frac{e^{\sqrt{n}}}{2\sqrt{n}}$ for $n > 1$, so $W_n = e^{\sqrt{n}}$. Then $\frac{w_n \log\log W_n}{W_n} \sim \frac{\log n}{4\sqrt{n}} \to 0$. Since $\sum_{k=1}^{\infty} \left(\frac{w_k}{W_k}\right)^2 = \infty$, Theorem 3.2 yields the existence of $\{X_n\}$ centered independent with $\sup_n \mathbf{E}[|X_n|^2] < \infty$ for which the weighted averages $\frac{1}{W_n}\sum_{k=1}^n w_k X_k$ diverge a.s.

In this example the counting function $N(t) := \#\{n : W_n/w_n \le t\}$ satisfies $N(t) \sim \frac{1}{4}t^2$, so Corollary 2.6 (see Remark 2 after) yields that for every $1 < p < 2$ there is a centered i.i.d. sequence $\{X_n\}$ with $\mathbf{E}[|X_1|^p] < \infty$ for which the weighted averages diverge a.s. On the other hand, since in the example $\sup_n \frac{\sqrt{n} w_n}{W_n} < \infty$, the weighted averages converge a.s. for every centered i.i.d. $\{X_n\}$ with $\mathbf{E}[|X_1|^3] < \infty$, by [**LW**, Theorem 4.4].

Theorem 4.13. *Let $\{w_n\}$ be a sequence of non-negative numbers, and put $M_n = \sum_{k=1}^n w_k^2$. Assume that $\sum_{k=1}^n w_k$ diverges and $\limsup_{n\to\infty} w_n^2/M_n < 1$. If*

$$\lim_{n\to\infty} \frac{M_n \log\log M_n}{W_n^2} = 0, \tag{21}$$

then for every sequence $\{X_n\}$ of uniformly bounded martingale differences, the weighted averages $\dfrac{1}{W_n}\displaystyle\sum_{k=1}^n w_k X_k$ converge a.s. to 0.

Proof. If $\sum_{k=1}^{\infty} w_k^2 < \infty$, Chow's extension [**Ch**, Corollary 5] to martingale differences of the Khintchine-Kolmogorov theorem yields that $\sum_{k=1}^n w_k X_k$ converges a.s. as $n \to \infty$, and the result follows from the assumption $W_n \to \infty$. Hence we have to prove the theorem when $M_n \to \infty$.

Since $\dfrac{M_n}{M_{n+1}} = 1 - \dfrac{w_{n+1}^2}{M_{n+1}}$, the assumption $\limsup w_n^2/M_n < 1$ is equivalent to $\limsup M_{n+1}/M_n < \infty$.

For a sequence $\{X_n\}$ of martingale differences with $|X_n| \le c$ a.s. for every n, put $Y_n = w_n X_n$. Then

$$\frac{\sum_{k=1}^n \|Y_k\|_\infty^2 \log\log\sqrt{M_n \log\log M_n}}{M_n \log\log M_n} \le c^2 \frac{\log\log\sqrt{M_n\log\log M_n}}{\log\log M_n}$$

shows that (17) holds with $A_n = \sqrt{M_n \log\log M_n}$, so the second part of Corollary 4.7 yields

$$\limsup_{n\to\infty} \frac{1}{\sqrt{M_n\log\log M_n}} \Big|\sum_{k=1}^n w_k X_k\Big| < \infty \quad a.s. \tag{22}$$

Now (21) yields the result. □

Remarks. 1. Obviously $\sum_{k=1}^n w_k^2 \le (\sum_{k=1}^n w_k)^2$, so $M_n \le W_n^2$ and thus (19) implies (21). Condition (8), which yields more than the theorem, also implies (21).

2. When $\{w_n\}$ is non-decreasing (with $w_1 > 0$), we have $M_n \geq w_1 W_n$, which shows, together with Lemma 4.11, that for non-decreasing weights the three conditions (20), (19), and (21) are equivalent. In this case, Corollary 4.12 applies, and the additional requirement $\limsup w_n^2/M_n < 1$ of Theorem 4.13 is not needed.

3. The proof shows that when $\sum_{k=1}^\infty w_k^2 = \infty$ and $\limsup \frac{w_n^2}{M_n} < 1$, (22) holds for every uniformly bounded martingale differences sequence $\{X_n\}$. When $\inf_n \|X_n\|_\infty > 0$, this follows also from Corollary 4.8, with $Y_n = w_n X_n$.

4. When $\{X_n\}$ is a martingale difference sequence with $|X_n| \leq c$ a.s. for every n, such that $\inf_n Var(X_n) = \alpha > 0$, the variance of the weighted sum $V_n := Var(\sum_{k=1}^n w_k X_k)$ satisfies $\alpha M_n \leq V_n \leq c^2 M_n$. In this case (22), which requires only $\limsup \frac{w_n^2}{M_n} < 1$ but not (21), yields

$$\limsup_{n\to\infty} \frac{1}{\sqrt{V_n \log\log V_n}} \Big| \sum_{k=1}^n w_k X_k \Big| < \infty \quad a.s. \tag{23}$$

5. When $\{X_n\}$ are centered independent uniformly bounded random variables with $\inf_n Var(X_n) > 0$, Egorov's LIL [**E**] yields (23) if $M_n \to \infty$ and

$$\limsup_{n\to\infty} \frac{w_n^2 \log\log M_n}{M_n} < \infty. \tag{24}$$

For $\{w_n\}$ non-decreasing, this condition implies $\limsup\limits_{n\to\infty} \dfrac{M_n \log\log M_n}{W_n^2} < \infty$.

6. The a.s. convergence of $\frac{1}{W_n}\sum_{k=1}^n w_k Z_k$, for $\{Z_k\}$ i.i.d. centered Gaussian and $\{w_n\}$ with $\sum_{n=1}^\infty w_n^2 = \infty$ satisfying (21), was deduced in [**LW**] from Hartman's LIL [**H**] (without requiring $\limsup_{n\to\infty} w_n^2/M_n < 1$).

5. A SLLN for modulated i.i.d.

In this section we study the SLLN for modulated averages of i.i.d. random variables: find conditions on a sequence $\{c_n\}$ which ensure a.s. convergence of $\frac{1}{n}\sum_{k=1}^n c_n X_n$ for every centered i.i.d. sequence $\{X_n\}$ with finite expectation. We also consider the case where $\{c_n\}$ comes from a realization of a dynamical system.

The following proposition is a refinement of a computation inside the proof of Assani [**A1**, Theorem 3].

Proposition 5.1. *Let $\varphi(t)$ be a positive non-decreasing function, and suppose that that $\int_1^\infty \frac{dt}{t\varphi(t)}$ is finite. Let $\{w_n\}$ be a sequence of positive numbers, and assume that*

$$M := \sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^n w_k < \infty\,. \tag{25}$$

Then, for each $k \geq 1$ we have

$$\sum_{n=k}^\infty \frac{w_n}{n\varphi(n)} \leq 2M\Big(\frac{1}{\varphi(k)} + \int_k^\infty \frac{dt}{t\varphi(t)}\Big)\,.$$

PROOF. Put $M_0 = 0$ and $M_n = \frac{1}{n}\sum_{k=1}^n w_k$, for any $n \geq 1$. Now,

$$\sum_{n=k}^\infty \frac{w_n}{n\varphi(n)} = \sum_{n=k}^\infty \frac{nM_n - (n-1)M_{n-1}}{n\varphi(n)} = \sum_{n=k}^\infty \frac{M_n - M_{n-1}}{\varphi(n)} + \sum_{n=k}^\infty \frac{M_{n-1}}{n\varphi(n)} =$$

$$\frac{M_k}{\varphi(k+1)} - \frac{M_{k-1}}{\varphi(k)} + \sum_{n=k}^\infty M_n\Big(\frac{1}{\varphi(n)} - \frac{1}{\varphi(n+1)}\Big) + \sum_{n=k}^\infty \frac{M_{n-1}}{n\varphi(n)} \leq$$

$$\frac{M}{\varphi(k)} + \frac{M}{\varphi(k)} + 2M\int_k^\infty \frac{dt}{t\varphi(t)}\,.$$

□

EXAMPLE 5.1. (i) Let $(S, \mathcal{A}, \mu)$ be a finite measure space, and let θ be μ-measure preserving transformation on S. Let $p > 1$, and let $0 \leq g \in L_p(\mu)$. For $x \in S$, put $w_k = (g(\theta^k x))^p$. Clearly, by the individual ergodic theorem, for a.e. $x \in S$ condition (25) holds. By Proposition 5.1, with $\varphi(t) = t^{p-1}$, and the maximal ergodic theorem we obtain

$$\mu\Big(\sup_{k\geq 1} k^{p-1}\sum_{n=k}^\infty \Big(\frac{g(\theta^n x)}{n}\Big)^p > \lambda\Big) \leq \frac{C}{\lambda}\int g^p d\mu\,,$$

for some finite constant C which depends only on p. In particular, we obtain that $\sup_{k\geq 1} k^{p-1}\sum_{n=k}^\infty \Big(\frac{g(\theta^n x)}{n}\Big)^p$ is finite a.e., a result which was already established in Assani [**A1**, proof of Th. 3] (see also [**A2**, Proposition 7]).

(ii) Take $0 \leq g \in L_1(\mu)$, and for any $p > 1$ apply (i) above to $g^{1/p} \in L_p(\mu)$, to obtain

$$\mu\Big(\sup_{k\geq 1} k^{p-1}\sum_{n=k}^\infty \frac{g(\theta^n x)}{n^p} > \lambda\Big) \leq \frac{C_p}{\lambda}\int g\,d\mu\,, \quad \text{for all} \quad p > 1,$$

with finite constants C_p which depend only on p.

(iii) Take $\varphi(t) = (\log^+(t))^\gamma$, $\gamma > 1$. Take $g \geq 0$ with $\int g(\log^+ g)^\gamma d\mu$ finite, and for $x \in S$ put $w_k = g(\theta^k x)(\log^+[g(\theta^k x)])^\gamma$. We obtain

$$\mu\Big(\sup_{k\geq 2}(\log k)^\gamma\sum_{n=k}^\infty \frac{g(\theta^n x)(\log^+[g(\theta^n x)])^\gamma}{n(\log n)^\gamma} > \lambda\Big) \leq \frac{C}{\lambda}\int g(\log^+ g)^\gamma d\mu\,,$$

for some finite constant C which depends only on γ.

Remark. Let $\{g_n\}$ be a sequence of non-negative identically distributed random variables, defined on $(S, \mathcal{A}, \mu)$, with $\int g_1 d\mu < \infty$, in particular, $\{g_n\}$ a stationary sequence. Sawyer [**S**, Lemma 3] showed that for every $p > 1$, the series $\sum_{n=1}^\infty \big(\frac{g_n}{n}\big)^p$ converges a.e. However, it is not true that $\sup_{k\geq 1} k^{p-1}\sum_{n=k}^\infty \big(\frac{g_n}{n}\big)^p$ is finite a.e. for every integrable, non-negative, identically distributed sequence $\{g_n\}$. If this were true, we will get in turn that (see the proof of Theorem 3 in [**A1**])

$$\sup_{n\geq 1}\frac{\#\{k \geq 1 : \frac{g_k}{k} \geq \frac{1}{n}\}}{n} < \infty \qquad \text{a.e.}$$

This is impossible in presence of Theorem 1 in Assani, Buczolich, and Mauldin [**ABM**]. It is interesting to know if there is any rate of a.e. convergence to zero of $\sum_{n=k}^\infty \big(\frac{g_n}{n}\big)^p$ as k tends to infinity.

THEOREM 5.2. *Let $\{c_n\}$ be a sequence of numbers, which for some $\gamma > 1$ satisfies*

$$\sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n} |c_k|(\log^+|c_k|)^\gamma < \infty\,. \tag{26}$$

Then for every centered i.i.d. sequence $\{X_n\} \subset L_1(\mathbf{P})$ the following hold:

(*i*) $\displaystyle\frac{1}{n}\sum_{k=1}^{n} c_k X_k$ *converges a.s. to* 0.

(*ii*) *If $\{X_n\}$ are symmetric, then* $\displaystyle\sum_{n=1}^{\infty}\frac{c_nX_n}{n}$ *converges a.s.*

(*iii*) *If $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$, then* $\displaystyle\sum_{n=1}^{\infty}\frac{c_nX_n}{n}$ *converges a.s.*

PROOF. Since $\sum_{n=1}^\infty \mathbf{P}(|X_n| > n) \leq \mathbf{E}|X_1|$, it is enough to prove the theorem for the sequence $\{X_n\mathbf{1}_{\{|X_n|\leq n\}}\}$. In order to do this, we use Chung's theorem [**Chu**, Theorem 2(ii)]. In the computations below, $\lfloor x\rfloor$ denotes the greatest integer smaller than x.

We have,

$$\sum_{n=2}^{\infty}\frac{\mathbf{E}\big[|c_n||X_n|\mathbf{1}_{\{|X_n|\leq n\}}(\log^+[|c_n||X_n|\mathbf{1}_{\{|X_n|\leq n\}}])^\gamma\big]}{n(\log n)^\gamma} =$$

$$\sum_{n=2}^{\infty}\frac{\mathbf{E}\big[|c_n||X_1|\mathbf{1}_{\{|X_1|\leq n\}}(\log^+[|c_n||X_1|\mathbf{1}_{\{|X_1|\leq n\}}])^\gamma\big]}{n(\log n)^\gamma} \leq$$

$$2\mathbf{E}\Big[\mathbf{1}_{\{|X_1|\leq 2\}}\sum_{n=2}^{\infty}\frac{|c_n|(\log^+[2|c_n|])^\gamma}{n(\log^+ n)^\gamma}\Big] +$$

$$\mathbf{E}\Big[|X_1|\mathbf{1}_{\{|X_1|>2\}}\sum_{n=\lfloor|X_1|\rfloor}^{\infty}\frac{|c_n|(\log^+[|c_n||X_1|])^\gamma}{n(\log n)^\gamma}\Big] = A + B\,.$$

The first term on the right, A, is finite thanks to (26), by applying Proposition 5.1 with $w_n = |c_n|(\log^+|c_n|)^\gamma$ and $\varphi(x) = (\log^+ x)^\gamma$. We use the inequality $(\log^+[ab])^\gamma \leq 2^{\gamma-1}((\log^+ a)^\gamma + (\log^+ b)^\gamma)$, for $a, b \geq 0$, to split the second term, B, into two additional terms,

$$B \leq 2^{\gamma-1}\mathbf{E}\Big[|X_1|\mathbf{1}_{\{|X_1|>2\}}\sum_{n=\lfloor|X_1|\rfloor}^{\infty}\frac{|c_n|(\log^+[|c_n|])^\gamma}{n(\log n)^\gamma}\Big] +$$

$$2^{\gamma-1}\mathbf{E}\Big[|X_1|(\log^+[|X_1|])^\gamma\mathbf{1}_{\{|X_1|>2\}}\sum_{n=\lfloor|X_1|\rfloor}^{\infty}\frac{|c_n|}{n(\log n)^\gamma}\Big] = C + D\,.$$

The term C is finite since by Proposition 5.1 $\sum_{n=2}^\infty \frac{|c_n|(\log^+[|c_n|])^\gamma}{n(\log n)^\gamma}$ is finite.

Since condition (26) holds, it is evident that $M := \sup_{n\geq 1}\frac{1}{n}\sum_{k=1}^n |c_k| < \infty$. Now, apply Proposition 5.1 with $w_n = |c_n|$ and $\varphi(x) = (\log^+ x)^\gamma$, and noting that $\int_y^\infty \frac{dx}{x(\log x)^\gamma} \leq \beta/(\log y)^\gamma$, for some appropriate constant $\beta > 0$. We obtain that

$$D \leq 2^\gamma M\mathbf{E}\Big[\frac{|X_1|(\log^+[|X_1|])^\gamma\mathbf{1}_{\{|X_1|>2\}}}{(\log^+[|X_1|])^\gamma}\Big] + 2^\gamma M\beta\mathbf{E}\Big[\frac{|X_1|(\log^+[|X_1|])^\gamma\mathbf{1}_{\{|X_1|>2\}}}{(\log^+[|X_1|])^\gamma}\Big]\,.$$

Hence, by Chung's theorem, mentioned above, we obtain that the series

$$\sum_{n=1}^{\infty} \frac{c_n X_n \mathbf{1}_{\{|X_n|\le n\}} - \mathbf{E}\big[c_n X_n \mathbf{1}_{\{|X_n|\le n\}}\big]}{n}$$

converges a.s. If $\{X_n\}$ are symmetric, this proves (ii). In particular,

$$\frac{1}{n}\sum_{k=1}^{n}\{c_k X_k \mathbf{1}_{\{|X_k|\le k\}} - \mathbf{E}\big[c_k X_k \mathbf{1}_{\{|X_k|\le k\}}\big]\} \to 0 \qquad \text{a.s.}$$

Since $\mathbf{E}\big[X_n \mathbf{1}_{\{|X_n|\le n\}}\big] \to 0$ and $\sup_{n\ge 1}\frac{1}{n}\sum_{k=1}^n |c_k| < \infty$, by summability arguments (see, e.g., the proof of Theorem 2.9) we obtain that $\frac{1}{n}\sum_{k=1}^n c_k \mathbf{E}\big[X_k \mathbf{1}_{\{|X_k|\le k\}}\big]$ converges to zero, and this proves (i).

To prove (iii) we show that $\sum_{n=1}^\infty \frac{\mathbf{E}[c_n X_n \mathbf{1}_{\{X_n\le n\}}]}{n}$ converges absolutely. Since $\mathbf{E}[X_n]=0$, it is the same like showing that $\sum_{n=1}^\infty \frac{|\mathbf{E}[c_n X_n \mathbf{1}_{\{X_n>n\}}]|}{n}$ converges. Indeed, recall that $M=\sup_{n\ge 1}\frac{1}{n}\sum_{k=1}^n |c_k| < \infty$, we have by Abel summation by parts (at the second line below),

$$\sum_{n=1}^{\infty}\frac{\big|\mathbf{E}[c_n X_n \mathbf{1}_{\{X_n>n\}}]\big|}{n} \le \sum_{n=1}^{\infty}\frac{|c_n|\mathbf{E}[|X_1|\mathbf{1}_{\{X_1>n\}}]}{n} \le \mathbf{E}\Big[|X_1| \sum_{n=1}^{\lfloor |X_1|\rfloor+1}\frac{|c_n|}{n}\Big] \le$$

$$\mathbf{E}\Big[|X_1|\big(M + M\sum_{n=1}^{\lfloor |X_1|\rfloor+1}\frac{1}{n}\big)\Big] \le M\mathbf{E}\big[|X_1| + |X_1|\log\big[|X_1|+1\big]\big] < \infty\,.$$

□

Remarks. 1. Applying the above theorem to the sequence $c_n \equiv 1$, we obtain Theorem 6 in Marcinkiewicz and Zygmund [**MZ-1**].

2. The above theorem extends Theorem 3.4 in Baxter *et al.* [**BJLO**], where the convergence was proved under the assumption $\sup_{n\ge 1}\frac{1}{n}\sum_{k=1}^n |c_k|^q < \infty$, for some $q>1$. It also gives a partial answer to the problem addressed at the end of [**BJLO**].

3. The above theorem, with an application of Proposition 2.5 ($p=1$ and $b_n = n/c_n$), shows that for *any* sequence $\{c_n\}$ which satisfies condition (26) with $\gamma > 1$, we have $\limsup_{t\to\infty} \#\{n\ge 1 : |c_n|/n \ge 1/t\}/t < \infty$. In particular this holds for *any* sequence $\{c_n\}$ with $\sup_{n\ge 1}\frac{1}{n}\sum_{k=1}^n |c_k|^q < \infty$, for some $q>1$.

4. If we could obtain $\limsup_{t\to\infty} \#\{n\ge 1 : |c_n|/n \ge 1/t\}/t < \infty$ *directly* from (26), we could deduce the theorem from Theorem 2.9 with $A_n = n$, since (26) implies also $\sup_n \frac{1}{n}\sum_{k=1}^n |c_k| < \infty$, which is (6).

5. In the case that the $c_n's$ are a "typical" realization of a dynamical system, i.e. of the form $c_n = g(\theta^n x)$, Corollary 2.10 requires a weaker moment condition on g than needed for applying Theorem 5.2.

6. Strong consistency in linear regression with i.i.d. noise

In one-dimensional linear regression models $\xi_k = \beta c_k + X_k,\ k=1,2,\ldots$, the least square estimator (LSE) of β, based on the first n measurements, is defined by $\hat\beta_n = \dfrac{\sum_{k=1}^n \overline{c_k}\,\xi_k}{\sum_{k=1}^n |c_k|^2}$. A natural question is in what circumstances the error of estimation $\dfrac{\sum_{k=1}^n \overline{c_k}\, X_k}{\sum_{k=1}^n |c_k|^2}$ tends a.s. to 0 as n tends to infinity (*strong consistency* of the LSE). For the case of $\{X_n\}$ i.i.d. with finite variance, see Drygas [**Dr**].

We might use Theorem 2.9 or Theorem 3.2 in order to obtain strong consistency results for linear regression models. As a specific example we can apply Corollary 3.3 with $A_n = \sum_{k=1}^n |a_n|^2$. Note that such an application does not assume a condition like (6) if we assume symmetry.

In Theorem 6.1 below a different approach is used for attacking the problem addressed to in Theorem 2.9 (with $a_n = c_n$ and $A_n = \sum_{k=1}^n |c_n|^2$), this time using also existence of moments of higher orders. We denote by $\lfloor x \rfloor$ the greatest integer smaller than x.

THEOREM 6.1. *Let $\{X_n\} \subset L_p(\mathbf{P})$, $1 \le p \le 2$, be an i.i.d. sequence of symmetric random variables, and let $\{c_n\}$ be a sequence of complex numbers, with $c_1 \neq 0$. If*

$$\int \frac{|X_1|^2 \mathbf{1}_{\{|X_1| \ge 1\}}}{\sum\limits_{k=1}^{\lfloor |X_1|^p \rfloor} |c_k|^2} d\mathbf{P} < \infty \,, \tag{27}$$

then the series $\displaystyle\sum_{n=1}^{\infty} \frac{c_n X_n}{\sum_{k=1}^n |c_k|^2}$ *converges a.s.*

PROOF. Since $\sum_{n=1}^\infty \mathbf{P}(|X_n| > n^{1/p}) = \sum_{n=1}^\infty \mathbf{P}(|X_1|^p > n) < \infty$, it is enough, by Borel-Cantelli lemma, to prove the theorem for the *centered* independent sequence $\{X_n \mathbf{1}_{\{|X_n| \le n^{1/p}\}}\}$. Put $X = X_1$.

Now, by Khintchine-Kolmogorov it is enough to show that

$$\sum_{n=3}^{\infty} \frac{|c_n|^2 \mathbf{E}[|X_n|^2 \mathbf{1}_{\{|X_n| \le n^{1/p}\}}]}{(\sum_{k=1}^n |c_k|^2)^2} < \infty \,.$$

Denote by $\{S_N\}$ the sequence of partial sums of the above series. Using the identity $(a-b)/(ab) = 1/b - 1/a$, and using Abel's summation by parts and after that Fubini, we obtain

$$S_N \le \sum_{n=3}^{N} \frac{|c_n|^2 \mathbf{E}[|X_n|^2 \mathbf{1}_{\{|X_n| \le n^{1/p}\}}]}{\sum_{k=1}^n |c_k|^2 \sum_{k=1}^{n-1} |c_k|^2} =$$

$$\sum_{n=3}^{N} \mathbf{E}[|X|^2 \mathbf{1}_{\{|X| \le n^{1/p}\}}] \Big(\frac{1}{\sum_{k=1}^{n-1} |c_k|^2} - \frac{1}{\sum_{k=1}^{n} |c_k|^2} \Big) =$$

$$\frac{\mathbf{E}[|X|^2 \mathbf{1}_{\{|X| \le 2^{1/p}\}}]}{|c_1|^2 + |c_2|^2} - \frac{\mathbf{E}[|X|^2 \mathbf{1}_{\{|X| \le N^{1/p}\}}]}{\sum_{k=1}^{N} |c_k|^2} + \sum_{n=2}^{N-1} \frac{\mathbf{E}[|X|^2 \mathbf{1}_{\{n^{1/p} < |X| \le (n+1)^{1/p}\}}]}{\sum_{k=1}^{n} |c_k|^2} \le$$

$$\frac{\mathbf{E}[|X|^2 \mathbf{1}_{\{|X| \le 2^{1/p}\}}]}{|c_1|^2 + |c_2|^2} + \mathbf{E}\Big[\frac{|X|^2 \mathbf{1}_{\{|X| > 2^{1/p}\}}}{\sum_{k=1}^{\lfloor |X|^p \rfloor} |c_k|^2} \Big] \,.$$

Now our condition implies that $\{S_N\}$ is bounded. □

COROLLARY 6.2. *Let $\{c_k\}$ (with $c_1 \neq 0$) satisfy*

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |c_k|^2 > 0 \,. \tag{28}$$

Then for every integrable centered i.i.d. $\{X_n\}$ *we have* $\dfrac{\sum_{k=1}^n c_k X_k}{\sum_{k=1}^n |c_k|^2}$ *converges a.s. to* 0*. Furthermore, the series* $\displaystyle\sum_{n=1}^\infty \frac{c_n X_n}{\sum_{k=1}^n |c_k|^2}$ *converges a.s. when* X_1 *is symmetric or* $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$*.*

Proof. Condition (28) implies $\sum_n |c_n|^2 = \infty$, and also condition (27) (with $p = 1$) for any X_1 with $\mathbf{E}[|X_1|] < \infty$. If X_1 is symmetric, the desired convergence of the series holds by Theorem 6.1. Hence condition (ii) of Proposition 2.5 is satisfied, with $b_n = \sum_{k=1}^n |c_k|^2/c_n$ (we interpret division by 0 as ∞), which yields that $\limsup_{t\to\infty} N(t)/t < \infty$. Hence for any centered i.i.d. sequence with finite expectation $\dfrac{\sum_{k=1}^n c_k X_k}{\sum_{k=1}^n |c_k|^2}$ converges a.s. to 0 by Theorem 2.9, since (28) implies $\sup_n \left(\sum_{k=1}^n |c_k|\right)/\sum_{k=1}^n |c_k|^2 < \infty$. Theorem 2.2$(ii)$ yields the desired convergence of the series when $\mathbf{E}[|X_1|\log^+|X_1|] < \infty$. $\square$

Remark. For $c_n \equiv 1$, condition (28) holds, and we obtain (again) a.s. convergence $\sum_{n=1}^\infty \frac{X_n}{n}$ when X_1 satisfies one of the additional assumptions of the corollary – a result of Marcinkiewicz and Zygmund [**MZ-1**, Theorem 6]. If we remove the additional assumption on X_1, the a.s. convergence of the series $\sum_{n=1}^\infty \frac{X_n}{n}$ may fail (see [**MZ-1**, Theorem 6(a)]). Hence, for $p = 1$ (at least) the symmetry assumption in Theorem 6.1 cannot be dropped.

Corollary 6.3. *Let* $\{c_k\}$ *(with* $c_1 \neq 0$*) satisfy* (28)*. Then for* $p > 1$ *and every* L_p*-bounded martingale difference sequence* $\{Y_n\}$ *the series* $\displaystyle\sum_{n=1}^\infty \frac{c_n Y_n}{\sum_{k=1}^n |c_k|^2}$ *converges a.s.*

Proof. We saw in the previous corollary that $\limsup_{t\to\infty} N(t)/t < \infty$, so we can apply Theorem 3.2. $\square$

Remark. In [**CLT**] condition (28) was used to obtain L_2-consistency of the LSE for stationary noise with finite variance and atomless spectral measure. Here there is no stationarity assumption on the martingale difference sequence.

Theorem 6.4. *Let* $1 < p \le 2$*, and assume that* $\{c_k\}$ *(with* $c_1 \neq 0$*) satisfies*

$$\liminf_{n\to\infty} \frac{1}{n^{(2-p)/p}} \sum_{k=1}^n |c_k|^2 > 0\,. \tag{29}$$

Then for every centered i.i.d. $\{X_n\}$ *with* $\mathbf{E}[|X_1|^p] < \infty$*, the series* $\displaystyle\sum_{n=1}^\infty \frac{c_n X_n}{\sum_{k=1}^n |c_k|^2}$ *converges a.s.*

Proof. For $p = 2$ condition (29) is trivially satisfied since $c_1 \neq 0$, and the assertion follows from Drygas [**Dr**, Lemma 4.1] (even for $\{X_n\}$ which are not necessarily identically distributed – see Remark 3 below).

Let $1 < p < 2$. Condition (29) implies condition (27) for any X_1 with $\mathbf{E}[|X_1|^p] < \infty$, so, by Theorem 6.1, the assertion of the theorem holds for every $\{X_n\} \subset L_p(\mathbf{P})$ i.i.d. with X_1 symmetric.

By $(ii) \Rightarrow (iii)$ in Corollary 2.6 with $b_n = \sum_{k=1}^n |c_k|^2/c_n$ (we interpret division by 0 as ∞), the assertion holds for any centered i.i.d. $\{X_n\} \subset L_p(\mathbf{P})$. $\square$

Remarks. 1. Condition (29) with $p = 1$ is (28), but in its generality the theorem is false for $p = 1$.

2. Condition (29), for some $1 \le p < 2$, implies that $\sum_{n=1}^\infty |c_n|^2 = \infty$, so by Kronecker's lemma the "averages" $\frac{\sum_{k=1}^n c_k X_k}{\sum_{k=1}^n |c_k|^2}$ converge a.s. to zero. This convergence of the averages (also for $p = 1$) is reported in [**JC**],[**C**] to have been proved by Zhu.

3. If $X_1 \in L_2(\mathbf{P})$, then condition (27) holds whatever the sequence $\{c_n\}$ is. In fact, in the proof of [**Dr**, Lemma 4.1] Drygas proved that *the convergence of the series asserted in the theorem holds for any centered independent sequence* $\{X_n\}$ *with* $M = \sup_{n\ge1} \mathbf{E}[|X_n|^2] < \infty$, (no symmetry assumption needed), as

$$\sum_{n=2}^N \frac{|c_n|^2 \|X_n\|_2^2}{(\sum_{k=1}^n |c_k|^2)^2} \le M^2 \sum_{n=2}^N \frac{|c_n|^2}{\sum_{k=1}^n |c_k|^2 \sum_{k=1}^{n-1} |c_k|^2} \le$$

$$\sum_{n=2}^N \Big(\frac{1}{\sum_{k=1}^{n-1} |c_k|^2} - \frac{1}{\sum_{k=1}^n |c_k|^2} \Big) \le \frac{1}{|c_1|^2} - \frac{1}{\sum_{k=1}^N |c_k|^2}\,.$$

If we want to have a.s. convergence to zero of the averages, we can assume that $\sum_{n=1}^\infty |c_n|^2 = \infty$. It was shown in [**Dr**, Lemma 4.1] that if for *some* centered independent $\{X_n\}$, with $\inf_{n\ge1} \mathbf{E}[|X_n|^2] > 0$, the averages converge a.s. to zero, we *must* have $\sum_{n=1}^\infty |c_n|^2 = \infty$.

4. Let $\Phi(x)$ be a positive non-decreasing function, with $\mathbf{E}[|X_1|\Phi(|X_1|)] < \infty$. If $\liminf_{n\to\infty} \frac{\Phi(n)}{n} \sum_{k=1}^n |c_k|^2 > 0$, then (27) holds. E.g., if $\mathbf{E}[|X_1| \log^+ |X_1|] < \infty$ and $\liminf_{n\to\infty} \frac{\log n}{n} \sum_{k=1}^n |c_k|^2 > 0$, then the conclusion of Theorem 6.1 holds (with the symmetry assumption). By Proposition 2.5 and Remark 2 after it, we conclude that $\limsup_{t\to\infty} N(t)/t \log t < \infty$ (for $b_n = \sum_{k=1}^n |c_k|^2/c_n$). Now, by the discussion preceding Proposition 2.4, the series $\displaystyle\sum_{n=1}^\infty \frac{c_n X_n}{\sum_{k=1}^n |c_k|^2}$ converges a.s. for every integrable centered i.i.d. sequence (not necessarily symmetric), with $\mathbf{E}[|X_1|(\log^+ |X_1|)^2] < \infty$.

5. If (27) holds, but $X_1 \notin L_2(\mathbf{P})$, then $\sum_{n=1}^\infty |c_n|^2 = \infty$. Otherwise, we have

$$\int \frac{|X_1|^2 \mathbf{1}_{\{|X_1|\ge1\}}}{\sum_{k=1}^{\lfloor |X_1| \rfloor} |c_k|^2} d\mathbf{P} \ge \frac{\int |X_1|^2 \mathbf{1}_{\{|X_1|\ge1\}} d\mathbf{P}}{\sum_{n=1}^\infty |c_n|^2} = \infty\,.$$

6. If for some $1 \le p < 2$, condition (27) holds for every $X_1 \in L_p(\mathbf{P})$, in particular, if condition (29) holds, then by Theorem 6.1 and Proposition 2.5 we have $\limsup_{t\to\infty} N(t)/t^p < \infty$ (for $b_n = \sum_{k=1}^n |c_k|^2/c_n$).

Since for $\{X_n\} \in L_2(\mathbf{P})$ centered i.i.d. we always have a.s. convergence of $\sum_{n=1}^\infty \frac{c_n X_n}{\sum_{k=1}^n |c_k|^2}$, for any non-zero $\{c_n\}$, Proposition 2.5 yields $\limsup_{t\to\infty} N(t)/t^2 < \infty$.

LEMMA 6.5. *Given a sequence of numbers $\{c_n\}$, for any $1 \le p \le 2$ we have the following equivalence:*

$$\sup_{\|X\|_p \le 1} \int \frac{|X|^2 \mathbf{1}_{\{|X|\ge 1\}}}{\sum_{k=1}^{\lfloor |X|^p \rfloor} |c_k|^2} d\mathbf{P} < \infty \quad \iff \quad \liminf_{n\to\infty} \frac{1}{n^{(2-p)/p}} \sum_{k=1}^{n} |c_k|^2 > 0\,.$$

PROOF. The direction $\Leftarrow$ is clear. We will show the opposite direction. Define a sequence of functions $X^{(n)}$ in the unit ball of $L_p(\mathbf{P})$ by the following distribution: $\mathbf{P}(X^{(n)} = \pm n) = 1/(2n^p)$. Now,

$$\int \frac{|X^{(n)}|^2 \mathbf{1}_{\{|X^{(n)}|\ge 1\}}}{\sum_{k=1}^{\lfloor |X^{(n)}|^p \rfloor} |c_k|^2} d\mathbf{P} = \frac{2}{2n^p} \frac{n^2}{\sum_{k=1}^{n^p} |c_k|^2}\,.$$

Hence, by taking the suprema over $n \ge 1$ we obtain the direction of equivalence $\Rightarrow$. □

Remark. Assume that $\{c_n\}$ is such that for some $1 \le p \le 2$, condition (27) holds for every $X \in L_p(\mathbf{P})$ (not necessarily uniformly). Let $\Phi(n)$ be a positive non-decreasing sequence for which $\sum_{n=1}^{\infty} \frac{1}{n\Phi(n)}$ converges. Take X which is defined by the law $\mathbf{P}(X = \pm n) = \frac{1}{2\alpha n^{p+1}\Phi(n)}$, for $\alpha = \sum_{n=1}^{\infty} \frac{1}{n^{p+1}\Phi(n)}$. Since condition (27) holds for X, we have that

$$\int \frac{|X|^2 \mathbf{1}_{\{|X|\ge 1\}}}{\sum_{k=1}^{\lfloor |X|^p \rfloor} |c_k|^2} d\mathbf{P} = \sum_{n=1}^{\infty} \frac{1}{\alpha n^{p+1}\Phi(n)} \frac{n^2}{\sum_{k=1}^{n^p} |c_k|^2} = \sum_{n=1}^{\infty} \frac{1}{\alpha n^{p-1}\Phi(n) \sum_{k=1}^{n^p} |c_k|^2},$$

is finite. Hence by monotonicity, $\liminf_{n\to\infty} \frac{\Phi(n^{1/p})}{n^{(2-p)/p}} \sum_{k=1}^{n} |c_k|^2 > \frac{1}{2\alpha}$. In particular, for every $\gamma > 1$, we have $\liminf_{n\to\infty} \frac{(\log n)^\gamma}{n^{(2-p)/p}} \sum_{k=1}^{n} |c_k|^2 > 0$.

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