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LAWS OF LARGE NUMBERS WITH RATES AND THE ONE-SIDED ERGODIC HILBERT TRANSFORM

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ABSTRACT. Let T be a power-bounded operator on $L_p(\mu)$, $1 < p < \infty$. We use a sublinear growth condition on the norms $\{\| \sum_{k=1}^{n} T^k f\|_p\}$ to obtain for f the pointwise ergodic theorem with rate, as well as a.e. convergence of the one-sided ergodic Hilbert transform. For μ finite and T a positive contraction, we give a sufficient condition for the a.e. convergence of the "rotated one-sided Hilbert transform"; the result holds also for $p = 1$ when T is ergodic with $T1 = 1$.

Our methods apply to norm-bounded sequences in L_p. Combining them with results of Marcus and Pisier, we show that if $\{g_n\}$ is independent with zero expectation and uniformly bounded, then almost surely any realization $\{b_n\}$ has the property that for every $\gamma > 3/4$, any contraction T on $L_2(\mu)$ and $f \in L_2(\mu)$, the series $\sum_{k=1}^{\infty} b_k T^k f(x)/k^\gamma$ converges μ-almost everywhere. Furthermore, for every Dunford-Schwartz contraction of $L_1(\mu)$ of a probability space and $f \in L_p(\mu)$, $1 < p < \infty$, the series $\sum_{k=1}^{\infty} b_k T^k f(x)/k^\gamma$ converges a.e. for $\gamma \in (\max\{\frac{3}{4}, \frac{p+1}{2p}\}, 1]$.

1. Introduction

The mean ergodic theorem for power-bounded operators in reflexive Banach spaces yields that for $1 < p < \infty$ and T power-bounded in $L_p(\mu)$ of a σ-finite measure space, the *ergodic averages* $\frac{1}{n}\sum_{k=1}^{n} T^k f$ converge in norm for every $f \in L_p$.

The celebrated theorem of Akcoglu [A] says that if T is a *positive contraction* in $L_p(\mu)$, $1 < p < \infty$, then for every $f \in L_p$ the ergodic averages converge a.e. Without positivity, the a.e. convergence need not hold (see [Kr, p. 191]).

In general, there is no *universal* speed of convergence in the pointwise ergodic theorem for probability preserving transformations, not even for bounded functions; see [Kr, pp. 14–15], [Pe, §3.2B], [K, p. 655–657]. Thus, we need additional assumptions, connecting the function f and the operator T induced by the transformation, in order to obtain rates of convergence.

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On the other hand, for a centered i.i.d. sequence $\{f_k\} \subset L_p(\mu)$ of a probability space, $1 < p < 2$, Marcinkiewicz and Zygmund [MaZ, Theorem 9] (see also [ChTe, p. 115]) proved that we have a.s. convergence of the series $\sum_{k=1}^\infty \frac{f_k}{k^{1/p}}$, which implies $\frac{1}{n^{1/p}}\sum_{k=1}^n f_k \to 0$ a.s. Hence for T induced by the shift and $f_k = T^k f_0$ with zero integral the ergodic averages have a pointwise rate $o(n^{1/p-1})$. Thus, the rate in this case is determined only by a moment condition. An equivalent formulation of the above SLLN is that for every $\epsilon > 0$ we have $\mu(\bigcup_{k=n}^\infty \{|\frac{1}{k^{1/p}}\sum_{j=1}^k T^j f| \ge \epsilon\}) \to 0$. In this case, rates of convergence to 0 of $\mu(\bigcup_{k=n}^\infty \{|\frac{1}{k^\alpha}\sum_{j=1}^k T^j f| \ge \epsilon\})$ for $\alpha > 1/p$, in terms of convergent series, were obtained by Baum and Katz [BauKat], who also showed that their results are no longer true for general stationary sequences. However, Peligrad [P-4] showed that some of their results do hold for ϕ-mixing stationary sequences (for earlier results see [P-2], [P-3], [Ber]). Integral tests for convergence rates for martingales were obtained in [JJoSt], extending earlier results of Strassen [Str].

By adapting the proof of Lemma 5.2.1 of [Kr], we obtain that if T is power-bounded in L_p and $f = (I-T)g$ (which is equivalent to $\sup_{n>0}\|\sum_{k=1}^n T^k f\|_p < \infty$), then $\frac{1}{n^\gamma}\sum_{k=1}^n T^k f \to 0$ a.e. for every $\gamma > 1/p$; thus, the rate $\|\sum_{k=1}^n T^k f\|_p = O(1)$ yields a.e. convergence (with rate) of the ergodic averages.

For T induced by an *invertible* probability preserving transformation and $f \in L_2$, Gaposhkin [G-1] showed that if $\|\sum_{k=1}^n T^k f\|_2 = O(n^{1-\beta})$ for some $\beta > 0$, then $\frac{1}{n^\gamma}\sum_{k=1}^n T^k f \to 0$ a.e. for appropriate $\gamma < 1$ (depending only on β). In [G-2] he proved (under the same assumption) the a.e. convergence of the series $\sum_{n=1}^\infty \frac{T^n f}{n^\gamma}$, which implies a.e. convergence of the *one-sided ergodic Hilbert transform* $\sum_{n=1}^\infty \frac{T^n f}{n}$. Derriennic and Lin [DL] used the same growth condition for the L_p-norms of the sums to obtain similar results, even for T a Dunford-Schwartz operator.

In this paper we develop an intermediate class of results—modulated ergodic theorems with rates; we look for sequences $\{a_k\}$ for which there is a $\gamma < 1$ such that for every Dunford-Schwartz contraction T of $L_1(\mu)$ and every $f \in L_p$ (or for every contraction of L_2) we have $\frac{1}{n^\gamma}\sum_{k=1}^n a_k T^k f \to 0$ a.e., or even a.e. convergence of $\sum_{n=1}^\infty \frac{a_n T^n f}{n^\gamma}$.

In the next section we show that obtaining a strong law of large numbers with rate from the rate of convergence to 0 of the norms of the averages is a very general result, applicable to L_p norm bounded sequences $\{f_n\}$, which yields also a.e. convergence of the series $\sum_{k=1}^\infty \frac{f_k}{k}$. Section 3 deals with modulated ergodic theorems with rates and a.e. convergence of the modulated one-sided ergodic Hilbert transform, for general L_2-contractions and for contractions induced on L_p by Dunford-Schwartz operators. In Section 5 we look at sequences $\{a_n\}$ which yield a.e. convergence of series of the form

$\sum_{k=1}^{\infty} \frac{a_k g_k}{k}$ for any L_2-norm bounded orthogonal sequence $\{g_n\}$. In Section 6 we study the a.e. convergence of the one-sided rotated Hilbert transform for T a positive contraction of L_p, $1 < p < \infty$. Examples of i.i.d. lead to a study of almost sure uniform convergence of certain random Fourier series. In Section 7 we combine our results to show that almost surely realizations of uniformly bounded centered independent random variables are universally good sequences for a.e. convergence of the modulated one-sided ergodic Hilbert transform of L_p-contractions induced by Dunford-Schwartz operators.

2. Strong laws of large numbers with rates

In this section we obtain a strong law of large numbers with rate from the rate of convergence to 0 of the norms of the averages, and apply the result to obtain a.e. convergence of certain series; for power-bounded operators on L_p $(1 < p < \infty)$ this yields a.e. (and norm) convergence of the one-sided ergodic Hilbert transform.

Proposition 1. *Let $1 < p < \infty$, and let $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$ with $\sup_{n>0} \|f_n\|_p < \infty$. If for some $0 < \beta \le 1$ we have*

$$\sup_{n>0} \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} f_k \right\|_p = K < \infty, \tag{1}$$

then $\frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k \to 0$ a.e. for every $0 \le \delta < \frac{p-1}{p}\beta$; hence $\frac{1}{n} \sum_{k=1}^{n} f_k \to 0$ a.e. Furthermore, for any $0 \le \delta < \frac{p-1}{p}\beta$ we have $\sup_n |\frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k| \in L_p$.

Proof. Let $r = 1/\beta$ and fix δ with $0 \le \delta < \beta(p-1)/p$. Then we have

$$(1 - r\delta)p = (\beta - \delta)rp > 1. \tag{i}$$

Define $n_m = [m^r] + 1$. By (1) we have

$$\left\| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right\|_p \le \frac{K}{n_m^{\beta-\delta}} \le \frac{K}{m^{r(\beta-\delta)}},$$

so

$$\int \sum_{m=1}^{\infty} \left| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right|^p d\mu = \sum_{m=1}^{\infty} \left\| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right\|_p^p \le K^p \sum_{m=1}^{\infty} \frac{1}{m^{pr(\beta-\delta)}},$$

which converges by (i). Hence

$$\sum_{m=1}^{\infty} \left| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right|^p < \infty \quad \text{a.e.},$$

so

$$\left| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right| \to 0 \quad \text{a.e.}$$

For $n_m \le n < n_{m+1}$ we have

$$\left|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}}\sum_{k=1}^{n_m} f_k\right| = \left|\frac{1}{n^{1-\delta}}\sum_{k=n_m+1}^{n} f_k\right|$$
$$\le \frac{1}{n^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} |f_k| \le \frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} |f_k|.$$

This yields, with $C := \sup_n \|f_n\|_p$,

$$\int \max_{n_m \le n < n_{m+1}} \left|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}}\sum_{k=1}^{n_m} f_k\right|^p d\mu$$
$$\le \int \left[\frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} |f_k|\right]^p d\mu = \left\|\frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} |f_k|\right\|_p^p$$
$$\le \left[\frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} \|f_k\|_p\right]^p \le C^p\left(\frac{n_{m+1}-n_m}{n_m^{1-\delta}}\right)^p.$$

Since for $r \ge 1$ and $t \ge 0$ we have $(t+2)^r \ge (t+1)^r + 1$ and $(t+2)^r - t^r \le 2r(t+2)^{r-1}$, we obtain

$$\frac{n_{m+1}-n_m}{n_m^{1-\delta}} \le \frac{(m+1)^r + 1 - m^r}{(m^r)^{1-\delta}} \le \frac{2r(m+2)^{r-1}}{m^{r(1-\delta)}} = 2r\left(\frac{m+2}{m}\right)^{r-1}\frac{1}{m^{1-r\delta}}.$$

Hence

$$\int \max_{n_m \le n < n_{m+1}} \left|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}}\sum_{k=1}^{n_m} f_k\right|^p d\mu$$
$$\le C^p(2r)^p\left(\frac{m+2}{m}\right)^{p(r-1)}\frac{1}{m^{(1-r\delta)p}}.$$

Since $(1-r\delta)p > 1$, we conclude as before that

$$\max_{n_m \le n < n_{m+1}} \left|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}}\sum_{k=1}^{n_m} f_k\right|^p \to 0 \quad \text{a.e.}$$

Since

$$\left|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n_m} f_k\right| \le \left|\frac{1}{n_m^{1-\delta}}\sum_{k=1}^{n_m} f_k\right| \to 0 \quad \text{a.e.},$$

the convergence part of the proposition is proved.

Put $S_n = \sum_{k=1}^{n} f_k$. For r and $\{n_m\}$ as above and fixed $\delta < \frac{p-1}{p}\beta$, we obtain as before

$$\int \sup_{m>0}\left|\frac{1}{n_m^{1-\delta}}S_{n_m}\right|^p d\mu \le \int \sum_{m=1}^{\infty}\left|\frac{1}{n_m^{1-\delta}}S_{n_m}\right|^p d\mu \le K^p \sum_{m=1}^{\infty}\frac{1}{m^{rp(\beta-\delta)}} < \infty,$$

so $\sup_{m>0}\left|\frac{1}{n_m^{1-\delta}}S_{n_m}\right| \in L_p$. For $n_m \le n < n_{m+1}$ we have

$$\left|\frac{S_n}{n^{1-\delta}}\right| \le \left|\frac{S_{n_m}}{n_m^{1-\delta}}\right| + \frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}}|f_k|,$$

so

$$\left\|\sup_n\left|\frac{S_n}{n^{1-\delta}}\right|\right\|_p \le \left\|\sup_{m>0}\left|\frac{S_{n_m}}{n_m^{1-\delta}}\right|\right\|_p + \left\|\sup_{m>0}\frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}}|f_k|\right\|_p.$$

The finiteness of the last term follows from

$$\begin{aligned}\left\|\sup_{m>0}\frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}}|f_k|\right\|_p^p &\le \int\sum_{m=1}^{\infty}\left[\frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}}|f_k|\right]^p d\mu \\ &\le C^p\sum_{m=1}^{\infty}\left(\frac{n_{m+1}-n_m}{n_m^{1-\delta}}\right)^p,\end{aligned}$$

with the last series converging by the previous estimates (since $(1-r\delta)p > 1$). $\square$

Example 1. *$\{f_n\}$ bounded in $L_1(\mu)$ satisfying (1), with $\frac{1}{n}\sum_{k=1}^n f_k$ diverging a.e.*

Let T be the positive contraction of $L_1(\mu)$ given by Chacon's example (see [Kr, p. 151]), for which there is a non-negative $0 \not\equiv f \in L_1$ with $\limsup_n \frac{1}{n}T^n f = \infty$ a.e. Let $f_n := T^{n-1}(I-T)f$. Then $\|\sum_{k=1}^n f_k\|_1 \le 2\|f\|_1$, so for any $0 < \beta \le 1$ (1) is satisfied, while $\frac{1}{n}\sum_{k=1}^n f_k = \frac{1}{n}(f - T^n f)$ is a.e. nonconvergent. This shows that the final conclusion of Proposition 1 fails when $p = 1$.

Remarks. (1) Let T be power-bounded on $L_p(\mu)$, $1 < p < \infty$ (so T is a contraction in an equivalent norm). For $0 < \beta < 1$, the power series expansion $(1-t)^\beta = 1 - \sum_{j=1}^\infty a_j^{(\beta)}t^j$ is used in [DL] to define the operator $(I-T)^\beta$, and it is shown there that $\overline{(I-T)^\beta L_p} = \overline{(I-T)L_p}$. When $(I-T)L_p$ is not closed, the linear manifolds $\{(I-T)^\beta L_p : 0 < \beta \le 1\}$ are all different, and decrease when β increases. Theorem 2.15 of [DL] yields that for every $f \in (I-T)^\beta L_p$, (1) is satisfied by $f_k = T^k f$, i.e.,

$$\sup_{n>0}\left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^n T^k f\right\|_p = K < \infty, \tag{1'}$$

and Theorem 2.17 there shows that (1′) implies that $f \in (I-T)^\delta L_p$ for every $0 < \delta < \beta$. Example 1 shows that for $p = 1$ and T a positive contraction, (1′) does not yield a.e. convergence of $\frac{1}{n}\sum_{k=1}^n T^k f$.

(2) If T is as above, and for some $\beta > 1$ (1′) holds, then $\|\sum_{k=1}^n T^k f\|$ converges to 0, and applying $I - T$ to the sums we obtain $Tf = 0$.

(3) The a.e. convergence to 0 of $\frac{1}{n}\sum_{k=1}^n T^k f$ under ($1'$) in the special case of T unitary on L_2, due to Loève [Lo-1] (in the continuous parameter case), is proved in Doob [Do, p. 492]. The rates of a.e. convergence obtained by Gaposhkin [G-1, Theorem 3] for this particular case are better than what Proposition 1 yields.

(4) For more precise information on the rate of a.e. convergence when T is induced on L_p ($p > 1$) by a Dunford-Schwartz operator (a contraction of L_1 which contracts also the L_∞-norm), see [DL], Theorem 3.2 (and also Corollary 3.7); Remark 1 following Theorem 3.1 of [DL] shows that for Dunford-Schwartz operators, ($1'$) in L_1-norm does not yield a rate in the ergodic theorem.

(5) Any sequence $\{f_n\}$ of i.i.d. random variables with zero expectation and finite variance satisfies (1) with $\beta = 1/2$.

Example 2. Let $\{f_n\} \subset L_2(\mu)$ be a mutually orthogonal sequence with $\sup_n \|f_n\|_2 < \infty$ (e.g., an L_2-bounded martingale difference sequence in a probability space). By orthogonality

$$\left\|\frac{1}{n}\sum_{k=1}^n f_k\right\|_2^2 = \frac{1}{n^2}\sum_{k=1}^n \|f_k\|_2^2 \le \frac{\sup_j \|f_j\|_2^2}{n}.$$

Hence $\{f_n\}$ satisfies (1) with $\beta = 1/2$, and therefore for every $0 \le \delta < 1/4$, $\frac{1}{n^{1-\delta}}\sum_{k=1}^n f_k \to 0$ a.e.

In Example 2 we may assume μ to be a probability (see [Kr, p. 189]), since an isometry of L_2 preserves the inner product, hence the orthogonality. The Menchoff-Rademacher theorem [Do, p. 157], [Z, vol. II, p. 193] then implies that $\sum_{n=1}^\infty \frac{f_n}{n^{1/2+\epsilon}}$ converges a.e. for every $\epsilon > 0$. Using Kronecker's lemma we thus obtain better rates of convergence than those of Proposition 1.

Cotlar [Co] (see also [Pe, §3.6]) proved that for T induced by an invertible probability preserving transformation, the *ergodic Hilbert transform* $Hf := \lim_{n\to\infty}\sum_{0<|k|\le n}\frac{T^k f}{k}$ converges a.e. for every $f \in L_1$. Jajte [Ja] proved that for T unitary on L_2, a.e. convergence of the ergodic averages for every $f \in L_2$ is equivalent to a.e. convergence for every $f \in L_2$ of the ergodic Hilbert transform (norm convergence of the ergodic Hilbert transform holds for every unitary operator [C]). For $1 < p < \infty$, Berkson, Bourgain and Gillespie [BBGi] extended Jajte's result to T invertible on (a closed subspace of) L_p with $\sup_{-\infty<n<\infty}\|T^n\| < \infty$; when T is also positive, this and De la Torre's theorem [De] yield a.e convergence of the ergodic Hilbert transform for every $f \in L_p$ (a result originally due to Sato [S-1], see also [S-2], [S-3]).

The Khinchine-Kolmogorov theorem for series of independent random variables (e.g., [Do, p. 108]) yields that for $\{f_n\}$ i.i.d. with zero expectation and finite variance $\sum_{k=1}^\infty \frac{f_k}{k}$ converges a.e.; moreover, for every $\gamma > 1/2$ the series

$\sum_{k=1}^{\infty} \frac{f_k}{k^\gamma}$ converges a.s., which yields a rate $\frac{1}{n^\gamma}\sum_{k=1}^{n} f_k \to 0$ in the SLLN. However, in general for T unitary on L_2 induced by a probability preserving transformation the *one-sided ergodic Hilbert transform* $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{T^k f}{k}$ need not always exist, neither in norm [H] nor a.e. [Pe, p. 94] (see also [DelR]). Theorems 2.17 and 2.11 of [DL] show that if (1′) is satisfied, then $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{T^k f}{k^{1-\delta}}$ exists in norm for every $0<\delta<\beta$, and hence also the one-sided ergodic Hilbert transform converges in norm.

THEOREM 1. *Let $1<p<\infty$, and let $\{f_n\}_{n=1}^{\infty}\subset L_p(\mu)$ with $\sup_{n>0}\|f_n\|_p<\infty$. If $\{f_n\}$ satisfies* (1) *for some $0<\beta\le 1$, then for every $0\le\delta<\frac{p-1}{p}\beta$, the series $\sum_{k=1}^{\infty}\frac{f_k}{k^{1-\delta}}$ converges a.e. and $\sup_{n>0}\left|\sum_{k=1}^{n}\frac{f_k}{k^{1-\delta}}\right|\in L_p$. Convergence of the series in L_p-norm holds for any $0\le\delta<\beta$.*

Proof. We can and do assume that μ is a probability measure (e.g., [Kr, p. 189]). For $\delta<\frac{p-1}{p}\beta$ denote $\gamma=1-\delta$. Put $S_0=0$ and $S_k=\sum_{j=1}^{k} f_j$. Abel's summation by parts yields the decomposition

$$\sum_{k=1}^{n}\frac{f_k}{k^\gamma}=\sum_{k=1}^{n}\frac{S_k-S_{k-1}}{k^\gamma}=\frac{S_n}{n^\gamma}+\sum_{k=1}^{n-1}\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)S_k.$$

By Proposition 1, $\frac{1}{n^\gamma}S_n\to 0$ a.e. For the series we have

$$\sum_{k=1}^{n}\left|\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)S_k\right|\le\gamma\sum_{k=1}^{n}\frac{1}{k^{1+\gamma}}|S_k|=\gamma\sum_{k=1}^{n}\frac{1}{k^{\beta+\gamma}}\left|\frac{1}{k^{1-\beta}}S_k\right|.$$

Since μ is a probability and $\gamma+\beta=1-\delta+\beta>1$, we obtain from (1)

$$\int\sum_{k=1}^{\infty}\frac{1}{k^{\beta+\gamma}}\left|\frac{1}{k^{1-\beta}}S_k\right|d\mu\le\sum_{k=1}^{\infty}\frac{1}{k^{\beta+\gamma}}\left\|\frac{1}{k^{1-\beta}}S_k\right\|_p<\infty.$$

Hence $\sum_{k=1}^{\infty}\frac{1}{k^{1+\gamma}}|S_k|<\infty$ a.e., which completes the proof of the a.e. convergence. For the maximal function, we have

$$\sup_{n>1}\left|\sum_{k=1}^{n}\frac{f_k}{k^\gamma}\right|\le\sup_{n>1}\left|\frac{S_n}{n^\gamma}\right|+\sup_{n>1}\left|\sum_{k=1}^{n-1}\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)S_k\right|.$$

By Proposition 1 and the previous estimates for the last term, we obtain

$$\left\|\sup_{n>1}\left|\sum_{k=1}^{n}\frac{f_k}{k^\gamma}\right|\right\|_p\le\left\|\sup_{n>1}\left|\frac{S_n}{n^\gamma}\right|\right\|_p+\left\|\gamma\sum_{k=1}^{\infty}\frac{1}{k^{\beta+\gamma}}\left|\frac{1}{k^{1-\beta}}S_k\right|\right\|_p<\infty.$$

The L_p-norm convergence holds in fact for any $\gamma > 1-\beta$: (1) implies $\left\|\frac{1}{n^\gamma}S_n\right\|_p \to 0$, so $\sum_{k=1}^n \frac{f_k}{k^\gamma}$ is Cauchy in L_p since

$$\left\|\sum_{k=j}^{n-1}\left|\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)S_k\right|\right\|_p \le \gamma\sum_{k=j}^{n-1}\frac{1}{k^{1+\gamma}}\|S_k\|_p$$

$$\le \gamma\sum_{k=j}^{\infty}\frac{1}{k^{\beta+\gamma}}\left\|\frac{1}{k^{1-\beta}}S_k\right\|_p. \qquad \square$$

Remarks. (1) Note that formally Proposition 1 follows from Theorem 1, by Kronecker's lemma, but the proposition is *used* in the proof of the theorem.

(2) When $p=1$, (1) yields a.e. convergence of $\sum_{k=1}^\infty \frac{f_k(x)}{k}$ if we *know* that $\frac{1}{n}\sum_{k=1}^n f_k$ converges a.e. (take $\gamma=1$ in the proof of Theorem 1).

(3) Fix $1<p\le 2$, and let $\{f_n\}\subset L_p(\mu)$ of a probability space be an L_p-bounded martingale difference sequence, with $\sup_n\|f_n\|_p = C<\infty$. Theorem 2 of [BaE] yields $\|\sum_{k=1}^n f_k\|_p \le 2^{1/p}Cn^{1/p}$, so (1) holds with $\beta = (p-1)/p$. In the special case of $\{f_n\}$ independent (with 0 expectations), the result can be deduced also from Theorem 13 of [MaZ] (see [ChTe, p. 356]); in this case Theorem 5' in [MaZ] (for a more general form, due to Loève [Lo-2] and based on the three series theorem, see [ChTe, p. 114]) implies that for every $0\le\delta<(p-1)/p$ the series $\sum_{k=1}^\infty \frac{f_k}{k^{1-\delta}}$ converges a.e., which is better (i.e., giving larger values of δ) than what Theorem 1 yields.

(4) Peligrad [P-1, Lemma 3.4] showed that if $\{f_n\}$ is an L_2-bounded centered ρ-mixing sequence with $\sum_i \rho(2^i)<\infty$, then (1) holds with $\beta = 1/2$. Hence Theorem 1 applies.

Corollary 1. *Let T be a power-bounded operator on $L_p(\mu)$, $1<p<\infty$. If $f\in L_p$ satisfies $(1')$ for $0<\beta\le 1$, then $\sum_{k=1}^\infty \frac{T^kf}{k^{1-\delta}}$ converges a.e. for every $0\le\delta<\frac{p-1}{p}\beta$ (and in L_p-norm for $0\le\delta<\beta$). For $0\le\delta<\frac{p-1}{p}\beta$ we also have $\sup_{n>0}\left|\sum_{k=1}^n\frac{T^kf}{k^{1-\delta}}\right|\in L_p$.*

Remarks. (1) The corollary improves considerably Theorem 3.12 of [DL].

(2) See Gaposhkin [G-2] for more precise information when T is unitary on L_2. For T a Dunford-Schwartz operator in L_p (in particular, T induced by a probability preserving transformation), see [DL, Theorem 3.6].

Modulated ergodic theorems are concerned with the convergence (a.e. or in norm) of $\frac{1}{n}\sum_{k=1}^n a_kT^kf$ for certain sequences $\{a_k\}$. We refer the reader to [LOT], where earlier references are given. Weighted strong laws of large numbers for i.i.d. sequences were studied by Jamison, Orey, and Pruitt [JOP].

COROLLARY 2. *Let $1 < p < \infty$, and $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$ with $\sup_{n>0} \|f_n\|_p < \infty$. Let $\{a_n\}$ be a bounded sequence, such that for some $0 < \beta \le 1$ we have*

$$\sup_{n>0} \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} a_k f_k \right\|_p = K < \infty. \tag{2}$$

Then for any $0 \le \delta < \frac{p-1}{p}\beta$ the series $\sum_{k=1}^{\infty} \frac{a_k f_k}{k^{1-\delta}}$ converges a.e. and $\sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k f_k}{k^{1-\delta}} \right| \in L_p$. Convergence of the series in L_p-norm holds for any $0 \le \delta < \beta$.

Proof. By (2), the sequence $f'_n = a_n f_n$ satisfies (1), so Theorem 1 applies. □

EXAMPLE 3. Let $a_{j^2} = 1$, and $a_k = 0$ if k is not a square. Then for every norm-bounded sequence in L_p, (1) holds with $\beta = 1/2$ and $K = \sup_n \|f_n\|_p$. Note that the sequence is supported on a set of density 0.

3. Modulated ergodic Hilbert transforms for Dunford-Schwartz operators

In this section we look at conditions on a modulating sequence $\{a_n\}$ which will yield a.e. convergence of the modulated one-sided ergodic Hilbert transform for every L_2 contraction and every $f \in L_2$. An interpolation yields a similar result for T induced on L_p $(1 < p \le 2)$ by a Dunford-Schwartz operator.

PROPOSITION 2. *Let $\{n_k\}$ be a non-decreasing sequence of positive integers and let $\{a_n\}$ be a bounded sequence of complex numbers such that for some $0 < \beta \le 1$ we have*

$$\sup_{n>0} \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} a_k \lambda^{n_k} \right| = K < \infty. \tag{3}$$

(i) *For every contraction T in $L_2(\mu)$ and $f \in L_2(\mu)$, the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^{1-\delta}}$ converges a.e. for any $0 \le \delta < \beta/2$, and in L_2-norm for $0 \le \delta < \beta$. Furthermore, $\sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k T^{n_k} f}{k^{1-\delta}} \right| \in L_2$ for any $0 \le \delta < \beta/2$.*

(ii) *For every Dunford-Schwartz operator T on $L_1(\mu)$ and $f \in L_p(\mu)$, $1 < p \le 2$, the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^{1-\delta}}$ converges a.e. for any $0 \le \delta < \frac{p-1}{p}\beta$, and in L_p-norm for $0 \le \delta < 2\frac{p-1}{p}\beta$. Furthermore, $\sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k T^{n_k} f}{k^{1-\delta}} \right| \in L_p$ for any $0 \le \delta < \frac{p-1}{p}\beta$.*

(iii) *In the case $n_k = k$, for every Dunford-Schwartz operator T on $L_1(\mu)$ and $f \in L_1(\mu)$, we have $\frac{1}{n} \sum_{k=1}^{n} a_k T^k f \to 0$ a.e., and in L_1-norm if μ is finite.*

Proof. (i) Theorem 2.1 of [BLRT] and the unitary dilation theorem yield that for any contraction T in a Hilbert space

$$\sup_{n>0}\left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^{n} a_k T^{n_k}\right\| \le K < \infty,$$

(for a different proof see [RiN, §153]). If T is a contraction of $L_2(\mu)$, and $f \in L_2$, then (2) holds with $f_k = T^{n_k}f$ and constant $K\|f\|_2$. Hence Corollary 2 yields that for every $0 \le \delta < \beta/2$ the series $\sum_{k=1}^{\infty}\frac{a_k T^{n_k} f}{k^{1-\delta}}$ converges a.e. with $\sup_{n>0}\left|\sum_{k=1}^{n}\frac{a_k T^{n_k} f}{k^{1-\delta}}\right| \in L_2$, and the series converges in L_2-norm for $0 \le \delta < \beta$. Inspection of the proofs of Proposition 1 and Theorem 1 yields an estimate on the norm of the maximal function in terms of $\sup_k \|f_k\|_p$ and the constant K there, which for $p = 2$ yields that there is a constant C, depending only on β and δ, such that $\left\|\sup_{n>0}\left|\sum_{k=1}^{n}\frac{a_k T^{n_k} f}{k^{1-\delta}}\right|\right\|_2 \le CK\|f\|_2$, with K here given in (3).

(ii) Let $\phi_n(\zeta) := \sum_{k=1}^{n} a_k \zeta^{n_k}$. By the maximum principle and (3), we have $|\phi_n(\zeta)| \le Kn^{1-\beta}$ for $|\zeta| \le 1$. Hence for every contraction T on a Hilbert space $\phi_n(T) = \sum_{k=1}^{n} a_k T^{n_k}$ satisfies $\|\phi_n(T)\| \le Kn^{1-\beta}$, by Theorem A in [RiN, §153] (for T unitary this inequality follows also from the spectral theorem, as in [BLRT], and the dilation theorem yields it for any contraction T). Now fix a Dunford-Schwartz operator T on $L_1(\mu)$, and put $T_n = \sum_{k=1}^{n} a_k T^{n_k}$. Then $\|T_n\|_2 \le Kn^{1-\beta}$, and obviously $\|T_n\|_1 \le n\|\{a_k\}\|_\infty$. The Riesz-Thorin interpolation theorem [Z, vol. II, p. 95] yields that for $1 < p < 2$ we have

$$\|T_n\|_p \le \|\{a_k\}\|_\infty^{2/p-1} K^{2-2/p} n^{1-\beta_p},$$

with $\beta_p := 2\beta\big(1 - \frac{1}{p}\big) > 0$. Thus, for $f \in L_p(\mu)$ (2) holds for $f_k = T^{n_k}f$ and β_p (with $K_p := \|\{a_k\}\|_\infty^{2/p-1}K^{2-2/p}$). Now Corollary 2 yields the L_p-norm convergence of the series for $0 \le \delta < \beta_p$, and the a.e. convergence for $\delta < \frac{p-1}{p}\beta_p = 2\beta(\frac{p-1}{p})^2$.

In order to improve the rate in the a.e. convergence (i.e., to allow larger values of δ), we will change the interpolation method, and following [R] we will use Stein's complex interpolation [Z, Theorem XII.1.39]. Since the condition on δ is satisfied also when β is replaced by $\beta' < \beta$ close enough to β, and also (3) will obviously hold for β', we may assume $\beta < 1$.

Claim. *If $\{a_k\}$ satisfies* (3), *then for any real η the sequence $\{a_k k^{i\eta}\}$ satisfies* (3).

With ϕ_n as above, Abel's summation by parts yields, uniformly in $|\lambda| = 1$,

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k k^{i\eta}\lambda^{n_k}\right| &\le |n^{i\eta}\phi_n(\lambda)| + \left|\sum_{k=1}^{n-1}[k^{i\eta} - (k+1)^{i\eta}]\phi_k(\lambda)\right| \\ &\le |\phi_n(\lambda)| + \sum_{k=1}^{n}|\eta|\frac{1}{k}|\phi_k(\lambda)| \\ &\le Kn^{1-\beta} + |\eta|\sum_{k=1}^{n-1}\frac{Kk^{1-\beta}}{k} \\ &\le K\left(1 + \frac{|\eta|}{1-\beta}\right)n^{1-\beta},\end{aligned}$$

which shows that (3) is satisfied, as claimed, with K replaced by $K(1+\frac{|\eta|}{1-\beta})$.

We now fix a Dunford-Schwartz operator T. Part (i) and the claim yield that for fixed $\alpha < \beta/2$ and $f \in L_2$, we have

$$\left\|\sup_{n>0}\left|\sum_{k=1}^{n}\frac{a_k k^{-i\eta\beta/2}T^{n_k}f}{k^{1-\alpha}}\right|\right\|_2 \le CK\left(1 + \frac{\beta|\eta|}{2(1-\beta)}\right)\|f\|_2$$

for every real η. For $\zeta = \xi + i\eta$ in the strip $B := 0 \le \operatorname{Re}\zeta \le 1$ we look at the operator $\Psi_{n,\zeta} := \sum_{k=1}^{n}\frac{a_k k^{-\zeta\beta/2}}{k^{1-\alpha}}T^{n_k}$, so we have

$$\|\sup_n|\Psi_{n,i\eta}f|\|_2 \le CK\left(1 + \frac{\beta|\eta|}{2(1-\beta)}\right)\|f\|_2.$$

For $\zeta = 1 + i\eta$ we have

$$\sup_{n>0}|\Psi_{n,1+i\eta}f| \le \sum_{k=1}^{\infty}\frac{|a_k|k^{-\beta/2}|T^{n_k}f|}{k^{1-\alpha}};$$

the theorem of Beppo Levi and $\alpha < \beta/2$ yield

$$\begin{aligned}\left\|\sup_{n>0}|\Psi_{n,1+i\eta}f|\right\|_1 &\le \sum_{k=1}^{\infty}\frac{|a_k|k^{-\beta/2}\|T^{n_k}f\|_1}{k^{1-\alpha}} \\ &\le \|\{a_k\}\|_\infty\|f\|_1\sum_{k=1}^{\infty}\frac{1}{k^{1-\alpha+\beta/2}} < \infty.\end{aligned}$$

For a bounded measurable positive integer-valued function I and ζ with $0 \le \operatorname{Re}\zeta \le 1$ we define the linear operator

$$\Psi_{I,\zeta}f(x) := \sum_{k=1}^{I(x)}\frac{a_k k^{-\zeta\beta/2}}{k^{1-\alpha}}T^{n_k}f(x) = \sum_{j=1}^{\max I}1_{\{I=j\}}(x)\sum_{k=1}^{j}\frac{a_k k^{-\zeta\beta/2}}{k^{1-\alpha}}T^{n_k}f(x),$$

which is defined on all the L_p spaces. It is easily checked that for any two integrable simple functions f and g, the function

$$\Phi(\zeta) = \int \Psi_{I,\zeta} f \cdot g d\mu = \sum_{j=1}^{\max I} \sum_{k=1}^{j} \int g(x) 1_{\{I=j\}}(x) \frac{a_k k^{-\zeta\beta/2}}{k^{1-\alpha}} T^{n_k} f(x) d\mu,$$

is continuous and bounded in the strip B and analytic in its interior. Clearly

$$\|\Psi_{I,i\eta} f\|_2 \leq \|\sup_n |\Psi_{n,i\eta} f|\|_2 \leq CK \left(1 + \frac{\beta|\eta|}{2(1-\beta)}\right) \|f\|_2 = M_1(\eta)\|f\|_2,$$

and

$$\|\Psi_{I,1+i\eta} f\|_1 \leq \|\sup_n |\Psi_{n,1+i\eta} f|\|_1 \leq C_1 \|f\|_1.$$

For $1 < p < 2$ let $t = \frac{2}{p} - 1$, so $\frac{1}{p} = (1-t) \cdot \frac{1}{2} + t \cdot 1$. Stein's interpolation theorem now yields that there exists a constant A_t, which depends only on t, M_1, and C_1, such that for every $f \in L_p$ we have

$$\begin{aligned}\left\| \sum_{k=1}^{I(x)} \frac{a_k}{k^{1-\alpha+(\frac{2}{p}-1)\beta/2}} T^{n_k} f(x) \right\|_p &= \left\| \sum_{k=1}^{I(x)} \frac{a_k k^{-t\beta/2}}{k^{1-\alpha}} T^{n_k} f(x) \right\|_p \\ &= \|\Psi_{I,t} f\|_p \leq A_t \|f\|_p.\end{aligned}$$

For an integer $N \geq 2$ let $I_N(x) = j$ for j the first integer with

$$\left| \sum_{k=1}^{j} \frac{a_k k^{-t\beta/2}}{k^{1-\alpha}} T^{n_k} f(x) \right| = \max_{1 \leq n \leq N} \left| \sum_{k=1}^{n} \frac{a_k k^{-t\beta/2}}{k^{1-\alpha}} T^{n_k} f(x) \right|.$$

Then for $f \in L_p$ (and our fixed $\alpha < \beta/2$) we have

$$\begin{aligned}&\left\| \max_{1 \leq n \leq N} \left| \sum_{k=1}^{n} \frac{a_k}{k^{1-\alpha+(\frac{2}{p}-1)\beta/2}} T^{n_k} f(x) \right| \right\|_p \\ &\qquad = \left\| \sum_{k=1}^{I_N(x)} \frac{a_k}{k^{1-\alpha+(\frac{2}{p}-1)\beta/2}} T^{n_k} f(x) \right\|_p \leq A_t \|f\|_p,\end{aligned}$$

and letting $N \to \infty$ we conclude that for $\gamma = 1 - \alpha + (\frac{2}{p} - 1)\beta/2 > 1 - \frac{p-1}{p}\beta$ we have

$$\left\| \sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k}{k^\gamma} T^{n_k} f(x) \right| \right\|_p < \infty.$$

Fix $1 < p < 2$ and $\delta < \frac{p-1}{p}\beta$, and put $\gamma := 1 - \delta$. Since $\gamma > 1 - \frac{p-1}{p}\beta$, we have $\sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k}{k^\gamma} T^{n_k} f(x) \right| \in L_p$ for every $f \in L_p$; part (i) yields a.e. convergence of the series $\sum_{k=1}^{\infty} \frac{a_k}{k^\gamma} T^{n_k} f(x)$ for every $f \in L_2$, so the Banach principle now yields the same a.e. convergence for any $f \in L_p$.

(iii) We now assume $n_k = k$. By (i) the claimed a.e. convergence holds for L_2-functions. The a.e. convergence to 0 for all L_1 functions follows from the Banach principle, since for every $f \in L_1(\mu)$ we have

$$\sup_n \left| \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right| \le \|\{a_j\}\|_\infty \sup_n \frac{1}{n} \sum_{k=1}^{n} \tau^k |f| < \infty,$$

by the pointwise ergodic theorem for τ, the linear modulus of T. When μ is finite we may assume it is a probability, so the L_1-norm convergence to 0 for L_2 functions follows from (i), and boundedness of $\{a_k\}$ yields the norm convergence for all L_1 functions. □

REMARKS. (1) Stein's theorem yields the L_p-norm convergence in (ii) for a smaller interval of δ than what we obtain from the Riesz-Thorin theorem, so both interpolations are needed.

(2) The assertions of Proposition 2 for a fixed sequence $\{n_k\}$ are true under the following weaker condition:

$$(3') \qquad \sup_{n>0} \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta'}} \sum_{k=1}^{n} a_k \lambda^{n_k} \right| < \infty, \qquad 0 < \beta' < \beta,$$

which is equivalent to

$$\lim_{n\to\infty} \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta'}} \sum_{k=1}^{n} a_k \lambda^{n_k} \right| = 0, \qquad 0 < \beta' < \beta.$$

The sequence defined by $a_n = \frac{\log n}{\sqrt{n}}$ satisfies $\sum_{k=1}^{n} a_k = O(\sqrt{n} \log n)$, so for any $\{n_k\}$ condition (3′) is satisfied with $\beta = 1/2$, while (3) is not.

(3) Theorem 2.1 of [BLRT] shows that if for *every* contraction T in $L_2(\mu)$ and *every* $f \in L_2(\mu)$, the sequence $\left\{\frac{1}{n^{1-\delta}} \sum_{k=1}^{n} a_k T^k f\right\}$ is bounded in L_2-norm for each $0 \le \delta < \beta$, then (3′) holds for $n_k = k$.

(4) The sequence $\{n_k\}$ need not really be monotone, but this will be the case in most applications. The terms need not be distinct.

PROPOSITION 3. *Let $\{n_k\}$ be a non-decreasing sequence of positive integers, and let $\{a_n\}$ be a sequence of complex numbers satisfying* (3) *for some $0 < \beta \le 1$ (no boundedness is assumed), and let $0 \le \delta < \beta$. Then for every contraction T on a Hilbert space, the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k}}{k^{1-\delta}}$ converges in operator norm, and this convergence is uniform in all contractions. In particular, the Fourier series $\sum_{k=1}^{\infty} \frac{a_k \lambda^{n_k}}{k^{1-\delta}}$ converges uniformly in $|\lambda| = 1$.*

Proof. For a contraction T on a Hilbert space, denote $s_n(T) = \sum_{k=1}^{n} a_k T^{n_k}$. The spectral theorem for unitary operators and the unitary dilation theorem yield $\|s_n(T)\| \le K n^{1-\beta}$, with the constant K, given by (3), independent of

T. Put $\gamma = 1 - \delta$. Then

$$\sum_{k=1}^{n} \frac{a_k T^{n_k}}{k^\gamma} = \sum_{k=1}^{n} \frac{s_k(T) - s_{k-1}(T)}{k^\gamma}$$
$$= \frac{s_n(T)}{n^\gamma} + \sum_{k=1}^{n-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma} \right) s_k(T).$$

By the above discussion, $\|\frac{1}{n^\gamma} s_n(T)\| \le \frac{K}{n^{\beta-\delta}}$, so we have uniform convergence to 0. For the sum on the right hand side, we have

$$\left\| \sum_{k=j}^{n-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma} \right) s_k(T) \right\| \le \gamma \sum_{k=j}^{n-1} \frac{1}{k^{1+\gamma}} \|s_k(T)\| \le \gamma \sum_{k=j}^{\infty} \frac{K}{k^{\beta+\gamma}},$$

which shows that the series is Cauchy in operator norm, uniformly in T. $\square$

Remarks. (1) When $\sup_n \frac{1}{n^{1-\beta}} \sum_{k=1}^n |a_k| < \infty$, condition (3) is obviously satisfied for every $\{n_k\}$. A simple example of $\{a_n\}$ unbounded satisfying (3) (with $\beta = 1/4$) is given by $a_{j^2} = \sqrt{j}$ and $a_k = 0$ for k not a square.

(2) If $\{a_n\}$ is bounded and satisfies (3) with a given $\{n_k\}$, then the proof of Proposition 3, combined with the proof of Proposition 2(ii), yields that for fixed $1 < p \le 2$ and $0 \le \delta < 2\beta(1 - \frac{1}{p})$, *the series* $\sum_{k=1}^\infty \frac{a_k T^{n_k}}{k^{1-\delta}}$ *converges in* L_p *operator norm for every Dunford-Schwartz operator* T*, and this convergence is uniform in all Dunford-Schwartz operators.*

Example 4. Let $\{\epsilon_n\}$ be the Rudin-Shapiro sequence [Ru] (see also [Ka-2, p. 75]): $\epsilon_n = \pm 1$, and for some K we have

$$\max_{|\lambda|=1} \left| \sum_{j=1}^{n} \epsilon_j \lambda^j \right| \le K\sqrt{n}.$$

Propositions 2 and 3 now apply with $\beta = 1/2$; for example, if T is a contraction of $L_2(\mu)$, and $f \in L_2$, then $\sum_{k=1}^\infty \frac{\epsilon_k T^k f}{k^{1-\delta}}$ converges a.e. for every $0 \le \delta < 1/4$.

Remarks. (1) For $n_k = k$, condition (3) is satisfied also by the Hardy-Littlewood sequence $\{e^{icn\log n}\}$ (with $\beta = 1/2$) [Z, vol. I, p. 199], and by the sequence $\{e^{in^\alpha}\}$ with $0 < \alpha < 1$ (when $\beta = \alpha/2$) [Z, vol. I, p. 200]. The convergence results for L_2 contractions, obtained in these cases from Propositions 2 and 3, are Theorem 14 of [R] (without the uniformity in all contractions of the operator norm convergence; the uniform convergence of the Fourier series for these sequences is proved already in [Z]). Adapting the methods of [Z, §§V.4-V.5], we can show that the sequence $\{e^{in^\alpha}\}$ with $1 < \alpha < 2$ satisfies (3) for $\beta = 1 - \alpha/2$, and our results include those of Remark 15 of [R].

(2) Examples of $\{a_n\}$ satisfying (3) for $n_k = k^2$ will be given later.

4. Additional examples for modulating sequences

The following lemma shows how to obtain additional examples for (2). Note that it applies also in the case $p = 1$.

LEMMA 1. *Let* $1 \leq p < \infty$, *and* $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$ *with* $\sup_{n>0} \|f_n\|_p < \infty$. *If* $\{f_n\}$ *satisfies* (1), *and* $\{a_k\}$ *satisfies*

$$\sum_{k=1}^{\infty} |a_k - a_{k+1}| < \infty, \tag{4}$$

then (2) *is satisfied.*

Proof. Since $a_n = a_1 + \sum_{k=1}^{n-1}(a_{k+1} - a_k)$, the sequence $\{a_n\}$ converges. With $S_0 = 0$ and $S_k = \sum_{j=1}^{k} f_j$, we obtain

$$\sum_{k=1}^{n} a_k f_k = \sum_{k=1}^{n} a_k (S_k - S_{k-1}) = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n.$$

Using (1), we obtain

$$\left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} a_k f_k \right\| \leq K \sum_{k=1}^{\infty} |a_k - a_{k+1}| + K \sup_j |a_j|$$

for every n. □

COROLLARY 3. *Let* $1 < p < \infty$, *and* $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$ *with* $\sup_{n>0} \|f_n\|_p < \infty$, *and let* $\{a_n\}$ *satisfy* (4). *If* $\{f_n\}$ *satisfies* (1) *for some* $0 < \beta \leq 1$, *then* $\sum_{k=1}^{\infty} \frac{a_k f_k}{k^{1-\delta}}$ *converges a.e. for every* $0 \leq \delta < \frac{p-1}{p}\beta$.

EXAMPLE 5. *T positive, f satisfies $(1')$, $\{a_k\}$ convergent, but* $\sum_{k=1}^{n} \frac{a_k T^k f}{k}$ *a.e. divergent.*

Let θ be a probability preserving ergodic invertible transformation on (Ω, μ) and $Tg = g \circ \theta$. Then T is a positive invertible isometry of $L_p(\mu)$, $1 \leq p < \infty$. We assume that there is $0 \neq f \in L_\infty$ such that $Tf = -f$ (e.g., $\Omega = [0, 2)$, τ an invertible measure preserving ergodic transformation of $[0, 1)$; define $\theta x = \tau x + 1$ for $0 \leq x < 1$ and $\theta x = \tau(x - 1)$ for $1 \leq x < 2$, and take $f = 1_{[0,1)} - 1_{[1,2)}$). Clearly $(1')$ is satisfied for any $p \geq 1$ and any $\beta \in (0, 1]$, but for the sequence $a_k = \frac{(-1)^k}{\log k}$ we have that $\sum_{k=1}^{n} \frac{a_k T^k f}{k} = \sum_{k=1}^{n} \frac{1}{k \log k} f$ is a.e. divergent. This example shows also that for $\lambda = -1$ the series $\sum_{k=1}^{n} \frac{\lambda^k T^k f}{k}$ is a.e. divergent.

THEOREM 2. *Let T be a contraction on $L_1(\mu)$ with mean ergodic modulus, and let $\{a_k\}$ satisfy* (4). *If $f \in L_1$ satisfies $(1')$ for some $0 < \beta \leq 1$, then* $\sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ *converges a.e.*

Proof. With $S_kf = \sum_{j=1}^k T^jf$, we have $\|S_kf/k\| \to 0$ by ($1'$), and the pointwise ergodic theorem for T [ÇL] yields $\frac{S_kf}{k} \to 0$ a.e. Defining $S_0f = 0$, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^n a_kT^kf &= \frac{1}{n}\sum_{k=1}^n a_k(S_kf - S_{k-1}f)\\ &= \frac{1}{n}\sum_{k=1}^{n-1}(a_k - a_{k+1})S_kf + a_n\frac{1}{n}S_nf.\end{aligned}$$

Since $\{a_n\}$ is bounded, the last term tends to 0 a.e. For $\epsilon > 0$ fix N such that $\sum_{k=N}^\infty |a_k - a_{k+1}| < \epsilon$. Since $\sup_n |\frac{S_nf(x)}{n}| < \infty$ a.e., the inequalities

$$\left|\frac{1}{n}\sum_{k=1}^{n-1}(a_k - a_{k+1})S_kf\right| \le \left|\frac{1}{n}\sum_{k=1}^{N-1}(a_k - a_{k+1})S_kf\right| + \sum_{k=N}^{n-1}|a_k - a_{k+1}|\left|\frac{S_kf}{k}\right|$$

yield

$$\limsup_n \left|\frac{1}{n}\sum_{k=1}^{n-1}(a_k - a_{k+1})S_kf\right| \le \epsilon\sup_n\left|\frac{S_nf(x)}{n}\right|.$$

Hence $\frac{1}{n}\sum_{k=1}^n a_kT^kf(x) \to 0$ a.e.

Since (2) holds by Lemma 1, we can use the proof of Theorem 1 for $\gamma = 1$, with $S_k = \sum_{j=1}^k a_jT^jf$, to obtain our theorem. □

The following was suggested by D. Çömez (for the case $f_k = T^kf$):

Let $\{f_n\} \subset L_p(\mu)$, $1 \le p < \infty$, with $\sup_n \|f_n\|_p < \infty$, and assume $\sum_{k=1}^\infty \frac{|a_k|}{k} < \infty$. Then the series $\sum_{k=1}^\infty \frac{a_kf_k}{k}$ is a.e. absolutely convergent.

Proof. We may and do assume that μ is a probability. Then the assertion follows from

$$\int\sum_{k=1}^\infty \frac{|a_kf_k|}{k}d\mu = \sum_{k=1}^\infty \frac{|a_k|\,\|f_k\|_1}{k} \le \sum_{k=1}^\infty \frac{|a_k|\,\|f_k\|_p}{k} < \infty.$$

Note that $\sum_{k=1}^\infty \frac{|a_k|}{k}\|f_k\|_p < \infty$, so we also have norm convergence. □

Remarks. (1) For the sequence $a_k = 1$, (4) holds; Corollary 1 and Theorem 2 show convergence of the one-sided Hilbert transform when ($1'$) is satisfied, although $\sum_{k=1}^\infty \frac{|a_k|}{k} = \infty$.

(2) Let $a_k = \frac{(-1)^k}{k}$. Then $\sum_{k=1}^\infty \frac{|a_k|}{k} < \infty$, but $\sum_{k=1}^\infty |a_k - a_{k+1}| = \infty$. Thus (4) is not necessary for a.e. convergence of $\sum_{k=1}^\infty \frac{a_kT^kf}{k}$ for every power-bounded T and f satisfying ($1'$).

Theorem 3. *Fix $\beta \le 1$, and let $\{a_k\}$ be a bounded sequence with $\sum_{k=1}^\infty \frac{|a_k-a_{k+1}|}{k^\beta} < \infty$. Let $\{f_n\} \subset L_p(\mu)$, $1<p<\infty$, with $\sup_n \|f_n\|_p < \infty$, and assume that $\{f_n\}$ satisfies (1). Then the series $\sum_{k=1}^\infty \frac{a_k f_k}{k}$ converges a.e.*

Proof. With $S_0 = 0$ and $S_k = \sum_{j=1}^k f_j$, we clearly have

$$\sum_{k=1}^n \frac{a_k f_k}{k} = \sum_{k=1}^{n-1} \left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right) S_k + \frac{a_n S_n}{n}.$$

The last term tends to 0 a.e., since $\{a_n\}$ is bounded, and $\frac{1}{k}S_k \to 0$ a.e. by Proposition 1.

The sum $\sum_{k=1}^n \left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right)S_k$ is a.e. absolutely convergent, since using (1) we obtain

$$\begin{aligned}\int \sum_{k=1}^n \left|\left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right) S_k\right| d\mu &\le \sum_{k=1}^n \left|\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right| \|S_k\|_p \\ &\le \sum_{k=1}^n \frac{|a_k - a_{k+1}|}{k+1}\|S_k\|_p + \sum_{k=1}^n \frac{|a_k|}{k(k+1)}\|S_k\|_p \\ &\le \sum_{k=1}^n \frac{|a_k - a_{k+1}|}{k^\beta} K + \sum_{k=1}^n \frac{|a_k|}{k^{1+\beta}} K. \qquad \square\end{aligned}$$

Theorem 4. *Fix $\beta \le 1$, and let $\{a_k\}$ be a bounded sequence with $\sum_{k=1}^\infty \frac{|a_k-a_{k+1}|}{k^\beta} < \infty$. Then for every T power-bounded in $L_p(\mu)$, $1<p<\infty$, or a contraction with mean ergodic modulus in L_1, and every f satisfying (1′), the series $\sum_{k=1}^\infty \frac{a_k T^k f}{k}$ converges a.e.*

Proof. We may and do assume that μ is a probability. For the power bounded case (with $p>1$) we apply Theorem 3 to the sequence $\{T^n f\}$. For the L_1-contraction case, we have $\frac{1}{k}S_k f \to 0$ a.e. by [ÇL], since (1′) implies $\|\frac{1}{k}S_k f\| \to 0$. The result now follows from the calculation in the proof of Theorem 3, this time with $p=1$. $\square$

Remark. Theorems 3 and 4 do not follow from the previous results. If we define $a_k = 1$ for k not a power of 2, and $a_{2^j} = -1$, then obviously (4) fails, and also $\sum_{k=1}^\infty \frac{|a_k|}{k} = \infty$. However, for any $\beta > 0$ we have

$$\sum_{k=1}^\infty \frac{|a_k - a_{k+1}|}{k^\beta} \le 2 + \sum_{j=1}^\infty \frac{4}{(2^j-1)^\beta} < \infty.$$

Note that *if $\{a_k\}$ is a (complex) sequence such that $\sum_{k=1}^\infty \frac{a_k T^k f}{k}$ converges a.e. (or in norm) for every T power-bounded on L_p and $f \in L_p$ satisfying (1′) (for some $\beta > 0$), then $\sum_{k=1}^\infty \frac{a_k}{k}\lambda^k$ converges for every complex $\lambda \ne 1$*

with $|\lambda| = 1$. To see this, note that for such λ there is an ergodic probability preserving transformation θ on $[0, 1)$ with a bounded function $f \neq 0$ such that $Tf = \lambda f$ (for λ a root of unity, proceed as in Example 5, for other λ let $\theta z = \lambda z$ on the unit circle). Then f satisfies ($1'$), so $\sum_{k=1}^{\infty} \frac{a_k \lambda^k}{k} f = \sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ converges.

5. Series of modulated L_2-bounded orthogonal sequences

LEMMA 2. *Given* $1 \leq p \leq 2$, *let* $\{a_n\}$ *satisfy*

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} |a_k|^p = A < \infty. \tag{5}$$

Then for every $\epsilon > 0$ *we have*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2/p+\epsilon}} < \infty$, *and*

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{|a_n|^p}{n^{1+\epsilon}} < \infty$.

Proof. Denote $S_n^{(p)} := \sum_{k=1}^{n} |a_n|^p$, and define similarly $S_n^{(2)}$. Summation by parts yields

$$\sum_{k=1}^{n} \frac{|a_k|^2}{k^{2/p+\epsilon}} = \sum_{k=1}^{n-1} \left(\frac{1}{k^{2/p+\epsilon}} - \frac{1}{(k+1)^{2/p+\epsilon}} \right) S_k^{(2)} + \frac{S_n^{(2)}}{n^{2/p+\epsilon}}.$$

Since $1 \leq p \leq 2$, we have $(S_n^{(2)})^{1/2} \leq (S_n^{(p)})^{1/p}$ (e.g., [HLP, p. 4]). Hence

$$\begin{aligned}
\sum_{k=1}^{n} \frac{|a_k|^2}{k^{2/p+\epsilon}} &\leq \sum_{k=1}^{n-1} \left| \frac{1}{k^{2/p+\epsilon}} - \frac{1}{(k+1)^{2/p+\epsilon}} \right| (S_k^{(p)})^{2/p} + \frac{(S_n^{(p)})^{2/p}}{n^{2/p+\epsilon}} \\
&\leq \left(\frac{2}{p} + \epsilon \right) \sum_{k=1}^{n-1} \frac{(S_k^{(p)})^{2/p}}{k^{2/p+1+\epsilon}} + \frac{(S_n^{(p)})^{2/p}}{n^{2/p+\epsilon}} \\
&\leq \left(\frac{2}{p} + \epsilon \right) A^{2/p} \sum_{k=1}^{n-1} \frac{1}{k^{1+\epsilon}} + \frac{A^{2/p}}{n^{\epsilon}},
\end{aligned}$$

which yields (i). Similar computations yield that if $\{c_k\}$ is a non-negative sequence with $\sup_n \frac{1}{n} \sum_{k=1}^{n} c_k < \infty$, then $\sum_{k=1}^{\infty} \frac{c_k}{k^{1+\epsilon}} < \infty$ for every $\epsilon > 0$ (see also [As-2, pp. 228–229]). When applied to $\{|a_k|^p\}$, this yields (ii). $\square$

THEOREM 5. *Let* $\{a_n\}$ *be a sequence of complex numbers satisfying* (5) *with* $1 < p \leq 2$, *and let* $\{g_n\}$ *be an orthogonal sequence in* $L_2(\Omega, \mu)$, *with* $\sup_n \|g_n\|_2 = K < \infty$. *Then for every* $\epsilon > 0$ *the series* $\sum_{n=1}^{\infty} \frac{a_n g_n}{n^{1/p+\epsilon}}$ *converges a.e. and in* L_2, *with* $\int \left[\sup_{n>0} \left|\sum_{k=1}^{n} \frac{a_k g_k}{k^{1/p+\epsilon}}\right|\right]^2 d\mu < \infty$. *Thus* $\sum_{n=1}^{\infty} \frac{a_n g_n}{n}$

converges a.e. (and in L_2). If, in addition, $\{g_n\}$ is uniformly bounded (i.e., $\sup_n \sup_{x\in\Omega} |g_n(x)| < \infty$), then $\sum_{n=1}^{\infty} \frac{a_n g_n}{n^{1/p+\epsilon}}$ is in $L_q(\mu)$ with $q = p/(p-1)$, and $\sum_{n=1}^{\infty} \frac{a_n g_n}{n} \in \bigcap_{q\le s<\infty} L_s(\mu)$.

Proof. For the first part we may assume, as mentioned before, that μ is a probability. By Lemma 2(i),

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 \|g_n\|_2^2}{n^{2/p+2\epsilon}} \log^2 n \le K^2 \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2/p+\epsilon}} \frac{\log^2 n}{n^{\epsilon}} < \infty.$$

Now, the L_2 convergence is immediate, and the a.e. convergence follows by applying the Menchoff-Rademacher theorem to the sequence $\{\frac{a_n g_n}{n^{1/p+\epsilon}}\}$. For the maximal function we will use the inequality given in [Z, XIII.10.23] (which improves Menchoff's original inequality). Let g_{k_j} be the j-th non-zero function in the sequence $\{g_k\}$. Put $\tilde{g}_j = \frac{1}{\|g_{k_j}\|_2} g_{k_j}$ and $\tilde{c}_j := a_{k_j} \|g_{k_j}\|_2 / k_j^{1/p+\epsilon}$. Then

$$\sum_{j=1}^{\infty} |\tilde{c}_j|^2 \log^2 j \le \sum_{j=1}^{\infty} |\tilde{c}_j|^2 \log^2 k_j = \sum_{n=1}^{\infty} \frac{|a_n|^2 \|g_n\|_2^2}{n^{2/p+2\epsilon}} \log^2 n < \infty,$$

so we can apply the inequality from [Z] to the orthonormal sequence $\{\tilde{g}_j\}$, to obtain

$$\int \left[\sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k g_k}{k^{1/p+\epsilon}} \right| \right]^2 d\mu = \int \left[\sup_{n>0} \left| \sum_{j=1}^{n} \tilde{c}_j \tilde{g}_j \right| \right]^2 d\mu < \infty.$$

We now assume that $\{g_n\}$ is also uniformly bounded (this is done in the original measure space, so μ is just σ-finite). By Lemma 2(ii) $\sum_{k=1}^{\infty} \left[\frac{|a_n|}{n^{1/p+\epsilon}} \right]^p < \infty$. Since $1 < p \le 2$, we can use the Riesz version of the Hausdorff-Young theorem [Z, Theorem XII.2.8] to conclude that $\sum_{n=1}^{\infty} \frac{a_n g_n}{n^{1/p+\epsilon}}$ is in $L_q(\mu)$ for every $\epsilon > 0$ (this part of the theorem does not require $\{g_n\}$ to be normalized, but only $\sup_n \|g_n\|_2 < \infty$); thus also $\sum_{n=1}^{\infty} \frac{a_n g_n}{n} \in L_q(\mu)$. For any $s > q$ let $r = s/(s-1)$, so $1 < r < p$ and (5) is satisfied also with p replaced by r, and we have $\sum_{n=1}^{\infty} \frac{a_n g_n}{n} \in L_s(\mu)$. □

COROLLARY 4. *Let $\Lambda = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ be the unit circle, and let $\{a_n\}$ be a sequence of complex numbers satisfying* (5) *with $1 < p \le 2$. Then for every $\gamma > 1/p$ the series $\sum_{k=1}^{\infty} \frac{a_k \lambda^k}{k^{\gamma}}$ converges a.e. and in $L_q(\Lambda, d\lambda)$, $q = \frac{p}{p-1}$, with $\int_{\Lambda} \left[\sup_{n>0} \left| \sum_{k=1}^{n} \frac{a_k \lambda^k}{k^{\gamma}} \right| \right]^q d\lambda < \infty$. Hence for a.e. λ on the unit circle, $\sum_{k=1}^{\infty} \frac{a_k \lambda^k}{k}$ converges and $\frac{1}{n} \sum_{k=1}^{n} a_k \lambda^k \to 0$, and $\sum_{k=1}^{\infty} \frac{a_k \lambda^k}{k} \in \bigcap_{2\le s<\infty} L_s(\Lambda, d\lambda)$.*

Proof. We apply Theorem 5. Its last part yields that the Fourier series $\sum_{k=1}^{\infty} \frac{a_k \lambda^k}{k^{\gamma}}$ is in $L_q(\Lambda)$, so the convergence is also in L_q-norm. The maximal function is in $L_q(\Lambda, d\lambda)$ by Hunt's strong maximal inequality [Hu]. □

Remarks. (1) When $a_k = 1$ for every k, Corollary 4 applies, but $\sum_{k=1}^\infty \frac{a_k\lambda^k}{k}$ is not in $L_\infty(\Lambda, d\lambda)$.

(2) Let $a_{2^j} = 2^j$, and $a_k = 0$ if k is not a power of 2. Then (5) is satisfied with $p = 1$, but $\frac{1}{n}\sum_{k=1}^n a_k\lambda^k$ does not converge for any λ, since $|a_n\lambda^n|/n$ does not converge to 0. Thus Theorem 5 and Corollary 4 fail when $p = 1$.

(3) Kahane [Ka-1] proved (his proof can be adapted from the continuous to discrete time) that if $\{a_n\}$ satisfies (5) with $p = 1$, and we *assume* that $\frac{1}{n}\sum_{k=1}^n a_k\lambda^k$ converges for every λ with $|\lambda| = 1$, then the limit is non-zero only for at most countably many λ.

(4) The L_2-norm boundedness assumption of Example 2 can be somewhat relaxed. Let $\{h_n\}$ be an orthogonal sequence in $L_2(\mu)$ with $\sup_n \frac{1}{n}\sum_{k=1}^n \|h_k\|_2^2 < \infty$. Let $a_k = \|h_k\|_2$, and put $g_k = h_k/a_k$ if $a_k \neq 0$, and $g_k = 0$ when $a_k = 0$. Theorem 5 then yields that $\sum_{n=1}^\infty \frac{h_n}{n^{1/2+\epsilon}}$ converges a.e. for every $\epsilon > 0$, and thus $\frac{1}{n^{1/2+\epsilon}}\sum_{k=1}^n h_k \to 0$ a.e.

(5) Let $\{g_n\} \subset L_2(\mu)$ of a probability space be a sequence of uncorrelated random variables, non-negative or pairwise independent, such that for some $1 < q \le 2$ we have $\sup_n \frac{1}{n}\sum_{k=1}^n \|g_k\|_q^q < \infty$. Landers and Rogge [LaRo] proved that $\frac{1}{n}\sum_{k=1}^n (g_k - Eg_k) \to 0$ a.e. Example 4 in [LaRo] shows that for $1 < q < 2$ the above convergence may fail without non-negativity; combined with the previous remark, it yields that in Theorem 5 one cannot replace the assumption $\sup_n \|g_n\|_2 < \infty$ by $\sup_n \|g_n\|_q < \infty$ for some $1 < q < 2$. The previous remark shows that for $q = 2$ the non-negativity assumption of [LaRo] can be dropped, and there is even a rate of convergence.

(6) In Corollary 4, $\sum_{k=1}^\infty \frac{a_k\lambda^k}{k^\gamma}$ can be replaced by $\sum_{k=1}^\infty \frac{a_k\lambda^{n_k}}{k^\gamma}$ for $\{n_k\}$ *strictly increasing.*

Definition. A contraction T of $L_p(\mu)$ is said to be *positively dominated* if there is a positive contraction τ on $L_p(\mu)$ such that $|Tf| \le \tau(|f|)$ a.e. for any $f \in L_p(\mu)$.

Thus, a positive contraction is obviously positively dominated. If T is a Dunford-Schwartz contraction on $L_1(\mu)$, its linear modulus τ [Kr, p. 159] is also a Dunford-Schwartz contraction, and thus induces a positive contraction of $L_p(\mu)$ [Kr, p. 65]; hence T is a positively dominated contraction of $L_p(\mu)$, for any $1 \le p \le \infty$.

Theorem 6. *Let T be a positively dominated contraction of $L_p(\Omega, \mu)$, $p > 1$, and $f \in L_p(\mu)$. Then for a.e. $x \in \Omega$, the sequence $a_k = T^k f(x)$ has the property that for every $\gamma > \max\{1/p, 1/2\}$ and for any orthogonal sequence $\{g_n\} \subset L_2(Y, m)$ with $\sup_n \|g_n\|_2 < \infty$, the series $\sum_{k=1}^\infty \frac{a_k g_k}{k^\gamma}$ converges m-a.e., and $\sup_{n>0} \left|\sum_{k=1}^n \frac{a_k g_k}{k^\gamma}\right| \in L_2(m)$. Hence $\sum_{k=1}^\infty \frac{a_k g_k}{k}$ converges m-a.e. (and in $L_2(m)$-norm). If in addition $\{g_n\}$ is uniformly bounded, then $\sum_{k=1}^\infty \frac{a_k g_k}{k}$ is in $\bigcap\{L_s(m) : \max\{p/(p-1), 2\} < s < \infty\}$.*

Proof. Let τ be the positive contraction of $L_p(\mu)$ which dominates T. For $1 < r < p$, we have

$$\frac{1}{n}\sum_{k=1}^{n}|T^k f|^r \leq \frac{1}{n}\sum_{k=1}^{n}[\tau^k(|f|)]^r,$$

with a.e. convergence of the right hand side by [Be], so for μ-almost every $x \in \Omega$ the sequence $a_k = T^k f(x)$ satisfies (5) with p replaced by r (the required boundedness $\sup_n \frac{1}{n}\sum_{k=1}^{n}[\tau^k(|f|)]^r < \infty$ a.e. can be proved along the lines of the proof of Theorem 3.10 of [LOT]—first for τ an isometry, and then for the general case with the help of a dilation). We now apply Theorem 5 with p replaced by r for $r < \min\{2, p\}$. $\square$

REMARK. For T Dunford-Schwartz we have [Kr, p. 65] $|Tf| \leq \tau|f| \leq [\tau(|f|^p)]^{1/p}$ a.e., so $\frac{1}{n}\sum_{k=1}|T^k f|^p \leq \frac{1}{n}\sum_{k=1}\tau^k(|f|^p)$, which converges a.e. if $f \in L_p(\mu)$.

When $p = 2$ we can assume T to be only power-bounded, as implied by the next result.

THEOREM 7. *Let $\{f_n\} \subset L_2(\Omega, \mu)$ with $\sup_n \|f_n\|_2 < \infty$. Then for a.e. $x \in \Omega$, the sequence $a_k = f_k(x)$ has the property that for every $\gamma > 1/2$ and for any orthogonal sequence $\{g_n\} \subset L_2(Y, m)$ with $\sup_n \|g_n\|_2 = K < \infty$, the series $\sum_{k=1}^{\infty}\frac{a_k g_k}{k^\gamma}$ converges m-a.e. and in $L_2(m)$, with $\sup_{n>0}\left|\sum_{k=1}^{n}\frac{a_k g_k}{k^\gamma}\right| \in L_2(m)$. Hence $\sum_{k=1}^{\infty}\frac{a_k g_k}{k}$ converges m-a.e. and in $L_2(m)$. If in addition $\{g_n\}$ is uniformly bounded, then $\sum_{k=1}^{\infty}\frac{a_k g_k}{k} \in \bigcap_{2 \leq s < \infty} L_s(m)$.*

Proof. Fix $\gamma = 1/2 + \epsilon$. Since

$$\int \sum_{n=1}^{\infty}\frac{|f_n(x)|^2}{n^{1+\epsilon}}d\mu = \sum_{n=1}^{\infty}\frac{\|f_n\|_2^2}{n^{1+\epsilon}} \leq (\sup_k \|f_k\|_2)^2 \sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}} < \infty,$$

for a.e. $x \in \Omega$, the sequence $a_k(x) = f_k(x)$ satisfies $\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{1+\epsilon}} < \infty$. Given a norm-bounded orthogonal sequence $\{g_n\} \subset L_2(Y, m)$, we have

$$\sum_{n=1}^{\infty}\frac{|a_n|^2\|g_n\|_2^2}{n^{2\gamma}}\log^2 n \leq K^2\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{1+\epsilon}}\frac{\log^2 n}{n^\epsilon} < \infty.$$

Since we may assume m to be a probability, the Menchoff-Rademacher theorem yields the result.

Assume now that $\{g_n\}$ is also bounded. Let $s > 2$ and $r = s/(s-1)$. Then $r < 2$, and the simple inequality $|a|^r \leq |a|^2 + 1$ yields $\sum_{n=1}^{\infty}\frac{|a_n|^r}{n^{1+\epsilon}} < \infty$ for any $\epsilon > 0$. Hence $\sum_{n=1}^{\infty}|\frac{a_n}{n}|^r < \infty$, and the Riesz-Hausdorff-Young theorem yields, as before, that $\sum_{n=1}^{\infty}\frac{a_n g_n}{n} \in L_s(m)$. $\square$

Corollary 5. *Let T be a positively dominated contraction of $L_p(\Omega,\mu)$, $p>1$, or only power-bounded when $p=2$, and let $f\in L_p(\mu)$. Then for a.e. λ with $|\lambda|=1$, the series $\sum_{k=1}^\infty \frac{T^kf(x)\lambda^k}{k}$ converges μ-a.e.*

Proof. Theorem 6, or Theorem 7 when $p=2$, and orthogonality of $f_n(\lambda)=\lambda^n$ yield that for μ-a.e. $x\in\Omega$ the series $\sum_{k=1}^\infty \frac{T^kf(x)\lambda^k}{k}$ converges for a.e. λ. Fubini's theorem yields the assertion. □

6. Rotated ergodic Hilbert transforms and random Fourier series

In this section we look at a positively dominated contraction T in L_p, $p>1$, and would like to obtain, for $f\in L_p$, that for μ-a.e. $x\in\Omega$ we have convergence of $\sum_{k=1}^\infty \frac{T^kf(x)\lambda^k}{k}$ for *every* λ on the unit circle. Thus, we are looking for a special type of random Fourier series, with dependent random coefficients (for random Fourier series, we refer the reader to [Ka-2]). We saw in the proof of Corollary 5 that for a.e. x the series converges for a.e. λ. In order to have the convergence for every λ, it is necessary that f be "orthogonal" to all the eigenfunctions of T^* with unimodular eigenvalues, i.e., $\|\frac{1}{n}\sum_{k=1}^n\lambda^kT^kf\|\to 0$ for every λ.

Lemma 3. *Let $\{a_k\}$ be a sequence of complex numbers. Assume that for every $\epsilon>0$ there exists $\{b_k\}$ with $\max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^n b_k\lambda^k\right|\to 0$, such that $\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n|a_k-b_k|<\epsilon$. Then $\max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^n a_k\lambda^k\right|\to 0$.*

Proof. Fix $\epsilon>0$, and take the corresponding $\{b_k\}$. For n large enough,

$$\max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^n a_k\lambda^k\right| \le \max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^n b_k\lambda^k\right| + \frac{1}{n}\sum_{k=1}^n|a_k-b_k|<2\epsilon. \qquad \square$$

For T induced by an ergodic probability preserving transformation on (Ω,μ) and $f\in L_1(\mu)$, the Wiener-Wintner theorem [WW] yields that for μ-a.e. $x\in\Omega$, we have convergence of $\frac{1}{n}\sum_{k=1}^n\lambda^kT^kf(x)$ for *every* λ. When $f\in L_1(\mu)$ is "orthogonal" to all eigenfunctions of T (which are those of T^*, and bounded by ergodicity), i.e., $\|\frac{1}{n}\sum_{k=1}^n\lambda^kT^kf\|_1\to 0$ for every $|\lambda|=1$, then for a.e. $x\in\Omega$ we have $\frac{1}{n}\sum_{k=1}^n\lambda^kT^kf(x)\to 0$ for *every* λ, and if $f\in L_2$ the convergence to 0 is in fact *uniform in* λ (e.g., [As-1]). Since the L_2 functions orthogonal to all the eigenfunctions are dense in the L_1 functions orthogonal to the eigenfunctions (see Proposition 2.6 of [LOT]), for such $f\in L_1$ and $\epsilon>0$ we have $g\in L_2$ orthogonal to the eigenfunctions with $\|f-g\|_1<\epsilon$. The pointwise ergodic theorem yields that for a.e. x we have

$$\lim_n\frac{1}{n}\sum_{k=1}^n|T^kf(x)-T^kg(x)| = \lim_n\frac{1}{n}\sum_{k=1}^n T^k|f-g|(x)=\|f-g\|_1<\epsilon.$$

The previous lemma now shows that for a.e. $x \in \Omega$ we have $\max_{|\lambda|=1} \left| \frac{1}{n} \sum_{k=1}^{n} \lambda^k T^k f(x) \right| \to 0$. By continuity in λ for each fixed $x \in \Omega$, we can compute $\max_{|\lambda|=1} \left| \frac{1}{n} \sum_{k=1}^{n} \lambda^k T^k f(x) \right|$ as the supremum over the countable dense subset of roots of unity, so it is measurable. For $f \in L \log^+ L$ orthogonal to the eigenfunctions this yields by Lebesgue's dominated convergence theorem (since $\sup_n \frac{1}{n} \sum_{k=1}^{n} T^k |f| \in L_1$ [Kr, p. 52]) that

$$\left\| \max_{|\lambda|=1} \left| \frac{1}{n} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_1 \underset{n\to\infty}{\longrightarrow} 0,$$

and

$$\left\| \max_{|\lambda|=1} \left| \frac{1}{n} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_p \underset{n\to\infty}{\longrightarrow} 0,$$

if $f \in L_p$, $p > 1$.

THEOREM 8. *Let (Ω, μ) be a probability space, and T be a positively dominated contraction of $L_p(\mu)$, $1 < p < \infty$, or an ergodic positive contraction of $L_1(\mu)$ with $T1 = 1$. If for some $0 < \beta \le 1$, the function $f \in L_p$ satisfies*

$$\sup_{n>0} \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_1 = K < \infty, \tag{6}$$

then for μ-a.e. $x \in \Omega$ the series $\sum_{k=1}^{\infty} \frac{T^k f(x) \lambda^k}{k}$ converges uniformly in λ on the unit circle (and is therefore a continuous function of λ).

Proof. Put $\phi_n(x, \lambda) = \sum_{k=1}^{n} T^k f(x) \lambda^k$, and $\psi_n(x) = \max_{|\lambda|=1} |\phi_n(x, \lambda)|$.

CLAIM. *$\psi_n(x)/n \to 0$ for μ-a.e. x.*

We first prove the claim when $p > 1$. Let r be an integer with $r\beta > 1$, and define $n_m = m^r$. Then (6) yields

$$\sum_{m=1}^{\infty} \left\| \frac{\psi_{n_m}}{n_m} \right\|_1 \le K \sum_{m=1}^{\infty} \frac{n_m^{1-\beta}}{n_m} = K \sum_{m=1}^{\infty} \frac{1}{m^{r\beta}} < \infty.$$

Hence $\psi_{n_m}(x)/n_m \to 0$ for μ-a.e. x.

For $n_m \le n < n_{m+1}$ we have

$$\frac{1}{n} \psi_n(x) \le \max_{|\lambda|=1} \left| \frac{\phi_n(x)}{n} - \frac{\phi_{n_m}(x)}{n} \right| + \frac{\psi_{n_m}(x)}{n_m}.$$

The last term tends to 0 for a.e. $x \in \Omega$. For a.e. x and any $1 < s < p$, the sequence $\{T^k f(x)\}$ satisfies (5) with p replaced by s (see proof of Theorem 6).

Using Hölder's inequality, with $s' = s/(s-1)$, we obtain for those $x \in \Omega$

$$\max_{|\lambda|=1}\left|\frac{\phi_n(x)}{n} - \frac{\phi_{n_m}(x)}{n}\right| = \max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=n_m+1}^{n} T^k f(x)\lambda^k\right| \le \frac{1}{n_m}\sum_{k=n_m+1}^{n}|T^k f(x)|$$

$$\le \left(\frac{1}{n_m}\sum_{k=1}^{n_{m+1}}|T^k f(x)|^s\right)^{1/s}\cdot\left(\frac{n_{m+1}-n_m}{n_m}\right)^{1/s'} \underset{n\to\infty}{\longrightarrow} 0.$$

Thus $\frac{1}{n}\psi_n(x) \to 0$ a.e., and the claim is proved when $p > 1$.

For T an ergodic contraction on L_1 with $T1 = 1$, μ is invariant. We will assume T induced by a transition probability $P(x, A)$ (see [ÇLO] for the reduction to this case). On the space of one-sided trajectories $\Omega^{\mathbf{N}}$, with coordinate projections $\{X_n\}$, the shift θ is ergodic, with invariant probability $\mathbf{P}_\mu$ induced by the initial distribution μ. For any $g \in L_2(\mu)$ the function $\tilde{g} := g \circ X_0$ is in $L_2(\mathbf{P}_\mu)$. When $\|\frac{1}{n}\sum_{k=1}^{n}\lambda^k T^k g\|_2 \to 0$, we have $\|\frac{1}{n}\sum_{k=1}\lambda^k \tilde{g}\circ\theta^k\|_{L_2(\mathbf{P}_\mu)} \to 0$. Thus, if $g \in L_2(\mu)$ is orthogonal to all eigenfunctions of T with unimodular eigenvalues, we have $\max_{|\lambda|=1}|\frac{1}{n}\sum_{k=1}^{n}\lambda^k\tilde{g}\circ\theta^k| \to 0$ $\mathbf{P}_\mu$ a.e., and therefore for a.e. x this convergence holds $\mathbf{P}_x$ a.e. By Lebesgue's dominated convergence theorem, for a.e. $x \in \Omega$ we have

$$\max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^{n}\lambda^k T^k g(x)\right| = \max_{|\lambda|=1}\left|\frac{1}{n}\int\sum_{k=1}^{n}\lambda^k\tilde{g}\circ\theta^k d\mathbf{P}_x\right|$$

$$\le \int \max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^{n}\lambda^k\tilde{g}\circ\theta^k\right| d\mathbf{P}_x \to 0.$$

By (6), f is orthogonal to all the eigenfunctions of unimodular eigenvalues. We proceed as in the discussion above (see also [LOT]): we approximate f in L_1 norm by $g \in L_2$ which is orthogonal to the eigenfunctions; we have $\max_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=1}^{n}\lambda^k T^k g(x)\right| \to 0$ a.e., and Hopf's pointwise ergodic theorem with ergodicity of T show that Lemma 3 can be applied. This proves the claim when $p = 1$.

Now (6) yields

$$\int\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\psi_k(x)d\mu \le \sum_{k=1}^{\infty}\frac{1}{k^{1+\beta}}\frac{\|\psi_k\|_1}{k^{1-\beta}} < \infty,$$

which yields $\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\psi_k(x) < \infty$ a.e. Since

$$\sum_{k=1}^{n}\frac{T^k f(x)\lambda^k}{k} = \frac{1}{n}\phi_n(x,\lambda) + \sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right)\phi_k(x,\lambda),$$

for a.e. $x \in \Omega$ we have the desired convergence uniformly in λ. $\square$

Remarks. (1) For T induced by an ergodic probability preserving transformation, Theorem 8 was proved in [As-4] (for $p = 2$). For such T, functions satisfying (6) were called there *Wiener-Wintner functions*.

(2) For $p \geq 2$ and T induced by a probability preserving transformation, Assani and Nicolaou [AsN] proved that under the rate condition (6) with $\frac{1}{p} < \beta < 1$, for μ-a.e. $x \in \Omega$ we have convergence of $\sum_{k=1}^{\infty} \frac{T^k f(x) \lambda^k}{k^\gamma}$ for every $1 + \frac{1}{2p} - \frac{\beta}{2} < \gamma \leq 1$ and *every* λ (and for fixed γ the convergence is uniform in λ). Even for such T, our theorem is new when $1 < p < 2$.

(3) Examples of ergodic dynamical systems with $f \in L_2$ satisfying (6) are given in [As-4] and [AsN]. For a spectral characterization of the rate condition (6) see [As-5].

Theorem 9. *Let $1 < p < \infty$, and let $\{f_n\}_{n=1}^{\infty} \subset L_p(\Omega, \mu)$ with $\sup_{n>0} \|f_n\|_p < \infty$. Let Y be a compact metric space and $\{g_n\} \subset C(Y)$ with $\sup_n \|g_n\|_\infty = C < \infty$. If for some $0 < \beta \leq 1$ we have*

$$\sup_{n>0} \left\| \max_{y \in Y} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} g_k(y) f_k \right| \right\|_p = K < \infty,$$

then there exists a set $\Omega' \subset \Omega$ with $\mu(\Omega') = 0$, such that for $x \notin \Omega'$ and every $0 \leq \delta < \frac{p-1}{p}\beta$, the series $\sum_{k=1}^{\infty} \frac{g_k(y) f_k(x)}{k^{1-\delta}}$ converges uniformly in $y \in Y$ (and is therefore a continuous function on Y), and $\sup_{n>0} \max_{y \in Y} \left| \sum_{k=1}^{n} \frac{g_k(y) f_k}{k^{1-\delta}} \right| \in L_p(\mu)$.

Proof. We may assume μ to be a probability. Fix $0 \leq \delta < \frac{p-1}{p}\beta$. The first step is to show that $\max_{y \in Y} \left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k(x) g_k(y) \right| \to 0$ a.e. The proof of this convergence is similar to that of Proposition 1, with $\sum_{k=1}^{n} f_k$ replaced by $\max_{y \in Y} |\sum_{k=1}^{n} f_k g_k(y)|$. We also obtain

$$\sup_{n>0} \max_{y \in Y} \left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k(x) g_k(y) \right| \in L_p(\mu).$$

Setting $\tilde{S}_0 \equiv 0$ and $\tilde{S}_k(x, y) := \sum_{j=1}^{k} f_k(x) g_k(y)$, we have, for $\gamma = 1 - \delta$,

$$\sum_{k=1}^{n} \frac{f_k(x) g_k(y)}{k^\gamma} = \frac{\tilde{S}_n(x, y)}{n^\gamma} + \sum_{k=1}^{n-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma} \right) \tilde{S}_k(x, y).$$

The first term tends to 0 uniformly in y as indicated above; for the series we obtain the a.e. convergence uniformly in y, similarly to the proof of Theorem 1, since

$$\max_{y \in Y} \sum_{k=n}^{\infty} \frac{1}{k^{\gamma+\beta}} \left| \frac{\tilde{S}_k(x, y)}{k^{1-\beta}} \right| \leq \sum_{k=n}^{\infty} \frac{1}{k^{\gamma+\beta}} \max_{y \in Y} \left| \frac{\tilde{S}_k(x, y)}{k^{1-\beta}} \right|,$$

and the last series converges to 0 for a.e. x, as using the assumption we have

$$\int \sum_{k=n}^{\infty} \frac{1}{k^{\gamma+\beta}} \max_{y \in Y} \left| \frac{\tilde{S}_k(x,y)}{k^{1-\beta}} \right| d\mu \leq \sum_{k=n}^{\infty} \frac{1}{k^{\gamma+\beta}} \left\| \max_{y \in Y} \frac{1}{k^{1-\beta}} \tilde{S}_k(\cdot, y) \right\|_p$$
$$\leq K \sum_{k=n}^{\infty} \frac{1}{k^{\gamma+\beta}} \underset{n\to\infty}{\longrightarrow} 0.$$

Similarly to the proof of Theorem 1, we obtain also

$$\sup_{n>0} \max_{y \in Y} \left| \sum_{k=1}^{n} \frac{f_k g_k(y)}{k^{1-\delta}} \right| \in L_p(\mu).$$

Taking $\delta_j > 0$ increasing to $\frac{p-1}{p}\beta$ we obtain the set Ω'. □

Remark. If each g_k is identically a constant a_k, we obtain Corollary 2.

Corollary 6. *Let $1 < p < \infty$, and let $\{f_n\}_{n=1}^{\infty} \subset L_p(\Omega, \mu)$ with $\sup_{n>0} \|f_n\|_p < \infty$. If for some sequence of integers $\{n_k\}$ and $0 < \beta \leq 1$ we have*

$$\sup_{n>0} \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} f_k \lambda^{n_k} \right| \right\|_p = K < \infty, \tag{7}$$

then there exists a set $\Omega' \subset \Omega$ with $\mu(\Omega') = 0$, such that for $x \notin \Omega'$ and every $0 \leq \delta < \frac{p-1}{p}\beta$ the series $\sum_{k=1}^{\infty} \frac{f_k(x)\lambda^{n_k}}{k^{1-\delta}}$ converges uniformly in $|\lambda| = 1$ (and is therefore a continuous function of λ), and

$$\sup_{n>0} \max_{|\lambda|=1} \left| \sum_{k=1}^{n} \frac{f_k(x)\lambda^{n_k}}{k^{1-\delta}} \right| \in L_p(\mu).$$

Corollary 7. *Let T be a power-bounded operator of $L_p(\Omega, \mu)$, $1 < p < \infty$. If $f \in L_p$ satisfies, for some $0 < \beta \leq 1$,*

$$\sup_{n>0} \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_p = K < \infty,$$

then there exists a set $\Omega' \subset \Omega$ with $\mu(\Omega') = 0$, such that for $x \notin \Omega'$ and every $0 \leq \delta < \frac{p-1}{p}\beta$ the series $\sum_{k=1}^{\infty} \frac{T^k f(x)\lambda^k}{k^{1-\delta}}$ converges uniformly in λ on the unit circle (and is therefore a continuous function of λ), and

$$\sup_{n>0} \max_{|\lambda|=1} \left| \sum_{k=1}^{n} \frac{T^k f(x)\lambda^k}{k^{1-\delta}} \right| \in L_p(\mu).$$

The following result was obtained by Assani [As-4].

THEOREM 10. *Let (Ω,μ) be a probability space, and let $\{f_n\}\subset L_2(\mu)$ be* independent *with $\int f_n d\mu = 0$ and $\sup_n \|f_n\|_2<\infty$. Then*

$$\left\| \max_{|\lambda|=1} \left| \frac{1}{n}\sum_{k=1}^{n} \lambda^k f_k \right| \right\|_2 \le \frac{(1+\sqrt{2})^{1/2}}{n^{1/4}} \sup_k \|f_k\|_2 .$$

Assani's proof is elementary; he remarked that the inequality follows also from the general (deep) results of [MPi] (without an estimate of the constant). We are grateful to him for providing us with his (unpublished) derivation of the inequality of Theorem 10 from [MPi]; his method is used below to obtain a more general result (with a better rate in Theorem 10).

THEOREM 11. *Let (Ω,μ) be a probability space, and let $\{f_n\}\subset L_2(\mu)$ be* independent *with $\int f_n d\mu = 0$ and $\sup_n \|f_n\|_2<\infty$. Let $\{n_k\}$ be a strictly increasing sequence with $n_k\le ck^r$ for some $r\ge 1$. Then for any $\beta<1/2$ there is a constant $K_{c,r,\beta}$ such that*

$$\left\| \max_{|\lambda|=1} \left| \frac{1}{n}\sum_{k=1}^{n} \lambda^{n_k} f_k \right| \right\|_2 \le \frac{K_{c,r,\beta}}{n^{\beta}} \sup_k \|f_k\|_2 .$$

Proof. Fix $0<\beta<1/2$ and put $\alpha=(1-2\beta)/r$. We will use Corollary 1.1.2 of [MPi], with the group G the unit circle, G the compact neighborhood, the set of characters $A:=\{n_k : k\ge 1\}$, and the independent random variables $\xi_{n_k}=f_k$.

For each n, we want to apply that result to the sequence $\{a_j\}$ defined on A by $a_{n_k}=1$ for $1\le k\le n$ and $a_{n_k}=0$ for $k>n$ (the sequence need not be defined outside A, but we put $a_j=0$ for $j\notin A$). It will be convenient to identify the unit circle with the interval $[0,2\pi]$, with addition modulo 2π. Let $t_1,t_2\in[0,2\pi]$ and define the corresponding translation invariant pseudo-metric $d_n(t_1,t_2)=\sigma_n(t_1-t_2)$, where

$$\begin{aligned}\sigma_n(t) &:= \left(\sum_{j\in A}|a_j|^2\cdot|1-e^{ijt}|^2\right)^{1/2} = \left(\sum_{k=1}^{n}|1-e^{in_k t}|^2\right)^{1/2}\\ &= 2\left(\sum_{k=1}^{n}\sin^2\frac{n_k t}{2}\right)^{1/2}.\end{aligned}$$

Since $|\sin t|\le 1$ and $|\sin t|\le |t|$, we obtain $\sin^2 t\le|\sin t|^\alpha\le|t|^\alpha$. This yields

$$\begin{aligned}\sigma_n(t) &\le 2\left(\sum_{k=1}^{n}\frac{n_k^\alpha t^\alpha}{2^\alpha}\right)^{1/2} \le 2c^{\alpha/2}\left(\sum_{k=1}^{n}\frac{k^{r\alpha}t^\alpha}{2^\alpha}\right)^{1/2}\\ &\le 2c^{\alpha/2}\frac{t^{\alpha/2}(n+1)^{\frac{r\alpha+1}{2}}}{2^{\alpha/2}\sqrt{r\alpha+1}} \le c^{\alpha/2}2^{1-\alpha/2}t^{\alpha/2}(n+1)^{\frac{r\alpha+1}{2}}.\end{aligned}$$

Denote by m the Lebesgue measure on $[0, 2\pi]$. Then the "distribution" of σ_n satisfies

$$m_{\sigma_n}(\epsilon) := m\{t \in [0, 2\pi] : \sigma_n(t) < \epsilon\} \geq 2^{1-2/\alpha} \frac{\epsilon^{2/\alpha}}{c(n+1)^{\frac{r\alpha+1}{\alpha}}};$$

hence the 'inverse' function defined on $[0, 2\pi]$ (which is the non-decreasing rearrangement of σ_n), satisfies

$$\overline{\sigma_n(s)} := \sup\{t > 0 : m_{\sigma_n}(t) < s\} \leq c^{\alpha/2} 2^{1-\alpha/2} s^{\alpha/2} (n+1)^{\frac{r\alpha+1}{2}}.$$

In order to apply inequality (1.15) of [MPi, p. 9] we estimate

$$\begin{aligned} I_n(\sigma) =: &\int_0^{2\pi} \frac{\overline{\sigma_n(s)} ds}{s(\log \frac{8\pi}{s})^{1/2}} \\ &\leq c^{\alpha/2} 2^{1-\alpha/2} (n+1)^{\frac{r\alpha+1}{2}} \int_0^{2\pi} \frac{ds}{s^{1-\alpha/2} (\log \frac{8\pi}{s})^{1/2}} = C_{c,r,\beta} (n+1)^{\frac{r\alpha+1}{2}}, \end{aligned}$$

with $C_{c,r,\beta} < \infty$ by the integrability of $\frac{1}{s^{1-\alpha/2}}$ for $\alpha > 0$.

Now inequality (1.15) of [MPi] (as modified in Corollary 1.1.2 there) yields

$$\begin{aligned} \left\| \max_{|\lambda|=1} \left| \sum_{k=1}^n \lambda^{n_k} f_k \right| \right\|_2 &\leq 4C \sup_k \|f_k\|_2 \left[\left(\sum_{j \in A} |a_j|^2 \right)^{1/2} + I_n(\sigma) \right] \\ &= 4C \sup_k \|f_k\|_2 \left[\left(\sum_{k=1}^\infty |a_{n_k}|^2 \right)^{1/2} + I_n(\sigma) \right] \\ &\leq 4C \sup_k \|f_k\|_2 \left[n^{1/2} + C_{c,r,\beta} (n+1)^{\frac{r\alpha+1}{2}} \right]; \end{aligned}$$

the constant C (which was not determined in [MPi]) is independent of the specific sequence $\{a_j\}$. Dividing the inequality by n, we obtain the assertion of the theorem, since $(r\alpha + 1)/2 = 1 - \beta > 1/2$. $\square$

REMARKS. (1) The additional condition $\inf_n \int |f_n| d\mu > 0$ in the statement of Corollary 1.1.2 of [MPi] is not needed for the proof of (1.15) there (see [MPi, p. 51]).

(2) The theorem applies to sequences $\{[k^r] : k \geq 1\}$ with $r \geq 1$.

(3) The sequence $\{n_k\}$ need not be monotone, but its terms must be *distinct* (in addition to the growth condition), to make it an *enumeration* of the *set* of characters A; hence the proof of Theorem 11 does not apply to the sequence $\{[\sqrt{k}]\}$.

THEOREM 12. *Let* (Ω, μ) *be a probability space, and let* $\{f_n\} \subset L_2(\mu)$ *be* independent *with* $\int f_n d\mu = 0$ *and* $\sup_n \|f_n\|_2 < \infty$. *Let* $\{n_k\}$ *be a strictly increasing sequence with* $n_k \leq ck^r$ *for some* $r \geq 1$. *Then for a.e.* x, *the* *series* $\sum_{k=1}^\infty \frac{f_k(x)}{k^{1-\delta}}\lambda^{n_k}$ *converges uniformly in* λ*, for any* $0 \le \delta < 1/2$*. For* $0 \le \delta < 1/4$*, we even have*

$$\sup_{n>0}\max_{|\lambda|=1}\left|\sum_{k=1}^n \frac{f_k(x)}{k^{1-\delta}}\lambda^{n_k}\right| \in L_2(\mu).$$

Proof. Fix $r \ge 1$ and $0 \le \delta < 1/4$. Taking β with $2\delta < \beta < 1/2$, Theorem 11 yields that (7) is satisfied by $\{n_k\}$ with $p = 2$, so Corollary 6 yields the claimed result for the maximal function, and also the required a.e. convergence for $\delta < 1/4$.

An appropriate use of [MPi] will yield the a.e. uniform convergence for δ in the larger interval $[0, 1/2)$ (without using Theorem 11). As in the proof of Theorem 11, take G the unit circle, $A := \{n_k\}$, and $\xi_{n_k} = f_k$. Fix $0 < \delta < 1/2$, and put $\alpha = (1-2\delta)/2r$, so $0 < \alpha < 1/2$. Define $a_{n_k} = \frac{1}{k^{1-\delta}}$ (and $a_j = 0$ for $j \notin A$), and consider the corresponding metric $d(t_1, t_2) = \sigma(t_1 - t_2)$ (which is uniformly convergent), where

$$\begin{aligned}\sigma(t) &:= \left(\sum_{j\in A}|a_j|^2|1-e^{ijt}|^2\right)^{1/2} = 2\left(\sum_{k=1}^\infty \frac{\sin^2\frac{n_k t}{2}}{k^{2-2\delta}}\right)^{1/2}\\ &\le 2\left(\sum_{k=1}^\infty \frac{c^\alpha k^{r\alpha}|t|^\alpha}{2^\alpha k^{2-2\delta}}\right)^{1/2} \le 2^{1-\alpha/2}c^{\alpha/2}|t|^{\alpha/2}\frac{\sqrt{\gamma}}{\sqrt{\gamma-1}},\end{aligned}$$

with $\gamma := 2 - 2\delta - r\alpha = 3/2 - \delta > 1$.

Estimations of m_σ and $\overline{\sigma}$ as in the previous proof show that $I(\sigma) < \infty$; now by Corollary 1.2 in [MPi, p. 10] for a.e. x the series $\sum_{k=1}^\infty \frac{f_k(x)}{k^{1-\delta}}\lambda^{n_k}$ converges uniformly in λ. Note that the condition $\inf_n \int |f_n| d\mu > 0$ is not needed for the convergence [MPi, p. 51]. □

Remarks. (1) Since Theorem 8 and [As-4, Theorem 9] require an operator (also in their proofs), they cannot be used to prove Theorem 12 in the case $n_k = k$.

(2) Let $n_k = k$. For $\{f_n\}$ independent identically distributed random variables with mean 0 and finite variance, Theorem 10 (see also [As-4]) shows that (7) is satisfied with $\beta = 1/4$, and Theorem 11 yields (7) with any $\beta < 1/2$. The result of [AsN] (with $p = 2$ and T induced by the shift of the i.i.d. sequence) cannot be applied in this case, since $\beta < 1/p$.

(3) We mention that for $\{f_n\}$ i.i.d., Cuzick and Lai [CuLa, Theorem 2(iv)] proved that if $E(f_1) = 0$ and $E(|f_1|\log^+|f_1|) < \infty$, then we have uniform convergence of $\sum_{k=1}^\infty \frac{f_k(x)}{k}\lambda^k$ for a.e. x. Furthermore, if $f_1 \in L_p, 1 < p < 2$, then for any $\gamma > 1/p$ the series $\sum_{k=1}^\infty \frac{f_k(x)}{k^\gamma}\lambda^k$ converges uniformly for a.e. x.

7. Convergence with random modulating sequences

In this section we show that random bounded sequences (realizations of certain independent uniformly bounded random variables) are almost surely universally good—they satisfy the assumptions of Section 3—and yield a.e. convergence of the modulated one-sided ergodic Hilbert transform for all Dunford-Schwartz operators and L_p functions.

THEOREM 13. *Let $\{n_k\}$ be a strictly increasing sequence of positive integers with $n_k \le ck^r$ for some $r \ge 1$, let (Y, m) be a probability space, and let $\{g_n\} \subset L_\infty(Y, m)$ be independent with $\int g_n dm = 0$ and $\sup_n \|g_n\|_\infty < \infty$. Then for a.e. $y \in Y$ the sequence $b_k := g_k(y)$ has the property that for any contraction T in $L_2(\Omega, \mu)$ and $f \in L_2(\mu)$, the series $\sum_{k=1}^\infty \frac{b_k T^{n_k} f}{k^\gamma}$ converges μ-a.e. for $\gamma > 3/4$, with $\sup_{n>0} \left|\sum_{k=1}^n \frac{b_k T^{n_k} f}{k^\gamma}\right| \in L_2(\mu)$. For $\gamma > 1/2$ the series converges in $L_2(\mu)$-norm.*

Proof. By Theorem 12 (applied to $\{g_n\}$) we have that for a.e. $y \in Y$, the bounded sequence $b_k = g_k(y)$ satisfies $\sup_n \max_{|\lambda|=1} |\sum_{k=1}^n \frac{b_k}{k^{1-\beta}} \lambda^{n_k}| < \infty$ for any $0 < \beta < 1/2$. By a variant of Kronecker's lemma, we obtain $\sup_n \sup_{|\lambda|=1} \left|\frac{1}{n^{1-\beta}} \sum_{k=1}^n b_k \lambda^{n_k}\right| < \infty$ for any $\beta < 1/2$. For $\gamma > 3/4$, Proposition 2(i) now yields that for T and f as in the assertion, the series $\sum_{k=1}^\infty \frac{b_k T^{n_k} f}{k^\gamma}$. converges a.e., with $\sup_{n>0} \left|\sum_{k=1}^n \frac{b_k T^{n_k} f}{k^\gamma}\right| \in L_2(\mu)$. The norm convergence of the series for $\gamma > 1/2$ also follows from Proposition 2. $\square$

REMARKS. (1) In fact, by Proposition 3, for $\gamma > 1/2$ the series $\sum_{k=1}^\infty \frac{b_k T^{n_k}}{k^\gamma}$ in Theorem 13 converges in operator norm, and this convergence is uniform in all L_2-contractions.

(2) Theorem 7 has more general assumptions, but using Fubini's theorem (as in Corollary 5), the null set outside which we get the "good modulating sequence" $\{g_k(y)\}$ depends on T and f. In Theorem 13 we obtain a *universally good modulating sequence*, but the rate is not as good as in Theorem 7.

EXAMPLE 6. Let $\{\phi_n\}$ be the Rademacher sequence on $[0, 1]$. It corresponds to i.i.d. with values 1 or -1 with probability 1/2. By Theorem 13, for a.e. $y \in [0, 1]$ the sequence of signs $\epsilon_n := \phi_n(y)$ is universally good: *for every $\gamma > 3/4$, any contraction T on $L_2(\mu)$ and $f \in L_2(\mu)$, the series $\sum_{k=1}^\infty \frac{\epsilon_k T^k f}{k^\gamma}$ converges a.e.* This result is Remark 12 and (part of) Theorem 23 of [R]. A concrete example of a universally good $\{\epsilon_n\}$ is provided by the Rudin-Shapiro sequence.

REMARKS. (1) Using different methods, Boukhari and Weber [BoWe] have obtained that if $\{g_n\}$ are symmetric i.i.d. with second moment (not necessarily bounded) and $n_k = k$, also the a.e. convergence assertion of Theorem 13 holds for $\gamma > 1/2$. This improves the result of Example 6. This improvement is due to the use in [BoWe] of all the information (identical distribution, symmetry), while our proof relies on the very general results of Theorem 1 (through Corollary 2); in L_2, the interval of δ obtained in Theorem 1 for a.e. convergence is $[0, \beta/2)$, while for norm convergence it is $[0, \beta)$. On the other hand, Theorem 13 applies in cases where the distributions are not the same.

(2) In Example 6, for any given $\{n_k\}$ with $n_k \le ck^r$ (e.g., $n_k = k^2$), a.e. random sequence of signs $\{\epsilon_n\}$ yields a.e. convergence of $\sum_{k=1}^{\infty} \frac{\epsilon_k T^{n_k} f}{k^\gamma}$ for $\gamma > 3/4$.

THEOREM 14. *Let $\{n_k\}$ be a strictly increasing sequence of positive integers with $n_k \le ck^r$ for some $r \ge 1$, let (Y, m) be a probability space, and let $\{g_n\} \subset L_\infty(Y, m)$ be independent with $\int g_n dm = 0$ and $\sup_n \|g_n\|_\infty < \infty$. Then for a.e. $y \in Y$ the sequence $b_k := g_k(y)$ has the following property:*

For every Dunford-Schwartz operator T on $L_1(\Omega, \mu)$ of a probability space and $f \in L_p(\mu)$, $1 < p < \infty$, the series $\sum_{k=1}^{\infty} \frac{b_k T^{n_k} f}{k^\gamma}$ converges a.e. for $\gamma \in (\max\{\frac{3}{4}, \frac{p+1}{2p}\}, 1]$, with $\sup_{n>0} \big| \sum_{k=1}^{n} \frac{b_k T^{n_k} f}{k^\gamma} \big| \in L_p(\mu)$ when $p \le 2$.

Proof. It was noted in the proof of Theorem 13 that $\{b_k\}$ satisfies (3) for any $\beta < 1/2$. Thus for $f \in L_p(\mu)$ with $1 < p \le 2$ and $\gamma > \frac{p+1}{2p}$, take $\beta < 1/2$ such that $\gamma > 1 - \frac{p-1}{p}\beta$, and apply Proposition 2(ii), which yields the a.e. convergence of $\sum_{k=1}^{\infty} \frac{b_k T^{n_k} f}{k^\gamma}$, and also that $\sup_{n>0} \big| \sum_{k=1}^{n} \frac{b_k T^{n_k} f}{k^\gamma} \big| \in L_p$. For $p > 2$ we have $f \in L_2$ since μ is a probability. $\square$

THEOREM 15. *Let (Y, m) be a probability space, and let $\{g_n\} \subset L_\infty(Y, m)$ be independent with $\int g_n dm = 0$ and $\sup_n \|g_n\|_\infty < \infty$. Then for a.e. $y \in Y$ the sequence $b_k := g_k(y)$ has the following properties:*

(i) *For every Dunford-Schwartz operator on $L_1(\Omega, \mu)$ and $f \in L_1(\mu)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} b_k T^k f = 0 \tag{8}$$

μ-almost everywhere, and in $L_1(\mu)$-norm when μ is finite.

(ii) *For every Dunford-Schwartz operator T on $L_1(\Omega, \mu)$ of a probability space and $f \in L_p(\mu)$, $1 < p < \infty$, the series $\sum_{k=1}^{\infty} \frac{b_k T^k f}{k^\gamma}$ converges a.e. for $\gamma \in (\max\{\frac{3}{4}, \frac{p+1}{2p}\}, 1]$, with $\sup_{n>0} \big| \sum_{k=1}^{n} \frac{b_k T^k f}{k^\gamma} \big| \in L_p(\mu)$ when $p \le 2$.*

(iii) *For every contraction T on $L_1(\Omega, \mu)$ with mean ergodic modulus and $f \in L_1(\mu)$, (8) holds μ a.e. and in $L_1(\mu)$-norm.*
(iv) *For every positively dominated contraction of $L_p(\Omega, \mu)$, $1 < p < \infty$, and $f \in L_p(\mu)$, (8) holds μ a.e. and in $L_p(\mu)$-norm.*

Proof. (i) Theorem 13 (for $n_k = k$) and Kronecker's lemma yield the convergence for $f \in L_2(\mu)$. The a.e. convergence now follows from the Banach principle (see proof of Proposition 2(iii)).

(ii) Apply Theorem 14 to $n_k = k$. For $f \in L_p$ this also yields a rate in (8).

(iii) and (iv) follow from (i), by [ÇLO], Theorems 2.3 and 2.4, respectively. □

Remarks. (1) The remark following Proposition 3 yields that for fixed $1 < p < 2$ and $\gamma > 1/p$, the series $\sum_{k=1}^{\infty} \frac{b_k T^k}{k^\gamma}$ in Theorem 15(ii) converges in the L_p-operator norm, and this convergence is uniform in all Dunford-Schwartz contractions.

(2) When the independent sequence $\{g_k\}$ is identically distributed, Theorem 15(i) follows from the "return times theorem" (see [ÇLO] for the passage from T induced by a probability preserving transformation to a general Dunford-Schwartz operator). If the i.i.d. $\{g_k\}$ are symmetric, one can also use the result of [As-3].

(3) Theorem 15(i) can be proved independently of [MPi], since the precise rates of convergence are not needed: in the proof of Theorem 13, we can use Theorem 10 and Corollary 6, instead of Theorem 12, to obtain the convergence of the series $\sum_{k=1}^{\infty} \frac{b_k T^k f}{k^\gamma}$ for some $\gamma < 1$.

(4) For the special case of $\{g_n\}$ the Rademacher functions, part (i) of Theorem 15 is Corollary 24 of [R], and part (ii) is in Theorems 18 and 25 of [R]. Theorem 14 provides a more general result.

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