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On strong laws of large numbers with rates

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Abstract. Let $\{f_n\} \subset L_p(\mu)$, $1 < p < \infty$, be a sequence of functions with $\sup_n ||f_n||_p < \infty$. We prove that if for some $0 < \beta \le 1$ we have $\sup_n \Big\| \frac{1}{n^{1-\beta}} \sum_{k=1}^n f_k \Big\|_p < \infty$, then for $\delta < \frac{p-1}{p}\beta$ the sequence $\{\frac{1}{n^{1-\delta}} \sum_{k=1}^n f_k\}$ has a.e. bounded p-variation, hence converges, and the p-variation norm function is in $L_p(\mu)$. If we replace $\sup_n ||f_n||_p < \infty$ by $\sup_n ||f_n||_\infty < \infty$, then the a.e. convergence holds for $\delta < \frac{p}{p+1}\beta$. Furthermore, in each case we also have a.e. convergence of the series $\sum_{k=1}^\infty \frac{f_k}{k^{1-\delta}}$ for the corresponding values of δ, and in the first case we even have that the sequence of partial sums has bounded p-variation.

Some applications are given. In particular, we show that if $\{g_n\}$ are centered independent (not necessarily identically distributed) random variables with $\sup_n ||g_n||_q < \infty$ for some $q \ge 2$, then almost every realization $a_n = g_n(y)$ has the property that for every Dunford-Schwartz operator T on a probability space (Ω, μ) and $f \in L_p(\mu)$, $p > \frac{q}{q-1}$ the series $\sum_{k=1}^\infty \frac{a_k T^k f}{k}$ converges a.e. The same result holds for $1 < q < 2$ if in addition the random varaibles $\{g_n\}$ are all symmetric. When the $\{g_n\}$ are i.i.d. the symmetry is not needed, and a.e. convergence of the above series holds also for $f \in L_{\frac{q}{q-1}}(\mu)$.

1. Introduction

It is known that there is no general speed of convergence in the pointwise ergodic theorem for ergodic measure preserving transformations; Krengel [**Kr1**] has shown that *for every measure preserving transformation θ of the unit circle with Lebesgue measure and for every sequence $\{a_n\}$ of positive numbers converging to 0 there exists a continuous function f with integral 0 such that* $\limsup_n |\frac{1}{n} \sum_{k=1}^n f \circ \theta^k|/a_n = \infty$ *a.e.* For further discussion see pp. 14-15 of [**Kr2**].

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Derriennic and Lin [**DL**] have used a rate of convergence in the mean to obtain pointwise rates of convergence: *Let T be a Dunford-Schwartz operator on $L_1(\mu)$ of a probability space, and let $f \in L_p$ for some (fixed) $p > 1$. Assume that for some $0 < \beta \leq 1$ we have*

$$\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k f \right\|_p < \infty. \tag{1}$$

(i) If $\beta > 1 - 1/p$, then the series $\sum_{k=1}^{\infty} T^k f / k^{1/p}$ converges a.e. and thus $(1/n^{1/p}) \sum_{k=1}^{n} T^k f \to 0$ a.e.

(ii) If $\beta \leq 1 - 1/p$, then for every $\gamma > 1 - \beta$ the series $\sum_{k=1}^{\infty} T^k f / k^{\gamma}$ converges a.e. and $(1/n^{\gamma}) \sum_{k=1}^{n} T^k f \to 0$ a.e.

Condition (1) had been previously used by Loève [**Lo**] (see [**Do**], p. 492) for T unitary on L_2 to obtain the strong law of large numbers. Rates of convergence in this case were obtained by Gaposhkin [**G**].

For T induced by an ergodic probability preserving transformation on (Ω, μ) and $f \in L_1(\mu)$ orthogonal to the eigenfunctions of T, the Wiener-Wintner theorem [**WW**] yields that for a.e. x we have $\lim_n \frac{1}{n} \sum_{k=1}^{n} \lambda^k T^k f(x) = 0$ for every λ on the unit circle; in fact, the convergence (for fixed x) is uniform in λ (see [**A1**] for $f \in L_2$, and [**CL**] for the extension to $f \in L_1$). This yields [**CL**] $\left\| \max_{|\lambda|=1} \left| \frac{1}{n} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_p \to 0$ when $f \in L_p$, $p > 1$. Independently of [**DL**], Assani [**A3**] studied the rate of convergence in the Wiener-Wintner theorem, and considered functions $f \in L_2$ which for some $\beta > 0$ satisfy

$$\sup_n \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_1 < \infty.$$

He showed the existence of such functions for K-automorphisms and other interesting systems, and proved that for x in a set of full measure the Fourier series $\sum_{k=1}^{\infty} \lambda^k T^k f(x)/k$ converges for every λ on the unit circle. When $f \in L_p$ with $p \geq 2$ and $\beta > \frac{1}{p}$, Assani and Nicolaou [**AN**] strenghtened the result, proving the uniform convergence of $\sum_{k=1}^{\infty} \lambda^k T^k f(x)/k^{\gamma}$ for any $\gamma > 1 - (\frac{\beta}{2} - \frac{1}{2p})$.

A different method of measuring the speed of convergence of a numerical sequence $x_n \to x$ is to check whether $\sum_{k=1}^{\infty} |x_n - x|^p < \infty$ (i.e., $\{x_n - x\} \in \ell_p$) for some $p \geq 1$. Note that if for $\epsilon > 0$ we define the *ϵ-deviation* of the convergent sequence by $D(\{x_n\}, \epsilon) := |\{n : \ |x_n - x| > \epsilon\}|$, we obtain

$$D(\{x_n\}, \epsilon) \leq \sum_{\{k: |x_k - x| > \epsilon\}} \left(\frac{|x_k - x|}{\epsilon} \right)^p \leq \frac{1}{\epsilon^p} ||\{x_n - x\}||_{\ell_p}^p.$$

The condition $\{x_n - x\} \in \ell_p$ is obviously very strong, and implies

$$\sup_{\{n_k\} \nearrow} \left[\sum_{k=1}^{\infty} |x_{n_{k+1}} - x_{n_k}|^p \right]^{1/p} \leq 2 ||\{x_n - x\}||_p < \infty.$$

A sequence $\{x_n\}$ of complex numbers is said to have *bounded p-variation* if it satisfies $||\{x_n\}||_{V_p} := \sup_{\{n_k\} \nearrow} \left[\sum_{k=1}^{\infty} |x_{n_{k+1}} - x_{n_k}|^p \right]^{1/p} < \infty$. For fixed $p \geq 1$ the sequences of bounded p-variation are a vector space, with $||\{x_n\}||_{V_p}$ a semi-norm. Since $|x_{n_{k+1}} - x_{n_k}| \le \sum_{j=n_k}^{n_{k+1}-1} |x_{j+1} - x_j|$, we have $||\{x_n\}||_{V_1} = \sum_{j=1}^{\infty} |x_{j+1} - x_j|$.

Lemma. *Every complex sequence of bounded p-variation converges.*

Proof. For $p = 1$ this is immediate, since $x_n = x_1 + \sum_{k=1}^{n-1}(x_{k+1} - x_k)$.

Fix $j > 1$, and take $n_1 = 1$, $n_2 = j$, and $n_k = k + j$ for $k > 2$. Then $|x_j| \le |x_j - x_1| + |x_1| \le ||\{x_n\}||_{V_p} + |x_1|$. Hence $\{x_n\}$ is bounded. Assume $\{x_n\}$ has two different limit points a and b. Then we can find an increasing subsequence $\{n_k\}$ with $a = \lim x_{n_{2k}}$ and $b = \lim x_{n_{2k+1}}$, so $|x_{n_{2k+1}} - x_{n_{2k}}| \ge |b - a|/2 > 0$ for large k, contradicting the convergence of the series of p-powers. □

The Lemma (which should be well-known) shows that $||\{x_n\}||_{V_p}$ is a norm (the p-variation norm) on the space BV_p^0 of all sequences of bounded p-variation converging to 0, which contains ℓ_p.

Definition. The *ϵ-jump* of a sequence $\{x_k\}$ is defined for $\epsilon > 0$ by

$$J(\epsilon) = \max\{n : \exists\, s_1 < t_1 \le s_2 < t_2 \cdots \le s_n < t_n \text{ with } |x_{t_j} - x_{s_j}| > \epsilon,\ 1 \le j \le n\}.$$

Note that $J(\epsilon) = J(\{x_k\}, \epsilon)$ is finite for every $\epsilon > 0$ if (and only if) $\{x_k\}$ converges; it counts the number of jumps of size ϵ that are observed along the sequence $\{x_k\}$. It is easy to check that $D(\{x_n\}, \epsilon/2) \ge J(\{x_n\}, \epsilon)/2$.

Let $\{x_n\}$ have bounded p-variation. If $J(\{x_n\}, \epsilon) = n$ and the jumps occur at the n pairs $s_j < t_j$, $1 \le j \le n$, as in the definition, then

$$J(\{x_n\}, \epsilon) \le \sum_{j=1}^{n} \left(\frac{|x_{t_j} - x_{s_j}|}{\epsilon} \right)^p \le \frac{1}{\epsilon^p} ||\{x_n\}||_{V_p}^p.$$

Bourgain [**B**] showed that for a probability preserving transformation θ on (Ω, μ) and $f \in L_2$ the sequence of ergodic averages $A_n f(x) := \frac{1}{n}\sum_{k=1}^{n} f(\theta^k x)$ satisfies $\|\, ||A_n f(x)||_{V_\rho} \|_2 \le c(\rho) ||f||_2$ for every $\rho > 2$. This was generalized to L_p, $1 < p < \infty$, by Jones, Kaufman, Rosenblatt, and Wierdl [**JKRW**], who proved for $\rho > 2$ the weak (1,1) inequality

$$\mu\left\{x :\ ||A_n f(x)||_{V_\rho} > \epsilon\right\} \le \frac{c(\rho)}{\epsilon} ||f||_1.$$

For further discussion and additional references, see [**CJRW**].

2. Strong laws of large numbers with rates

Our main results give more precise information on the SLLN with rate obtained in Cohen and Lin [**CL**]. Throughout this section we assume that(Ω, μ) is a σ-finite measure space. We start with a rather simple result.

Theorem 1. *Let $1 < p < \infty$. Let $\{f_n\} \subset L_p(\mu)$, and assume that for some $\frac{1}{p} < \beta \le 1$ we have*

$$\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} f_k \right\|_p = B < \infty. \tag{2}$$

Then for $0 \le \delta < \beta - \frac{1}{p}$ *we have* $\{\frac{1}{n^{1-\delta}} \sum_{k=1}^n f_k(x)\} \in \ell_p$ *a.e. Moreover, the series* $\displaystyle\sum_{k=1}^\infty \frac{f_k(x)}{k^{1-\delta}}$ *converges a.e.,* $\displaystyle\left\{\sum_{k=n}^\infty \frac{f_k(x)}{k^{1-\delta}}\right\} \in \ell_p$ *a.e., and* $\displaystyle\left\|\sum_{k=n}^\infty \frac{f_k(x)}{k^{1-\delta}}\right\|_{\ell_p}$ *is in* $L_p(\mu)$.

Proof. Denote $s_n = \sum_{k=1}^n f_k$. Then

$$\int \sum_{n=1}^\infty \left|\frac{s_n}{n^{1-\delta}}\right|^p d\mu = \sum_{n=1}^\infty \int \left|\frac{s_n}{n^{1-\beta}}\right|^p \frac{1}{n^{(\beta-\delta)p}} d\mu \le B^p \sum_{n=1}^\infty \frac{1}{n^{(\beta-\delta)p}} < \infty. \tag{3}$$

Hence $\sum_{n=1}^\infty \left|\frac{s_n}{n^{1-\delta}}\right|^p < \infty$ a.e.

Denote $\gamma = 1-\delta$. For $1 \le n < m$, Abel's summation by parts (with $s_0 = 0$) yields

$$\sum_{k=n}^m \frac{f_k}{k^\gamma} = \sum_{k=n}^m \frac{s_k - s_{k-1}}{k^\gamma} = \frac{s_m}{m^\gamma} - \frac{s_{n-1}}{n^\gamma} + \sum_{k=n}^{m-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\right) s_k. \tag{4}$$

The a.e. convergence of $\displaystyle\sum_{k=1}^\infty \frac{f_k(x)}{k^{1-\delta}}$ is proved as in Theorem 1 of [**CL**], where the boundedness of $\{||f_n||_p\}$ is not used for the a.e. convergence of the series on the right hand side of (4), so letting $m \to \infty$ in (4) we obtain

$$\sum_{k=n}^\infty \frac{f_k}{k^\gamma} = -\frac{s_{n-1}}{n^\gamma} + \sum_{k=n}^\infty \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\right) s_k. \tag{5}$$

By the first part, for a.e. x the sequence $\left\{\frac{s_n(x)}{n^\gamma}\right\}$ is in ℓ_p. Since $\sum_{k=n}^\infty \frac{1}{k^{\beta+\gamma}} = O(n^{\delta-\beta})$, and $p(\beta-\delta) > 1$ by assumption, Minkowski's inequality yields

$$\int \sum_{n=1}^\infty \left|\sum_{k=n}^\infty \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\right) s_k\right|^p d\mu \le \sum_{n=1}^\infty \int \left(\gamma \sum_{k=n}^\infty \frac{1}{k^{\beta+\gamma}} \left|\frac{1}{k^{1-\beta}} s_k\right|\right)^p d\mu$$

$$\le \sum_{n=1}^\infty \gamma^p \left(\sum_{k=n}^\infty \frac{1}{k^{\beta+\gamma}} \left\|\frac{1}{k^{1-\beta}} s_k\right\|_p\right)^p \le \gamma^p B^p \sum_{n=1}^\infty \frac{c}{n^{p(\beta-\delta)}} < \infty.$$

Hence $\displaystyle\sum_{n=1}^\infty \left|\sum_{k=n}^\infty \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\right) s_k\right|^p < \infty$ a.e., so by (5) $\displaystyle\left\{\sum_{k=n}^\infty \frac{f_k(x)}{k^{1-\delta}}\right\} \in \ell_p$ for a.e. x, and

$$\left\| \left\|\sum_{k=n}^\infty \frac{f_k(x)}{k^{1-\delta}}\right\|_{\ell_p} \right\|_{L_p(\mu)} \le C(p,\beta,\delta)B. \quad \square$$

Remarks. 1. Unlike the result of [**CL**], Theorem 1 does not require that $\sup_n ||f_n||_p$ be finite. This is due to the restriction on β and the small range for δ.

2. For $\delta = \beta - \frac{1}{p}$ the above result is no longer valid. Fix $1 < p < \infty$ and $\frac{1}{p} < \beta \le 1$. Put $f_k = k^{1-\beta} - (k-1)^{1-\beta}$, so (2) is satisfied, but for $\delta = \beta - \frac{1}{p}$ we have $\{\frac{1}{n^{1-\delta}} \sum_{k=1}^n f_k\} = \{\frac{1}{n^{1/p}}\}$ which is not in ℓ_p.

Definition. The ϵ-*deviation* function of a sequence of functions $\{g_n\}$ is defined for $\epsilon > 0$ by $D(\{g_n\},\epsilon)(x) = D(\{g_n(x)\},\epsilon)$, i.e., for each point x we look at the ϵ-deviation of the sequence of values $\{g_n(x)\}$.

Corollary. *Under the hypothesis of Theorem 1 we have*

$$\left\| D\left(\left\{\frac{1}{n^{1-\delta}}\sum_{k=1}^n f_k\right\},\epsilon\right)\right\|_1^{\frac{1}{p}} \le \frac{c}{\epsilon}\sup_n \left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^n f_k\right\|_p .$$

proof. For every point x we have (see the introduction)

$$D\left(\left\{\frac{1}{n^{1-\delta}}\sum_{k=1}^n f_k(x)\right\},\epsilon\right) \le \frac{1}{\epsilon^p}\left\|\left\{\frac{1}{n^{1-\delta}}\sum_{k=1}^n f_k(x)\right\}\right\|_{\ell_p}^p ,$$

and the result follows by integrating and applying (3).

Theorem 2. *Let $1 < p < \infty$. Let $\{f_n\} \subset L_p$ such that $\sup_n \|f_n\|_p < \infty$, and assume that (2) holds for some $0 < \beta \le 1$. For fixed $0 \le \delta < \beta(p-1)/p$, define the "averages"*

$$A_n^{(1-\delta)} := \frac{1}{n^{1-\delta}}\sum_{k=1}^n f_k.$$

Then for a.e x the sequence $\{A_n^{(1-\delta)}(x)\}$ has bounded p-variation and converges to 0. Moreover, the p-variation norm of $\{A_n^{(1-\delta)}(x)\}$ is in L_p, and satisfies the p-variational inequality

$$\left\|\sup_{\{n_k\}\nearrow}\left(\sum_{k=1}^\infty \left|A_{n_k}^{(1-\delta)} - A_{n_{k+1}}^{(1-\delta)}\right|^p\right)^{\frac{1}{p}}\right\|_p \le c\left(\sup_n\left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^n f_k\right\|_p + \sup_n \|f_n\|_p\right),$$

and thus $\sup_n |A_n^{(1-\delta)}| \in L_p$.

proof. In view of Theorem 1 and (3), we have to prove the theorem only when either $\beta \le \frac{1}{p}$, or $\beta > \frac{1}{p}$ and $\delta \ge \beta - \frac{1}{p}$, which will be assumed henceforth.

The measurability of the variation norm that occurs in the left hand side of the p-variational inequality above is handled by first restricting the supremum to all finite increasing sequences of length N (and then the series are summed for $k \le N$); this supremum is clearly measurable. These restricted suprema are monotone increasing in N, and hence the limit will also be measurable.

Throughout the arguments, c and C will denote constants that may depend on α,β,δ and p, but will not depend on x, nor even on $\{f_n\}$. The values of these constants may vary from one occurance to the next. We put $q = p/(p-1)$, the dual index of p.

Fix $\delta < \beta(p-1)/p$; this is equivalent to $\dfrac{\delta}{p(\beta-\delta)} < \dfrac{1}{q}$, so for $\epsilon > 0$ small enough we have $\dfrac{(1+\epsilon)\delta}{p(\beta-\delta)} < \dfrac{1}{q}$. For such $\epsilon > 0$ fixed, put $\alpha = \frac{1+\epsilon}{p(\beta-\delta)}$, so $\alpha\delta < \frac{1}{q}$. Note that if $\beta \le \frac{1}{p}$ then $\alpha > \frac{1}{p\beta} \ge 1$, and if $\beta > \frac{1}{p}$ and $\delta \ge \beta - \frac{1}{p}$, then $p(\beta-\delta) \le 1$; thus in any case $\alpha > 1$. Let $m_k = [k^\alpha] + 1$, which is strictly increasing since $\alpha > 1$. We first prove that $\sum_{k=1}^{\infty}\left|A_{m_k}^{(1-\delta)}(x)\right|^p$ converges a.e. to an integrable function. Since $m_k \geq k^\alpha$, we have

$$\left\|A_{m_k}^{(1-\delta)}\right\|_p^p = \left\|\frac{1}{m_k^{1-\delta}}\sum_{j=1}^{m_k} f_j\right\|_p^p \leq \left(\frac{m_k^{1-\beta}}{m_k^{1-\delta}}\right)^p \sup_n \left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^{n} f_k\right\|_p^p$$

$$= B^p\left(\frac{m_k^{1-\beta}}{m_k^{1-\delta}}\right)^p \leq B^p\frac{c}{k^{p\alpha(\beta-\delta)}} = B^p\frac{c}{k^{1+\epsilon}}\ ,$$

which yields

$$\int \sum_{k=1}^{\infty}\left|A_{m_k}^{(1-\delta)}(x)\right|^p\, d\mu = \sum_{k=1}^{\infty}\left\|A_{m_k}^{(1-\delta)}\right\|_p^p \leq B^p\sum_{k=1}^{\infty}\frac{c}{k^{1+\epsilon}} < CB^p. \tag{6}$$

Hence the series $\sum_{k=1}^{\infty}\left|A_{m_k}^{(1-\delta)}(x)\right|^p$ converges a.e.

As is now standard in such arguments, we break the variation along any given strictly increasing sequence $\{n_j\}$ into two parts, the "long variation" and the "short variation", described below. For the "long variation" we will later use the variation at times from the above sequence $\{m_k\}$. First note that for each x we have

$$\left(\sum_{k=1}^{\infty}\left|A_{m_{n_k}}^{(1-\delta)}(x) - A_{m_{n_{k+1}}}^{(1-\delta)}(x)\right|^p\right)^{\frac{1}{p}} \tag{7}$$

$$\leq 2\left(\sum_{k=1}^{\infty}\left|A_{m_{n_k}}^{(1-\delta)}(x)\right|^p\right)^{\frac{1}{p}} \leq 2\left(\sum_{k=1}^{\infty}\left|A_{m_k}^{(1-\delta)}(x)\right|^p\right)^{\frac{1}{p}}.$$

In order to handle the short variation, for each k we put $I_k = [m_k, m_{k+1}]$. For the given subsequence $\{n_j\}$, let J_k denote the set of j such that $[n_j, n_{j+1}] \subset I_k$, and let L be the set of j such that for some i we have $n_j < m_i < n_{j+1}$. Of course, J_k and L depend on $\{n_j\}$. In the series $\sum_{j=1}^{\infty}\left|A_{n_j}^{(1-\delta)}(x) - A_{n_{j+1}}^{(1-\delta)}(x)\right|^p$, the *long variation* is the sum over the indices in L, and the *short variation* is the sum over the indices in $J := \bigcup_{k\geq 1} J_k$. In order to estimate the short variation, define

$$S_k(x) := \left(\sum_{j\in J_k}\left|A_{n_j}^{(1-\delta)}(x) - A_{n_{j+1}}^{(1-\delta)}(x)\right|^p\right)^{\frac{1}{p}}.$$

Clearly, S_k depends on $\{n_j\}$. Using the inequality $|a+b+c|^p \leq 3^{p-1}(|a|^p+|b|^p+|c|^p)$, we obtain

$$S_k^p = \sum_{j\in J_k}\left|A_{n_j}^{(1-\delta)} - A_{n_{j+1}}^{(1-\delta)}\right|^p = \sum_{j\in J_k}\left|\frac{1}{n_j^{1-\delta}}\sum_{i=1}^{n_j} f_i - \frac{1}{n_{j+1}^{1-\delta}}\sum_{i=1}^{n_{j+1}} f_i\right|^p =$$

$$\sum_{j\in J_k}\left|\left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}}\right)\sum_{i=1}^{m_k} f_i + \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}}\right)\sum_{i=m_k+1}^{n_j} f_i - \frac{1}{n_{j+1}^{1-\delta}}\sum_{i=n_j+1}^{n_{j+1}} f_i\right|^p$$

$$\leq 3^{p-1}(U_k^p + V_k^p + W_k^p)$$

where

$$U_k^p(x) = \sum_{j\in J_k} \left| \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) \sum_{i=1}^{m_k} f_i(x) \right|^p,$$

$$V_k^p(x) = \sum_{j\in J_k} \left| \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) \sum_{i=m_k+1}^{n_j} f_i(x) \right|^p,$$

and

$$W_k^p(x) = \sum_{j\in J_k} \left| \frac{1}{n_{j+1}^{1-\delta}} \sum_{i=n_j+1}^{n_{j+1}} f_i(x) \right|^p.$$

Using the fact that $\|\cdot\|_{\ell_p} \le \|\cdot\|_{\ell_1}$, we obtain

$$U_k(x) = \left(\sum_{j\in J_k} \left| \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) \sum_{i=1}^{m_k} f_i(x) \right|^p \right)^{\frac{1}{p}}$$

$$\le \sum_{j\in J_k} \left| \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) \sum_{i=1}^{m_k} f_i(x) \right| = \sum_{j\in J_k} \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) m_k^{1-\beta} \left| \frac{1}{m_k^{1-\beta}} \sum_{i=1}^{m_k} f_i(x) \right|$$

$$= \sum_{j\in J_k} \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) m_k^{1-\beta} \left| A_{m_k}^{(1-\beta)}(x) \right|$$

$$\le m_k^{1-\beta} \left| A_{m_k}^{(1-\beta)}(x) \right| \left(\frac{1}{{m_k}^{1-\delta}} - \frac{1}{m_{k+1}^{1-\delta}} \right) \le \left| A_{m_k}^{(1-\beta)}(x) \right| \left(\frac{m_{k+1}^{1-\delta} - m_k^{1-\delta}}{{m_k}^{2-2\delta}} \right) m_k^{1-\beta}.$$

Since $1+t^\alpha \le (1+t)^\alpha$ for $t \ge 0$ and $\alpha \ge 1$, the definition of m_k yields

$$m_{k+1}^{1-\delta} - m_k^{1-\delta} \le ((k+2)^\alpha)^{1-\delta} - (k^\alpha)^{1-\delta} \le c k^{\alpha(1-\delta)-1},$$

and we obtain

$$\frac{m_{k+1}^{1-\delta} - m_k^{1-\delta}}{{m_k}^{2-2\delta}} m_k^{1-\beta} \le c \frac{k^{\alpha(1-\delta)-1}}{k^{\alpha(2-2\delta)}} k^{\alpha(1-\beta)} \le \frac{c}{k^{\alpha(\beta-\delta)+1}} \le \frac{c}{k}. \tag{8}$$

Hence $U_k(x) \le \frac{c}{k} \left| A_{m_k}^{(1-\beta)}(x) \right|$.

Using again the fact that $\|\cdot\|_{\ell_p} \le \|\cdot\|_{\ell_1}$, we see that

$$V_k(x) \le \sum_{j\in J_k} \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) \left| \sum_{i=m_k+1}^{n_j} f_i(x) \right|$$

$$\le \sum_{j\in J_k} \left(\frac{1}{n_j^{1-\delta}} - \frac{1}{n_{j+1}^{1-\delta}} \right) \sum_{i=m_k+1}^{m_{k+1}} |f_i(x)|$$

$$\le \left(\frac{1}{m_k^{1-\delta}} - \frac{1}{m_{k+1}^{1-\delta}} \right) \sum_{i=m_k+1}^{m_{k+1}} |f_i(x)| \le \frac{m_{k+1}^{1-\delta} - m_k^{1-\delta}}{m_k^{2-2\delta}} \sum_{i=m_k+1}^{m_{k+1}} |f_i(x)|.$$

The estimate of the first factor in (8) yields $V_k(x) \le \frac{c}{k^{1+\alpha-\alpha\delta}} \sum_{i=m_k+1}^{m_{k+1}} |f_i(x)|$.

For the third term in $S_k^p(x)$, we use $||\cdot||_{\ell_p} \le ||\cdot||_{\ell_1}$ and Hölder's inequality, and obtain

$$W_k^p(x) = \sum_{j\in J_k} \left| \frac{1}{n_{j+1}^{1-\delta}} \sum_{i=n_j+1}^{n_{j+1}} f_i(x) \right|^p \le \left(\frac{1}{m_k^{1-\delta}}\right)^p \sum_{j\in J_k} \left(\sum_{i=n_j+1}^{n_{j+1}} |f_i(x)| \right)^p$$

$$\le \left(\frac{1}{m_k^{1-\delta}}\right)^p \left(\sum_{j\in J_k} \left| \sum_{i=n_j+1}^{n_{j+1}} |f_i(x)| \right| \right)^p \le \left(\frac{1}{m_k^{1-\delta}}\right)^p \left(\sum_{i=m_k+1}^{m_{k+1}} |f_i(x)| \right)^p$$

$$\le \left(\frac{1}{k^{\alpha(1-\delta)}}\right)^p (m_{k+1}-m_k)^{p/q} \sum_{i=m_k+1}^{m_{k+1}} |f_i(x)|^p.$$

For fixed k define the following functions (which do not depend on $\{n_j\}$):

$$F_k(x) := \frac{1}{k^p} \left| A_{m_k}^{(1-\beta)}(x) \right|^p ,$$

$$G_k(x) := \frac{1}{k^{(1+\alpha-\alpha\delta)p}} \left(\sum_{i=m_k+1}^{m_{k+1}} |f_i(x)| \right)^p ,$$

and

$$H_k(x) := \frac{1}{k^{\alpha(1-\delta)p}} (m_{k+1}-m_k)^{p/q} \sum_{i=m_k+1}^{m_{k+1}} |f_i(x)|^p.$$

We have shown that $S_k^p(x) \le c_1 F_k(x) + c_2 G_k(x) + H_k(x)$. Putting $F = \sum_{k=1}^\infty F_k$, $G = \sum_{k=1}^\infty G_k$, and $H = \sum_{k=1}^\infty H_k$, we conclude that

The "short p-variation" relative to any increasing sequence $\{n_j\}$ satisfies

$$\sum_{j\in J} |A_{n_j}^{(1-\delta)}(x) - A_{n_{j+1}}^{(1-\delta)}(x)|^p \le c_1 F(x) + c_2 G(x) + H(x). \tag{9}$$

In order to finally estimate the p-variation of a given sequence $\{n_j\}$, fix $j \in L$, and let $i_1 = i_1(j)$ be the smallest i with $n_j < m_i$, and let $i_2 = i_2(j)$ be the largest i with $m_i < n_{j+1}$. We then have $m_{i_1-1} \le n_j < m_{i_1} \le m_{i_2} < n_{j+1}$, and obtain

$$|A_{n_j}^{(1-\delta)} - A_{n_{j+1}}^{(1-\delta)}|^p \tag{10}$$

$$\le 3^{p-1}(|A_{n_j}^{(1-\delta)} - A_{m_{i_1}}^{(1-\delta)}|^p + |A_{m_{i_1}}^{(1-\delta)} - A_{m_{i_2}}^{(1-\delta)}|^p + |A_{m_{i_2}}^{(1-\delta)} - A_{n_{j+1}}^{(1-\delta)}|^p).$$

We now define a new increasing sequence of integers $\{n'_j\}$ which is the refinement of $\{n_j\}$ by joining all the integers $\{m_{i_1(j)}, m_{i_2(j)} : j \in L\}$ (if $i_1(j) = i_2(j)$ we add only $m_{i_1(j)}$). Similarly to the definition of J and L for the original sequence $\{n_j\}$, we define $J'_k := \{j : [n'_j, n'_{j+1}] \subset I_k\}$, $J' := \bigcup J'_k$, and $L' := \{j : n'_j < m_i < n'_{j+1}$ for some $i\}$. Let $j \in J_k$; we have $n_j = n'_{j'}$ for some j', and the definition of J_k yields that $j' \in J'_k$; hence $\{n_j : j \in J\} \subset \{n'_j : j \in J'\}$. When $j \in L$, there is no element of $\{m_k\}$ between n_j and $m_{i_1(j)}$, while $n_{j+1} > m_{i_1(j)}$ and $m_{i_1(j)-1} \le n_j$, so if $n_j = n'_{j'}$, then $[n'_{j'}, n'_{j'+1}] \subset I_{i_1(j)-1}$, so $j' \in J'$. All this means that the short variation of $\{n'_j\}$ contains *all* the variation of the original $\{n_j\}$. Furthermore, for $j \in L$ we always have $m_{i_2(j)} \in \{n'_{j'} : j' \in J'\}$; if $i_2(j) = i_1(j)+1$, then also $m_{i_1(j)} \in \{n'_{j'} : j' \in J'\}$; when $i_2(j) > i_1(j)+1$, then $m_{i_1(j)}$ is in $\{n'_{j'} : j' \in L'\}$, so

$\{n'_{j'} : j' \in L'\} = \{m_{i_1(j)} : j \in L,\, i_1(j)+1 < i_2(j)\}$. Using (10), and then applying (9) to the short variation of $\{n'_j\}$ and (7) to the long one, we have

$$\sum_{j=1}^{\infty} |A_{n_j}^{(1-\delta)}(x) - A_{n_{j+1}}^{(1-\delta)}(x)|^p \le 3^{p-1} \sum_{j=1}^{\infty} |A_{n'_j}^{(1-\delta)}(x) - A_{n'_{j+1}}^{(1-\delta)}(x)|^p$$

$$\le 3^{p-1}\left[c_1 F(x) + c_2 G(x) + H(x)\right] + 3^{p-1} 2^p \sum_{k=1}^{\infty} |A_{m_k}^{(1-\delta)}(x)|^p.$$

Since the estimate does not depend on the sequence $\{n_j\}$, we have

$$\text{(11)} \quad \sup_{\{n_j\}\nearrow} \sum_{j=1}^{\infty} |A_{n_j}^{(1-\delta)} - A_{n_{j+1}}^{(1-\delta)}|^p \le 3^{p-1}(c_1 F + c_2 G + H) + 3^{p-1} 2^p \sum_{k=1}^{\infty} |A_{m_k}^{(1-\delta)}|^p.$$

In order to prove the claimed p-variational inequality, we have to show the integrability of the right-hand side of (11), with an appropriate estimate. For the last term we use (6). For the integrals of F, G, and H we look at their summands.

$$\int \sum_{k=1}^{\infty} F_k(x) d\mu = \sum_{k=1}^{\infty} \int \frac{1}{k^p} \left| A_{m_k}^{(1-\beta)}(x) \right|^p d\mu \le B^p \sum_{k=1}^{\infty} \frac{1}{k^p} < \infty.$$

With $K := \sup_n ||f_n||_p$ and using Minkowski's inequality, we obtain

$$\int G_k d\mu = \frac{1}{k^{(1+\alpha-\alpha\delta)p}} \int \Big(\sum_{i=m_k+1}^{m_{k+1}} |f_i| \Big)^p d\mu \le \frac{1}{k^{(1+\alpha-\alpha\delta)p}} (m_{k+1} - m_k)^p K^p$$

$$\le cK^p \frac{k^{(\alpha-1)p}}{k^{(1+\alpha-\alpha\delta)p}} = \frac{cK^p}{k^{p(2-\alpha\delta)}}.$$

Thus, $G = \sum_k G_k$ will be integrable, with the desired estimate, if $p(2-\alpha\delta) > 1$. This is equivalent to $1 - \alpha\delta > \frac{1}{p} - 1$, or $\alpha\delta < 1 + \frac{1}{q}$, which certainly holds, since $\alpha\delta < \frac{1}{q}$ by the definition of α.

Using $p/q = p-1$ and the estimate $m_{k+1} - m_k \le ck^{\alpha-1}$, we obtain

$$\int H_k(x)\, d\mu \le \frac{1}{k^{\alpha(1-\delta)p}} (m_{k+1} - m_k)^{p/q+1}\; K^p \le \frac{CK^p}{k^{\alpha(1-\delta)p - (\alpha-1)p}}.$$

Thus, $H = \sum_k H_k$ is integrable, with the desired estimate, since $p(1-\alpha\delta) > 1$, which is equivalent to $\alpha\delta < \frac{1}{q}$, holds.

We therefore have the required p-variational inequality, by (11), which implies the a.e convergence of $\{A_n^{(1-\delta)}\}$, and since (2) yields norm convergence to 0, the limit in the a.e. convergence is 0. The inequality $\sup_j |x_j| \le ||\{x_n\}||_{V_p} + |x_1|$ proved in the Lemma yields that $\sup_n\{|A_n^{(1-\delta)}|\}$ is in L_p. □

Definition. The *ϵ-jump* function of a sequence of functions $\{g_n\}$ is defined for $\epsilon > 0$ by $J(\{g_n\}, \epsilon)(x) = J(\{g_n(x)\}, \epsilon)$, i.e., for each point x we we look at the ϵ-jump of the sequence of values $\{g_n(x)\}$.

Corollary. *Under the hypothesis of Theorem 2 we have*

$$\left\| J(\{A_n^{(1-\delta)}\},\epsilon) \right\|_1^{\frac{1}{p}} \le \frac{c}{\epsilon}\left(\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^n f_k \right\|_p + \sup_n \|f_n\|_p \right).$$

proof. For every point x we have (see the introduction)

$$J(\{A_n^{(1-\delta)}(x)\},\epsilon)^{\frac{1}{p}} \le \frac{||\{A_n^{(1-\delta)}(x)||_{V_p}}{\epsilon} = \frac{1}{\epsilon}\left(\sup_{(n_k)\nearrow} \sum_{k=1}^\infty \left| A_{n_k}^{(1-\delta)}(x) - A_{n_{k+1}}^{(1-\delta)}(x) \right|^p \right)^{\frac{1}{p}}$$

So the result follows by taking the L_p-norm of each side and applying Theorem 2.

Theorem 3. *Let $1 < p < \infty$. Let $\{f_n\} \subset L_p$ such that $\sup_n \|f_n\|_p = K < \infty$, and assume that (2) holds for some $0 < \beta \le 1$. Then for fixed $0 \le \delta < \beta(p-1)/p$, the sequence of finite sums $\left\{ \sum_{k=1}^n \frac{f_k(x)}{k^{1-\delta}} \right\}$ has a.e. bounded p-variation, hence the series converges. Moreover, we have*

$$\left\| \sup_{\{n_j\}\nearrow} \left(\sum_{j=1}^\infty \left| \sum_{k=n_j+1}^{n_{j+1}} \frac{f_k}{k^{1-\delta}} \right|^p \right)^{\frac{1}{p}} \right\|_p \le C\left(\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^n f_k \right\|_p + \sup_n \|f_n\|_p \right),$$

Proof. As before, we use the notations $s_n := \sum_{k=1}^n f_k$ and $A_n^{(1-\delta)} := \frac{1}{n^{1-\delta}} s_n$, and put $\gamma := 1-\delta$. For every increasing sequence $\{n_j\}$ we use (4) with $n = 1$ and $m = n_j$, and after subtracting we obtain

$$\sum_{k=n_j+1}^{n_{j+1}} \frac{f_k}{k^\gamma} = \left(A_{n_{j+1}}^{(1-\delta)} - A_{n_j}^{(1-\delta)} \right) + \sum_{k=n_j}^{n_{j+1}-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma} \right) s_k.$$

Together with Minkowski's inequality in ℓ_p, this yields

$$\Big(\sum_{j=1}^\infty \Big| \sum_{k=n_j+1}^{n_{j+1}} \frac{f_k}{k^\gamma} \Big|^p \Big)^{\frac{1}{p}}$$

$$= \left(\sum_{j=1}^\infty \left| \left(A_{n_{j+1}}^{(1-\delta)} - A_{n_j}^{(1-\delta)} \right) + \sum_{k=n_j}^{n_{j+1}-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma} \right) s_k \right|^p \right)^{\frac{1}{p}}$$

$$\le \left(\sum_{j=1}^\infty \left| \left(A_{n_{j+1}}^{(1-\delta)} - A_{n_j}^{(1-\delta)} \right) \right|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^\infty \left| \sum_{k=n_j}^{n_{j+1}-1} \left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma} \right) s_k \right|^p \right)^{\frac{1}{p}}.$$

Hence

$$\sup_{\{n_j\}\nearrow} \left(\sum_{j=1}^\infty \left| \sum_{k=n_j+1}^{n_{j+1}} \frac{f_k}{k^\gamma} \right|^p \right)^{\frac{1}{p}} \le \tag{12}$$

$$\sup_{\{n_j\}\nearrow}\Big(\sum_{j=1}^{\infty}\Big|\big(A^{(1-\delta)}_{n_{j+1}}-A^{(1-\delta)}_{n_j}\big)\Big|^p\Big)^{\frac{1}{p}}+\sup_{\{n_j\}\nearrow}\left(\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1}\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)s_k\right|^p\right)^{\frac{1}{p}}$$

with first term on the right in $L_p(\mu)$, with an appropriate estimate of the norm, by Theorem 2. It remains to check the last term. For this put $S(\{n_j\}) := \left(\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1}\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)s_k\right|^p\right)^{\frac{1}{p}}$. Then the norm inequality $||\cdot||_{\ell_p}\le||\cdot||_{\ell_1}$ and obvious estimations yield

$$S(\{n_j\})\le\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1}\left(\frac{1}{k^\gamma}-\frac{1}{(k+1)^\gamma}\right)s_k\right|$$

$$\le c\sum_{j=1}^{\infty}\sum_{k=n_j}^{n_{j+1}-1}\frac{|s_k|}{k^{1-\beta}}\frac{1}{k^{\beta+\gamma}}=c\sum_{k=1}^{\infty}\frac{|s_k|}{k^{1-\beta}}\frac{1}{k^{\beta+\gamma}}\ .$$

Since the right hand side does not depend on $\{n_j\}$, and $\beta+\gamma>1$, we obtain

$$\left\|\sup_{\{n_j\}\nearrow}S(\{n_j\})\right\|_p\le c\left\|\sum_{k=1}^{\infty}\frac{|s_k|}{k^{1-\beta}}\frac{1}{k^{\beta+\gamma}}\right\|_p\le cB\sum_{k=1}^{\infty}\frac{1}{k^{\beta+\gamma}}=CB,$$

which shows that also the last term in (12) is in $L_p(\mu)$ with the desired estimate of the norm, and the theorem is proved. □

Corollary. *Under the hypothesis of Theorem 2 we have*

$$\left\|J\left(\left\{\sum_{k=1}^{n}\frac{f_k}{k^{(1-\delta)}}\right\},\epsilon\right)\right\|_1^{\frac{1}{p}}\le\frac{c}{\epsilon}\left(\sup_n\left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^{n}f_k\right\|_p+\sup_n\|f_n\|_p\right).$$

Remarks. 1. The a.e. convergence obtained in Theorems 2 and 3 was first proved in[**CL**].

2. The results of Theorems 2 and 3 (in fact, even the a.e. convergence proved in [**CL**]) cannot be improved in general, as the following example shows.

Example 1. *Under the assumptions of Theorem 2, the a.e. convergence of* $\{\frac{1}{n^{1-\delta}}\sum_{k=1}^{n}f_k\}$ *can fail if* $\delta\ge\beta(p-1)/p$.

We will work on $[0,1)$ with Lebesgue measure, thought of as the unit circle. Fix $p>1$ and $\beta<1$. Let $n_k=[k^\alpha]$ with $\alpha=\frac{1}{\beta}$. For each k, let I_k be a half open interval of length $\frac{1}{k}$, such that I_{k+1} is adjacent to, and to the right of I_k, mod 1 (i.e., I_k corresponds to a half open arc). I_1 is the whole space, and for $k>1$ the intervals (arcs) I_k and I_{k+1} are clearly disjoint. Also note that each $x\in[0,1)$ will be in infinitely many of the I_k.

Let $\tilde{f}_j(x)=k^{1/p}\chi_{I_k}(x)$ if $n_k<j\le n_{k+1}$. Note that $\|\tilde{f}_j\|_p=k^{1/p}(1/k)^{1/p}=1$ where $n_k<j\le n_{k+1}$. Also note that

$$\left\|\sum_{j=n_k+1}^{n_{k+1}}\tilde{f}_j\right\|_p=\left\|(n_{k+1}-n_k)k^{1/p}\chi_{I_k}\right\|_p=n_{k+1}-n_k\approx k^{\alpha-1}.$$

Since $\alpha > 1$ we have $\{n_{k+1} - n_k\}$ increasing. Define $f_j(x) = \tilde{f}_j(x) - \tilde{f}_{n_k}(x)$ if $n_k < j \le n_k + (n_k - n_{k-1})$ and $f_j = \tilde{f}_j(x)$ when $n_k + (n_k - n_{k-1}) < j \le n_{k+1}$.

The idea is that for the first few terms of the k-th block, we both put positive mass on the interval I_k and put negative mass on the interval I_{k-1}. We stop putting negative mass on I_{k-1} after we have cancelled all the previous positive masses on it, but continue to put mass on I_k until we reach n_{k+1}.

Thus $\|f_j\|_p \le 2$ for each j, and by the definitions

$$\sum_{j=1}^{n_{k+1}} f_j = \sum_{j=n_k+1}^{n_{k+1}} \tilde{f}_j = (n_{k+1} - n_k) k^{1/p} \chi_{I_k}(x).$$

Using our choice of n_k, we see that

$$\left\| \frac{1}{n_{k+1}^{1-\beta}} \sum_{j=1}^{n_{k+1}} f_j \right\|_p = \frac{n_{k+1} - n_k}{n_{k+1}^{1-\beta}} \approx \frac{k^{\alpha-1}}{k^{\alpha(1-\beta)}} = \frac{1}{k^{1-\alpha\beta}} \ . \tag{13}$$

For any n, let $n_k \le n < n_{k+1}$. Since $||f_j||_p \le 2$, (13) yields

$$\left\| \frac{1}{n^{1-\beta}} \sum_{j=1}^{n} f_j \right\|_p \le \left\| \frac{1}{n_k^{1-\beta}} \sum_{j=1}^{n_k} f_j \right\|_p + \frac{2(n_{k+1} - n_k)}{n_k^{1-\beta}} \approx \frac{3}{(k-1)^{1-\alpha\beta}}.$$

Since we selected $\alpha = \frac{1}{\beta}$, we have $1 - \alpha\beta = 0$, so (2) is satisfied.

However, on I_k the height of the "average" is

$$\frac{1}{n_{k+1}^{1-\delta}} \sum_{j=1}^{n_{k+1}} f_j = \frac{n_{k+1} - n_k}{n_{k+1}^{1-\delta}} k^{1/p} \approx \frac{k^{\alpha-1} k^{1/p}}{k^{\alpha(1-\delta)}} = \frac{1}{k^{1-\alpha\delta-1/p}} \ .$$

Hence on I_k we will have height greater than some fixed positive constant provided $1 - \delta/\beta - 1/p = 1 - \alpha\delta - 1/p \le 0$, which is $\delta \ge \beta(p-1)/p$. Since every $x \in [0, 1)$ is in infinitely many I_k, we obtain $\limsup_k \frac{1}{n_{k+1}^{1-\delta}} \sum_{j=1}^{n_{k+1}} f_j(x) > 0$ for every x. Since $\sum_{j=1}^{n_{k+1}} f_j(x) = 0$ for $x \notin I_k$, and each x is outside infinitely many I_k, we have $\liminf_k \frac{1}{n_{k+1}^{1-\delta}} \sum_{j=1}^{n_{k+1}} f_j(x) = 0$ for every x. Hence $\{\frac{1}{n^{1-\delta}} \sum_{j=1}^{n} f_j(x)\}$ is everywhere divergent.

THEOREM 4. *Let $1 \le p < \infty$ and $1 < q < \infty$. Let $\{f_n\} \subset L_p(\mu) \cap L_q(\mu)$ such that $\sup_n \|f_n\|_q < \infty$, and assume that (2) holds for some $0 < \beta \le 1$. Then for $0 \le \delta < \max\{\beta - \frac{1}{p}, \frac{(q-1)p\beta}{q+(q-1)p}\}$ the sequence $\{\frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k\}$ converges to 0 a.e., and the series $\displaystyle\sum_{k=1}^{\infty} \frac{f_k(x)}{k^{1-\delta}}$ converges a.e.*

PROOF. If $\beta > \frac{1}{p}$, we can apply Theorem 1. We first check when, in this case, the assertion of the theorem enlarges the interval for δ; it turns out that $\beta - \frac{1}{p} > \frac{(q-1)p\beta}{q+(q-1)p}$ is equivalent to $pq\beta > q + (q-1)p$. Put $r := \frac{q+(q-1)p}{pq\beta}$; we have to deal only with the case $r \ge 1$ (which is obviously satisfied also when $\beta \le \frac{1}{p}$).

Fix $\delta \in [0, \frac{(q-1)p\beta}{q+(q-1)p})$. We first prove that $\frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k(x) \to 0$ a.e., by modifying the proof of Proposition 1 of [**CL**] (which treats the case $q = p$). The assumption on δ yields

$$\text{(i) } (\beta-\delta)rp > 1 \quad \text{and} \quad \text{(ii) } (1-r\delta)q > 1$$
since we have equality for the above value of r when $\delta = \frac{(q-1)p\beta}{q+(q-1)p}$.

Define $n_m = [m^r]+1$ (which is strictly increasing since $r \ge 1$). Then (i) yields

$$\int \sum_{m=1}^{\infty} \left| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right|^p d\mu = \sum_{m=1}^{\infty} \left\| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right\|_p^p \le B^p \sum_{m=1}^{\infty} \frac{1}{m^{rp(\beta-\delta)}} < \infty \, , \tag{14}$$

so $\sum_{m=1}^{\infty} \left| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \right|^p$ converges a.e., which implies $\frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k(x) \to 0$ a.e.

For $n_m \le n < n_{m+1}$ we have [**CL**]

$$\left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}} \sum_{k=1}^{n_m} f_k \right| \le \frac{1}{n_m^{1-\delta}} \sum_{k=n_m+1}^{n_{m+1}} |f_k|. \tag{15}$$

With $C := \sup_n ||f_n||_q$ we obtain, as in [**CL**],

$$\int \max_{n_m \le n < n_{m+1}} \left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}} \sum_{k=1}^{n_m} f_k \right|^q d\mu \le \int \left[\frac{1}{n_m^{1-\delta}} \sum_{k=n_m+1}^{n_{m+1}} |f_k| \right]^q d\mu$$

$$\le \left[\frac{1}{n_m^{1-\delta}} \sum_{k=n_m+1}^{n_{m+1}} ||f_k||_q \right]^q \le C^q \left(\frac{n_{m+1}-n_m}{n_m^{1-\delta}} \right)^q \le C^q \left(\frac{m+2}{m} \right)^{(r-1)q} \frac{(2r)^q}{m^{(1-r\delta)q}}.$$

Since $(1-r\delta)q > 1$ by (ii), we have a convergent series, which proves that

$$\max_{n_m \le n < n_{m+1}} \left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k - \frac{1}{n^{1-\delta}} \sum_{k=1}^{n_m} f_k \right|^q \underset{m\to\infty}{\longrightarrow} 0 \quad \text{a.s.}$$

Since $|\frac{1}{n^{1-\delta}} \sum_{k=1}^{n_m} f_k| \le |\frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k| \to 0$ a.e., we have $|\frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k| \to 0$ a.e.

The a.e. convergence of the series $\displaystyle\sum_{k=1}^{\infty} \frac{f_k(x)}{k^{1-\delta}}$ is proved, using (4) (with $n=1$), as in Theorem 1 of [**CL**]; see the proof of our Theorem 1. □

Remarks. 1. Note that we may have $1 < q < p$, so when μ is finite no convergence follows from Theorem 2.

2. When μ is finite and $q > p$, we obviously have also $\sup_n ||f_n||_p < \infty$, the assumption of Theorem 2; however, Theorem 4 yields a larger interval for δ. In any case, for fixed p, the larger q is, the larger the interval for δ is.

3. When μ is finite, we can also prove (as in [**CL**]) that $\displaystyle\sup_{n>0} \left| \sum_{k=1}^{n} \frac{f_k}{k^{1-\delta}} \right|$ is in $L_{\min\{p,q\}}$.

When μ is finite and $\sup_n ||f_n||_\infty < \infty$, we can apply the previous theorem, and let $q \to \infty$ to obtain the interval for δ, given in the case $a_k \equiv 1$ of the next theorem. However, when μ is not finite this cannot be done. For example, on $[0,\infty)$ with Lebesgue's measure let $A_n := [0,n)$ and $f_n = (-1)^n \chi_{A_n}$; then $\sup ||f_n||_q = \infty$ for any $1 < q < \infty$, while for $1 < p < \infty$ (2) is satisfied with $\beta = 1 - 1/p$.

Definition. Let $\{a_k\}$ be a sequence of (complex) numbers, and let $1 \le t < \infty$; we say that $\{a_k\} \in W_t$ if $\sup_{n>0} \frac{1}{n}\sum_{k=1}^n |a_k|^t < \infty$. If $\{a_k\}$ is bounded we say that $\{a_k\} \in W_\infty$.

Theorem 5. *Let $1 \le p < \infty$, and let $\{f_n\} \subset L_p(\mu)$ such that $\sup_n \|f_n\|_\infty < \infty$. Let $1 < t \le \infty$ with dual index $s := t/(t-1)$, and let $\{a_k\} \in W_t$. If for some $0 < \beta \le 1$ we have*

$$\sup_{n>0} \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^n a_k f_k \right\|_p < \infty,$$

then for $0 \le \delta < \max\{\frac{p}{p+s}\beta,\, \beta - \frac{1}{p}\}$ the sequence $\{\frac{1}{n^{1-\delta}}\sum_{k=1}^n a_k f_k\}$ converges to 0 a.e., and the series $\displaystyle\sum_{k=1}^\infty \frac{a_k f_k(x)}{k^{1-\delta}}$ converges a.e. When μ is finite, for δ as above we also have $\sup_{n>0} |\frac{1}{n^{1-\delta}}\sum_{k=1}^n a_k f_k| \in L_p$ and $\displaystyle\sup_{n>0}\left|\sum_{k=1}^n \frac{a_k f_k(x)}{k^{1-\delta}}\right| \in L_p$.

Proof. We want to check when the value of the upper limit for δ is $\beta - \frac{1}{p}$. This requires first that $\beta > \frac{1}{p}$ (in which case Theorem 1 applies to $\{a_k f_k\}$). The inequality $\beta - \frac{1}{p} > \frac{p}{p+s}\beta$ is equivalent to $ps\beta/(p+s) > 1$. We therefore have to prove the theorem only when $r := \frac{p+s}{ps\beta} \ge 1$. Then for fixed δ with $0 \le \delta < \frac{p}{p+s}\beta$ we have (since for $\delta = \frac{p}{p+s}\beta = \frac{1}{rs}$ equality holds)

(i) $rp(\beta - \delta) > 1$ and (ii) $1 - rs\delta > 0$.

Let $n_m = [m^r] + 1$, which is strictly increasing since $r \ge 1$. Replacing f_k in (14) by $a_k f_k$ we obtain by (i), as in the previous proof, that $\displaystyle\frac{1}{n_m^{1-\delta}}\sum_{k=1}^{n_m} a_k f_k \to 0$ a.e.

Put $K_1 := \sup_n \|f_n\|_\infty$. Let $K_2 := \sup_n(\frac{1}{n}\sum_{k=1}^n |a_k|^t)^{1/t}$ if $t < \infty$, and $K_2 := \sup_n |a_n|$ if $t = \infty$. For $n_m \le n < n_{m+1}$ we obtain, using (15) with f_k replaced by $a_k f_k$, and then Hölder's inequality in case $t < \infty$ (i.e., $s > 1$),

$$\left|\frac{1}{n^{1-\delta}}\sum_{k=1}^n a_k f_k - \frac{1}{n^{1-\delta}}\sum_{k=1}^{n_m} a_k f_k\right| \le \frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} |a_k f_k| \le K_1 \frac{1}{n_m^{1-\delta}}\sum_{k=n_m+1}^{n_{m+1}} |a_k|$$

$$\overset{\text{if } s>1}{\le} K_1 \frac{1}{n_m^{1-\delta}} \left(\sum_{k=n_m+1}^{n_{m+1}} |a_k|^t\right)^{\frac{1}{t}} (n_{m+1} - n_m)^{\frac{1}{s}}$$

$$\le K_1 K_2 n_m^\delta \left(\frac{n_{m+1}}{n_m}\right)^{\frac{1}{t}} \left(\frac{n_{m+1} - n_m}{n_m}\right)^{\frac{1}{s}} \le (2^r + 1)^{\frac{1}{t}} K_1 K_2 \left(\frac{n_{m+1} - n_m}{n_m^{1-s\delta}}\right)^{\frac{1}{s}}.$$

(If $t = \infty$ we take $s = 1$, and skip the middle line above). We now use $r \ge 1$ and the definiton of n_m to obtain, as in [**CL**] (see proof of Theorem 4)

$$\frac{n_{m+1} - n_m}{n_m^{1-s\delta}} \le \frac{2r(m+2)^{r-1}}{m^{r-1} m^{1-rs\delta}} = 2r\left(\frac{m+2}{m}\right)^{r-1} \frac{1}{m^{1-rs\delta}}.$$

Since $1 - rs\delta > 0$ by (ii), we conclude (with $K := K_1 K_2 (2^r+1)^{1/t}(2r)^{1/s}$) that

$$\left|\frac{1}{n^{1-\delta}}\sum_{k=1}^n a_k f_k\right| \le \left|\frac{1}{n_m^{1-\delta}}\sum_{k=1}^{n_m} a_k f_k\right| + K\Big(\frac{m+2}{m}\Big)^{\frac{r-1}{s}} \left(\frac{1}{m^{1-rs\delta}}\right)^{\frac{1}{s}} \underset{m\to\infty}{\longrightarrow} 0. \tag{16}$$

The a.e. convergence of the series $\sum_{k=1}^{\infty}\frac{a_k f_k(x)}{k^{1-\delta}}$ is proved as in [**CL**]; see the proof of our Theorem 1.

When μ is finite, the constant functions are in $L_p(\mu)$; using (14) with $\{f_k\}$ replaced by $\{a_k f_k\}$, we obtain from (16)

$$\sup_{n>0}\left|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} a_k f_k\right| \le \sup_{m>0}\left|\frac{1}{n_m^{1-\delta}}\sum_{k=1}^{n_m} a_k f_k\right| + K\cdot 3^{(r-1)/s} \in L_p(\mu).$$

When μ is finite, $\sup_{n>0}\left|\sum_{k=1}^{n}\frac{a_k f_k(x)}{k^{1-\delta}}\right| \in L_p$ is proved as in Theorem 1 of [**CL**]. □

Corollary. *Let $1 \le p < \infty$. Let $\{f_n\} \subset L_p(\mu)$ such that (2) holds for some $0 < \beta \le 1$. In addition, assume that $\sup_n ||f_n||_\infty < \infty$. Then for $0 \le \delta < \beta p/(p+1)$ the sequence $\{\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k\}$ converges to 0 a.e., and the series $\sum_{k=1}^{\infty}\frac{f_k(x)}{k^{1-\delta}}$ converges a.e. When μ is finite, for δ as above we also have $\sup_{n>0}|\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k| \in L_p$ and $\sup_{n>0}\left|\sum_{k=1}^{n}\frac{f_k(x)}{k^{1-\delta}}\right| \in L_p$.*

Proof. Note that $\frac{p}{p+1}\beta > \beta - \frac{1}{p}$, and apply Theorem 5 with $a_k = 1$. □

Remarks. 1. The proof of the a.e. convergence in the corollary does not require that $\{||f_n||_p\}$ be bounded, but when μ is finite this follows from the boundedness of the L_∞-norms.

2. Note that Theorem 4 and the previous corollary hold also for $p = 1$, while in general for $p = 1$ condition (2) does not imply a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} f_k$ – see Example 1 in [**CL**] (the condition $0 \le \delta < \beta(p-1)/p$ cannot be satisfied when $p = 1$, so Theorems 1 and 2 are meaningless for $p = 1$).

3. The speed of convergence obtained in Theorem 2, namely the bounded p-variation of $\{\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k(x)\}$, may fail in the corollary when $\delta \ge \beta(p-1)/p$ (although the sequence converges), as shown by the following simple example: let μ be finite and $p > 1$, and let $f_k = (-1)^{k+1}$ be constant functions. Then (2) is satisfied with $\beta = 1$, but for $\delta \ge (p-1)/p$ we have

$$\sum_{n=1}^{\infty}\left|A_n^{(1-\delta)} - A_{n+1}^{(1-\delta)}\right|^p \ge \sum_{n=1}^{\infty}\frac{1}{(n+1)^{p(1-\delta)}} = \infty.$$

Example 2. *Under the assumptions of the corollary, the a.e. convergence of $\{\frac{1}{n^{1-\delta}}\sum_{k=1}^{n} f_k\}$ can fail if $\delta \ge \beta p/(p+1)$.*

We modify Example 1. We still work on $[0, 1)$ and define the same sets $\{I_k\}$, but we now take $n_k = [k^\alpha]$ with $\alpha = (p+1)/p\beta$. Put $\tilde{f}_j = \chi_{I_k}$ when $n_k < j \le n_{k+1}$, and define $\{f_j\}$ as before: $f_j = \tilde{f}_j - \tilde{f}_{n_k}$ when $n_k < j \le n_k + (n_k - n_{k-1})$, and $f_j = \tilde{f}_j$ when $n_k + (n_k - n_{k-1}) < j \le n_{k+1}$. Thus $||f_j||_\infty = 1$ for every j, and by the definitions

$$\sum_{j=1}^{n_{k+1}} f_j = \sum_{j=n_k+1}^{n_{k+1}} \tilde{f}_j = (n_{k+1} - n_k)\chi_{I_k}.$$

Since $||\chi_{I_k}||_p = k^{-1/p}$, the definition of n_k yields

$$\left\| \frac{1}{n_{k+1}^{1-\beta}} \sum_{j=1}^{n_{k+1}} f_j \right\|_p = \frac{n_{k+1} - n_k}{n_{k+1}^{1-\beta}} k^{-1/p} \approx \frac{k^{\alpha-1}}{k^{\alpha(1-\beta)+1/p}} = \frac{1}{k^{1-\alpha\beta+1/p}} \ . \tag{17}$$

For any n, let $n_k \le n < n_{k+1}$. Since $||f_j||_p \le ||\chi_{I_k}||_p + ||\chi_{I_{k-1}}||_p < 2(k-1)^{-1/p}$ when $n_k < j \le n_{k+1}$, (17) yields

$$\left\| \frac{1}{n^{1-\beta}} \sum_{j=1}^{n} f_j \right\|_p \le \left\| \frac{1}{n_k^{1-\beta}} \sum_{j=1}^{n_k} f_j \right\|_p + \frac{(n_{k+1} - n_k)}{n_k^{1-\beta}} 2(k-1)^{-1/p} \approx \frac{3}{(k-1)^{1-\alpha\beta+1/p}}.$$

By our choice of α we have $1 - \alpha\beta + 1/p = 0$, so (2) is satisfied.

However, on I_k the height of the "average" is

$$\frac{1}{n_{k+1}^{1-\delta}} \sum_{j=1}^{n_{k+1}} f_j = \frac{n_{k+1} - n_k}{n_{k+1}^{1-\delta}} \approx \frac{k^{\alpha-1}}{k^{\alpha(1-\delta)}} = \frac{1}{k^{1-\alpha\delta}} \ .$$

Hence on I_k we will have height greater than some fixed positive constant provided $1-\alpha\delta \le 0$, which is $\delta \ge \beta p/(p+1)$. Since every $x \in [0,1)$ is in infinitely many I_k, we obtain $\limsup_k \frac{1}{n_{k+1}^{1-\delta}} \sum_{j=1}^{n_{k+1}} f_j(x) > 0$ for every x. Since $\sum_{j=1}^{n_{k+1}} f_j(x) = 0$ for $x \notin I_k$, and each x is outside infinitely many I_k, we have $\liminf_k \frac{1}{n_{k+1}^{1-\delta}} \sum_{j=1}^{n_{k+1}} f_j(x) = 0$ for every x. Hence $\{\frac{1}{n^{1-\delta}} \sum_{j=1}^{n} f_j(x)\}$ is everywhere divergent.

3. Applications

In this section we apply our previous results, especially Theorems 4 and 5, to obtain additional information in some special cases of the results of [**CL**].

Proposition 6. *Let $\{n_k\}$ be a non-decreasing sequence of positive integers, and let $\{a_k\}$ be a sequence of complex numbers such that for some $0 < \beta \le 1$ we have*

$$\sup_{n>0} \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} a_k \lambda^{n_k} \right| = K < \infty \ . \tag{18}$$

(i) If $\{a_k\}$ is bounded, then for every Dunford-Schwartz operator T on $L_1(\mu)$ of a probability space and every $f \in L_p(\mu)$, $2 < p < \infty$, the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^{1-\delta}}$ converges a.e. for any $0 \le \delta < \frac{2p-2}{3p-2}\beta$. If $f \in L_\infty$, then the convergence holds for $0 \le \delta < \frac{2}{3}\beta$, and also $\sup_n |\sum_{k=1}^{n} \frac{a_k T^{n_k} f}{k^{1-\delta}}| \in L_2(\mu)$.

(ii) If $\{a_k\} \in W_t$ for $1 < t < \infty$ with dual index s, then for $f \in L_\infty$ the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^{1-\delta}}$ converges a.e. for every $0 \le \delta < \max\{\frac{2}{2+s}\beta,\ \beta - \frac{1}{2}\}$.

(iii) If $\{a_k\} \in W_t$ and (18) holds for $n_k = k$, then for any $f \in L_s(\mu)$ the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ converges a.e., and thus $\displaystyle\frac{1}{n} \sum_{k=1}^{n} a_k T^k f \underset{n\to\infty}{\longrightarrow} 0$ a.e.

Proof. As in [**CL**], we note that (18) implies (by applying the spectral theorem for unitary operators and the unitary dilation theorem for contractions) that for any contraction T on a Hilbert space we have

$$\sup_{n>0}\left\|\frac{1}{n^{1-\beta}}\sum_{k=1}^{n} a_k T^{n_k}\right\| \le K. \tag{19}$$

(i): Putting $f_k = a_k T^{n_k} f$, the sequence $\{f_n\}$ is in $L_2(\mu)$ and satisfies (2). We now apply Theorem 4 (with q replaced by p), noting that for $p > 2$ and $\beta \le 1$ we always have $\frac{2p-2}{3p-2}\beta > \beta - \frac{1}{2}$. For $f \in L_\infty$ apply the corollary to Theorem 5 (with $p = 2$).

(ii) follows from applying Theorem 5 with $p = 2$ to $f_k = T^{n_k} f$.

(iii): By (ii) we have the a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ for bounded functions, which are dense in $L_s(\mu)$. For any $f \in L_s(\mu)$, Hölder's inequality yields

$$\sup_n \left|\frac{1}{n}\sum_{k=1}^{n} a_k T^k f\right| \le \sup_n \frac{1}{n}\sum_{k=1}^{n} |a_k T^k f| \le \sup_n \left\{\Big(\frac{1}{n}\sum_{k=1}^{n}|a_k|^t\Big)^{\frac{1}{t}}\Big(\frac{1}{n}\sum_{k=1}^{n}|T^k f|^s\Big)^{\frac{1}{s}}\right\}.$$

But $|T|$, the linear modulus of T, satisfies $|T^k f|^s \le (|T|^k|f|)^s \le |T|^k(|f|^s)$ (e.g., p. 65 of [**Kr2**]). Since $\{a_k\} \in W_t$, the pointwise ergodic theorem for $|T|$ applied to $|f|^s \in L_1(\mu)$ yields $\sup_n |\frac{1}{n}\sum_{k=1}^{n} a_k T^k f| < \infty$ a.e.; now the Banach principle yields $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f \underset{n\to\infty}{\longrightarrow} 0$ a.e. for every $f \in L_s(\mu)$.

For $f \in L_s(\mu)$, put $S_n f = \sum_{k=1}^{n} a_k T^k f$. Abel's summation by parts yields $\displaystyle\sum_{k=1}^{n}\frac{a_k T^k f}{k} = \frac{S_n f}{n} + \sum_{k=1}^{n-1}\frac{1}{k^2}S_k f$. We have shown that $S_n f/n \to 0$ a.e., so it remains to check the series. When $s \ge 2$ (i.e., $1 < t \le 2$), we have $f \in L_2(\mu)$, and $||S_n f||_2 \le K n^{1-\beta}||f||_2$ by (19). Since μ is a probability, we obtain

$$\int \sum_{k=1}^{\infty}\frac{|S_k f|}{k^2}d\mu = \sum_{k=1}^{\infty}\frac{||S_k f||_1}{k^2} \le \sum_{k=1}^{\infty}\frac{||S_k f||_2}{k^2} \le K||f||_2\sum_{k=1}^{\infty}\frac{1}{k^{1+\beta}} < \infty,$$

showing that $\sum_{k=1}^{\infty}\frac{|S_k f|}{k^2}$ converges a.e., which proves (iii) when $s \ge 2$.

Assume now $1 < s < 2$. The operator $S_n = \sum_{k=1}^{n} a_k T^k$ maps L_2 into itself with norm $||S_n||_2 \le K n^{1-\beta}$ by (19), and it maps $L_1(\mu)$ into itself with norm $||S_n||_1 \le \sum_{k=1}^{n}|a_k|$. Since $1 < s < 2$, the Riesz-Thorin theorem ([**Z**], vol. II p. 95) yields that S_n maps $L_s(\mu)$ into itself with norm $||S_n||_s \le ||S_n||_2^{\alpha}||S_n||_1^{1-\alpha}$, where $0 < \alpha < 1$ is defined by $\frac{1}{s} = \alpha\cdot\frac{1}{2} + (1-\alpha)\cdot 1$. Hölder's inequality yields $||S_n||_1 \le (\sum_{k=1}^{n}|a_k|^t)^{1/t} n^{1/s}$. Hence

$$||S_n||_s \le K^{\alpha} n^{(1-\beta)\alpha}\Big(\sum_{k=1}^{n}|a_k|^t\Big)^{\frac{1-\alpha}{t}} n^{\frac{1-\alpha}{s}} \le K^{\alpha} n^{(1-\beta)\alpha} n^{\frac{1-\alpha}{t}}\Big(\frac{1}{n}\sum_{k=1}^{n}|a_k|^t\Big)^{\frac{1-\alpha}{t}} n^{\frac{1-\alpha}{s}}.$$

Since $\{a_k\} \in W_t$ and $\frac{1}{t} + \frac{1}{s} = 1$, we obtain

$$||S_n||_s \le C\cdot n^{(1-\beta)\alpha} n^{(1-\alpha)(\frac{1}{t}+\frac{1}{s})} = C\cdot n^{1-\alpha\beta}.$$

This yields $\displaystyle\int\sum_{k=1}^{\infty}\frac{|S_k f|}{k^2}d\mu \le \sum_{k=1}^{\infty}\frac{||S_k f||_s}{k^2}d\mu \le C||f||_s\sum_{k=1}^{\infty}\frac{1}{k^{1+\alpha\beta}} < \infty$. Now the previous arguments yield (iii) also in the case $s < 2$. □

REMARKS. 1. Proposition 6(i) complements Proposition 2(ii) of [**CL**], which deals with $f \in L_p$ for $1 < p \le 2$.

2. Since μ is assumed finite, $f \in L_p(\mu)$ with $p > 2$ is in L_2, and Proposition 2(ii) of [**CL**] can be applied; however, we obtain here a larger interval for δ than that given in [**CL**] for L_2 functions (which is the interval for which Theorems 2 and 3 hold).

3. Proposition 2 of [**CL**] gives additional results under the assumption (18). These can be improved by applying Theorems 1 or 3, according to the value of δ. We omit the statements of these improvements.

4. Examples of sequences $\{a_n\}$ satisfying (18) for $n_k = k$ were given in [**CL**]. Another example (not mentioned there) is $a_n = exp[2\pi in(\log n)^\gamma]$ with $\gamma > 0$; by [**I**] the series $\sum_{k=1}^{\infty} \frac{a_k}{k^{1/2}(\log k)^\delta}\lambda^k$ converges uniformly on the unit circle for large enough δ, so (18) is satisfied with any $\beta < 1/2$.

5. For $\{a_k\}$ bounded satisfying (18), Proposition 2(ii) of [**CL**] applies also when μ is *not finite*. It yields, for $1 < p \le 2$, the estimate of the L_p-norm of the operators $\left\| \frac{1}{n}\sum_{k=1}^{n} a_k T^{n_k} \right\|_p = O(n^{\beta_p})$ with $\beta_p = 2\beta\frac{p-1}{p}$. For $f \in L_\infty \bigcap L_p$, we can now apply the corollary to Theorem 5, with $f_k = a_k T^{n_k} f$, to obtain the a.e. convergence of the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^{1-\delta}}$ when $0 \le \delta < \frac{p}{p+1}\beta_p = \frac{p-1}{p+1}2\beta$. For bounded L_p functions, this improves the interval $\delta < \frac{p-1}{p}\beta$ obtained in Proposition 2(ii) of [**CL**].

THEOREM 7. *Fix $1 < q < \infty$, and let $\{g_n\}$ be i.i.d. on a probability space (Y, m), with $||g_1||_q < \infty$ and $\int g_1 dm = 0$. Then for a.e. $y \in Y$ the sequence $a_k := g_k(y)$ has the following property:*

For every Dunford-Schwartz operator T on $L_1(\mu)$ of a probability space and $f \in L_{\frac{q}{q-1}}(\mu)$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ converges a.e.

PROOF. We first note that by the strong law of large numbers, $\frac{1}{n}\sum_{k=1}^{n} |g_k|^q$ converges a.s. to $\int |g_1|^q dm$. Hence for a.e. $y \in Y$ the sequence $\{a_k\}$ is in W_q.

If $q > 2$ then also $\int |g_1|^2 dm < \infty$, so putting $q_1 := \min\{2, q\}$ we have $\{g_n\}$ centerd i.i.d. with finite absolute moment of order $q_1 \le 2$. Let $\alpha \in (q_1^{-1}, 1)$, so $\alpha \in (\frac{1}{2}, 1)$, and $1 < 1/\alpha < q_1$ yields

$$E\Big(|g_1|^{1/\alpha}(\log^+ |g_1|)^{\frac{1}{\alpha}-1+\epsilon}\Big) < \infty \quad \text{for every } \epsilon > 0.$$

By the result of Cuzick and Lai [**CuLa**] we now have that for a.e. $y \in Y$ the series $\displaystyle\sum_{k=1}^{\infty} \frac{g_k(y)}{k^\alpha}\lambda^k$ converges uniformly in $|\lambda| = 1$. For such y, put $a_k = g_k(y)$. A variant of Kronecker's lemma (a Banach space version, in the space of continuous functions) yields that $\frac{1}{n^\alpha}\sum_{k=1}^{n} a_k\lambda^k$ converges uniformly to 0, so $\{a_k\}$ satisfies (18) with $n_k = k$ and $\beta = 1 - \alpha$ (note that $\beta < \frac{1}{2}$). The theorem now follows from Proposition 6(iii). □

REMARKS. 1. The convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ under the assumptions of the theorem follows from the "return times theorem" (Appendix of [**B**], see also [**Ru**]; for the passage from measure preserving transformations to Dunford-Schwartz operators see [**ÇLO**]). Our result improves this convergence (in the particular i.i.d. case).

2. Assani [**A4**] showed that Theorem 7 fails for $q = 1$, although the "return times theorem" holds.

3. For an i.i.d. sequence as in the theorem, with the additional assumption that g_1 is symmetric, Assani [**A2**] obtained the a.e convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ for every $f \in L_p(\mu)$ with $p > 1$ (even if $p < \frac{q}{q-1}$). We do not know if in this case also the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ converges a.e. for every Dunford-Schwartz operator and every $f \in L_p(\mu)$ when $1 < p < \frac{q}{q-1}$.

THEOREM 8. *Let (Ω, μ) be a probability space, and let $\{f_n\} \subset L_p(\mu)$, $1 \le p < \infty$, such that $\sup_n \|f_n\|_q < \infty$ for some $1 < q < \infty$. Let $\{n_k\}$ be a sequence of integers such that for some $0 < \beta \le 1$ we have*

$$\sup_n \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} f_k \lambda^{n_k} \right| \right\|_p = K < \infty \ . \tag{20}$$

If $\frac{(q-1)p\beta}{q+(q-1)p} \ge \beta - \frac{1}{p}$ (e.g., $\beta \le \frac{1}{p}$ or $q \ge p$), then there exists a set $\Omega' \subset \Omega$ with $\mu(\Omega') = 0$ such that for $x \notin \Omega'$ and every $0 \le \delta < \frac{(q-1)p\beta}{q+(q-1)p}$ the series $\displaystyle\sum_{k=1}^{\infty} \frac{f_k(x)}{k^{1-\delta}} \lambda^{n_k}$ converges uniformly in $|\lambda| = 1$.

PROOF. The proof is similar to that of Theorem 4, with the same notations. Instead of (14) we obtain $\displaystyle\int \sum_{m=1}^{\infty} \max_{|\lambda|=1} \left| \frac{1}{n_m^{1-\delta}} \sum_{k=1}^{n_m} f_k \lambda^{n_k} \right|^p < \infty$, and instead of (15) we have

$$\max_{|\lambda|=1} \left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k \lambda^{n_k} - \frac{1}{n^{1-\delta}} \sum_{k=1}^{n_m} f_k \lambda^{n_k} \right| \le \frac{1}{n_m^{1-\delta}} \sum_{k=n_m+1}^{n_{m+1}} |f_k|.$$

¿From these we deduce $\displaystyle\max_{|\lambda|=1} \left| \frac{1}{n^{1-\delta}} \sum_{k=1}^{n} f_k(x) \lambda^{n_k} \right| \to 0$ for a.e. x. For the proof of the uniform convergence of the series, see the proof of Theorem 9 of [**CL**]. □

REMARKS. 1. Since μ is finite, for $q = \infty$ (i.e., when $\sup \|f_n\|_\infty < \infty$), we have the above result for $\delta < \frac{p}{p+1}\beta$, by using finite q tending to ∞.

2. Theorem 8 extends Corollary 6 of [**CL**]. Theorem 9 there could be similarly extended.

COROLLARY. *Let (Ω, μ) be a probability space, and let T be a power-bounded operator on $L_q(\mu)$, $1 < q < \infty$. If $f \in L_q(\mu)$ satisfies, for some $\beta > 0$,*

$$\sup_n \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} \lambda^k T^k f \right| \right\|_1 = K < \infty \ ,$$

then there exists a set $\Omega' \subset \Omega$ with $\mu(\Omega') = 0$ such that for $x \notin \Omega'$ and every $\gamma \in (1 - \frac{(q-1)\beta}{2q-1}, 1]$ the series $\displaystyle\sum_{k=1}^{\infty} \frac{T^k f(x) \lambda^k}{k^\gamma}$ converges uniformly in $|\lambda| = 1$.

PROOF. Apply Theorem 8 with $f_n := T^n f$ and $p = 1$.

REMARKS. 1. For $q \geq 2$ and T induced on $L_q(\mu)$ by a probability preserving transformation, the corollary was proved in [**AN**]. Since $\frac{\beta}{2} - \frac{1}{2q} < \frac{\beta(q-1)}{2q-1}$, our result yields the convergence for a wider range of γ. However, if f is bounded, the limit as $q \to \infty$ in the corollary yields the same range as in Theorem 5 of [**AN**]. Existence functions satisfying the assumption of the corollary was shown in [**A3**] and [**AN**].

2. For T a positively dominated contraction on L_q, $1 < q < \infty$, the a.e. uniform convergence of the random Fourier series $\sum_{k=1}^{\infty} \frac{T^k f(x)\lambda^k}{k}$ under the assumption of the corollary was proved in Theorem 8 of [**CL**] by a different method.

3. For T a positive contraction of $L_1(\mu)$ with $T1 = 1$ and $f \in L_1$ satisfying the hypothesis of the corollary, the a.e. uniform convergence of the random Fourier series $\sum_{k=1}^{\infty} \frac{T^k f(x)\lambda^k}{k}$ was proved in Theorem 8 of [**CL**]; this does not follow from our Theorem 8.

THEOREM 9. *Let (Ω, μ) be a probability space and $2 \leq p \leq \infty$. Let $\{f_n\} \subset L_p(\mu)$ be independent, with $\int f_n d\mu = 0$ and $\sup_n ||f_n||_p < \infty$. Then*

$$\sup_{n>0} \left\| \max_{|\lambda|=1} \left| \frac{1}{n^{3/4}} \sum_{k=1}^{n} f_k \lambda^{[\sqrt{k}]} \right| \right\|_2 < \infty \tag{21}$$

and for a.e. $x \in \Omega$ and $\delta < \frac{p-1}{6p-4}$ the series $\sum_{k=1}^{\infty} \frac{f_k(x)}{k^{1-\delta}} \lambda^{[\sqrt{k}]}$ converges uniformly in $|\lambda| = 1$.

PROOF. We first prove (21). The assumption yields $\sup_n ||f_n||_2 = K < \infty$. Put $S_n = \sum_{k=1}^{n} \lambda^{[\sqrt{k}]} f_k$. Then

$$|S_{n^2-1}|^2 = \left| \sum_{j=1}^{n-1} \lambda^j \sum_{k=j^2}^{(j+1)^2-1} f_k \right|^2 = \left(\sum_{j=1}^{n-1} \lambda^j \sum_{k=j^2}^{(j+1)^2-1} f_k \right) \left(\sum_{j=1}^{n-1} \lambda^{-j} \sum_{k=j^2}^{(j+1)^2-1} \bar{f}_k \right)$$

$$= \sum_{j,m=1}^{n-1} \lambda^{j-m} \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=m^2}^{(m+1)^2-1} f_k \bar{f}_\ell$$

$$= \sum_{j=1}^{n-1} \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=j^2}^{(j+1)^2-1} f_k \bar{f}_\ell + \sum_{\substack{j,m=1 \\ j \neq m}}^{n-1} \lambda^{j-m} \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=m^2}^{(m+1)^2-1} f_k \bar{f}_\ell \ .$$

Denote the last two summands by G_n and H_n. Then G_n does not depend on λ, and satisfies

$$||G_n||_1 \leq \sum_{j=1}^{n-1} \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=j^2}^{(j+1)^2-1} ||f_k \bar{f}_\ell||_1 \leq K^2 \sum_{j=1}^{n-1} (2j+1)(2j+1) \leq 4K^2(n+1)^3/3 \ .$$

Since H_n does depend on λ, we have

$$\int \max_{|\lambda|=1} |H_n|\, d\mu \le \int \sum_{\substack{j,m=1\\ j\ne m}}^{n-1} \left| \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=m^2}^{(m+1)^2-1} f_k \bar{f}_\ell \right| d\mu$$

$$\le \left\{ \int \left[\sum_{\substack{j,m=1\\ j\ne m}}^{n-1} \left| \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=m^2}^{(m+1)^2-1} f_k \bar{f}_\ell \right| \right]^2 d\mu \right\}^{\frac12}$$

$$\le \left\{ \int [(n-1)^2-(n-1)] \sum_{\substack{j,m=1\\ j\ne m}}^{n-1} \left| \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=m^2}^{(m+1)^2-1} f_k \bar{f}_\ell \right|^2 d\mu \right\}^{\frac12} \le$$

$$n\left\{ \int \Big(\sum_{\substack{j,m=1\\ j\ne m}}^{n-1} \sum_{k=j^2}^{(j+1)^2-1} \sum_{\ell=m^2}^{(m+1)^2-1} |f_k|^2 |\bar{f}_\ell|^2 + \sum_{\substack{j,m=1\\ j\ne m}}^{n-1} \sum_{k,r=j^2}^{(j+1)^2-1} \sum_{\substack{\ell,s=m^2\\ (k,\ell)\ne(r,s)}}^{(m+1)^2-1} f_k \bar{f}_\ell f_r \bar{f}_s \Big) d\mu \right\}^{\frac12}.$$

The restriction $j \ne m$ puts k and r in one block of integers, while ℓ and s are in another one; thus when $(k,\ell) \ne (r,s)$ the independence yields $\int f_k \bar{f}_\ell f_r \bar{f}_s d\mu = 0$. Hence the independence of $|f_k|^2$ and $|f_\ell|^2$ yields

$$\int \max_{|\lambda|=1} |H_n|\, d\mu \le n\{\sup_k ||f_k||_2^4 \sum_{j,m=1}^{n-1} (2j)(2m)\}^{1/2} \le nK^2n^2 = K^2n^3.$$

We conclude that

$$\left\| \max_{|\lambda|=1} \left| \frac{1}{n^2-1} S_{n^2-1} \right| \right\|_2^2 \le \frac{1}{(n^2-1)^2} \Big(||G_n||_1 + || \max_{|\lambda|=1} |H_n| \, ||_1 \Big) \le \frac{C}{n}.$$

Now let n satisfy $m^2 \le n < (m+1)^2$. Then the previous inequality yields

$$\left\| \frac1n \max_{|\lambda|=1} |S_n| \right\|_2 \le \frac{1}{m^2-1} \left\| \max_{|\lambda|=1} \left| \sum_{k=1}^{m^2-1} \lambda^{[\sqrt{k}]} f_k \right| \right\|_2 + \frac{1}{m^2} \left\| \max_{|\lambda|=1} \left| \sum_{k=m^2}^{n} \lambda^{[\sqrt{k}]} f_k \right| \right\|_2$$

$$\le \sqrt{\frac{C}{m}} + \frac{2m+1}{m^2} K \le \frac{C'}{\sqrt{m+1}} \le \frac{C'}{n^{1/4}}\ ,$$

which proves inequality (21).

The claimed a.e. convergence assertion now follows from Theorem 8, with $\beta = \frac14$, p replaced by 2, and q replaced by p. $\square$

Remark. The method of [**CL**], based on the deep results of Marcus and Pisier [**MP1**], cannot be applied here since the terms in $\{[\sqrt{k}]\}$ are not distinct; regrouping terms according to powers of λ and then following the method of [**CL**] yields a worse estimate (i.e., a smaller value of β).

PROPOSITION 10. *Let (Ω, μ) be a probability space and let $\{f_n\} \subset L_p(\mu)$, $1 < p \le \infty$, be independent with $\sup_n ||f_n||_p < \infty$. Then for $1 \le t < p$ we have $\sup_{n>0} \frac{1}{n} \sum_{k=1}^{n} |f_k|^t < \infty$ a.e. (i.e., for a.e. $x \in \Omega$ the sequence $\{f_k(x)\}$ is in W_t).*

PROOF. We first prove that the assumptions imply $\sup_n \frac{1}{n} \sum_{k=1}^n |f_k| < \infty$ a.e. (the case $t = 1$). It is clearly sufficient to prove for $\{f_k\}$ non-negative, and we may certainly assume in this part that $1 < p \le 2$. We then have $E(f_n) = ||f_n||_1 \le ||f_n||_p$, and the centering $g_n = f_n - E(f_n)$ satisfies $||g_n||_p \le 2||f_n||_p$. Hence $\sum_{n=1}^{\infty} E(|g_n|^p)/n^p < \infty$. By the Marcinkiewicz-Zygmund theorem ([**MaZ**], Theorem 5'; see also [**S**], Theorem 2.12.2), the series $\sum_{n=1}^{\infty} \frac{g_n}{n}$ converges a.e., so by Kronecker's lemma $\frac{1}{n} \sum_{k=1}^n g_k \to 0$ a.e. The claim now follows from

$$\frac{1}{n} \sum_{k=1}^{n} f_k \le \left| \frac{1}{n} \sum_{k=1}^{n} g_k \right| + \frac{1}{n} \sum_{k=1}^{n} E(f_k) \le \left| \frac{1}{n} \sum_{k=1}^{n} g_k \right| + \sup_j ||f_j||_p.$$

We now prove the proposition. The functions $h_n = |f_n|^t \in L_{p/t}$ are independent, with $\sup_n ||h_n||_{p/t} < \infty$. Since $p/t > 1$, we can apply the first part of the proof to $\{h_n\} \subset L_{p/t}(\mu)$, and obtain

$$\sup_{n>0} \frac{1}{n} \sum_{k=1}^{n} |f_k|^t = \sup_{n>0} \frac{1}{n} \sum_{k=1}^{n} h_n < \infty \quad \text{a.e.} \qquad \square$$

REMARK. Note that $\{f_k(x)\}$ need not be in W_p. Let $\{A_n\}$ be independent sets in non-atomic (Ω, μ) with $\mu(A_n) = \frac{1}{n \log n}$ and $f_n := (n \log n)^{1/p} \chi_{A_n}$. By Borel-Cantelli a.e. x is in infinitely many A_n, and for $x \in A_{n_j}$ we have $\frac{1}{n_j} \sum_{k=1}^{n_j} |f_k(x)|^p \ge \log n_j$.

THEOREM 11. *Let $\{n_k\}$ be a strictly increasing sequence of integers with $n_k \le ck^r$ for some $r \ge 1$, let (Y, m) be a probability space, and let $\{g_n\} \subset L_q(Y, m)$, $2 \le q < \infty$, be independent with $\sup ||g_n||_q < \infty$ and $\int g_n dm = 0$. Then for a.e. $y \in Y$ the sequence $a_k := g_k(y)$ has the following property:*

For every Dunford-Schwartz operator T on $L_1(\mu)$ of a probability space and $f \in L_\infty(\mu)$, the series $\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^\gamma}$ converges a.e. for $\gamma \in (\frac{2q-1}{3q-2}, 1]$.

PROOF. Since $q \ge 2$, we have $\sup_n ||g_n||_2 < \infty$. It follows from Theorem 12 of [**CL**] (by a variant of Kronecker's lemma) that for a.e. $y \in Y$ the sequence $\{a_k\}$ satisfies (18) for any $\beta < \frac{1}{2}$. By Proposition 10 $\{a_k\} \in W_t$ for $1 \le t < q$. We can now apply Proposition 6(ii) (letting $t \to q$ and $\beta \to 1/2$). $\square$

THEOREM 12. *Let $\{n_k\}$ be a strictly increasing sequence of integers with $n_k \le ck^r$ for some $r \ge 1$, let (Y, m) be a probability space, and let $\{g_n\} \subset L_\infty(Y, m)$ be independent with $\sup ||g_n||_\infty < \infty$ and $\int g_n dm = 0$. Then for a.e. $y \in Y$ the sequence $a_k := g_k(y)$ has the following property:*

For every Dunford-Schwartz operator T on $L_1(\mu)$ of a probability space and $f \in L_p(\mu)$, $2 \le p < \infty$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k T^{n_k} f}{k^\gamma}$ *converges a.e. for* $\gamma \in (\frac{2p-1}{3p-2}, 1]$.

PROOF. As before, $\{a_k\}$ satisfies (18) for any $\beta < \frac{1}{2}$. For $p = 2$ we apply Proposition 2(i) of [**CL**], and for $p > 2$ we apply Proposition 6(i). □

REMARKS. 1. When $f \in L_\infty$ and $\sup_n ||g_n||_\infty < \infty$, the lower limit for γ is $2/3$, either by letting $q \to \infty$ in Theorem 11 or by letting $p \to \infty$ in Theorem 12.

2. Theorem 12 complements Theorem 14 of [**CL**], which gives the result for $p = 2$, with $\gamma > 3/4$, and uses it also when $f \in L_p(\mu)$ with $p > 2$. Theorem 12 gives a better lower bound for γ.

THEOREM 13. *Let (Y, m) be a probability space, and let $\{g_n\} \subset L_q(Y, m)$, $2 \le q < \infty$, be independent with* $\sup ||g_n||_q < \infty$ *and* $\int g_n dm = 0$. *Then for a.e. $y \in Y$ the sequence $a_k := g_k(y)$ has the following property:*

For every Dunford-Schwartz operator T on $L_1(\mu)$ of a probability space and $f \in L_p(\mu)$, $p > \frac{q}{q-1}$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ *converges a.e. and* $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f \to 0$ *a.e.*

PROOF. As in the proof of Theorem 11, $\{a_k\} \in W_t$ for $t < q$, and $\{a_k\}$ satisfies (18), with $n_k = k$, for any $\beta < \frac{1}{2}$. For a given p, if $p > \frac{q}{q-1}$ then its dual index t is less than q, and we apply Proposition 6(iii) (with $s = p$). □

REMARKS. 1. When $q = 2$ we obtain the convergence for all $f \in L_p$, $p > 2$. When $q > 2$ we obtain convergence for all $f \in L_2$.

2. If the sequence $\{g_n\}$ in Theorem 13 is i.i.d., then Theorem 7 gives the convergence of the series also for $p = \frac{q}{q-1}$, since the SLLN can be used instead of Proposition 10. Moreover, for $\{g_n\}$ i.i.d. Theorem 7 does not require a finite second moment.

In order to extend the previous theorem to the case $q < 2$, we need the following theorem, which complements Theorem 12 of [**CL**]. Note that we have an additional assumption of symmetry.

THEOREM 14. *Let (Ω, μ) be a probability space. Let $1 < p < 2$, and $\{f_n\} \subset L_p(\mu)$ be* symmetric *and* independent *with* $\int f_n d\mu = 0$, *and* $\sup_n ||f_n||_p < \infty$. *Let $\{n_k\}$ be a strictly increasing sequence with $n_k \le ck^r$ for some $r \ge 1$. Then for a.e. x, the series* $\displaystyle\sum_{k=1}^{\infty} \frac{f_k(x)}{k^{1-\delta}} \lambda^{n_k}$ *converges uniformly in λ, for any* $0 \le \delta < \frac{p-1}{p}$.

PROOF. We will use Theorem B(i) of [**MP2**], with the group G the unit circle, G the compact neighborhood, the set of characters $A := \{n_k : k \ge 1\}$, and the independent random variables $\xi_{n_k} = f_k$.

By linearity of the model we may and do assume that $\sup_n ||f_n||_p \le 1$; this clearly implies that $P(|f_n| > c) \le c^{-p}$ for every n and $c > 0$, the assumption in [**MP2**], p. 247. Fix $0 < \delta < (p-1)/p$, and put $\alpha = \frac{p(1-\delta)-1}{pr}$, so $0 < \alpha < (p-1)/p$. Define $\{a_j\}$ on A by $a_{n_k} = \frac{1}{k^{1-\delta}}$ (the sequence need not be defined outside A, but we put $a_j = 0$ for $j \notin A$). It will be convenient to identify the unit circle with the interval $[0, 2\pi]$, with addition modulo 2π. Let $t_1, t_2 \in [0, 2\pi]$ and define the corresponding translation invariant pseudo-metric $d(t_1, t_2) = \sigma(t_1 - t_2)$ (which is uniformly convergent), where

$$\sigma(t) := \Big(\sum_{j\in A} |a_j|^p |1 - e^{ijt}|^p\Big)^{1/p} = 2\Big(\sum_{k=1}^{\infty} \frac{|\sin \frac{n_k t}{2}|^p}{k^{p-p\delta}}\Big)^{1/p}.$$

Since $|\sin t| \le 1$ and $|\sin t| \le |t|$, we obtain $|\sin t|^p \le |\sin t|^\alpha \le |t|^\alpha$. This yields

$$\sigma(t) \le 2\Big(\sum_{k=1}^{\infty} \frac{c^\alpha k^{r\alpha}|t|^\alpha}{2^\alpha k^{p-p\delta}}\Big)^{1/p} \le 2^{1-\frac{\alpha}{p}} c^{\frac{\alpha}{p}} |t|^{\frac{\alpha}{p}} \Big(\frac{\gamma}{\gamma - 1}\Big)^{1/p} \le C_\alpha |t|^{\frac{\alpha}{p}}$$

with $\gamma := p - p\delta - r\alpha > p - p\delta - p(1-\delta) + 1 = 1$.

Denote by m the Lebesgue measure on $[0, 2\pi]$. Then the "distribution" of σ satisfies

$$m_\sigma(\epsilon) := m\{t \in [0, 2\pi] : \sigma(t) < \epsilon\} \ge C_\alpha^{-\frac{p}{\alpha}} \epsilon^{\frac{p}{\alpha}} \ ;$$

hence the 'inverse' function defined on $[0, 2\pi]$ (which is the non-decreasing rearrangement of σ), satisfies

$$\overline{\sigma(s)} := \sup\{t > 0 : m_\sigma(t) < s\} \le C_\alpha s^{\frac{\alpha}{p}}.$$

In order to apply Theorem B(i) of [**MP2**] (in the form described in the discussion beginning at the end of p. 248 there), we estimate

$$I_p(\sigma) := \int_0^{2\pi} \frac{\overline{\sigma(s)}ds}{s(\log \frac{b(p)}{s})^{1/p}} \le C_\alpha \int_0^{2\pi} \frac{ds}{s^{1-\frac{\alpha}{p}}(\log \frac{b(p)}{s})^{1/p}} \quad ,$$

where $b(p) > 2\pi$ is a constant depending only on p (see p. 290 of [**MP2**]). The finiteness of $I_p(\sigma)$ follows from the integrability of $\frac{1}{s^{1-\frac{\alpha}{p}}}$ for $\alpha > 0$. Now the claimed convergence follows from [**MP2**]. □

Remarks. 1. The theorem applies to sequences $\{[k^r] : k \ge 1\}$ with $r \ge 1$.

2. The integers in the sequence $\{n_k\}$ must be *distinct* (in addition to the growth condition), to make it an *enumeration* of the *set* of characters A; hence the proof of the theorem does not apply to the sequence $\{[\sqrt{k}]\}$.

Theorem 15. *Let (Y, m) be a probability space, and let $\{g_n\} \subset L_q(Y, m)$, $1 < q < 2$, be independent and symmetric with* $\sup ||g_n||_q < \infty$ *and* $\int g_n dm = 0$. *Then for a.e. $y \in Y$ the sequence $a_k := g_k(y)$ has the following property:*

For every Dunford-Schwartz operator T on $L_1(\mu)$ of a probability space and $f \in L_p(\mu)$, $p > \frac{q}{q-1}$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k T^k f}{k}$ *converges a.e. and* $\frac{1}{n}\sum_{k=1}^n a_k T^k f \to 0$ *a.e.*

Proof. The proof is similar to that of Theorem 13, but uses Theorem 14 instead of Theorem 12 of [**CL**]: $\{a_k\} \in W_t$ for $1 \le t < q$, and by Theorem 14 (and a variant of Kronecker's lemma) $\{a_k\}$ satisfies (18) with $n_k = k$ for any $0 < \beta < \frac{q-1}{q}$. For $p > \frac{q}{q-1}$ the dual index t is less than q and we apply Proposition 6(iii) (with $s = p$). □

REMARK. Note that in the i.i.d. case (Theorem 7) symmetry is not required, and the convergence holds also for $f \in L_{\frac{q}{q-1}}$.

THEOREM 16. *Let* (Y, m) *be a probability space, and let* $\{g_n\} \subset L_q(Y, m)$, $2 \le q < \infty$, *be independent with* $\sup ||g_n||_q < \infty$ *and* $\int g_n dm = 0$. *Then for a.e.* $y \in Y$ *the sequence* $a_k := g_k(y)$ *has the following property:*

For every Dunford-Schwartz operator T *on* $L_1(\mu)$ *of a probability space and* $f \in L_\infty(\mu)$, *the series* $\sum_{k=1}^{\infty} \frac{a_k T^{[\sqrt{k}]} f}{k^\gamma}$ *converges a.e. for* $\gamma \in (1 - \left(\frac{q-1}{3q-2}\right)^2, 1]$.

PROOF. By Theorem 9 (and a variant of Kronecker's lemma), $\{a_k\}$ satisfies (18), with $n_k = [\sqrt{k}]$, for any $\beta < \frac{q-1}{6q-4}$. By Proposition 10 $\{a_k\} \in W_t$ for any $t < q$. We now apply Proposition 6(ii) with $\beta \to \frac{q-1}{6q-4}$ and $t \to q$. $\square$

THEOREM 17. *Let* (Y, m) *be a probability space, and let* $\{g_n\} \subset L_\infty(Y, m)$ *be independent, with* $\sup ||g_n||_\infty < \infty$ *and* $\int g_n dm = 0$. *Then for a.e.* $y \in Y$ *the sequence* $a_k := g_k(y)$ *has the following property:*

For every Dunford-Schwartz operator T *on* $L_1(\mu)$ *of a probability space and* $f \in L_p(\mu)$, $2 \le p < \infty$, *the series* $\sum_{k=1}^{\infty} \frac{a_k T^{[\sqrt{k}]} f}{k^\gamma}$ *converges a.e. for* $\gamma \in (\frac{8p-5}{9p-6}, 1]$.

PROOF. As before, Theorem 9 implies that the sequence $\{a_k\}$ satisfies (18) for $n_k = [\sqrt{k}]$, this time for any $\beta < \frac{1}{6}$ (by letting $p \to \infty$ in the result). Since $\{a_k\}$ is bounded, we apply Proposition 6(i), letting $\beta \to \frac{1}{6}$. $\square$

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References

[A1] I. Assani, *A Wiener-Wintner property for the helical transform*, Ergodic Th. & Dyn. Syst. **12** (1992), 185-194.

[A2] I. Assani, *A weighted pointwise ergodic theorem*, Ann. Inst. Poincaré Proba. Stat. **34** (1998), 139-150.

[A3] I. Assani, *Wiener Wintner dynamical systems, Preprint 1998,*, Ergodic Th. & Dyn. Syst., to appear.

[A4] I. Assani, *Duality and the one-sided ergodic Hilbert transform, preprint.*

[AN] I. Assani and K. Nicolaou, *Properties of Wiener Wintner dynamical systems,*, Bull. Soc. Math. France **129** (2001), 361-377.

[B] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Publ. Math. IHES **69** (1989), 5-45.

[CJRW] J. Campbell, R. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation in singular integrals in higher dimensions*, Trans. Amer. Math. Soc., to appear.

[CL] G. Cohen and M. Lin, *Laws of large numbers with rates and the one-sided ergodic Hilbert transform*, Illinois J. Math., to appear.

[ÇLO] D. Çömez, M. Lin, and J. Olsen, *Weighted ergodic theorems for mean ergodic L_1 contractions*, Trans. Amer. Math. Soc. **350** (1998), 101-117.

[CuLa] J. Cuzick and T.L. Lai, *On random Fourier series*, Trans. Amer. Math. Soc. **261** (1980), 53-80.

[DL] Y. Derriennic and M. Lin, *Fractional Poisson equations and ergodic theorems for fractional coboundaries*, Israel J. Math. **123** (2001), 93–130.

[Do] J. Doob, *Stochastic processes*, John Wiley, New York, 1953.

[G] V. Gaposhkin, *On the dependence of the convergence rate in the SLLN for stationary processes on the rate of decay of the correlation function*, Theory of probability and its applications **26** (1981), 706–720.

[I] A. E. Ingham, *Note on a certain power series*, Ann. Math. **31** (1930), 241–245.

[JKRW] R. Jones, R. Kaufman, J. Rosenblatt, and M. Wierdl, *Oscillation in ergodic theory*, Ergodic Th. & Dyn. Syst. **18** (1998), 889–935.

[Kr1] U. Krengel, *On the speed of convergence in the ergodic theorem*, Monatshef. Math. **86** (1978), 3–6.

[Kr2] U. Krengel, *Ergodic Theorems*, De Gruyter, Berlin, 1985.

[Lo] M. Loève, *Sur les fonctions aléatoires de second ordre*, Rev. Sci. **83** (1945), 297–303.

[MaZ] J. Marcinkiewicz and A. Zygmund, *Sur les fonctions indépendentes*, Fun. Math. **29** (1937), 60-90.

[MP1] M. B. Marcus and G. Pisier, *Random Fourier series with applications to harmonic analysis*, Princeton University Press, Princeton, 1981.

[MP2] M. B. Marcus and G. Pisier, *Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes*, Acta Math. **152** (1984), 245-301.

[Ru] D. Rudolph, *A joinings proof of Bourgain's return time theorem*, Ergodic Th. & Dyn. Syst. **14** (1994), 197-203.

[S] W. Stout, *Almost sure convergence*, Academic Press, New York, 1974.

[WW] N. Wiener and A, Wintner, *Harmonic analysis and ergodic theory*, American J. Math. **63** (1941), 415-426.

[Z] A. Zygmund, *Trigonometric Series*, vol. I-II, corrected second edition, Cambridge University Press, Cambridge, 1968.

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