On Billard's theorem for random Fourier series

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Abstract

In this note we show that Billard's theorem on a.s. uniform convergence of random Fourier series with independent symmetric coefficients is not true when the coefficients are only assumed to be centered independent. We give some necessary or sufficient conditions to ensure the validity of Billard's theorem in the centered case.

1 Introduction

In this paper we deal with a.s. uniform convergence of random Fourier series. Let Γ be the unit circle, and denote by $C(\Gamma)$ the Banach space of continuous functions on Γ, with the sup-norm that we denote by $\|\cdot\|_{C(\Gamma)}$. Given a sequence of independent centered random variables $\{X_n\}$ defined on a probability space $(\Omega, \mathbb{P})$, one wonders whether, for $\mathbb{P}$ almost every $x \in \Omega$, the series $\sum_{n=1}^{\infty} X_n(x)\lambda^n$ is uniformly convergent on Γ.

Since the early study of Paley and Zygmund [14], this matter benefited from many works by Salem and Zygmund [16], Billard [1], Kahane [9], Marcus [11], Cuzick and Lai [2], Marcus and Pisier [12] [13], Talagrand [17], Fernique [4], and Weber [18].

Most of the time, the sequence $\{X_n\}$ is first assumed to be symmetric, so that, by a theorem of Billard [1], it suffices to prove that, a.s., $\sum_{n=1}^{\infty} X_n(x)\lambda^n$ represents a continuous function. Then, one may hope to reach the general centered case by symmetrization.

Our purpose is to prove that this procedure may fail sometimes. We show that Billard's theorem is no longer true when the independent sequence $\{X_n\}$ is only assumed to be centered. To illustrate our result, we recall some more or less known moment conditions which ensure the validity of Billard's theorem, and give also necessary or sufficient conditions.

2 Conditions for validity of Billard's theorem

Let us first recall Billard's theorem, as it appears in Kahane [9, Theorem 3, p. 58].

Theorem 2.1 (Billard). *Let $\{X_n\}$ be a sequence of independent* symmetric *complex valued random variables. Then the following conditions are equivalent for the random Fourier series $\sum_{n=1}^{\infty} X_n\lambda^n$:*

(i) Almost surely the series represents a bounded function.

(ii) Almost surely the series represents a continuous function.

(iii) Almost surely the series is convergent at every point.

(iv) Almost surely the series is uniformly convergent.

We would like to know what happens in the centered case. We discuss conditions which allow to proceed by symmetrization. We denote by $L_1(\mathbb{P}, C(\Gamma))$ the Banach space of all $C(\Gamma)$-valued random variables ξ on Ω with the norm $\int_\Omega \|\xi\|_{C(\Gamma)}\, d\mathbb{P}$. The following result is a simple combination of a theorem of Itô and Nisio [7] (see also Ledoux and Talagrand [6, Theorem 6.1]), and a theorem of Hoffman-Jørgensen [5] (see also Jain and Marcus [8]).

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Theorem 2.2. *Let $\{X_n\} \subset L_1(\mathbb{P})$ be a sequence of independent complex valued centered random variables on $(\Omega, \mathbb{P})$. Assume that $\mathbb{E}(\sup_{n\geq 1} |X_n|) < \infty$. Let $\{X'_n\}$ be an independent copy of $\{X_n\}$, defined on $(\Omega', \mathbb{P}')$. The following conditions are equivalent:*

(i) Almost surely the series $\sum_{n=1}^{\infty} X_n \lambda^n$ converges in $C(\Gamma)$.

(ii) Almost surely in $\Omega \times \Omega'$ the series $\sum_{n=1}^{\infty} (X_n - X'_n)\lambda^n$ converges in $C(\Gamma)$.

(iii) The series $\sum_{n=1}^{\infty} (X_n - X'_n)\lambda^n$ converges in $L_1(\mathbb{P} \otimes \mathbb{P}', C(\Gamma))$.

(iv) The series $\sum_{n=1}^{\infty} X_n \lambda^n$ converges in $L_1(\mathbb{P}, C(\Gamma))$.

Furthermore, if any of the above holds, then $\sup_{n\geq 1} \max_{|\lambda|=1} |\sum_{k=1}^{n} (X_k - X'_k)\lambda^k|$ and $\sup_{n\geq 1} \max_{|\lambda|=1} |\sum_{k=1}^{n} X_k \lambda^k|$ are integrable.

Proof. Clearly $(i) \Rightarrow (ii)$. By assumption $\mathbb{E}(\sup_{n\geq 1} |X_n - X'_n|) < \infty$, hence $(ii) \Rightarrow (iii)$ by [5, Corollary 3.3]. The implication $(iii) \Rightarrow (iv)$ follows by independence and convexity (see [12, Lemma 4.2, p. 42]). The implication $(iv) \Rightarrow (i)$ follows by Itô and Nisio theorem [7]. If one of the conditions $(i)-(iv)$ holds, then the integrability of the maximal functions follow by [5, Corollary 3.3]. □

Remark. In the above theorem, if we assume that $\{X_n\} \subset L_p(\mathbb{P})$ for some $1 \leq p < \infty$, and $\mathbb{E}(\sup_{n\geq 1} |X_n|^p) < \infty$, then we have, in fact, convergence in $L_p(\mathbb{P}, C(\Gamma))$ and $L_p(\mathbb{P}, C(\Gamma))$-integrability of the maximal functions (see [5, Corollary 3.3]).

The next lemma is a simple case of Lemma 3.1 in [8].

Lemma 2.3. *Let $\{X_n\} \subset L_1(\mathbb{P})$ be a sequence of independent random variables. Then $\mathbb{E}(\sup_{n\geq 1} |X_n|) < \infty$ if and only if $\sum_{n=1}^{\infty} \mathbb{E}(|X_n| \mathbf{1}_{\{|X_n|\geq a\}}) < \infty$ for some $a > 0$.*

Corollary 2.4. *Let $1 < p < 2$, and let $\{a_n\}$ be a sequence of complex numbers, such that $\sum_{n=1}^{\infty} \frac{(\sum_{k=n}^{\infty} |a_k|^p)^{1/p}}{n(\log n)^{1/p}} < \infty$. Let $\{X_n\} \subset L_1(\mathbb{P})$ be a sequence of centered independent random variables. Assume that for some $M > 0$ we have $\mathbb{P}(|X_n| \geq u) \leq \frac{M}{u^p}$, for every $u \geq 1$ and $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n X_n \lambda^n$ is a.s. uniformly convergent.*

Proof. It follows from Fernique [4] (see also Marcus and Pisier [13], Theorem B and the discussion in Remark 4.4) , that the result is true for *symmetric* independent $\{X_n\}$. Now, by the condition on the tails of the variables $\{X_n\}$, we have, for $a_n \neq 0$,

$$\mathbb{E}(|a_n X_n| \mathbf{1}_{\{|a_n X_n|\geq 1\}}) = |a_n| \int_{1/|a_n|}^{\infty} \mathbb{P}(|X_n| > u) du + \mathbb{P}(|a_n X_n| \geq 1) \leq \frac{p}{p-1} M |a_n|^p.$$

Hence, by Lemma 2.3 we have $\mathbb{E}(\sup_{n\geq 1} |a_n X_n|) < \infty$. So, Theorem 2.2 yields the result. □

Remark. In a similar way, the results of Fernique for $2 \leq p < \infty$ (see [4, Examples (d) and (e)]) can be extended to the centered case.

Corollary 2.5. *Let $1 < p \leq 2$, and let $\{X_n\} \subset L_p(\mathbb{P})$ be a sequence of centered independent random variables. If $\displaystyle\sum_{n=2}^{\infty} \frac{(\sum_{k=n}^{\infty} \mathbb{E}|X_k|^p)^{1/p}}{n(\log n)^{1/p}} < \infty$, in particular, if for some $\epsilon > 0$ the series $\sum_{n=2}^{\infty} \mathbb{E}|X_n|^p (\log n)^{p-1} (\log\log n)^{p+\epsilon}$ converges, then the random Fourier series $\sum_{n=1}^{\infty} X_n \lambda^n$ is a.s. convergent in $C(\Gamma)$.*

Proof. First we prove the case $p = 2$. By Theorem 5.1.5 of Salem and Zygmund [16], and by the *principle of reduction* [9, p. 9] (see also [12, Ch. VII, §1]), we obtain the assertion for the case where $\{X_n\}$ are symmetric random variables. Since the condition of the theorem implies $\mathbb{E}(\sup_{n\geq 1} |X_n|^p) < \infty$, the non-symmetric case follows from the symmetrization argument of Theorem 2.2.

In the case $1 < p < 2$ we do the following: for $X_n \neq 0$ put $a_n = (\mathbb{E}|X_n|^p)^{1/p}$ and $Y_n = X_n/(\mathbb{E}|X_n|^p)^{1/p}$. Otherwise put $a_n = 0$ and $Y_n = 0$. Clearly, we have $\mathbb{P}(|Y_n| > u) \leq \frac{1}{u^p}$. By application of Corollary 2.4 to $\{a_n\}$ and $\{Y_n\}$, we obtain the result. □

Remark. Under the condition of the corollary above, we have $\mathbb{E}(\sup_{n\geq 1}|X_n|^p) < \infty$. Hence, Theorem 2.2 yields also convergence in $L_p(\mathbb{P}, C(\Gamma))$ and L_p-integrability of the maximal function (see the remark after Theorem 2.2).

Corollary 2.6 (Cuzick and Lai, [2])**.** *Let $\{Z_n\}$ be centered i.i.d. random variables, with $\int_\Omega |Z_1| \log^+ |Z_1| \, d\mathbb{P} < \infty$. Then the series $\sum_{n=1}^\infty \frac{Z_n}{n}\lambda^n$ is a.s. uniformly convergent.*

Proof. It was proved in Cuzick and Lai [2] and Talagrand [17], that for a symmetric i.i.d. sequence $\{Z_n\}$ with $\int_\Omega |Z_1| \log^+ \log^+ |Z_1| \, d\mathbb{P} < \infty$, the conclusion of the corollary holds. Now, since $\int_\Omega |Z_1| \log^+ |Z_1| \, d\mathbb{P} < \infty$ we have,

$$\sum_{n=1}^\infty \mathbb{E}\left(\frac{|Z_n|}{n}\mathbf{1}_{\{|Z_n|\geq n\}}\right) \leq \int_\Omega |Z_1| \sum_{n=1}^\infty \frac{\mathbf{1}_{\{|Z_1|\geq n\}}}{n} d\mathbb{P} \leq$$

$$\int_\Omega |Z_1| \sum_{n=1}^{[|Z_1|]+1} \frac{1}{n} d\mathbb{P} \leq \int_\Omega |Z_1| \log(|Z_1|+1) d\mathbb{P} < \infty\,,$$

where $[|Z_1|]$ denotes the integral part of $|Z_1|$. Hence, Lemma 2.3 and Theorem 2.2 yield the result. □

Remarks. 1. Corollary 2.6 is already proven in [2]. We gave it to illustrate the use of Theorem 2.2, and also because the statement of this corollary is very close to the situation of the counterexample given in Theorem 3.1 below. Indeed, modulo a rearrangement, the sequence $\{a_n\}$ defined in the proof of Theorem 3.1 is essentially the sequence $\{\frac{1}{n}\}$, and by Lemma 2.3, the condition $\mathbb{E}(\sup \frac{|Z_n|}{n}) < \infty$ is actually equivalent to $\int |Z_1| \log^+ |Z_1| d\mathbb{P} < \infty$. In Theorem 3.1 we have $\int |Z_1| (\log^+ |Z_1|)^{1-\epsilon} d\mathbb{P} < \infty$ for every $\epsilon > 0$, while $\int |Z_1| \log^+ |Z_1| d\mathbb{P} = \infty$.

2. The optimality of the condition $\int |Z_1| \log^+ |Z_1| d\mathbb{P} < \infty$ is also justified by the discussion in Marcinkiewicz and Zygmund [10, p. 78].

3. When $\{Z_n\}$ are symmetric i.i.d., Talagrand [17] proved that the condition $\int |Z_1| \log^+ \log^+ |Z_1| d\mathbb{P} < \infty$ is necessary and sufficient for the a.s. uniform convergence of $\sum_{n=1}^\infty \frac{Z_n \lambda^n}{n}$.

Using Billard's theorem and the symmetrization argument of Theorem 2.2, we obtain the following extension of Billard's theorem in the centered case.

Theorem 2.7. *Let $\{X_n\} \subset L_1(\mathbb{P})$ be a complex sequence of centered independent random variables. If $\mathbb{E}(\sup_{n\geq 1}|X_n|) < \infty$, then the following conditions are equivalent for the random Fourier series $\sum_{n=1}^\infty X_n \lambda^n$:*

(i) Almost surely the series represents a bounded function

(ii) Almost surely the series represents a continuous function

(iii) Almost surely the series converges everywhere.

(iv) Almost surely the series converges uniformly.

Definition 2.1. We say that the random Fourier series $\sum_{n=1}^\infty X_n \lambda^n$ has property $\mathcal{B}$, if conditions $(i) - (iv)$ of Theorem 2.7 are all equivalent, i.e., either all of them hold or all of them false.

Remark. The condition $\sup_{n\geq 1}\|\sum_{k=1}^n X_k\lambda^k\|_{C(\Gamma)}<\infty$ a.s. yields that, almost surely, $\sum_{k=1}^n X_n\lambda^n$ represents a bounded function (see [19, Theorem 4.2(ii), p. 136]). Hence, under the assumption $\mathbb{E}(\sup_{n\geq 1}|X_n|)<\infty$ the assertions $(i)-(iv)$ are all equivalent to $\sup_{n\geq 1}\|\sum_{k=1}^n X_k\lambda^k\|_{C(\Gamma)}<\infty$ a.s., for centered independent variables. In the symmetric case, it remains true without the assumption $\mathbb{E}(\sup_{n\geq 1}|X_n|)<\infty$, by Billard's theorem (see also the discusssions in [9, Theorem 1, p.13]).

Here we give a different type of moment condition.

Corollary 2.8. *Let $\{X_n\}\subset L_4(\mathbb{P})$ be a sequence of independent centered random variables, and assume that $\sup_{n\geq 1}\mathbb{E}(|X_n|^4)/(\mathbb{E}|X_n|^2)^2<\infty$. Then the series $\sum_{n=1}^\infty X_n\lambda^n$ has property $\mathcal{B}$.*

Proof. Let $\{X_n'\}$ be an independent copy of $\{X_n\}$, and denote by $\mathbb{E}'$ the corresponding expectation. If one of the above conditions $(i)-(iv)$ holds, then it holds also for $\sum_{n=1}^\infty(X_n-X_n')\lambda^n$. Billard's theorem yields that a.s. $\sum_{n=1}^\infty(X_n-X_n')\lambda^n$ converges uniformly. By the Riesz-Fischer theorem we have that $\sum_{n=1}^\infty|X_n-X_n'|^2$ converges a.s. Since $\mathbb{E}\mathbb{E}'|X_n-X_n'|^4\leq 16\mathbb{E}|X_n|^4$ and $\mathbb{E}\mathbb{E}'|X_n-X_n'|^2=2\mathbb{E}|X_n|^2$, we have that $\sup_{n\geq 1}\mathbb{E}\mathbb{E}'(|X_n-X_n'|^4)/(\mathbb{E}\mathbb{E}'|X_n-X_n'|^2)^2<\infty$. Theorem 5 in [9, Ch. 3, p. 33], applied to $\{|X_n-X_n'|^2\}$, yields that $\sum_{n=1}^\infty\mathbb{E}\mathbb{E}'|X_n-X_n'|^2<\infty$, and that yields $\mathbb{E}(\sup_{n\geq 1}|X_n|^2)\leq\sum_{n=1}^\infty\mathbb{E}|X_n|^2<\infty$. Theorem 2.7 yields the result. □

Corollary 2.9. *Let $\{a_n\}$ be a sequence of complex numbers, and let $\{X_n\}\subset L_1(\mathbb{P})$ be a sequence of centered independent random variables. Assume that there exists $C>0$, such that $\int_u^\infty\mathbb{P}(|X_n|>v)dv\leq Cu\mathbb{P}(|X_n|>u)$, for every $n\geq 1$ and $u\geq 1$. Then the random Fourier series $\sum_{n=1}^\infty a_nX_n\lambda^n$ has property $\mathcal{B}$.*

Remarks. 1. If there exists $p>1$, such that for every $n\geq 1$ the function $u\mapsto u^p\mathbb{P}(|X_n|>u)$ is non-increasing, then the condition of the corollary is satisfied.

2. The condition is also satisfied in the case where the $\{X_n\}$ are centered i.i.d., and the tail $\mathbb{P}(|X_1|>u)$ is regularly varying, with exponent $\rho<-1$ (see the definition in Feller [3, p. 276], and use [3, Theorem 1(a), p. 281]).

Proof. Assume that the series $\sum_{n=1}^\infty a_nX_n\lambda^n$ satisfies one of the conditions $(i)-(iv)$ above. Conditions (iii) or (iv) yield $a_nX_n\to 0$ a.s. Conditions (i) or (ii) and the Riemann-Lebesgue lemma yield that $a_nX_n\to 0$ a.s. Hence, the events $\{|a_nX_n|\geq 1\}$ take place a finite number of time a.s., and by the Borel-Cantelli lemma, we have that the series $\sum_{n=1}^\infty\mathbb{P}(|a_nX_n|\geq 1)$ converges. By the assumption on the tail, applied with $u=1/|a_n|$ (when $a_n\neq 0$), we obtain

$$\mathbb{E}(|a_nX_n|\mathbf{1}_{\{|a_nX_n|\geq 1\}})=$$

$$|a_n|\int_{1/|a_n|}^\infty\mathbb{P}(|X_n|>v)dv+\mathbb{P}(|a_nX_n|\geq 1)\leq(C+1)\mathbb{P}(|a_nX_n|\geq 1).$$

Hence by Lemma 2.3 we have $\mathbb{E}(\sup_{n\geq 1}|a_nX_n|)<\infty$, and the result follows by Theorem 2.7. □

Theorem 2.10. *Let $\{X_n\}$ be a sequence of centered independent random variables. If the deterministic series $\sum_{n=1}^\infty\mathbb{E}(X_n\mathbf{1}_{\{|X_n|\leq 1\}})\lambda^n$ is uniformly convergent, then the random Fourier series $\sum_{n=1}^\infty X_n\lambda^n$ has property $\mathcal{B}$.*

Proof. Assume the series $\sum_{n=1}^\infty X_n\lambda^n$, satisfies one of the conditions $(i)-(iv)$ above, say $(j)\in\{(i),..,(iv)\}$. As shown in the proof of Corollary 2.9, we have $X_n\to 0$ a.s., so $\sum_{n=1}^\infty X_n\mathbf{1}_{\{|X_n|\leq 1\}}\lambda^n$ satisfies condition (j). Since $\sum_{n=1}^\infty\mathbb{E}(X_n\mathbf{1}_{\{|X_n|\leq 1\}})\lambda^n$ is uniformly convergent, the series $\sum_{n=1}^\infty[X_n\mathbf{1}_{\{|X_n|\leq 1\}}-\mathbb{E}(X_n\mathbf{1}_{\{|X_n|\leq 1\}})]\lambda^n$ satisfies condition (j). Theorem 2.7 yields that it is a.s. uniformly convergent, so, the series $\sum_{n=1}^\infty X_n\mathbf{1}_{\{|X_n|\leq 1\}}\lambda^n$ is a.s. uniformly convergent. Since $X_n\to 0$ a.s., the series $\sum_{n=1}^\infty X_n\lambda^n$ satisfies $(i)-(iv)$. □

Theorem 2.11. *Let $\{X_n\}$ be a sequence of centered independent random variables. If the random Fourier series $\sum_{n=1}^{\infty} X_n\lambda^n$ satisfies one of the above conditions $(i)-(iv)$, then also the deterministic series $\sum_{n=1}^{\infty} \mathbb{E}(X_n\mathbf{1}_{\{|X_n|\le 1\}})\lambda^n$ satisfies this condition.*

Proof. Assume the series $\sum_{n=1}^{\infty} X_n\lambda^n$, satisfies one of the conditions $(i)-(iv)$ above, say $(j)\in\{(i),..,(iv)\}$. So, $X_n\to 0$ a.s., and the series $\sum_{n=1}^{\infty} X_n\mathbf{1}_{\{|X_n|\le 1\}}\lambda^n$ satisfies condition (j). Let $\{X_n'\}$ be an independent copy of $\{X_n\}$, and let $\mathbb{E}'$ be the corresponding expectation in the probability space of $\{X_n'\}$. For every $n\ge 1$, define the *symmetric* random variables $Y_n = X_n\mathbf{1}_{\{|X_n|\le 1\}} - X_n'\mathbf{1}_{\{|X_n'|\le 1\}}$. Now, the series $\sum_{n=1}^{\infty} Y_n\lambda^n$ satisfies condition (j), so by Billard's theorem it is a.s. uniformly convergent. Hence, by Theorem 2.2, the series $\sum_{n=1}^{\infty}[X_n\mathbf{1}_{\{|X_n|\le 1\}} - \mathbb{E}(X_n\mathbf{1}_{\{|X_n|\le 1\}})]\lambda^n$ is a.s. uniformly convergent, and the result follows. □

3 A counterexample in the centered case

Theorem 3.1. *There exist a sequence of independent identically distributed centered (hence in L^1) real random variables $\{Z_n\}$, and a sequence of complex numbers $\{a_n\}\in\ell_2$ such that almost surely the Fourier series $\sum_{n=1}^{\infty} a_nZ_n\lambda^n$ converges at every $\lambda\in\Gamma$ and represents a continuous function, while $\sum_{n=1}^{\infty} a_nZ_n\lambda^n$ is almost surely non uniformly convergent. Moreover, almost surely, the partial sums of the series are uniformly bounded on the circle.*

Remarks. 1. Cuzick and Lai [2, p. 10] constructed an example of random Fourier series, for which a.s. there is convergence at each point, *except one* for which, in fact, the random Fourier series diverges to infinity. So, the limit function is not continuous.

2. The sequence $\{a_nZ_n\}$ in the theorem is complex. We do not know of any counterexample with $\{a_nZ_n\}$ being real valued.

Theorem 3.1 is based on the following basic result.

Proposition 3.2. *There exists $C>0$ such that for all $l\ge 2$, we have*

$$\sup_{n\ge 1}\sup_{x\in\mathbb{R}}\Big|\sum_{k=1}^{n}\frac{\sin kx}{(k+l)\log(k+l)}\Big|\le\frac{C}{\log l}.$$

Proof. Define $S_n(x)=\sum_{k=1}^{n}\frac{\sin kx}{k}$, for every $n\ge 1$. It is well known (see e.g. Zygmund [19, Vol I, p. 61]), that $\{S_n\}$ is uniformly bounded, i.e., $K:=\sup_{n\ge 1}\sup_{x\in\mathbf{R}}|S_n(x)|<\infty$. Define $R_{l,n}(x)=\sum_{k=1}^{n}\frac{\sin kx}{(k+l)\log(k+l)}$, for every $n\ge 1$, and put $S_0(x)\equiv 0$. Also, for any $l\ge 2$ and $k\ge 1$ define $u_l(k)=\frac{1}{\log(k+l)}$ and $v_l(k)=\frac{l}{(k+l)\log(k+l)}$.

By Abel's summation by parts we have

$$R_{l,n}(x)=\sum_{k=1}^{n}[S_k(x)-S_{k-1}(x)]\frac{k}{(k+l)\log(k+l)}=\sum_{k=1}^{n}[S_k(x)-S_{k-1}(x)][u_l(k)-v_l(k)]$$

$$=\sum_{k=1}^{n}S_k(x)[u_l(k)-v_l(k)-u_l(k+1)+v_l(k+1)]+S_n(x)[u_l(n+1)-v_l(n+1)].$$

Hence, by monotonicity (in k) of $\{u_l(k)\}$ and $\{v_l(k)\}$ we have

$$|R_{l,n}(x)|\le\sum_{k=1}^{n}|S_k(x)|[u_l(k)-u_l(k+1)]+\sum_{k=1}^{n}|S_k(x)|[v_l(k)-v_l(k+1)]+$$

$$|S_n(x)||u_l(n+1)-v_l(n+1)|\le$$

$$K[u_l(1)-u_l(n+1)]+K[v_l(1)-v_l(n+1)]+K[u_l(n+1)+v_l(n+1)]\leq$$
$$K[u_l(1)+v_l(1)]\leq K\Big[\frac{1}{\log(l+1)}+\frac{l}{(l+1)\log(l+1)}\Big]\leq\frac{2K}{\log(l+1)}\,.$$

□

Proof of Theorem 3.1. For any $n\geq 1$ put $u_n=2^{2^n}$ and $v_n=\frac{u_n+u_{n+1}}{2}$. In addition put $a_k=\frac{1}{u_n+v_n-k}e^{\frac{ik}{n}}$ if $u_n+1\leq k\leq v_n-1$, and $a_k=-\frac{1}{u_n-v_n+k}e^{\frac{ik}{n}}$ if $v_n+1\leq k\leq u_{n+1}-1$. Otherwise put $a_k=0$.

Let Z be a real random variable with distribution law given by:

$$\mathbb{P}(Z=-1)=\alpha,\quad \text{and for } t\geq 2\quad \mathbb{P}(Z\geq t)=\beta\int_t^\infty f(x)dx,$$

where $f(x)=\frac{1}{x^2(\log\ x)^2}$, and α and β are chosen such that Z be a centered random variable.

Let $\{Z_n\}$ be a sequence of independent copies of Z. We want to prove that the series $\sum_{n=1}^\infty a_nZ_n\lambda^n$ satisfies the assertions of the theorem.

As suggested by Theorem 2.10 and Theorem 2.11, we will proceed by truncation in order to reduce our proof to the case of a deterministic Fourier series.

For any $n\geq 1$ put $b_k=\frac{u_n+v_n-k}{\log(u_n+v_n-k)}$ if $u_n\leq k\leq v_n-1$, and put $b_k=\frac{u_n-v_n+k}{\log(u_n-v_n+k)}$ if $v_n\leq k\leq u_{n+1}-1$. Now we define $Y_n=Z_n\mathbf{1}_{\{Z_n\leq b_n\}}$, and we show that the series $\sum_{n=1}^\infty|a_n||Z_n-Y_n|$ converges a.s.

(i) Since $\mathbb{E}|Z|<\infty$,

$$\sum_{k=5}^\infty\mathbb{P}(|a_k||Z_k-Y_k|\geq 1)=$$
$$\sum_{n=1}^\infty\Big(\sum_{k=u_n+1}^{v_n-1}\mathbb{P}(|Z_k-Y_k|\geq u_n+v_n-k)+\sum_{k=v_n+1}^{u_{n+1}-1}\mathbb{P}(|Z_k-Y_k|\geq u_n-v_n+k)\Big)=$$
$$\sum_{n=1}^\infty\Big(\sum_{k=u_n+1}^{v_n-1}\mathbb{P}(Z_k\geq u_n+v_n-k)+\sum_{k=v_n+1}^{u_{n+1}-1}\mathbb{P}(Z_k\geq u_n-v_n+k)\Big)=$$
$$2\sum_{n=1}^\infty\sum_{k=u_n+1}^{v_n-1}\mathbb{P}(Z\geq k)\leq 2\sum_{n=5}^\infty\mathbb{P}(Z\geq n)<\infty$$

(ii) In a similar way of breaking the sums we have,

$$\sum_{k=5}^\infty\mathbb{E}\big(|a_k||Z_k-Y_k|\mathbf{1}_{\{|a_k||Z_k-Y_k|\leq 1\}}\big)$$
$$=\sum_{n=1}^\infty\Big(\sum_{k=u_n+1}^{v_n-1}|a_k|\mathbb{E}\big(Z_k\mathbf{1}_{\{b_k<Z_k\leq 1/|a_k|\}}\big)+\sum_{k=v_n+1}^{u_{n+1}-1}|a_k|\mathbb{E}\big(Z_k\mathbf{1}_{\{b_k<Z_k\leq 1/|a_k|\}}\big)\Big)$$
$$=\sum_{n=1}^\infty\sum_{k=u_n+1}^{v_n-1}\frac{1}{u_n+v_n-k}\mathbb{E}\big(Z\mathbf{1}_{\{\frac{u_n+v_n-k}{\log(u_n+v_n-k}\leq Z\leq u_n+v_n-k\}}\big)+$$
$$\sum_{n=1}^\infty\sum_{k=v_n+1}^{u_{n+1}-1}\frac{1}{u_n-v_n+k}\mathbb{E}\big(Z\mathbf{1}_{\{\frac{u_n-v_n+k}{\log(u_n-v_n+k)}\leq Z\leq u_n-v_n+k\}}\big)=$$

$$2\sum_{n=1}^{\infty}\sum_{k=u_n+1}^{v_n-1}\frac{1}{k}\mathbb{E}\big(Z\mathbf{1}_{\{\frac{k}{\log k}\le Z\le k\}}\big)\le 2\sum_{n=5}^{\infty}\frac{1}{n}\mathbb{E}\big(Z\mathbf{1}_{\{\frac{n}{\log n}\le Z\le n\}}\big)$$

$$=2\beta\sum_{n=5}^{\infty}\frac{1}{n}\int_{\frac{n}{\log n}}^{n}xf(x)dx=2\beta\sum_{n=5}^{\infty}\frac{1}{n}\big(\frac{1}{\log n-\log\log n}-\frac{1}{\log n}\big)\le C\sum_{n=5}^{\infty}\frac{\log\log n}{n(\log n)^2}<\infty.$$

Combining (i) and (ii) together yields that $\sum_{n=1}^{\infty}|a_n||Z_n-Y_n|$ converges a.s.

We have

$$\sum_{n=1}^{\infty}2^{n/2}\big(\sum_{k=u_n+1}^{u_{n+1}}\mathbb{E}|a_kY_k|^2\big)^{1/2}\le\sum_{n=1}^{\infty}2^{n/2}\big(2\sum_{k=u_n+1}^{v_n-1}\frac{1}{k^2}\mathbb{E}(Z^2\mathbf{1}_{\{|Z|\le\frac{k}{\log k}\}})\big)^{1/2}\le$$

$$\sqrt{2}\sum_{n=1}^{\infty}2^{n/2}\Big(\sum_{k=u_n+1}^{\infty}\frac{1}{k^2}(\alpha+\int_2^{\frac{k}{\log k}}x^2f(x)dx)\Big)^{1/2}\le C\sum_{n=1}^{\infty}2^{n/2}\big(\sum_{k\ge u_n+1}\frac{1}{k(\log k)^3}\big)^{1/2}\le$$

$$C'\sum_{n=1}^{\infty}2^{-n/2}<\infty\,,$$

for some positive constants C and C'.

Let $\{Y_n'\}$ be an independent copy of $\{Y_n\}$ and write $\hat{Y}_n=Y_n-Y_n'$. Then

$$\sum_{n\ge1}2^{n/2}\big(\sum_{k=u_n+1}^{u_{n+1}}\mathbb{E}|a_k\hat{Y}_k|^2\big)^{1/2}<\infty.$$

By the proof of Theorem 1 in [9, Ch. 7, p. 84] (see also [18, Theorem 10]), almost surely, the random Fourier series $\sum_{n=1}^{\infty}a_n\hat{Y}_n\lambda^n$ is uniformly convergent. Since by construction we have $\sup_{n\ge1}|a_nY_n|<\infty$ everywhere, Theorem 2.2 yields that the random Fourier series $\sum_{n=1}^{\infty}a_n(Y_n-\mathbb{E}Y_n)\lambda^n$ is a.s. uniformly convergent. Since by (i) and (ii) above the series $\sum_{n=1}^{\infty}a_n(Y_n-Z_n)\lambda^n$ is a.s. absolutely convergent, the random Fourier series $\sum_{n=1}^{\infty}a_nZ_n\lambda^n$ a.s. converges everywhere (or uniformly), or represents a continuous function if and only if the *deterministic* series $\sum_{n=1}^{\infty}a_n\mathbb{E}Y_n\lambda^n$ converges everywhere (or uniformly), or represents a continuous function.

For any $k\ge5$ we have

$$\mathbb{E}\big(Z\mathbf{1}_{\{Z\le\frac{k}{\log k}\}}\big)=-\mathbb{E}\big(Z\mathbf{1}_{\{Z>\frac{k}{\log k}\}}\big)=-\beta\int_{\frac{k}{\log k}}^{\infty}xf(x)dx=-\frac{\beta}{\log k-\log\log k}.$$

Hence,

$$\mathbb{E}Y_k=-\frac{\beta}{\log|a_k|-\log\log|a_k|},$$

and we obtain

$$\mathbb{E}Y_k=-\frac{\beta}{\log|a_k|}+O\Big(\frac{\log\log|a_k|}{\log^2|a_k|}\Big).\qquad(*)$$

Using the convention $0/\log 0=0$, define $T(\lambda)=\sum_{k=1}^{\infty}\frac{a_k}{\log|a_k|}\lambda^k$ in $L_2(d\lambda)$. By (*) we have $\sum_{k=1}^{\infty}|a_k\mathbb{E}Y_k+\frac{\beta a_k}{\log|a_k|}|<\infty$. Hence, it is enough for our purpose to study the behavior of $T(\lambda)$.

For any $n\ge1$, define

$$Q_n(\lambda)=\sum_{k=u_n+1}^{v_n-1}\frac{\lambda^k-\lambda^{2v_n-k}}{(u_n+v_n-k)\log(u_n+v_n-k)}.$$

We have

$$T(\lambda)=\sum_{n=1}^{\infty}\sum_{k=u_n+1}^{u_{n+1}-1}\frac{a_k}{\log|a_k|}\lambda^k=\sum_{n=1}^{\infty}\sum_{k=u_n+1}^{v_n-1}\frac{(e^{\frac{i}{n}}\lambda)^k-(e^{\frac{i}{n}}\lambda)^{2v_n-k}}{(u_n+v_n-k)\log(u_n+v_n-k)}=\sum_{n=1}^{\infty}Q_n(e^{\frac{i}{n}}\lambda).$$

Since

$$Q_n(\lambda)=\sum_{k=u_n+1}^{v_n-1}\frac{\lambda^k-\lambda^{2v_n-k}}{(u_n+v_n-k)\log(u_n+v_n-k)}=\lambda^{v_n}\sum_{k=1}^{v_n-u_n-1}\frac{\lambda^{-k}-\lambda^k}{(k+u_n)\log(k+u_n)},$$

Proposition 3.2 yields that

$$\|Q_n\|_{C(\Gamma)}\le\frac{2C}{\log u_n}\le\frac{2C}{2^n}.$$

Hence the partial sums $\sum_{k=1}^{u_{n+1}}\frac{a_k\lambda^k}{\log|a_k|}=\sum_{k=1}^{n}Q_k(e^{i/k}\lambda)$ converge uniformly , necessarily to $T(\lambda)$, so T represents a continuous function.

Now we show that $\sum_{k=1}^{\infty}\frac{a_k\lambda^k}{\log|a_k|}$ converges pointwise to $T(\lambda)$. For any $u_n<m\le u_{n+1}$ we have

$$\Big|\sum_{k=1}^{m}\frac{a_k\lambda^k}{\log|a_k|}-T(\lambda)\Big|\le\Big|\sum_{k=1}^{u_n}\frac{a_k\lambda^k}{\log|a_k|}-T(\lambda)\Big|+\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|.$$

As we already showed, the first term on the right hand side converges uniformly to zero, so it is enough to show that pointwise $\max\limits_{u_n<m\le u_{n+1}}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|\to_n 0$. Since we clearly have

$$\max_{u_n<m\le u_{n+1}}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|\le\max_{u_n<m\le v_n}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|+\max_{v_n<m\le u_{n+1}}\Big|\sum_{k=v_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|,$$

it is enough to show that each of the terms on the right hand side goes to zero pointwise. Put $D_n(t):=\sum_{k=1}^{n}\mathrm{e}^{int}$. Simple computation yields that $|D_n(t)|\le 4/|t|$ for $0<|t|\le\pi$. Fix $\lambda_0=e^{it_0}$. Using Abel's summation by parts we have

$$\sum_{k=u_n+1}^{m}\frac{a_k\lambda_0^k}{\log|a_k|}=\sum_{k=u_n+1}^{m}\frac{|a_k|\mathrm{e}^{ik(1/n+t_0)}}{\log|a_k|}=\sum_{k=u_n+1}^{m}\frac{|a_k|[D_k(1/n+t_0)-D_{k-1}(1/n+t_0)]}{\log|a_k|}=$$

$$\sum_{k=u_n+1}^{m}\Big(\frac{|a_k|}{\log|a_k|}-\frac{|a_{k+1}|}{\log|a_{k+1}|}\Big)D_k(1/n+t_0)-\frac{|a_{u_n+1}|D_{u_n}(1/n+t_0)}{\log|a_{u_n+1}|}+\frac{|a_{m+1}|D_m(1/n+t_0)}{\log|a_{m+1}|}.$$

For any t_0 we have eventually $|1/n+t_0|>0$. Using monotonicity of $\{-|a_k|/\log|a_k|\}$ we have

$$\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda_0^k}{\log|a_k|}\Big|\le\frac{8}{|1/n+t_0|}\Big(\frac{1}{u_n\log u_n}-\frac{1}{v_n\log v_n}\Big).$$

Hence $\max\limits_{u_n<m\le v_n}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda_0^k}{\log|a_k|}\Big|\to_n 0$. Similarly $\max\limits_{v_n<m\le u_{n+1}}\Big|\sum_{k=v_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|\to_n 0$ pointwise.

Now, by evaluating certain blocks of the series $\sum_{k=1}^{\infty}\frac{a_k\lambda^k}{\log|a_k|}$ along the sequence $\{\mathrm{e}^{-i/n}\}$, we show that $\sum_{k=1}^{n}\frac{a_k\lambda^k}{\log|a_k|}$ is not Cauchy in $C(\Gamma)$, hence does not uniformly converge. This is established by the following:

$$\Big|\sum_{k=u_n+1}^{v_n-1}\frac{a_k\mathrm{e}^{-ik/n}}{\log|a_k|}\Big|=\sum_{k=u_n+1}^{v_n-1}\frac{1}{(u_n+v_n-k)\log(u_n+v_n-k)}=\sum_{k=u_n+1}^{v_n-1}\frac{1}{k\log k}\ge$$

$$\log\log(v_n-1)-\log\log(u_n+1)\geq\log\log(u_{n+1}/2)-\log\log(2u_n)\geq$$
$$\log((2^{n+1}-1)\log 2)-\log((2^n+1)\log 2)\geq\log 2+o(1/2^n).$$

On the other hand, since

$$\sup_{m\geq 1}\max_{|\lambda|=1}\Big|\sum_{k=1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|\leq\sup_{n\geq 1}\max_{|\lambda|=1}\Big|\sum_{k=1}^{u_n}\frac{a_k\lambda^k}{\log|a_k|}\Big|+\sup_{n\geq 1}\Big\{\sum_{k=u_n+1}^{u_{n+1}}\frac{|a_k|}{-\log|a_k|}\Big\}\leq$$

$$\sum_{k=1}^{\infty}\|Q_k(e^{i/k}\lambda)\|_{C(\Gamma)}+2\sup_{n\geq 1}\Big\{\sum_{k=u_n+1}^{v_n-1}\frac{1}{k\log k}\Big\}\leq 4C+2\sup_{n\geq 1}(\log\log(v_n)-\log\log(u_n))\leq$$

$$4C+2\sup_{n\geq 1}(\log\log(2^{2^{n+1}})-\log\log(2^{2^n}))=4C+2\log 2,$$

the partial sums $\sum_{k=1}^{n}\frac{a_k\lambda^k}{\log|a_k|}$ are uniformly bounded.

□

Remark. The construction in Theorem 3.1 is inspired by the counterexample of Fejer as presented in Zygmund [19, Vol. I, p. 299].

Corollary 3.3. *There exist a sequence of independent identically distributed centered real random variables $\{Z_n\}$ and a sequence of complex numbers $\{a_n\}\in\ell_2$, such that almost surely, for any contraction T on a Hilbert space $\mathcal{H}$ the series $\sum_{n=1}^{\infty}a_nZ_nT^nh$ converges in norm for every $h\in\mathcal{H}$, but the Fourier series $\sum_{n=1}^{\infty}a_nZ_n\lambda^n$ does not converge uniformly.*

Proof. For the sequence which was constructed in the previous theorem, a.s. the partial sums $\sum_{k=1}^{n}a_kZ_k\lambda^k$ are uniformly bounded. We use the spectral theorem and the dominated convergence theorem to establish the corollary for unitary operators. Then we apply the unitary dilation theorem to pass to the contraction case (see [15, Appendix 4, §153]). □

Remark. Let $\{b_n\}$ be a series of complex numbers, and let $\{\gamma_n\}$ be a dense sequence in Γ. On ℓ_2 we define a unitary operator U by the following rule: $U\mathbf{c}=\{\gamma_nc_n\}$ for every $\mathbf{c}:=\{c_n\}\in\ell_2$. It is easy to check that uniform convergence of $\sum_{n=1}^{\infty}b_n\lambda^n$ is equivalent to operator norm convergence of $\sum_{n=1}^{\infty}b_nU^n$. Hence, in the above corollary we cannot have operator norm convergence for all contractions T (even only all unitary operators) on some Hilbert space.

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