A CLT FOR MULTI-DIMENSIONAL MARTINGALE DIFFERENCES IN A LEXICOGRAPHIC ORDER

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Dedicated to the memory of Mikhail Gordin

Abstract. We prove a central limit theorem for a square-integrable ergodic stationary multi-dimensional random field of martingale differences with respect to a lexicographic order.

1. Introduction

M. Rosenblatt [10] stated a central limit theorem (CLT) for *ergodic* square-integrable stationary two-dimensional random fields of martingale differences, with lexicographic order, which was a step towards a CLT for random fields satisfying some strong mixing conditions. In order to formulate the assertion we start with the following notations.

Notations. On $\mathbb{N}^d$, $d \geq 2$, we take a *lexicographic order* as follows: $\boldsymbol{n} = (n^1, \ldots, n^d) < \boldsymbol{m} = (m^1, \ldots, m^d)$ if and only if $n^d < m^d$ or there exists $i = 1, \ldots, d-1$, such that $n^j = m^j$ for $i < j \leq d$ and $n^i < m^i$. Let $\{\zeta_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$ be a square integrable array of random variables and let $\mathcal{F}_{\boldsymbol{n}}$ be the σ-field generated by $\{\zeta_{\boldsymbol{m}} : \boldsymbol{m} \leq \boldsymbol{n}\}$. We say that $\{\zeta_{\boldsymbol{n}}\}$ is a *d-dimensional martingale difference with respect to* $\{\mathcal{F}_{\boldsymbol{n}}\}$ if $\mathbb{E}[\zeta_{\boldsymbol{n}} | \mathcal{F}_{\boldsymbol{m}}] = 0$ for every $\boldsymbol{m} < \boldsymbol{n}$.

Rosenblatt's assertion for the ergodic stationary martingale differences was that, for $d = 2$, $\frac{1}{\sqrt{mn}} \sum_{j=0}^{m} \sum_{\ell=0}^{n} \zeta_{j,\ell}$ converges in distribution to a normal law, as $\min\{m, n\} \to \infty$. An indication of proof was mentioned in [10]. We are interested in convergence of the above expression as $\max\{m, n\} \to \infty$. As we will explain, one needs ergodicity of the individual shifts; mere ergodicity of the random field is not sufficient in general for this convergence.

Huang [8] proved the CLT for ergodic stationary two-dimensional square-integrable martingale differences with the lexicographic order, in the particular case of $m = n$.

Dedecker [3] has a CLT for multi-dimensional stationary random fields with averaging along Følner sequences. Clearly $m \times n$ rectangles with $mn \to \infty$ need not have the Følner property, so Dedecker's result does not implies the above mode of convergence, even when corrected to assume ergodicity of the individual shifts. It is worth mentioning that Dedecker's result is quite general and yields a CLT without requiring ergodicity – in that case the limiting distribution is (as usual) a mixture of normal distributions.

The purpose of the following note is to give a simple proof of the CLT for a d-dimensional martingale difference as above $\{\zeta_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$. Specifically, for $d = 2$,

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to give a sufficient condition for convergence under the assumption $\max\{m,n\} \to \infty$. Under some moment conditions, a rate in the CLT is also given.

2. The CLT for multi-dimensional martingale differences

We prove below a CLT for the random field $\{\zeta_{\boldsymbol{n}}, \mathcal{F}_{\boldsymbol{n}}\}$ described above. The second part of our CLT below is new. The first part of the theorem is included for the sake of completeness.

From now on we use the following notation: $D_{\boldsymbol{n}} := \{\boldsymbol{m} : 0 \le m^i < n^i, 1 \le i \le d\}$.

Since the adaptation of the notation and proofs from dimension two to any finite dimension $d > 2$ is straightforward, for the sake of clarity and in order to avoid too long expressions, we prove the relevant statements in dimension two.

Theorem 2.1. *Let $\{\zeta_{\boldsymbol{n}}, \mathcal{F}_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$ be a square-integrable ergodic stationary d-dimensional random field of real martingale differences. Then $\frac{1}{\sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m} \in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}$ converges in distribution to $\mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2)$ as $\min\{n^1, n^2, \dots, n^d\} \to \infty$.*

If the d shifts of the random field are ergodic, then $\frac{1}{\sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m} \in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}$ converges in distribution to $\mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2)$ as $n^1 \cdot n^2 \cdots n^d \to \infty$ (equivalently, as $\max\{n^1, n^2, \dots, n^d\} \to \infty$).

Proof. The first assertion is a consequence of Theorem 1 of Dedecker [3] (a result about stationary random fields with averaging along Følner sequences). We mention that this assertion can be proved also along the same lines of the proof of the second assertion given below. Only the multi-dimensional mean ergodic theorem is needed, instead of Lemma 2.2.

We prove the second assertion. For the sake of clarity we prove it for $d = 2$. First we make the following observation. A two-dimensional sequence of random variables $\{Z_{m,n}\}_{m,n \ge 1}$ converges in distribution, as $mn \to \infty$, to a random variable Z if and only if for every subsequences $\{m_k\}$, $\{n_k\}$ with $m_k n_k \to_k \infty$, $\{Z_{m_k,n_k}\}$ converges in distribution to Z. Indeed, this claim is about numerical sequences ($\{\mathbb{P}(Z_{m,n} \le t)\}$ for fixed t a point of continuity of Z), and can be easily verified by definition.

Let $\{m_k\}$, $\{n_k\}$ with $m_k n_k \to_k \infty$. In order to prove that $\frac{1}{\sqrt{m_k n_k}} \sum_{j=0}^{m_k-1} \sum_{\ell=0}^{n_k-1} \zeta_{j,\ell}$ converges in distribution to $\mathcal{N}(0, \mathbb{E}|\zeta_{0,0}|^2)$ we check the conditions of the CLT of McLeish [9, Theorem 2.3]. For every k we order the $m_k \times n_k$ "rectangle" of random variables $\{\zeta_{j,\ell} : 0 \le j < m_k,\ 0 \le \ell < n_k\}$ as a normalized martingale difference row $\{\eta_{k,j} : 1 \le j \le m_k n_k\}$ ordered according to the lexicographic order $<$ on $\mathbb{N}^2$, as follows:

$$\eta_{k,1} = \frac{\zeta_{0,0}}{\sqrt{m_k n_k}},\ \eta_{k,2} = \frac{\zeta_{1,0}}{\sqrt{m_k n_k}},\ \dots, \eta_{k,n_k} = \frac{\zeta_{n_k-1,0}}{\sqrt{m_k n_k}},$$

$$\eta_{k,n_k+1} = \frac{\zeta_{0,1}}{\sqrt{m_k n_k}}\ \dots,\ \eta_{k,2n_k} = \frac{\zeta_{n_k-1,1}}{\sqrt{m_k n_k}}, \dots, \eta_{k,m_k n_k} = \frac{\zeta_{m_k-1,n_k-1}}{\sqrt{m_k n_k}}.$$

Let $\mathcal{F}_i^k$ (sub σ-algebra of $\mathcal{F}_{m_k,n_k}$) be the σ-field generated by $\eta_{k,1}, \dots, \eta_{k,i}$, $i = 1, 2, \dots, m_k n_k$. By construction and the assumptions, it is easy to see that $\mathbb{E}[\eta_{k,i} | \mathcal{F}_{i-1}^k] = 0$ for $i =$

$1, 2, \dots, m_k n_k$ (where we put $\mathcal{F}_0^k$ the trivial σ-field). Clearly, by stationarity,

$$\sup_k \| \max_{i \le m_k n_k} |\eta_{k,i}| \|_2^2 \le \sup_k \frac{1}{m_k n_k} \sum_{j=0}^{m_k-1} \sum_{\ell=0}^{n_k-1} \mathbb{E}|\zeta_{j,\ell}|^2 = \mathbb{E}[|\zeta_{0,0}|^2] < \infty.$$

Also, for $\varepsilon > 0$ we have by stationarity

$$\left\| \frac{1}{m_k n_k} \sum_{j=0}^{m_k-1} \sum_{\ell=0}^{n_k-1} |\zeta_{j,\ell}|^2 \mathbf{1}_{\{|\zeta_{j,\ell}| > \sqrt{m_k n_k}\,\varepsilon\}} \right\|_1 =$$

$$\frac{1}{m_k n_k} \sum_{j=0}^{m_k-1} \sum_{\ell=0}^{n_k-1} \mathbb{E}\big[|\zeta_{0,0}|^2 \mathbf{1}_{\{|\zeta_{0,0}| > \sqrt{m_k n_k}\,\varepsilon\}}\big] = \mathbb{E}\big[|\zeta_{0,0}|^2 \mathbf{1}_{\{|\zeta_{0,0}| > \sqrt{m_k n_k}\,\varepsilon\}}\big] \to_{k \to \infty} 0.$$

The above convergence to zero implies $\max_{i \le m_k n_k} |\eta_{k,i}| \to_k 0$ in probability, since

$$\big\{ \max_{i \le m_k n_k} |\eta_{k,i}| > \varepsilon \big\} = \big\{ \max_{j \le m_k, \ell \le n_k} |\zeta_{j,\ell}|^2 > m_k n_k \varepsilon^2 \big\} \subset$$

$$\big\{ \frac{1}{m_k n_k} \sum_{j=0}^{m_k-1} \sum_{\ell=0}^{n_k-1} |\zeta_{j,\ell}|^2 \mathbf{1}_{\{|\zeta_{j,\ell}| > \sqrt{m_k n_k}\varepsilon\}} > \varepsilon^2 \big\}.$$

Lastly, ergodicity of the two shifts yields, by Lemma 2.2 below, that

$$\sum_{i=1}^{m_k n_k} |\eta_{k,i}|^2 = \frac{1}{m_k n_k} \sum_{j=0}^{m_k-1} \sum_{\ell=0}^{n_k-1} |\zeta_{j,\ell}|^2 \to_k \mathbb{E}[|\zeta_{0,0}|^2]. \tag{1}$$

Hence conditions (a), (b) and (c) of McLeish [9, Theorem 2.3] hold and the assertion follows. □

Lemma 2.2. *Let $T_1, \dots, T_d$ be the operators induced on L_p $(1 \le p < \infty)$ by commuting probability preserving ergodic transformations of (S, Σ, μ). Put $A_n(T_j) := \frac{1}{n} \sum_{k=1}^n T_j^k$. Then*

$$\lim_{\pi_{j=1}^d n_j \to \infty} A_{n_1}(T_1) \cdots A_{n_d}(T_d) f = \int f \, d\mu$$

in L_p-norm for every $f \in L_p$.

Proof. Put $Ef = \int f \, d\mu$ for $f \in L_1$. Then by the mean ergodic theorem $A_n(T_j)f$ converges to Ef in L_p-norm for every $f \in L_p$ $(j = 1, \dots, d)$. Fix p and $f \in L_p$. For $\epsilon > 0$ there is N such that $\|A_n(T_j)f - Ef\|_p < \epsilon$ for $n > N$ and $j = 1, \dots, d$. Hence for $\prod_{j=1}^d n_j > N^d$ there is an i with $n_i > N$, and since $A_n(T_j)E = E$ we obtain

$$\|A_{n_1}(T_1) \cdots A_{n_d}(T_d) f - Ef\|_p \le \prod_{j \ne i} \|A_{n_j}(T_j)\| \cdot \|A_{n_i}(T_i) f - Ef\|_p < \epsilon, \tag{2}$$

which proves the lemma. Here $\|A\| = \sup\{\|Af\|_p : \|f\|_p = 1\}$ is the operator norm of a linear operator A acting on L^p. □

Remarks. 1. As (2) shows, Lemma 2.2 holds if only one of the T_i's is ergodic, as long as its corresponding coordinate $n_i \to \infty$.

An inspection of the proof of the second part of Theorem 2.1 yields that its conclusion holds if only one of the shifts is ergodic, as long as its corresponding coordinate tends to infinity. Of course, this result does not imply that of the theorem, since $\max\{n^1, n^2, \dots, n^d\} \to \infty$ does not imply that one of the n^i's tends to infinity.

2. The ergodic theorem does not hold in general (for $mn \to \infty$) without assuming, e.g., that the two actions are separately ergodic (see Lemma 2.2), so it seems that when only the random field is ergodic (i.e. the $\mathbb{N}^2$-action of the shifts is), the second assertion of our theorem will not hold in its generality. If for example the shift T is not ergodic, we fix $m \equiv 1$ and then we have a one-dimensional stationary sequence of square-integrable martingale differences which is not ergodic; in our proof, with $n_k = k$, the limit of $\sum_{i=1}^{k} |\eta_{k,i}|^2$ (for condition (c) of [9]) will be a non-constant random variable, and then [5, Theorem 3.2] yields that the limiting distribution of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \zeta_{0.k}$ is not Gaussian.

3. Theorem 1 of Dedecker [3] is about stationary random fields and the averages are along sets which satisfy Følner's condition. It is clear that $n^1 \times n^2 \times \dots n^d$ rectangular boxes $D_{\boldsymbol{n}}$ with $|D_{\boldsymbol{n}}| = n^1 \cdot n^2 \cdots n^d \to \infty$ do not always have the Følner property, which is the reason we can not deduce the second assertion of the theorem from Dedecker's result.

4. In the proof of Theorem 2.1 we conclude the L^1-norm convergence in (1) either by the mean ergodic theorem or by Lemma 2.2. This yields condition (c) of [9]. To conclude, only convergence in probability is needed. In particular, if in (1) we have convergence in probability toward some a.s. finite random variable η^2, the limiting distribution would have been a random variable Z whose characteristic function is $\mathbb{E}[\mathrm{e}^{-\frac{1}{2}\eta^2 t^2}]$ (see [5, Theorem 3.2]). In the ergodic case η^2 is the a.s. constant $\mathbb{E}|\zeta_{\mathbf{0}}|^2$.

5. An equivalent form of the second part of the theorem is

$$\frac{1}{\sqrt{|D_{\boldsymbol{n}}|}} \sum_{\boldsymbol{m} \in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}} \overset{dist.}{\Rightarrow} \mathcal{N}(0, \mathbb{E}|\zeta_{\mathbf{0}}|^2) \quad \text{as} \quad |D_{\boldsymbol{n}}| \to \infty.$$

The question is whether we can replace the sequence of boxes $D_{\boldsymbol{n}}$ by other increasing sequences of finite sets in $\mathbb{N}^d$. If the sequence of sets has the Følner property, then the answer is positive, as a special case of Theorem 1 in Dedecker [3]. An inspection of the proof of the second part of Theorem 2.1 shows that the mode of convergence is determined by (1), deduced from Lemma 2.2. Because of the total ordering it is easy to see that the process of reducing the dimension of the field and ordering it as an array of one dimensional martingale differences holds for any choice of sequence of finite sets in $\mathbb{N}^d$, possibly only partially ordered. The only problem is whether (1) holds (enough in probability) along these sets.

6. We show an example, in $d = 2$, in which the technique of our proof applies to certain "convex" non-rectangular domains, which do not have the Følner property. Define the trapezoidal domain $C_n := \{(j, 0) : 0 \le j \le 2n-1\} \cup \{(j, 1) : 0 \le j \le n-1\}$. Clearly, C_n does not satisfy the Følner property. Using the fact that the two shifts are ergodic and using stationarity, the convergence in (1) holds:

$$\frac{1}{|C_n|} \sum_{(j,\ell) \in C_n} |\zeta_{j,\ell}|^2 = \frac{1}{3n} \sum_{j=0}^{2n-1} |\zeta_{j,0}|^2 + \frac{1}{3n} \sum_{j=0}^{n-1} |\zeta_{j,1}|^2 \to \frac{2}{3}\mathbb{E}|\zeta_{0,0}|^2 + \frac{1}{3}\mathbb{E}|\zeta_{0,1}|^2 = \mathbb{E}|\zeta_{0,0}|^2.$$

Combining with all remarks we have made, we obtain

$$\frac{1}{\sqrt{|C_n|}} \sum_{\boldsymbol{m} \in C_n} \zeta_{\boldsymbol{m}} \overset{dist.}{\Rightarrow} \mathcal{N}(0, \mathbb{E}|\zeta_{\mathbf{0}}|^2) \quad \text{as} \quad |C_n| \to \infty.$$

7. Day [2, Lemma 5] proved that a sequence (C_n) of convex sets in $\mathbb{R}^d$ is Følner if the radius $r(C_n)$ of the maximal ball contained in C_n tends to infinity. Tempelman [11,

Example 2.11, p.180] observed that in that case the intersections $C_n \cap \mathbb{Z}^d$ yield a Følner sequence in $\mathbb{Z}^d$. An example of a sequence of non-convex subsets, which is growing to infinity in each direction and does not satisfy the Følner property, is given (in $d = 2$) by the boundary of the square $[0, n) \times [0, n)$. That is, we put $E_n = \{(j, 0) : 0 \le j < n\} \cup \{(0, j) : 0 \le j < n\} \cup \{(j, n-1) : 0 \le j < n\} \cup \{(n-1, j) : 0 \le j < n\}$. The convergence in (1) holds since

$$\frac{1}{|E_n|} \sum_{(j,\ell)\in E_n} |\zeta_{j,\ell}|^2 = \frac{1}{4(n-1)} \Big(\sum_{j=0}^{n-1} |\zeta_{j,0}|^2 + \sum_{j=1}^{n-1} |\zeta_{0,j}|^2 + \sum_{j=1}^{n-1} |\zeta_{j,n-1}|^2 + \sum_{j=1}^{n-2} |\zeta_{n-1,j}|^2 \Big).$$

Indeed, the first and the second summand (after normalization) converge, each to $\frac{1}{4}\mathbb{E}|\zeta_{0,0}|^2$. Now, denote the first coordinate shift by T and the second by S. Put $M_n = \frac{1}{4(n-1)} \sum_{j=1}^{n-1} |\zeta_{j,0}|^2$. The third term above is $S^{n-1}M_n$. By norm preserving and invariance of constants we conclude that

$$\|S^{n-1}M_n - \frac{1}{4}\mathbb{E}|\zeta_{0,0}|^2\|_1 = \|S^{n-1}(M_n - \frac{1}{4}\mathbb{E}|\zeta_{0,0}|^2)\|_1 = \|(M_n - \frac{1}{4}\mathbb{E}|\zeta_{0,0}|^2)\|_1 \to 0.$$

Similarly for the fourth summand. Hence we conclude $\frac{1}{|E_n|}\sum_{(j,\ell)\in E_n} |\zeta_{j,\ell}|^2 \to_n \mathbb{E}|\zeta_{0,0}|^2$. Combining with all remarks we have made, we obtain

$$\frac{1}{\sqrt{|E_n|}} \sum_{\boldsymbol{m}\in E_n} \zeta_{\boldsymbol{m}} \overset{dist.}{\Rightarrow} \mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2) \quad \text{as} \quad |E_n| \to \infty.$$

8. Huang's result [8] can be deduced from [5, Theorem 3.4] or Theorem 2.1 of Dvoretzky [4]. It is of course a consequence of Theorem 2.1.

9. Basu and Dorea [1] proved a CLT for d dimensional stationary square-integrable martingale differences with respect to a non-lexicographic partial order on $\mathbb{N}^d$. This partial order seems to be less suitable for certain applications. For $d = 2$ their CLT is similar to the first part of Theorem 2.1 (convergence as $\min\{m, n\} \to \infty$).

10. Our theorem is valid, with the same proof, also when we replace $\mathbb{N}^d$ by $\mathbb{Z}^d$.

When we have for the stationary random field of martingale differences a moment of order higher than 2, we can even obtain a rate in the above CLT. Our main tool is the following theorem of Heyde and Brown [7].

Theorem 2.3. *Let $\{\xi_n, \mathcal{F}_n, n = 0, 1, \cdots\}$ be a real martingale with $\xi_0 = 0$ and $\mathcal{F}_n$ the σ-field generated by $\xi_0, \xi_1, \ldots, \xi_n$, and put $\zeta_n := \xi_n - \xi_{n-1}$ for $n \ge 1$. Suppose that for some $0 < \delta \le 1$ we have $\mathbb{E}|\zeta_n|^{2+2\delta} < \infty$ for $n \ge 1$, and put $s_n^2 = \sum_{i=1}^n \mathbb{E}|\zeta_i|^2$.*

Then there exists a constant C, depending only on δ, such that for every $n \ge 1$ we have

$$\sup_t \Big| \mathbb{P}\{\xi_n/s_n \le t\} - \Phi(t) \Big| \le C\Big\{ s_n^{-2-2\delta} \Big(\sum_{i=1}^n \mathbb{E}|\zeta_i|^{2+2\delta} + \mathbb{E}\Big| \sum_{i=1}^n \zeta_i^2 - s_n^2 \Big|^{1+\delta} \Big) \Big\}^{1/(3+2\delta)}$$

Where Φ is the standard normal distribution.

Remark. It is easy to see that this statement is about a finite sequence of martingale differences (for $n > N$ put $\zeta_n = 0$ and $\mathcal{F}_n = \mathcal{F}_N$), and that the conclusion does not depend on the conditioning sequence $\{\mathcal{F}_n\}$. We note that Haeusler [6] proved the theorem for every $\delta > 0$ (without the restriction $\delta \le 1$).

Theorem 2.4. *Let* $\{\zeta_{\boldsymbol{n}}, \mathcal{F}_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$ *be a real d-dimensional martingale difference. Assume that for some* $0 < \delta \le 1$ *we have* $\mathbb{E}|\zeta_{\boldsymbol{n}}|^{2+2\delta} < \infty$ *for* $\boldsymbol{n} \in \mathbb{N}^d$. *Put* $s_{\boldsymbol{n}}^2 = \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \mathbb{E}|\zeta_{\boldsymbol{m}}|^2$. *Then there exists a constant* C, *depending only on* δ, *such that for every* $\boldsymbol{n} \in \mathbb{N}^d$ *we have*

$$\sup_t \Big|\mathbb{P}\{\frac{1}{s_{\boldsymbol{n}}} \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}} \le t\} - \Phi(t)\Big| \le$$
$$C\big\{s_{\boldsymbol{n}}^{-2-2\delta}\big(\sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \mathbb{E}|\zeta_{\boldsymbol{m}}|^{2+2\delta} + \mathbb{E}\big|\sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}^2 - s_{\boldsymbol{n}}^2\big|^{1+\delta}\big)\big\}^{1/(3+2\delta)}.$$

In particular (similarly to [7]*), if*

$$s_{\boldsymbol{n}}^{-2-2\delta} \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \mathbb{E}|\zeta_{\boldsymbol{m}}|^{2+2\delta} \to 0 \tag{3}$$

and

$$\mathbb{E}\big|s_{\boldsymbol{n}}^{-2}\big(\sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}^2\big) - 1\big|^{1+\delta} \to 0, \tag{4}$$

under a certain mode of convergence to ∞ *of* $\boldsymbol{n}$, *then* $\mathbb{P}\{\frac{1}{s_{\boldsymbol{n}}} \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}} \le t\}$ *converges uniformly to* $\Phi(t)$ *with the given rate above, in the same mode of convergence.*

Proof. For the proof we consider only the case $d = 2$. We apply the Heyde-Brown theorem in the following manner. Fix $m, n \ge 1$ and consider the $m \times n$ rectangle of random variables $\{\zeta_{j,k} : 0 \le j \le m-1,\ 0 \le k \le n-1\}$. Order these random variables as a one-dimensional sequence $\{\zeta_i, i = 1, 2, \dots, mn\}$ according to the lexicographic order $<$ on $\mathbb{N}^2$. That is,

$$\zeta_1 = \zeta_{0,0}, \zeta_2 = \zeta_{1,0}, \dots, \zeta_n = \zeta_{n-1,0}, \zeta_{n+1} = \zeta_{0,1}\ \dots, \zeta_{2n} = \zeta_{n-1,1}, \dots, \zeta_{mn} = \zeta_{m-1,n-1}.$$

Let $\{\mathcal{F}_i\}$ be the σ-field generated by $\zeta_1, \dots, \zeta_i$, $i = 1, 2, \dots, mn$. By construction and the assumptions, it is easy to see that $\mathbb{E}[\zeta_i|\mathcal{F}_{i-1}] = 0$ for $i = 1, 2, \dots, mn$ (where we put $\mathcal{F}_0$ the trivial σ-field). By our construction $\sum_{i=1}^{mn} \zeta_i = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \zeta_{j,k}$,

$$s_{m,n}^2 = \sum_{i=1}^{mn} \mathbb{E}|\zeta_i|^2 = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \mathbb{E}|\zeta_{j,k}|^2$$

and so on for the other expressions in the Heyde-Brown theorem above. Under our assumptions the theorem above yields the results. $\square$

Remark. The assertion of the above theorem holds for any sequence of finite sets in $\mathbb{N}^d$, not necessarily $D_{\boldsymbol{n}}$, see Remarks above.

Corollary 2.5. *Let* $\zeta_{\boldsymbol{n}}$ *be an ergodic stationary real martingale difference with respect to the filtration* $\mathcal{F}_{\boldsymbol{n}}$ *ordered by a lexicographic order on* $\mathbb{N}^d$. *Assume* $\mathbb{E}|\zeta_{\boldsymbol{0}}|^p < \infty$ *for some* $p > 2$. *Then* $\frac{1}{\sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}$ *converges in distribution to* $\mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2)$ *as* $\min\{n^1, n^2, \dots, n^d\} \to \infty$.

If the d shifts of the random field are ergodic, then $\frac{1}{\sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}$ *converges in distribution to* $\mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2)$ *as* $n^1 \cdot n^2 \cdots n^d \to \infty$.

In both cases the convergence is uniform at the rate implied by Theorem 2.4:

$$\sup_t \Big|\mathbb{P}\{\frac{1}{\|\zeta_{\boldsymbol{0}}\|_2 \sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}} \le t\} - \Phi(t)\Big| \le$$

$$C\Big\{\frac{\|\zeta_{\mathbf{0}}\|_p^p}{(n^1\cdot n^2\cdots n^d)^{(p-2)/2}\|\zeta_{\mathbf{0}}\|_2^p}+\big(\mathbb{E}\big|\frac{1}{n^1\cdot n^2\cdots n^d}\sum_{\boldsymbol{m}\in D_{\boldsymbol{n}}}\zeta_{\boldsymbol{m}}^2-\mathbb{E}\zeta_{\mathbf{0}}^2\big|^{p/2}\big)\Big\}^{1/(1+p)}.$$

Proof. We prove only the case $d = 2$. We may assume $p \le 4$ and put $\delta = (p-2)/2$. Stationarity yields that $\mathbb{E}|\zeta_{j,k}|^s = \mathbb{E}|\zeta_{0,0}|^s$ for $1 \le s \le p$, so $s_{m,n}^2 = mn \cdot \mathbb{E}|\zeta_{0,0}|^2$. We substitute this into the estimate of Theorem 2.4 and obtain the asserted estimate. We also obtain that (3) holds as $mn \to \infty$.

Let S and T be the two commuting isometries induced (in the L_s-spaces) by the two directional shifts. Since $(\zeta_{j,k})^2 = (S^jT^k\zeta_{0,0})^2 = S^jT^k\zeta_{0,0}^2$, (4) holds, as $\min\{m,n\} \to \infty$, by the two-dimensional mean ergodic theorem (in $L_{p/2}$) and ergodicity of $\{\zeta_{m,n}\}$ (which means that the only common fixed points of S and T are the constants). The first part of the corollary now follows from Theorem 2.4.

When both shifts S and T are ergodic, (4) holds, as $mn \to \infty$, by Lemma 2.2, and the second part of the corollary also follows from Theorem 2.4. □

Remarks. 1. The rate of the uniform convergence of the distribution functions depends on the rate of convergence in the d-dimensional mean ergodic theorem (or in Lemma 2.2 when the d shifts are ergodic and $n^1 \cdot n^2 \cdots n^d \to \infty$), which depends on $\zeta_{\mathbf{0}}$ since there is no uniform rate in the mean ergodic theorem.

2. The estimation in the above corollary holds for any sequence of finite sets in $\mathbb{N}^d$, not necessarily $D_{\boldsymbol{n}}$. However, in order to conclude the convergence in distribution we need to guarantee that we have the norm convergence of the corresponding averages along these set, e.g., Følner type sets.

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