

THE ONE-SIDED ERGODIC HILBERT TRANSFORM IN BANACH SPACES

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ABSTRACT. Let T be a power-bounded operator on a (real or complex) Banach space. We study the convergence of the one-sided ergodic Hilbert transform $\lim_n \sum_{k=1}^n \frac{T^k x}{k}$. We prove that weak and strong convergence are equivalent, and in a reflexive space also $\sup_n \|\sum_{k=1}^n \frac{T^k x}{k}\| < \infty$ is equivalent to the convergence. We also show that $-\sum_{k=1}^{\infty} \frac{T^k}{k}$ (which converges on $(I - T)X$) is precisely the infinitesimal generator of the semi-group $(I - T)^r \Big|_{\overline{(I - T)X}}$.

1. INTRODUCTION

Izumi [12] raised the question of the a.e. convergence of the one-sided ergodic Hilbert transform (EHT) $\sum_{k=1}^{\infty} \frac{f \circ \theta^k}{k}$ associated to a probability preserving transformation θ and functions in $L_2(S, \Sigma, \mu)$ (which, by Kronecker's lemma, would be a strengthening of Birkhoff's pointwise ergodic theorem). Halmos [10] proved that for every ergodic probability preserving transformation on a non-atomic space there always exists a centered $f \in L_2$ such that the one-sided EHT fails to converge in L_2 -norm.

On the other hand, Cotlar [5] proved that when T is the operator induced by an invertible probability preserving transformation, for every $f \in L_1$ the two-sided EHT $\sum_{k=1}^{\infty} \frac{T^k f - T^{-k} f}{k}$ converges a.e., and in L_p -norm when $f \in L_p$, $1 \leq p < \infty$. Campbell [2] proved that for a unitary operator T on a complex Hilbert space H the two-sided EHT converges in norm for every $f \in H$.

For T unitary on a complex Hilbert space H , Gaposhkin [9] obtained a spectral characterization of the norm convergence of the one-sided EHT $\sum_{k=1}^{\infty} \frac{T^k f}{k}$. For a normal contraction T , several additional characterizations were recently obtained by Cohen and Lin [3], who proved that norm convergence is equivalent to weak convergence; this had been proved by Assani and Lin [1] for T unitary or self-adjoint.

Recently, Cuny [6] proved that for T a Dunford-Schwartz operator on a σ -finite measure space or a positive contraction of an L_p space, $1 < p < \infty$,

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norm convergence of $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ implies a.e. convergence. For additional results and references concerning pointwise convergence see [9] and [1].

In this paper we consider a power-bounded operator T on a Banach space X (which is therefore a contraction in an equivalent norm), and for $x \in X$ we study the convergence of the series

$$(1) \quad \sum_{k=1}^{\infty} \frac{T^k x}{k}$$

Obviously, weak convergence of (1) implies that $x^*(x) = 0$ whenever $T^*x^* = x^*$, so necessarily $x \in \overline{(I - T)X}$ (e.g. [13, p. 73]). Derriennic and Lin [7, Theorem 2.23] proved that the series (1) converges in norm for *every* $x \in \overline{(I - T)X}$ if and only if $(I - T)X$ is closed. It was noted in [3] that the same holds if norm convergence is replaced by weak convergence.

We obtain several necessary and sufficient conditions for the convergence of the one-sided ergodic Hilbert transform (1). One of our main results is that weak convergence of the series (1) always implies its norm convergence.

2. ON SOME OPERATOR SERIES

Motivated by conditions for the central limit theorem for stationary ergodic Markov chains, Derriennic and Lin [7] defined for $0 < \alpha < 1$ and T power-bounded on X the operator $(I - T)^\alpha$ by the series $(I - T)^\alpha = I - \sum_{n=1}^{\infty} a_n^{(\alpha)} T^n$, where the coefficients are those of the expansion of $(1 - t)^\alpha$ in the interval $[-1, 1]$, with $a_n^{(\alpha)} > 0$ and $\sum_{n=1}^{\infty} a_n^{(\alpha)} = 1$. It is not difficult to show that $(I - T)^\alpha X \subset \overline{(I - T)X}$. It is proved in [7] that when T is mean-ergodic, $x \in (I - T)^\alpha X$ if and only if the series $\sum_{n=1}^{\infty} \frac{T^n x}{n^{1-\alpha}}$ converges strongly.

The proof uses the series representation of $1/(1 - t)^\alpha$, which converges for $|t| < 1$. This suggests the idea in [6] that in order to study the one-sided EHT, we try to connect it with the inverse of an analytic function on the open unit disk D .

Let $D := \{z \in \mathbb{C} : |z| < 1\}$. For $z \in D$, define

$$H(z) := \log\left(\frac{e}{1 - z}\right) = 1 + \sum_{n \geq 1} \frac{z^n}{n} = \sum_{n \geq 0} \beta_n z^n.$$

Since $1 - e \notin D$, $H \neq 0$ on D . Hence $G := \frac{1}{H}$ is well defined and analytic on D . Hence there exists $\{\alpha_n\}_{n \geq 0}$, such that

$$G(z) = \sum_{n \geq 0} \alpha_n z^n \quad \forall z \in D.$$

One can see that $\alpha_0 = 1$, and it follows from the identity $G(z)H(z) = 1$ that

$$(2) \quad \beta_n + \sum_{k=1}^n \alpha_k \beta_{n-k} = \beta_n + \sum_{k=0}^{n-1} \beta_k \alpha_{n-k} = 0 \quad \forall n \geq 1.$$

Consequently, the coefficients $\{\alpha_n\}$ are real, and it follows from [17, Theorem 2.31 p. 192] that $\alpha_n \sim \frac{-1}{n(\log n)^2}$. For convenience we denote $\gamma_n := \frac{K}{n(\log(n+1))^2}$, $n \geq 1$, where $K := \sup_{n \geq 1} n(\log(n+1))^2 |\alpha_n|$, so that $|\alpha_n| \leq \gamma_n$ for $n \geq 1$. Since $G(t) = 1/H(t)$ for $t \in (0, 1)$ and $G(1)$ is defined by the absolutely convergent power series, letting $t \rightarrow 1^-$ we obtain $G(1) = 0$.

We now fix a power-bounded operator T on a (real or complex) Banach space X . Since $\sum_{n \geq 0} |\alpha_n| < \infty$ and the coefficients are real, the operator series $\sum_{n \geq 0} \alpha_n T^n$ converges in operator norm, and defines a bounded operator on X , denoted by $G(T)$.

For every $n \geq 1$ we define

$$(3) \quad H_n(T) := I + \sum_{k=1}^n \frac{T^k}{k} = \sum_{k=0}^n \beta_k T^k$$

and put $H(T)x = \lim_{n \rightarrow \infty} H_n(T)x$ whenever the limit exists in norm.

Proposition 2.1. *Let T be a power-bounded operator on a Banach space X . Then*

$$(4) \quad \sup_{n \geq 1} \|H_n(T)G(T)\| < \infty.$$

Proof. Since the series $\sum_{n \geq 0} \alpha_n$ is absolutely convergent, we have

$$\begin{aligned} H_n(T)G(T) &= \sum_{k=0}^n \beta_k T^k + \sum_{k=0}^n \beta_k \sum_{m \geq 1} \alpha_m T^{m+k} \\ &= \sum_{k=0}^n \beta_k T^k + \sum_{k=0}^n \beta_k \sum_{m \geq k+1} \alpha_{m-k} T^m \\ &= \sum_{k=0}^n \beta_k T^k + \sum_{m=1}^n \left(\sum_{k=0}^{m-1} \beta_k \alpha_{m-k} \right) T^m + \sum_{m \geq n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m \\ (5) \quad &= I + \sum_{m \geq n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m, \end{aligned}$$

where we used $\beta_0 = 1$ and (2) for the last equality. It suffices to deal with the series of the last equality. We have

$$(6) \quad \sum_{m \geq n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m = \sum_{m=n+1}^{2n} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m + \sum_{m \geq 2n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m.$$

For the last sum monotonicity of $\{\gamma_n\}$ and $\sup_n \|T^n\| = M < \infty$ yield

$$\begin{aligned} \left\| \sum_{m \geq 2n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m \right\| &\leq \sum_{m \geq 2n+1} \gamma_{m-n} \left(\sum_{k=0}^n \beta_k \right) \|T^m\| \\ &\leq M(2 + \log n) \sum_{m \geq 2n+1} \gamma_{m-n} \leq C \log n \sum_{j \geq n+1} \frac{K}{j(\log(j+1))^2} \leq C'. \end{aligned}$$

Let us deal with the first sum on the right-hand side of (6). By (2), we have

$$\begin{aligned} \left\| \sum_{m=n+1}^{2n} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m \right\| &= \left\| \sum_{m=n+1}^{2n} \beta_m T^m + \sum_{m=n+1}^{2n} \left(\sum_{k=n+1}^{m-1} \beta_k \alpha_{m-k} \right) T^m \right\| \\ &\leq \sum_{m=n+1}^{2n} \beta_m \|T^m\| + \sum_{m=n+1}^{2n} \left(\sum_{k=n+1}^{m-1} \beta_k \gamma_{m-k} \right) \|T^m\| \\ &\leq Mn \frac{1}{n+1} + Mn \frac{1}{n+1} \sum_{j \geq 1} \gamma_j, \end{aligned}$$

using the definition of $\{\beta_n\}$ and its monotonicity for the last inequality. \square

The next proposition provides the main tool for our results. It was proved in [6] in the context of power-bounded operators on L_p spaces.

Proposition 2.2. *Let T be a power-bounded operator on a Banach space X and let $x \in \overline{(I-T)X}$. Then*

$$\lim_{n \rightarrow \infty} \|x - H_n(T)G(T)x\| = 0.$$

Proof. By the previous proposition, it is enough to prove the convergence for $x \in (I-T)X$. By (5), the assertion is that for $x \in (I-T)X$

$$\left\| \sum_{m \geq n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m x \right\| \xrightarrow{n \rightarrow +\infty} 0.$$

We denote $M := \sup_{n \geq 0} \|T^n\|$. For $u \in X$ we have

$$\begin{aligned}
 & \sum_{m \geq n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m (u - Tu) \\
 &= \sum_{m \geq n+1} \left(\sum_{k=0}^n \beta_k \alpha_{m-k} \right) T^m u - \sum_{m \geq n+2} \left(\sum_{k=0}^n \beta_k \alpha_{m-k-1} \right) T^m u \\
 (7) \quad &= \sum_{m \geq n+2} \left(\sum_{k=1}^n (\beta_k - \beta_{k-1}) \alpha_{m-k} \right) T^m u \\
 &+ \left(\sum_{k=0}^n \beta_k \alpha_{n+1-k} \right) T^{n+1} u + \sum_{m \geq n+1} \alpha_m T^m u - \sum_{m \geq n+2} \beta_n \alpha_{m-n-1} T^m u.
 \end{aligned}$$

We estimate the norms of the last three terms using power-boundedness, (2) and monotonicity of $\{\beta_n\}$ and obtain

$$\begin{aligned}
 & \left\| \sum_{k=0}^n \beta_k \alpha_{n+1-k} T^{n+1} u + \sum_{m \geq n+1} \alpha_m T^m u - \sum_{m \geq n+2} \beta_n \alpha_{m-n} T^m u \right\| \\
 & \leq M \|u\| \left(\sum_{j=[n/2]+1}^{n+1} \gamma_j + \frac{1}{[n/2]+1} \sum_{j=1}^{[n/2]+1} \gamma_j + \sum_{m \geq n+1} \gamma_m + \frac{1}{n} \sum_{m \geq 2} \gamma_m \right) \xrightarrow{n \rightarrow +\infty} 0.
 \end{aligned}$$

It remains to deal with the first term in (7). Splitting the inner sum according to $k \leq [n/2]$ and using monotonicity of $\{\gamma_n\}$, we obtain

$$\begin{aligned}
 & \left\| \sum_{m \geq n+2} \left(\sum_{k=1}^n (\beta_k - \beta_{k-1}) \alpha_{m-k} \right) T^m u \right\| \leq M \sum_{m \geq n+2} \left(\sum_{k=1}^n |\beta_k - \beta_{k-1}| \gamma_{m-k} \right) \|u\| \\
 & \leq M \sum_{k=1}^{[n/2]} (\beta_{k-1} - \beta_k) \|u\| \sum_{m \geq n+2} \gamma_{m-[n/2]} + \frac{M \|u\|}{[n/2]} \sum_{m \geq n+2} \gamma_{m-n} \\
 & \leq \frac{KM}{\log n} \|u\|,
 \end{aligned}$$

for a constant $K > 0$ independent of u (and T), which proves the proposition. \square

3. CONVERGENCE OF THE ONE-SIDED ERGODIC HILBERT TRANSFORM

In this section we obtain several criteria for the convergence of the one-sided ergodic Hilbert transform of a power-bounded operator. Additional criteria are obtained when the Banach space is reflexive. For normal contractions in a *complex* Hilbert space the main theorems were proved in [3]; for unitary operators and self-adjoint contractions they had been obtained in [1].

Lemma 3.1. *Let T be a power-bounded operator on a Banach space X and let $x \in X$. If $\liminf_{n \rightarrow \infty} \left\| \sum_{k=1}^n \frac{T^k x}{k} \right\| < \infty$, then $x \in \overline{(I - T)X}$.*

Proof. The condition implies that $\sup_j \|H_{n_j}(T)x\| < \infty$ for some $\{n_j\}$. Hence $x^*(x) = 0$ whenever $T^*x^* = x^*$, which implies $x \in \overline{(I - T)X}$. \square

Theorem 3.2. *Let T be a power-bounded operator on a Banach space X and let $x \in X$. Then the following are equivalent:*

- (i) *There is an increasing $\{n_j\}$ such that $\sum_{k=1}^{n_j} \frac{T^k x}{k}$ converges weakly.*
- (ii) *The series $\sum_{k=1}^{\infty} \frac{T^k x}{k}$ converges weakly.*
- (iii) *The series $\sum_{k=1}^{\infty} \frac{T^k x}{k}$ converges in norm.*

Proof. Obviously we have to prove only that (i) implies (iii). We assume that $H_{n_j}(T)x$ converges weakly to some z ; this implies that $\{\|H_{n_j}(T)x\|\}$ is bounded, so x (by Lemma 3.1), and hence z , are in $\overline{(I - T)X}$ (since it is weakly closed [8, Theorem V.3.13]), and Proposition 2.2 yields $x = H(T)G(T)x$. Using weak continuity of $G(T)$ we then obtain

$$\begin{aligned} G(T)z &= \text{weak-lim} G(T)H_{n_j}(T)x = \\ &= \text{weak-lim} H_{n_j}(T)G(T)x = H(T)G(T)x = x. \end{aligned}$$

Since $z \in \overline{(I - T)X}$, we can apply to it Proposition 2.2, and obtain that $H_n(T)x = H_n(T)G(T)z$ converges in norm (to z). \square

Remark. The theorem is the analogue, for the one-sided EHT, of Yosida's theorem for the Cesàro averages (e.g. [13, p. 72]).

Proposition 3.3. *Let T be power-bounded on X . The series $\sum_{k=1}^{\infty} \frac{T^k x}{k}$ converges if and only if $x \in G(T)[\overline{(I - T)X}]$.*

Proof. Denote $X_0 := G(T)[\overline{(I - T)X}]$. Proposition 2.2 shows that the limit operator $H(T)$, defined for all those x for which the one-sided EHT converges (in norm), has X_0 included in its domain of definition, and $H(T)$ maps its domain of definition onto all of $\overline{(I - T)X}$. By the previous proof, convergence of $H_n(T)x$ implies that $x \in X_0$, so X_0 is the precise domain of definition of the one-sided EHT. \square

Remark. Note that when T is mean ergodic the ergodic decomposition yields $X_0 = G(T)X$, since $G(1) = 0$ implies $G(T)z = 0$ whenever $Tz = z$.

Corollary 3.4. *Let T be power-bounded on X . For $0 < \delta < 1$ we have*

$$(I - T)X \subset (I - T)^\delta X \subset G(T)[\overline{(I - T)X}] \subset G(T)X \subset \overline{(I - T)X}$$

Proof. Combining (4) with Lemma 3.1 we obtain $G(T)X \subset \overline{(I-T)X}$. The first inclusion and the convergence of $H_n(T)x$ for $x \in (I-T)^\delta X$ are proved in [7]. \square

Remark. When $(I-T)X$ is not closed, $(I-T)^{\delta_1}X \subsetneq (I-T)^{\delta_2}X$ for every $\delta_2 < \delta_1 \leq 1$ by [7], hence $(I-T)^\delta X \neq G(T)[\overline{(I-T)X}]$ for $0 < \delta \leq 1$. By [7, Theorem 2.23], $G(T)[\overline{(I-T)X}] \neq \overline{(I-T)X}$ in this case.

Proposition 3.5. *Let $X = Y^*$ be a dual Banach space and let $T = S^*$ be a power-bounded dual operator on X . If*

$$(8) \quad \liminf_{n \rightarrow \infty} \left\| \sum_{k=1}^n \frac{T^k x}{k} \right\| < \infty$$

then $x \in G(T)X$. If in addition T is mean ergodic, then $\sum_{k=1}^{\infty} \frac{T^k x}{k}$ converges in norm.

Proof. By Lemma 3.1, (8) implies that $x \in \overline{(I-T)X}$. By assumption we have an increasing $\{n_j\}$ such that $\sup_{j \geq 1} \left\| \sum_{k=1}^{n_j} \frac{T^k x}{k} \right\| < \infty$. Let LIM denote a fixed Banach limit (e.g. [13, p. 135]), and for $f \in Y$ (Y is the predual of X) define

$$\psi(f) = \text{LIM} \left\{ \sum_{k=1}^{n_j} \frac{x(S^k f)}{k} \right\} = \text{LIM} \left\{ \sum_{k=1}^{n_j} \frac{(T^k x)(f)}{k} \right\} = \text{LIM} \{ [H_{n_j}(T)x](f) \}.$$

By (8) ψ is well-defined, linear and bounded, so $\psi \in Y^* = X$. Since $x \in \overline{(I-T)X}$, we have $G(T)H_n(T)x = H_n(T)G(T)x \rightarrow x$ in norm, by Proposition 2.2. Since Banach limits preserve convergence and $G(T) = G(S)^*$, we obtain

$$\begin{aligned} [G(T)\psi](f) &= \psi(G(S)f) = \text{LIM} \{ [H_{n_j}(T)x](G(S)f) \} \\ &= \text{LIM} \{ [G(T)H_{n_j}(T)x](f) \} = x(f) \end{aligned}$$

for every $f \in Y$, so $G(T)\psi = x$.

If in addition T is mean ergodic, $x \in G(T)X = G(T)\overline{(I-T)X} = X_0$, so $H_n(T)x$ converges by Proposition 3.3. \square

Remark. The use of a Banach limit is not necessary when the predual Y is separable; in this case the weak-* topology on bounded sets in $X = Y^*$ is metrizable, and there exists an increasing sequence of integers such that $H_{n_j}(T)x$ converges weak-* to some $\psi \in X$. Since $x \in \overline{(I-T)X}$, weak-* continuity of $G(T)$ and Proposition 2.2 yield $G(T)\psi = \text{weak-*} \lim G(T)H_{n_j}(T)x = x$.

Corollary 3.6. *Let T be a power-bounded operator on a reflexive Banach space. Then $\sum_{k=1}^{\infty} \frac{T^k x}{k}$ converges in norm if and only if*

$$\liminf_{n \rightarrow \infty} \left\| \sum_{k=1}^n \frac{T^k x}{k} \right\| < \infty.$$

Proof. Since X is reflexive, T is a dual mean ergodic operator. □

Combining the previous corollary with [6, Theorem 1.3] we obtain the following.

Corollary 3.7. *Let $1 < p < \infty$ and let T be a positive contraction of $L_p(S, \Sigma, \mu)$ of a σ -finite measure space. If $f \in L_p$ satisfies*

$$\liminf_{n \rightarrow \infty} \left\| \sum_{k=1}^n \frac{T^k f}{k} \right\|_p < \infty,$$

then $\sum_{k=1}^n \frac{T^k f}{k}$ converges almost everywhere and

$$\sup_{n \geq 1} \left| \sum_{k=1}^n \frac{T^k f}{k} \right| \in L_p(S, \mu).$$

Remark. Positivity is not needed when T is a Dunford-Schwartz operator – we apply [6, Theorem 1.2] instead of [6, Theorem 1.3].

Example 1. *A dual operator T and $x \in X$ with $\{H_n(T)x\}$ bounded and non-convergent.*

Define S on $Y = \ell_1$ by

$$S(a_0, a_1, a_2, \dots, a_n, \dots) = (a_1, a_2, a_3, \dots, a_{n+1}, \dots).$$

Then $T = S^*$ on $X = \ell_\infty$ is given by

$$T(b_0, b_1, b_2, \dots, b_n, \dots) = (0, b_0, b_1, b_2, \dots, b_{n-1}, \dots).$$

For $e = (1, 1, 1, \dots, 1, \dots) \in \ell_\infty$ we have $\|\frac{1}{n} \sum_{k=1}^n T^k e\|_\infty = 1$ for every n , so T is not mean ergodic since it has no fixed points. We put $x = G(T)e$; Proposition 2.1 yields that $\sup_n \|H_n(T)x\| < \infty$.

We show that $\{H_n(T)x\}$ is not Cauchy. We use $G(1) = 0$ and the asymptotic behaviour of α_n to conclude that for $m \geq 0$ the m th coordinate of $x = G(T)e$ is

$$x_m = \sum_{k=0}^m \alpha_k = - \sum_{k \geq m+1} \alpha_k \sim \frac{C}{\log(m+2)}.$$

For $k \geq 0$ the m th coordinate of $T^k x$ is 0 for $m < k$ and x_{m-k} for $m \geq k$.

Let $1 < p < q$. For $m > q$ we have

$$\left(\sum_{k=p}^q \beta_k T^k x\right)_m = \sum_{k=p}^q \beta_k x_{m-k} \geq \sum_{k=p}^q \frac{C\beta_k}{\log(m+2-k)}.$$

If we take $q = p^2$, for $m = p^2 + p$ we obtain

$$\begin{aligned} \|H_q(T)x - H_p(T)x\| &\geq \sum_{k=p}^{p^2} \frac{C\beta_k}{\log(m+2-k)} \geq \frac{C}{\log(p+2)} \sum_{k=p}^{p^2} \frac{1}{k} \\ &\geq \frac{K}{\log(p+2)} (\log(p^2) - \log p) \not\rightarrow 0. \end{aligned}$$

Proposition 3.8. *Let T be a power-bounded operator on a Banach space X such that T^{**} is mean ergodic on X^{**} (e.g. X is reflexive). Then $\sum_{k=1}^{\infty} \frac{T^k x}{k}$ converges if and only if $\liminf_{n \rightarrow \infty} \left\| \sum_{k=1}^n \frac{T^k x}{k} \right\| < \infty$.*

Proof. Assume the condition (8) holds. We identify X as a closed subspace of X^{**} by the canonical embedding, which is T^{**} -invariant.

We apply Proposition 3.5 to T^{**} and x and obtain the existence of $x^{**} \in X^{**}$ such that $G(T^{**})x^{**} = x$. As noted earlier, the mean ergodicity of T^{**} implies the existence of $z^{**} \in \overline{(I - T^{**})X^{**}}$ such that $G(T^{**})z^{**} = x$. By Proposition 2.2 we then have

$$\|H_n(T)x - z^{**}\| = \|H_n(T^{**})x - z^{**}\| = \|H_n(T^{**})G(T^{**})z^{**} - z^{**}\| \rightarrow 0,$$

which shows that $z^{**} \in X$ and proves the desired convergence. \square

Remark. The proof shows that (8) always implies that $x \in G(T^{**})X^{**}$, but we need $x \in G(T)[\overline{(I - T)X}] \subset G(T)X$ for convergence of $\{H_n(T)x\}$.

Example 2. *A mean ergodic power-bounded T and $x \in X$ with $\{H_n(T)x\}$ bounded and non-convergent.*

On $X = c_0$ we define the isometry

$$T(b_0, b_1, b_2, \dots, b_n, \dots) = (0, b_0, b_1, b_2, \dots, b_{n-1}, \dots),$$

Since $T^n z \rightarrow 0$ weakly for every $z \in c_0$, the operator T is mean ergodic. Note that the dual T^* on ℓ_1 is the operator S of Example 1, and is also mean ergodic. The operator T^{**} on ℓ_∞ is in fact the operator of Example 1, and we put $x = G(T^{**})e$. As shown in Example 1, the m th coordinate of x behaves asymptotically like $C/\log(m+2)$, so $x \in c_0$. The computations in Example 1 prove boundedness and non-convergence of $\{H_n(T)x\}$; in particular, $x \notin G(T)X$.

Derriennic and Lin [7] proved that for T power-bounded on X the family of operators $\{(I - T)^r := (I - T)^{[r]}(I - T)^{r-[r]} : r \geq 0\}$ satisfies the semi-group property, and in [7, Theorem 2.22] it is proved that the restriction of this semi-group to the closed subspace $\overline{(I - T)X}$ is strongly continuous at 0 (with the identity as limit at 0). The following answers a question raised in [1].

Theorem 3.9. *Let T be power-bounded on X , and let A be the infinitesimal generator of the semi-group $\{(I - T)^r : r \geq 0\}$ defined on $\overline{(I - T)X}$. Then $x \in \mathcal{D}(A)$ if and only if the series $\sum_{n=1}^{\infty} \frac{T^n x}{n}$ converges.*

The generator is then given by $Ax = -\sum_{n=1}^{\infty} \frac{T^n x}{n}$, $x \in \mathcal{D}(A)$.

Proof. If the one-sided EHT converges in norm, then $x \in \mathcal{D}(A)$ and $Ax = -\sum_{k=1}^{\infty} \frac{T^k x}{k}$, by [7, Proposition 2.21].

Now let $x \in \mathcal{D}(A) \subset \overline{(I - T)X}$. Then $Ax \in \overline{(I - T)X}$ and Proposition 2.2 yields that $z := G(T)(I - A)x$ is in the domain of the one-sided EHT and satisfies $\lim_n H_n(T)z = (I - A)x$. But since z is in the domain of the EHT, by [7] $z \in \mathcal{D}(A)$ and $\lim(H_n(T) - I)z = -Az$, so $(I - A)(x - z) = 0$.

When $Ay = y$, the semi-group continuity yields that $(I - T)^r y = e^r y$ (see the first part of the proof of [15, Theorem IX.4.1, p. 240], which applies also to real Banach spaces), so in particular $(I - T)y = ey$, yielding that if $y \neq 0$ then $\|T^n y\| = (e - 1)^n \|y\| \rightarrow \infty$, contradicting the power-boundedness of T . Hence $x = z$, so $\sum_{n=1}^{\infty} \frac{T^n x}{n}$ converges (to $-Ax$). \square

Remark. If in the proof we use [1, Proposition 4.1] instead of [7], we can obtain Theorem 3.2 as a corollary; we have preferred to prove Theorem 3.2 directly, independently of results from semi-group theory (used in [1] and in Theorem 3.9).

Given T power-bounded on a Banach space X , for every $\epsilon > 0$ the series $\sum_{k=1}^{\infty} \frac{T^k}{k^{1+\epsilon}}$ converges in operator norm. Combining the theorem with Corollary 4.5 of Assani and Lin [1] we obtain the following.

Corollary 3.10. *Let T be a power-bounded operator on a Banach space X and let $x \in \overline{(I - T)X}$. Then*

$$\sum_{n=1}^{\infty} \frac{T^n x}{n} \text{ converges} \iff \lim_{\epsilon \rightarrow 0^+} \sum_{k=1}^{\infty} \frac{T^k x}{k^{1+\epsilon}} \text{ exists}$$

and the two limits are equal.

The following was proved in [3] for normal contractions in a Hilbert space.

Corollary 3.11. *Let T be power-bounded on a reflexive Banach space X and let $x \in \overline{(I - T)X}$. Then*

$$\sum_{n=1}^{\infty} \frac{T^n x}{n} \text{ converges} \iff \sup_{0 < \epsilon < \frac{1}{2}} \left\| \sum_{k=1}^{\infty} \frac{T^k x}{k^{1+\epsilon}} \right\| < \infty$$

Proof. We first assume that $\sup_{0 < \epsilon < \frac{1}{2}} \left\| \sum_{k=1}^{\infty} \frac{T^k x}{k^{1+\epsilon}} \right\| < \infty$. By weak sequential compactness of bounded sets, there is a sequence $\epsilon_j \rightarrow 0$ such that $\sum_{k=1}^{\infty} \frac{T^k x}{k^{1+\epsilon_j}}$ converges weakly, say to z . By the proof of [1, Corollary 4.5] $\sum_{k=2}^{\infty} \frac{T^k x}{k(k-1)^{\epsilon_j}}$ converges weakly to $z - Tx$. Combining this with Lemma 4.4 of [1] we obtain

$$\text{weak-} \lim_{\epsilon_j \rightarrow 0} \frac{(I - T)^{\epsilon_j} x - x}{\epsilon_j} = -Tx - \text{weak-} \lim_{\epsilon_j \rightarrow 0} \sum_{k=2}^{\infty} \frac{T^k x}{k(k-1)^{\epsilon_j}} = -z.$$

By a result of Yosida [11, Theorem 10.5.4, p. 318] this already implies $x \in \mathcal{D}(A)$ (and then $Ax = -z$), so by Theorem 3.9 $\sum_{n=1}^{\infty} \frac{T^n x}{n}$ converges strongly.

The reverse implication follows from the previous corollary. \square

4. THE ONE-SIDED ERGODIC HILBERT TRANSFORM OF L_1 ISOMETRIES

Lin and Sine [14] proved that for T a contraction of $L_1(S, \Sigma, \mu)$, a function f is in $(I - T)L_1$ if (and obviously only if) $\sup_n \left\| \sum_{k=1}^n T^k f \right\|_1 < \infty$. Unfortunately, their method cannot be used to prove that for L_1 -contractions $f \in G(T)L_1$ if $\sup_n \left\| \sum_{k=1}^n \frac{T^k f}{k} \right\|_1 < \infty$, since the coefficients $\{\alpha_n\}_{n \geq 1}$ of $G(z)$ are not all negative (though they eventually are). However, we can still deal with isometries of L_1 .

Recall that for a complete finite measure space, $L_{\infty}^* = L_1^{**}$ is identified with the space $ba(S, \Sigma, \mu)$ of bounded finitely additive (signed) measures, called charges (see [8, IV.8.16]), and by the canonical embedding L_1 is identified with the space $M(S, \Sigma, \mu)$ of countably additive signed measures absolutely continuous with respect to μ . A charge $\eta \in ba(S, \Sigma, \mu)$ is called a "pure charge" if $|\eta|$ does not bound any non-negative measure, and then $\|\eta + \nu\| = \|\eta\| + \|\nu\|$ for any countably additive ν . Every $\eta \in L_1^{**}$ can be decomposed as $\eta = \eta_1 + \eta_0$ with η_1 countably additive and η_0 a pure charge [16]. The following lemma must be known (its first part is essentially [16, Theorem 2.6]).

Lemma 4.1. (i) *Let $\eta \in ba(S, \Sigma, \mu)$ satisfy $\|\eta + \nu\| = \|\eta\| + \|\nu\|$ for every countably additive ν . Then η is a pure charge.*

(ii) *The set of pure charges is norm-closed.*

Proof. (i) Write $\eta = \eta_1 + \eta_0$ with η_1 countably additive and η_0 a pure charge. Putting $\nu = -\eta_1$ and using the assumption, we obtain

$$\|\eta_0\| = \|\eta + \nu\| = \|\eta\| + \|\eta_1\| = \|\eta_0\| + 2\|\eta_1\|$$

which implies $\eta_1 = 0$.

(ii) Let $\{\eta_n\}$ be a sequence of pure charges converging in norm to η and use (i). \square

Lemma 4.2. *Let T be an invertible isometry of $L_1(S, \Sigma, \mu)$. Then $T^{**}\eta$ is a pure charge if η is.*

Proof. For any $\nu \in M(S, \Sigma, \mu)$ we have $(T^{**})^{-1}\nu \in M(S, \Sigma, \mu)$ (with $\frac{d}{d\mu}(T^{**})^{-1}\nu = T^{-1}\frac{d\nu}{d\mu}$). Hence

$$\|T^{**}\eta + \nu\| = \|\eta + (T^{**})^{-1}\nu\| = \|\eta\| + \|(T^{**})^{-1}\nu\| = \|T^{**}\eta\| + \|\nu\|$$

and we apply lemma 4.1(i). \square

Proposition 4.3. *Let T be an invertible isometry of $L_1(S, \Sigma, \mu)$. If $f \in L_1$ satisfies $\sup_n \left\| \sum_{k=1}^n \frac{T^k f}{k} \right\|_1 < \infty$, then $f \in G(T)L_1$.*

If T is also mean ergodic, then $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converges in L_1 -norm.

Proof. We identify f with the measure it defines. We apply Proposition 3.5 to T^{**} and obtain an element $\eta \in L_1^{**}$ with $G(T^{**})\eta = f$. We decompose $\eta = \eta_1 + \eta_0$ with η_1 countably additive and η_0 a pure charge. Since T is an invertible isometry, $(T^{**})^k \eta_0$ is a pure charge for every k , so Lemma 4.1(ii) yields that $G(T^{**})\eta_0$ is a pure charge. Since $G(T^{**})\eta_1$ is countably additive, $f = G(T^{**})\eta_1 + G(T^{**})\eta_0$ implies $f = G(T)g$ with $g = \frac{d\eta_1}{d\mu}$.

If T is also mean ergodic, $G(T)L_1 = \overline{G(T)(I - T)L_1}$ and Proposition 3.3 yields the convergence. \square

Corollary 4.4. *Let θ be an invertible measure preserving transformation of a probability space (S, Σ, μ) . If $f \in L_1(S, \mu)$ satisfies $\sup_n \left\| \sum_{k=1}^n \frac{f \circ \theta^k}{k} \right\|_1 < \infty$, then the one-sided EHT $\sum_{k=1}^n \frac{f \circ \theta^k}{k}$ converges a.e. and in L_1 -norm.*

Proof. the operator $Tf = f \circ \theta$ is a mean ergodic invertible isometry, so the result follows by combining Proposition 4.3 with [6, Theorem 1.2]. \square

The following theorem deals with general probability preserving transformations, which may not be invertible. To deal with this contingency, we first assume that the measure space in question is a Lebesgue space; in this setting the existence of an invertible probability preserving transformation (also on a Lebesgue space) which has the original one as a factor (the *natural extension*) is well-known (e.g. [4, p. 240]).

Theorem 4.5. *Let θ be a measure preserving transformation of a Lebesgue probability space (S, Σ, μ) with $\theta(S) = S$. If $f \in L_1(S, \mu)$ satisfies*

$$(9) \quad \sup_n \left\| \sum_{k=1}^n \frac{f \circ \theta^k}{k} \right\|_1 < \infty,$$

then the one-sided EHT $\sum_{k=1}^n \frac{f \circ \theta^k}{k}$ converges a.e. and in L_1 -norm.

Proof. For $f \in L_1(\mu)$ put $Tf := f \circ \theta$. Then T is an isometry of $L_1(\mu)$, mean ergodic because μ is finite.

When θ not invertible, there exists an *invertible* probability preserving $\hat{\theta}$ on $(\hat{S}, \hat{\Sigma}, \hat{\mu})$ with a *factor map* $\pi : \hat{S} \rightarrow S$ such that $\hat{\mu}(\pi^{-1}A) = \mu(A)$ and $\pi \circ \hat{\theta} = \theta \circ \pi$, which yields $\pi \circ \hat{\theta}^n = \theta^n \circ \pi$ for $n \geq 0$. We put $\hat{T}\hat{g} = \hat{g} \circ \hat{\theta}$ for $\hat{g} \in L_1(\hat{S}, \hat{\mu})$. For $f \in L_1(S, \mu)$ define $\tilde{f} = f \circ \pi$; the identity $\widetilde{1_A} = 1_{\pi^{-1}A}$ yields that $\|\tilde{f}\|_1 = \|f\|_1$, and we obtain

$$\widetilde{T^k f} = \widetilde{f \circ \theta^k} = (f \circ \theta^k) \circ \pi = (f \circ \pi) \circ \hat{\theta}^k = \tilde{f} \circ \hat{\theta}^k = \hat{T}^k \tilde{f},$$

so $\|\sum_{k=1}^n a_k T^k f\|_1 = \|\sum_{k=1}^n a_k \widetilde{T^k f}\|_1 = \|\sum_{k=1}^n a_k \hat{T}^k \tilde{f}\|_1$.

It follows that if $f \in L_1(S, \mu)$ satisfies (9), then $\sup_n \|\sum_{k=1}^n \frac{\hat{T}^k \tilde{f}}{k}\|_1 < \infty$, and by Proposition 4.3 $\sum_{k=1}^n \frac{\hat{T}^k \tilde{f}}{k}$ converges in $L_1(\hat{\mu})$ -norm. Now $\|\sum_{k=j}^n \frac{T^k f}{k}\|_1 = \|\sum_{k=j}^n \frac{\hat{T}^k \tilde{f}}{k}\|_1$ shows that $\{\sum_{k=1}^n \frac{T^k f}{k}\}$ is Cauchy in $L_1(\mu)$, so converges in norm. The a.e. convergence follows the norm convergence by [6, Theorem 1.2]. \square

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