ALMOST EVERYWHERE CONVERGENCE OF POWERS OF SOME POSITIVE L_p CONTRACTIONS

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ABSTRACT. We extend the solution of Burkholder's conjecture for products of conditional expectations, obtained by Delyon and Delyon for L_2 and by Cohen for L_p, $1 < p < \infty$, to the context of Badea and Lyubich: *Let T be a finite convex combination of operators T_j which are products of finitely many conditional expectations. Then $T^n f$ converges a.e. for every $f \in L_p$, $1 < p < \infty$, with $\sup_n |T^n f| \in L_p$.* The proof uses the work of Le Merdy and Xu on positive L_p contractions satisfying Ritt's resolvent condition. As another application of the work of Le Merdy and Xu, we extend a result of Bellow, Jones and Rosenblatt, proving that *if a probability $\{a_k\}_{k\in\mathbb{Z}}$ has bounded angular ratio, then for every positive invertible isometry S of an L_p space ($1 < p < \infty$), the operator $T = \sum_{k\in\mathbb{Z}} a_k S^k$ is a positive L_p contraction such that for every $f \in L_p$, $T^n f$ converges a.e. and $\sup_n |T^n f| \in L_p$. If $\{a_k\}$ is supported on $\mathbb{N}$, the same result is true when S is only a positive contraction of L_p.* Similar results are obtained for μ-averages of bounded continuous representations of a σ-compact LCA group by positive operators in one L_p space, $1 < p < \infty$. For a positive contraction T on L_p which satisfies Ritt's condition and $f \in (I - T)^\alpha L_p$ $(0 < \alpha < 1)$ we prove that $n^\alpha T^n f \to 0$ a.e., and $\sup_n n^\alpha |T^n f| \in L_p$.

1. INTRODUCTION

Let $(\mathbb{S}, \mathcal{B})$ be a measurable space and $P(x, A) : \mathbb{S} \times \mathcal{B} \longrightarrow [0, 1]$ a transition probability, with Markov operator $Pf(x) = \int f(y)P(x, dy)$ defined for bounded f. When m is a σ-finite measure on $\mathcal{B}$ which is P-invariant, the operator P can be extended to a contraction of $L_1(\mathbb{S}, m)$. Moreover P becomes a contraction in each $L_p(\mathbb{S}, m)$ space, $1 \le p \le \infty$ [38].

Hopf's pointwise ergodic theorem yields that for $f \in L_1(m)$ the Cesàro averages $\frac{1}{n}\sum_{k=1}^n P^k f$ converge a.e., and also in L_1-norm when m is finite. When m is a probability and P is ergodic in L_1, i.e. when $Pf = f$ a.e. for $f \in L_1$ holds only for f constant a.e., the limit is $\int f\, dm$. When m is infinite and P is conservative and ergodic, the limit is 0.

It is therefore a natural question to study the convergence of the unaveraged sequence $\{P^n f\}$, in norm or a.e. The following general results for a.e. convergence are known:

1. If $P^* = P$ and -1 is not an eigenvalue, then $P^n f$ converges a.e. for every $f \in L_p$, $p > 1$ (Stein-Rota theorem [58] [57]; Rota's proof yields the convergence also for $f \in L \log^+ L$ [11], but in general convergence may fail for $f \in L_1$ [52]).

2. If P is an aperiodic Harris recurrent operator with invariant probability m, then $P^n f \to \int f\, dm$ a.e. for every $f \in L_1(\mathbb{S}, m)$ by S. Horowitz [34].

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3. If P is a Harris recurrent operator with infinite (σ-finite) invariant measure m, then $P^n f \to 0$ a.e. for every $f \in L_p(\mathbb{S}, m)$, $1 \le p < \infty$ [34].

4. If $PP^* = P^*P$ and the spectrum of P as a contraction of $L_2(\mathbb{S}, m)$ is contained in a Stolz region, then $P^n f$ converges a.e. for every $f \in L_p$, $1 < p < \infty$, by Bellow, Jones and Rosenblatt [6, Theorem 14]. This can be applied to convolutions on compact Abelian groups [18].

In the proof of [6, Theorem 14], it is shown that $\sup_n n\|P^n - P^{n+1}\|_2 < \infty$ under their assumptions. The purpose of this paper is to study examples of positive contractions T on $L_p(\mathbb{S}, m)$ ($1 < p < \infty$ fixed) satisfying the condition $\sup_n n\|T^n - T^{n+1}\| < \infty$. We will then apply the work of Le Merdy and Xu [44], [45], which yields for such operators the maximal inequality $\sup |T^n f| \in L_p$ and the a.e. convergence of $\{T^n f\}$, for every $f \in L_p$.

2. Powers of positive Ritt contractions of L_p

The proof of Stein's theorem uses spectral theory and the pointwise ergodic theorem. Combining it with Akcoglu's pointwise ergodic theorem [2] we obtain that if T is a positive self-adjoint contraction of the complex $L_2(\mathbb{S}, m)$ with -1 not an eigenvalue, then $T^n f$ converges a.e. Gaposhkin [29] extended Stein's result to normal contractions with spectrum in a Stolz region; see [6, Theorem 14].

In this introductory section we look at the a.e. convergence of $T^n f$ for every $f \in L_p(\mathbb{S}, m)$, where $1 < p < \infty$ is fixed and T is a positive contraction on $L_p(\mathbb{S}, m)$ satisfying

$$\sup_n n\|T^n - T^{n+1}\| = C < \infty. \tag{1}$$

We also study equivalent conditions for (1). Much of this section is based on the work of Le Merdy and Xu [44], [45] and Le Merdy [42]; it is included for the reader's convenience, in order to provide some completeness, as it is the basis for the next sections.

Nagy and Zemánek [51, p. 146] proved that if the resolvent $R(\lambda, T)$ of an operator T on a complex Banach space X satisfies Ritt's condition

$$\sup_{|\lambda|>1} |\lambda - 1| \cdot \|R(\lambda, T)\| < \infty, \tag{2}$$

then its spectrum $\sigma(T)$ is contained in a Stolz region. Their main theorem is that T *satisfies Ritt's condition if and only if T is power-bounded with* $\sup_n n\|T^n - T^{n+1}\| < \infty$; see also [49]. Hence a power-bounded operator T satisfying (1) will be called a *Ritt* operator.

Lemma 2.1. *The following are equivalent for a bounded linear operator T on a Banach space X.*

(i) T is power-bounded and $\sup_n n\|T^n - T^{n+1}\| = K < \infty$.

(ii) T is mean-bounded and there exists $C > 0$ such that

$$\|T^n x\| \le \frac{C}{n} \max_{1\le k\le n} \|\sum_{j=0}^{k-1} T^j x\| \quad \forall x \in X, \quad \forall n \ge 1. \tag{3}$$

Proof. The implication (ii) $\Longrightarrow$ (i) is easy.

If (i) holds, then obviously T is mean-bounded. For $n \ge 1$ and $x \in X$ we have

$$x = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{k-1}(I-T)T^j x + \frac{1}{n}\sum_{j=1}^{n} T^j x.$$

Applying T^n we obtain

$$\|T^n x\| \le \|T^n(I-T)\|\frac{1}{n}\sum_{k=1}^{n}\|\sum_{j=0}^{k-1}T^j x\| + \|T^{n+1}\|\cdot\|\frac{1}{n}\sum_{j=0}^{n-1}T^j x\|,$$

which yields (3) with $C = K + \sup_n \|T^n\|$. $\square$

Definition. Let T be a bounded linear operator on a complex Banach space X. A closed set $F \subset \mathbb{C}$ is called a *K-spectral set* for T if

$$\text{(4)} \qquad \|u(T)\| \le K \sup_{z\in F}|u(z)| \quad \text{for every rational function } u(z) \text{ with poles outside F.}$$

A K-spectral set necessarily contains the spectrum $\sigma(T)$.

Proposition 2.2. *Let T be a bounded operator on a Banach space X. If a closed Stolz region $\mathcal{S}$ is a K-spectral set for T, then T is power-bounded, and $\sup_n n\|T^n - T^{n+1}\| < \infty$. In addition, there exist $C > 0$ such that*

$$\text{(5)} \qquad n\|T^n x\| \le C\|\sum_{j=0}^{n-1}T^j x\| \quad \forall x \in X,\ \forall n \ge 1.$$

Proof. Since $\mathcal{S}$ is a subset of the unit disk, (4) yields $\|T^n\| \le K\sup_{z\in\mathcal{S}}|z^n| = K$. For a Stolz region $\mathcal{S}$ we have (as observed in [6]) $\sup_{1\ne z\in\mathcal{S}}\frac{|1-z|}{1-|z|} = C < \infty$. For $z \in \mathcal{S}$ we then have

$$|z^n - z^{n+1}| = |z|^n|1-z| \le C|z|^n(1-|z|).$$

Since $\max_{0\le t\le 1} t^n(1-t) = \left(\frac{n}{n+1}\right)^n\frac{1}{n+1}$, we have $\max_{z\in\mathcal{S}} n|z^n(1-z)| \le C$. Since $\mathcal{S}$ is a K-spectral set, (4) yields $\|n(T^n - T^{n+1})\| \le C\cdot K$ for every $n \ge 1$.

For $z \in \mathcal{S}$, $\sum_{j=0}^{n-1} z^j \ne 0$, since a Stolz region does not contain non-trivial roots of unity. Fix $1 \le k < n$ and define $u(z) = \sum_{j=0}^{k-1} z^j / \sum_{j=0}^{n-1} z^j$. Then $u(z)$ is a rational function bounded on $\mathcal{S}$. For $z \in \mathcal{S}$ we have

$$|1-z|\cdot|\sum_{j=0}^{k-1}z^j| \le C(1-|z|^k) \le C(1-|z|^n) \le C|1-z^n| = C|1-z|\cdot|\sum_{j=0}^{n-1}z^j|.$$

Hence $|\sum_{j=0}^{k-1}z^j| \le C|\sum_{j=0}^{n-1}z^j|$ for $1 \le k < n$ and $z \in \mathcal{S}$, so $\sup_{z\in\mathcal{S}}|u(z)| \le C$. By the functional calculus and (4),

$$\|\sum_{j=0}^{k-1}T^j x\| = \|u(T)\sum_{j=0}^{n-1}T^j x\| \le \|u(T)\|\cdot\|\sum_{j=0}^{n-1}T^j x\| \le K(\sup_{z\in\mathcal{S}}|u(z)|)\|\sum_{j=0}^{n-1}T^j x\| \le KC\|\sum_{j=0}^{n-1}T^j x\|.$$

Using Lemma 2.1, we combine this with (3) and obtain (5). $\square$

Remarks. 1. For proving (1) we have used (4) only for polynomials, but for proving (5) we used the full definition. An easy adaptation of Lebow's lemma in [40, p. 66] shows that when F is a compact set which does not separate the plane, it is a K-spectral set as soon as $\|P(T)\| \le K \sup_{z\in F} |P(z)|$ for every polynomial $P(z)$.

2. The converse of the proposition may fail; Lancien and Le Merdy [39] gave an example of a power-bounded Ritt operator T for which no Stolz region is a K-spectral set.

The following corollary was observed in [45, Remark 6.7].

Proposition 2.3. *Let T be a power-bounded operator on a complex Hilbert space H. If its numerical range $W(T) := \{\langle Tf, f\rangle : \ \|f\| = 1\}$ is contained in a Stolz region, then $\sup_n n\|T^n - T^{n+1}\| < \infty$.*

Proof. Let $\mathcal{S}$ be the closure of the Stolz region containing the numerical range. Since $\mathcal{S}$ is convex, by Delyon and Delyon [22, Theorem 3] (see also Putinar and Sandberg [54]), there is a $K_S > 0$ such that for every rational function $u(z)$ with poles outside $\mathcal{S}$ we have $\|u(T)\| \le K_S \sup_{z\in\mathcal{S}} |u(z)|$, i.e. $\mathcal{S}$ is a K_S-spectral set. We now apply Proposition 2.2. □

Remarks. 1. The numerical radius $\sup\{|z| : z \in W(T)\}$ of a power-bounded operator in H may be larger than 1. The assumption of the proposition implies a numerical radius not exceeding 1, for which it is necessary that $\sup_n \|T^n\| \le 2$ (see [59]).

2. For a contraction T, the proposition was proved (independently of [22]) in [12].

Definition. A bounded linear operator on a Banach space X is called *polynomially bounded* if there exists $K > 0$ such that $\|P(T)\| \le K \sup_{|z|\le 1} |P(z)|$ for every polynomial. Obviously polynomial boundedness implies power boundedness. If the closure of the open unit disk $\mathbb{D}$ is a K-spectral set, then T is polynomially bounded.

Theorem 2.4. *The following are equivalent for a bounded linear operator T on a Hilbert space H.*

(i) T is polynomially bounded and $\sup_n n\|T^n - T^{n+1}\| < \infty$.

(ii) There is a closed Stolz region $\mathcal{S}$ which is a K-spectral set for T.

(iii) T is polynomially bounded, and there exists $K > 0$ such that

$$n\|T^n x\| \le K \Big\| \sum_{j=0}^{n-1} T^j x \Big\| \quad \forall x \in H,\ \forall n \ge 1. \tag{6}$$

(iv) T is similar to a Ritt contraction.

Proof. By Proposition 2.2, (ii) implies both (i) and (iii). (iii) implies (i) is easy, by putting $x = (I - T)y$ in (6).

(iv) implies that T is polynomially bounded by von-Neumann's inequality [55, Section 153]. The similarity to a Ritt contraction yields that also T is Ritt.

If (i) holds, then by [51] T satisfies (2). By a result of deLaubenfels [21, Theorem 4.4], T is similar to a contraction S, which is necessarily also Ritt, so (iv) holds.

Finally, we prove that (i) implies (ii). By the above, T is similar to a Ritt contraction S. By Le Merdy [42, Theorem 8.1] (who uses the fact that in H Ritt operators are R-Ritt), S has a Stolz region which is a K-spectral set; this easily implies (ii). □

Remarks. 1. An alternative proof of (i) implies (ii) can be obtained by showing that the polygon $\mathcal{P}$ constructed in the proof of (a)$\Longrightarrow$(c) of [21, Theorem 4.4] can be taken to have all its vertices except 1 inside $\mathbb{D}$, with the Stolz region containing $\sigma(T)$ contained in its interior, and then including $\mathcal{P}$ in another Stolz region.

2. In an unpublished note [43], Le Merdy proved that if T satisfies the conditions in Theorem 2.4, then it is similar to a contraction with numerical range in a Stolz region.

Proposition 2.5. *Let T be a normal contraction on a complex Hilbert space H. Then the following are equivalent:*

(i) The spectrum $\sigma(T)$ is contained in a Stolz region.

(ii) The numerical range $W(T)$ is contained in a Stolz region.

(iii) $\sup_n n\|T^n - T^{n+1}\| < \infty$.

Proof. (ii) implies (iii) by Proposition 2.3, and (iii) implies (i) by the above cited results of [51] and [49].

Assume (i). Since T is normal, the closure of $W(T)$ is the convex hull of $\sigma(T)$ [8], so $W(T)$ is in the same Stolz region as the spectrum. □

Remarks. 1. A direct proof that (i) implies (iii) is given in [6, p. 111] (and also in [45, Lemma 6.3]).

2. Without normality (i) does not imply (iii). Let $Vf(x) = \int_0^x f(t)dt$ be the Volterra operator on $L_2[0,1]$. Let $S = (I+V)^{-1}$ and $T = I - V$. Then $\|S\| \le 1$ [31, Problem 150], and T is similar to S [1, p. 15], so T is power-bounded. By Lyubich [50], T does not satisfy Ritt's resolvent condition, so neither does S, while $\sigma(S) = \sigma(T) = \{1\}$.

3. We do not know if without normality (iii) implies (ii).

In view of Proposition 2.5, we rephrase the main consequence of Gaposhkin's result: *If T is a positive normal contraction on $L_2(\mathbb{S}, m)$ with $\sup_n n\|T^n - T^{n+1}\| < \infty$, then for every $f \in L_2(\mathbb{S}, m)$, $T^n f$ converges a.e.* We show below that with this formulation normality is not needed, and the result extends to L_p, $1 < p < \infty$.

Lemma 2.6. *Fix $1 < p < \infty$ and let T be a power-bounded operator on $L_p(\mathbb{S}, m)$ of a σ-finite measure space. If $\sup_n n^\alpha \|T^n - T^{n+1}\| < \infty$ for some $\alpha > 1/p$, then $T^n f$ converges a.e. for f in a dense subspace, and $n^\beta T^n f \to 0$ a.e. for $0 \le \beta < \alpha - 1/p$ and $f \in (I-T)L_p$.*

Proof. Put $F(T) := \{f \in L_p : Tf = f\}$. The mean ergodic theorem yields the ergodic decomposition $L_p = F(T) \oplus \overline{(I-T)L_p}$. For $f = (I-T)g$ and $0 \le \beta < \alpha - 1/p$, the assumption yields $n^\beta \|T^n f\| \le C\|g\|/n^{\alpha-\beta}$, so $\sum_{n=0}^\infty (n^\beta \|T^n f\|)^p < \infty$. By Beppo Levi's theorem $\sum_{n=0}^\infty n^{\beta p} |T^n f|^p$ converges a.e., so $n^\beta |T^n f| \to 0$ a.e. Hence $T^n f$ converges a.e. for $f \in F(T) \oplus (I-T)L_p$. □

Remarks. 1. Léka [41] constructed for every $\alpha \in (1/2, 1)$ a contraction T on a Hilbert space (which can be $L_2(\mathbb{S}, m)$) such that $\sigma(T) = \{1\}$ and $\|T^n - T^{n+1}\| \asymp n^\alpha$. We are grateful to J. Zemánek for this reference.

2. For T a positive contraction satisfying the assumptions with $\alpha = 1/2$, there can be functions f for which $T^n f$ does not converge a.e. For an example see [56, Theorem 10].

The following theorem is due to Le Merdy and Xu [45, Theorem 4.4 and Corollary 5.2].

Theorem 2.7. *Fix $1 < p < \infty$ and let T be a positive contraction of $L_p(\mathbb{S}, m)$ of a σ-finite measure space. If $\sup_n n\|T^n - T^{n+1}\| = C < \infty$, then $T^n f$ converges a.e. for every $f \in L_p(\mathbb{S}, m)$, and for some $c > 0$, $\big\| \sup_n |T^n f| \big\|_p \le c\|f\|_p$ for every $f \in L_p(\mathbb{S}, m)$. Moreover,*

$$\| \sup_{\{n_k\}\uparrow} \big\{ |T^{n_0} f|^q + \sum_{k=1}^{\infty} |T^{n_k} f - T^{n_{k-1}} f|^q \big\}^{1/q} \|_p < C_{p,q} \|f\|_p \quad \text{for} \ \ 2 < q < \infty \tag{7}$$

Remark. The a.e. convergence follows already from combining Lemma 2.6 with the maximal inequality of Le Merdy and Xu [44, Theorem 4.1].

Corollary 2.8. *Let T be a positive contraction of $L_p(\mathbb{S}, m)$. If T^n converges weakly in L_p and for some integer $d > 1$ we have*

$$\sup_n n\|T^{nd} - T^{nd+d}\| < \infty \ ,$$

then $T^n f$ converges a.e. for every $f \in L_p(\mathbb{S}, m)$, and $\sup_n |T^n f| \in L_p(\mathbb{S}, m)$.

Proof. Since $T^n f$ converges weakly, the fixed points of T^d are only those of T, and $\lim_n T^{nd} f = \lim_n T^{n+r} f = Ef$ (weakly) for every $f \in L_p$ and $0 \le r < d$. By Theorem 2.7 $T^{nd} T^r f$ converges a.e. for every $f \in L_p$ and $0 \le r < d$, and the limit is Ef, which easily yields the assertion. □

Example 1. *Convolution powers on compact Abelian groups*
Let G be a compact Abelian group with normalized Haar measure m, and μ a probability measure on G which is assumed strictly aperiodic, i.e. $|\hat{\mu}(\gamma)| < 1$ for every character $\gamma \ne 1$. Put $Tf = \mu * f$ for $f \in L_2(G, m)$. Conze and Lin [18, Theorem 5.3] proved that $\mu^n * f$ converges a.e. for every $f \in L_p$, $1 < p < \infty$, if and only if for some $d > 0$ we have

$$\sup_{\gamma \ne 0} \frac{|1 - \hat{\mu}(\gamma)^d|}{1 - |\hat{\mu}(\gamma)|^d} < \infty. \tag{8}$$

Since the characters are an orthonormal basis of eigenvectors, it follows easily that $\sigma(T)$ is the closure $\{\hat{\mu}(\gamma) : \gamma \in \hat{G}\}$. Condition (8) means that $\sigma(T^d)$ is contained in a Stolz region [6], so by Proposition 2.5 the condition of Corollary 2.8 holds. The example in [18, p. 558] shows that the conditions of Corollary 2.8 may hold with $d = 2$, but fail for $d = 1$. Thus, (1) is not necessary for the a.e. convergence.

Theorem 2.9. *Fix $1 < p < \infty$ and let T be a positive contraction of $L_p(\mathbb{S}, m)$ of a σ-finite measure space. Then the following are equivalent:*

(i) $\sup_n n\|T^n - T^{n+1}\| < \infty$.

(ii) There exists $C_p > 0$ *such that*

$$\Big\| \Big(\sum_{n=1}^{\infty} n|T^n(I-T)f|^2 \Big)^{1/2} \Big\|_p \le C_p \|f\|_p \quad \text{for every} \quad f \in L_p. \tag{9}$$

(iii) There exists a closed Stolz region $\mathcal{S}$ *and a constant* $K_S > 0$ *such that*

$$\|u(T)\|_p \le K_S \sup_{z\in\mathcal{S}} |u(z)| \quad \text{for every rational function} \quad u(z) \quad \text{with poles outside } \mathcal{S}. \tag{10}$$

(iv) There exists a constant $K > 0$ *such that* $n\|T^n f\| \le K \| \sum_{j=0}^{n-1} T^j f\|$ *for every* $f \in L_p$ *and* $n \ge 1$.

Proof. If T satisfies (i), then (9) holds by putting $m = 1$ in [44, Theorem 3.3(2)].

Assume now that (9) holds. By [42, Lemma 5.4] there is a $C > 0$ such that

$$\Big\| \Big(\sum_{n=1}^{\infty} n^3 |(I-T)^2 T^n f|^2 \Big)^{1/2} \Big\|_p \le C\|f\|_p \quad \forall f \in L_p. \tag{11}$$

The identity (see beginning of the proof of [13, Proposition 2.2])

$$n(I-T)T^{n-1} - \frac{1}{n+1}\sum_{k=2}^{n} k(k-1)(I-T)^2 T^{k-2} = 2T^n - \frac{2}{n+1}\sum_{k=0}^{n} T^k \tag{12}$$

together with the estimate (by the Cauchy-Schwarz inequality)

$$\Big|\frac{1}{n+1}\sum_{k=2}^{n} k(k-1)(I-T)^2 T^k f\Big|^2 \le \sum_{k=1}^{\infty} k^3 |(I-T)^2 T^k f|^2 \tag{13}$$

and (11) yield (i).

(iii) implies (i) and (iv) by Proposition 2.2. (iv) implies (i) by the argument used to prove that (iii) implies (i) in Theorem 2.4.

Assume (i). By Le Merdy [42, Theorem 8.3] T has a bounded $H^\infty(\mathcal{S})$ functional calculus for some closed Stolz region $\mathcal{S}$, i.e. (iii) holds. □

Remark. For $p = 2$, see also Theorem 2.4, where positivity is not needed.

Corollary 2.10. *Let* $T_1, \dots, T_N$ *be positive contractions on* $L_2(\mathbb{S}, m)$ *such that the numerical range of each* T_j *is included in a Stolz region, and put* $T = \sum_{j=1}^{N} \alpha_j T_j$ *with* $\alpha_j > 0$ *and* $\sum_{j=1}^{N} \alpha_j = 1$. *Then* T *satisfies all the properties of Theorem 2.9 with* $p = 2$, *and* $T^n f$ *converges a.e. for every* $f \in L_2(\mathbb{S}, m)$, *with* L_2 *integrability of the strong q-variation for* $2 < q < \infty$.

Proof. We show that when T_j are operators in a Hilbert space (not necessarily positive, nor contractions) such that $W(T_j)$ is contained in a closed Stolz region $\mathcal{S}_j$, then $W(T)$ is contained in the Stolz region $\mathcal{S} = \cup_{j=1}^{N} \mathcal{S}_j$. Since Stolz regions are comparable, we have that indeed $\mathcal{S}$ is a Stolz region, and $W(T_j) \subset \mathcal{S}$ for every j. Then for $\|f\| = 1$ we have

$$\langle Tf, f\rangle = \sum_{j=1}^{N} \alpha_j \langle T_j f, f\rangle \in \mathcal{S}$$

by convexity of Stolz regions. This implies, by Proposition 2.3, that when the T_j are positive contractions, T satisfies (i)-(iii) of Theorem 2.9. The a.e. convergence follows from Theorem 2.7. □

Proposition 2.11. *Fix $1 < p < \infty$ and let T be a positive contraction of $L_p(\mathbb{S}, m)$ of a σ-finite measure space satisfying $\sup_n n\|T^n - T^{n+1}\| < \infty$. If $f \in (I - T)L_p$, then:*
(i) $\sum_{n=0}^{\infty} T^n f$ converges a.e. and in L_p-norm, with $\sup_n |\sum_{k=0}^{n} T^k f| \in L_p$.
(ii) $nT^n f \to 0$ m-a.e. and in L_p-norm.

Proof. For $f = (I - T)g$ we can assume $Eg := \lim T^n g = 0$, by the ergodic decomposition. The assertions in (i) follow directly from Theorem 2.7, and $\sum_{k=0}^{n} T^k f$ converges to g.

By the proof of Theorem 2.9, we have that $\sum_{k=1}^{\infty} k^3 |(I - T)^2 T^k g|^2 < \infty$ a.e., so by the Cauchy-Schwarz inequality and Kronecker's lemma

$$|\frac{1}{n}\sum_{k=1}^{n} k(k-1)(I-T)^2 T^k g|^2 \le \frac{1}{n}\sum_{k=1}^{n} k^4 |(I-T)^2 T^k g|^2 \to 0 \quad a.e.$$

Since $Eg = 0$, we obtain the a.e. convergence in (ii) by applying the identity (12) to g. Now (13) and (11) yield

$$\|\frac{1}{n}\sum_{k=1}^{n} k(k-1)(I-T)^2 T^k g\| \to 0$$

by Lebesgue's dominated convergence theorem, so $\|nT^n f\| \to 0$ using (12). □

Example 2. *Some convex combinations of powers of a contraction.*
Let $1 < p < \infty$ and let S be a positive contraction of $L_p(\mathbb{S}, m)$. For $\alpha \in (0, 1)$ we define $T = \sum_{n=1}^{\infty} \frac{1}{c_\alpha n^{1+\alpha}} S^n$, where $c_\alpha = \sum \frac{1}{n^{1+\alpha}}$. In the power series expansion $(1-t)^\alpha = 1 - \sum_{k=1}^{\infty} a_k^{(\alpha)} t^k$ for $|t| < 1$, the coefficients are $a_1^{(\alpha)} = \alpha$ and

$$a_k^{(\alpha)} = \frac{\alpha}{k!}\prod_{j=1}^{k-1}(j - \alpha) \quad \text{for } k > 1. \tag{14}$$

Since $(k+1)a_{k+1}^{(\alpha)} = \frac{\alpha k^{-\alpha}}{\Gamma(1-\alpha)}[1 + O(\frac{1}{k})]$ [61, vol. I, p. 77], we have

$$\sum_{k=1}^{\infty} k|\frac{1}{c_\alpha k^{1+\alpha}} - C_\alpha a_k^{(\alpha)}| < \infty \tag{15}$$

with $C_\alpha = \Gamma(1-\alpha)/\alpha c_\alpha$. Combining Dungey's [28, Theorems 1.1 and 4.1(II)], we obtain that $\sup_n n\|T^n - T^{n+1}\| < \infty$. By Theorem 2.7, $T^n f$ converges a.e. for every $f \in L_p$.

Remarks. 1. The case $\alpha = \frac{1}{2}$ of the example was presented in [6, p. 116] for normal contractions S in L_2, showing the spectrum is in a Stolz region and deducing a.e. convergence. Proposition 2.5 applies in that case.

2. Theorem 1.3 of [28] shows that a power-bounded T satisfying (1) must be of the form in the example.

3. Dungey [28, Theorem 3.1] proved that if for a probability distribution $\{a_k : k \geq 0\}$ on $\mathbb{N}$ with $a_0 < 1$ the operator $T := \sum_{k=0}^{\infty} a_k S^k$ with S power-bounded always satisfies (1), then $\sum_{k=0}^{\infty} k a_k = \infty$, hence $\{a_k\}$ has infinite support. The situation is different for probabilities on $\mathbb{Z}$ and S unitary – see [6].

3. Convergence of powers of positive Dunford-Schwartz contractions

As mentioned in the introduction, Markov operators with σ-finite invariant measures can be extended to become contractions of all the L_p spaces. An operator T on $L_1(\mathbb{S}, m)$ which extends to a contraction of each of the $L_p(\mathbb{S}, m)$ spaces is called a *Dunford-Schwartz operator*. The Dunford-Schwartz theorem is that if T is a Dunford-Schwartz operator, then $\frac{1}{n}\sum_{k=1}^{n} T^k f$ converges a.e. for every $f \in L_p$, $1 \leq p < \infty$.

The following theorem is a special case of Blunck's interpolation theorem [10, Theorem 1.1].

Theorem 3.1. *Let T be a Dunford-Schwartz operator. If $\sup_n n\|T^n - T^{n+1}\|_r < \infty$ for some $1 \leq r \leq \infty$, then $\sup_n n\|T^n - T^{n+1}\|_p < \infty$ for every $1 < p < \infty$.*

Combining Theorems 3.1 and 2.7 we obtain the following.

Corollary 3.2. *Let T be a positive Dunford-Schwartz operator. If $\sup_n n\|T^n - T^{n+1}\|_r < \infty$ for some $1 \leq r \leq \infty$, then for any $f \in L_p(\mathbb{S}, m)$, $1 < p < \infty$, we have $\sup_n |T^n f| \in L_p$ and $T^n f$ converges a.e.; moreover, for $2 < q < \infty$ the strong q-variation is in L_p.*

Example 1 (continued)
If μ satisfies (8), $Tf = \mu^d * f$ is a positive Dunford-Schwartz operator on $L_1(G, m)$, so by Theorems 3.1 and 2.9 for every $1 < p < \infty$ the spectrum $\sigma(T_p^d)$ is included in a Stolz region.

Example 3. *Products of conditional expectations*
Let $(\mathbb{S}, \mathcal{B}, m)$ be a probability space, let $E_1, \ldots, E_d$ be conditional expectations, and put $T = E_1 \cdot E_2 \cdots E_d$. The maximal inequality in [13, Proposition 2.2], with $r = 1$, yields that $\sup n\|(I - T)T^n\|_2 < \infty$. Hence, by Corollary 3.2, we obtain Cohen's result [13, Theorem 2.7]: *for every $1 < p < \infty$ and $f \in L_p(\mathbb{S}, \mathcal{B}, m)$ the sequence $\{T^n f\}$ converges a.e., with $\sup_n |T^n f| \in L_p(\mathbb{S}, \mathcal{B}, m)$.*

Remark. The maximal inequality of Le Merdy and Xu used in Theorem 2.7 was not available when [13] was written, though Blunck's interpolation was.

Theorem 3.3. *Let $(\mathbb{S}, \mathcal{B}, m)$ be a probability space, let $T = \sum_{j=1}^{N} \alpha_j T_j$ with $\alpha_j > 0$, $\sum_{j=1}^{N} \alpha_j = 1$, and each T_j is a product of d_j conditional expectation operators $E_{j,1}, \ldots, E_{j,d_j}$. Then for every $f \in L_p(\mathbb{S}, m)$, $1 < p < \infty$, the sequence $\{T^n f\}$ converges a.e. and in L_p-norm, with $\sup_n |T^n f| \in L_p(\mathbb{S}, \mathcal{B}, m)$, and (7) holds.*

Proof. Since each T_j is a product of orthogonal projections, it follows from [22, p. 39] that for each T_j the numerical range is contained in a closed Stolz region, so also $W(T)$ is contained in a closed Stolz region, by the proof of Corollary 2.10. Each T_j is a positive contraction of all the $L_p(\mathbb{S},m)$ spaces, and so is T. By Corollary 2.10 we have $\sup_n n\|T^n(I-T)\|_2 < \infty$. Now we obtain the maximal inequality and the a.e. convergence for every $f \in L_p$ from Corollary 3.2; together they yield the L_p-norm convergence. □

Remark. The proof shows that if $T = \sum_{j=1}^N \alpha_j T_j$ is a convex combination with each T_j a product of finitely many orthogonal projections on a Hilbert space H, then $\sup_n n\|T^n(I-T)\|_2 < \infty$. This yields strong convergence (in H) of T^n, a special case of the main result of Badea and Lyubich [4], who also describe the limit (a projection by the mean ergodic theorem). Our proof adds in this case the rate $\mathcal{O}(\frac{1}{n})$ for the convergence $\|T^n(I-T)\|_2 \to 0$, proved in [4].

Example 4. *Convolution powers on* $\mathbb{R}$
Let G be a locally compact non-compact σ-compact Abelian group with Haar measure m, and let μ be a strictly aperiodic probability on G (defined as in Example 1). If the random walk generated by μ is transient, then $\sum_{n=0}^\infty \mu^n * f$ converges a.e. for any $f \in L_1(G,m)$ by the definition of transience, so $\mu^n * f \to 0$ a.e. Since for any transition probability $P(x,A)$ we have $|P^n f(x)|^p \le P^n(|f(x)|^p)$ for $1 < p < \infty$, we obtain that $\mu^n * f \to 0$ a.e. for any $f \in L_p(G,m)$, $1 \le p < \infty$. Thus the problem of a.e. convergence of convolution powers in the non-compact case is only for recurrent random walks. If the random walk is recurrent and some power of μ is non-singular with respect to m, then the Markov chain is Harris recurrent and $\mu^n * f \to 0$ a.e. for every $f \in L_p(G,m)$, $1 \le p < \infty$, by [34]; moreover, if in addition μ is centered, has a finite second moment, and $\sigma(T_2) \subset \mathbb{D} \cup \{1\}$ (where $T_2 f = \mu * f$ for $f \in L_2(G,m)$), then by Dungey [27] $\sup_n n\|T^n(I-T)\|_p < \infty$ for $1 < p < \infty$. Thus the a.e. convergence problem, say for $\mathbb{R}$ or $\mathbb{R}^2$, is for μ recurrent strictly aperiodic with all its convolution powers singular (e.g. discrete). It is known that on $\mathbb{R}$ $\sigma(T_2)$ is the closure of the range of the Fourier-Stieltjes transform of μ (see Proposition 5.1 below), so when μ on $\mathbb{R}$ is strictly aperiodic and satisfies (8), $\sigma(T_2^d)$ is contained in a closed Stolz region, and we obtain that $\mu^n * f \to 0$ a.e. for every $f \in L_p(G,m)$, $1 < p < \infty$, using Proposition 2.5, Corollary 2.8 and Corollary 3.2. The finitely supported μ' of [27, p. 439] is recurrent (being centered) and strictly aperiodic, but since $\sigma(T_2)$ contains the unit circle, so does $\sigma(T_2^d)$; hence μ' does not satisfy (8) for any $d > 0$.

4. Convex combinations of powers of positive L_p-contractions

In this section we study the a.e. convergence of the powers of $T = \sum_{k=-\infty}^\infty a_k S^k$, where $\{a_k\}_{k\in\mathbb{Z}}$ is a probability on $\mathbb{Z}$ and S is a positive invertible isometry of an L_p space, $1 < p < \infty$. This problem was studied by Bellow, Jones and Rosenblatt [6],[7] in the case of S induced by an invertible ergodic probability preserving transformation. When $\{a_k\}$ is supported on $\mathbb{N}$, we can define T for any positive contraction of L_p (see Example 2).

Definitions [6]. Let $\mu = \{a_k : k \in \mathbb{Z}\}$ be a probability on $\mathbb{Z}$. It is called *strictly aperiodic* if $|\hat{\mu}(\lambda)| < 1$ for any $|\lambda| = 1,\ 1 \neq \lambda \in \mathbb{C}$. A probability μ has *bounded angular ratio* if

$$\sup_{|\lambda|=1,\lambda\neq 1} \frac{|1-\hat{\mu}(\lambda)|}{1-|\hat{\mu}(\lambda)|} < \infty.$$

If μ has bounded angular ratio, then it must be strictly aperiodic. On the other hand, if μ is strictly aperiodic, then $\frac{|1-\hat{\mu}(\lambda)|}{1-|\hat{\mu}(\lambda)|}$ is finite and continuous on any arc $\{|\lambda| = 1, |\lambda - 1| \geq \epsilon\}$, so it is bounded on each such arc. Hence μ strictly aperiodic has bounded angular ratio if and only if

$$\limsup_{1\neq\lambda\to 1} \frac{|1-\hat{\mu}(\lambda)|}{1-|\hat{\mu}(\lambda)|} < \infty.$$

For $\mu = \{a_k\}$ strictly aperiodic supported on $\mathbb{N}$ and S a power-bounded operator on a Banach space, we put $T = T_\mu(S) := \sum_{k=0}^\infty a_k S^k$. Then (see [6, Proposition 5] for the convergence)

$$\|T^n - T^{n+1}\| \leq (\sup_n \|S^n\|) \cdot \|\mu^n - \mu^{n+1}\|_{\ell_1} \to 0.$$

Dungey [28, Theorem 2.1] proved a "spectral mapping theorem" for $T_\mu(S)$, namely

$$\sigma(T_\mu(S)) = \{\sum_{k=0}^\infty a_k z^k :\ z \in \sigma(S)\}. \tag{16}$$

Theorem 4.1. *Let $\mu = \{a_k\}$ be a probability supported on $\mathbb{N}$ with bounded angular ratio. Then for every contraction S on a Hilbert space the operator $T = T_\mu(S)$ has its numerical range in a Stolz region, and satisfies $\sup_n n\|T^n - T^{n+1}\| < \infty$.*

Proof. Since μ has bounded angular ratio, the range of its Fourier-Stieltjes transform is in a Stolz region $\mathcal{S}$. Let U be unitary. By the spectral mapping theorem (16) (which for unitary operators is implicit in [6, p. 115])

$$\sigma(T_\mu(U)) = \{\sum_{n=0}^\infty a_k \lambda^k : \lambda \in \sigma(U)\} = \{\hat{\mu}(\bar{\lambda}) :\ \lambda \in \sigma(U)\} \subset \mathcal{S}.$$

By Proposition 2.5, $\sup_n n\|T_\mu^n(U) - T_\mu^{n+1}(U)\| < \infty$ and $W(T_\mu(U)) \subset \mathcal{S}$. Now let S be a contraction on H, and let U be a unitary dilation of S on a larger space H_1 containing H, so $S^n = EU^n$. Then $n\|T^n - T^{n+1}\| \leq n\|E(T_\mu^n(U) - T_\mu^{n+1}(U)\,)\|$, and $W(T_\mu(S)) \subset W(T_\mu(U)) \subset \mathcal{S}$, which proves the theorem. □

Theorem 4.2. *Let $\mu = \{a_k\}$ be a probability supported on $\mathbb{N}$ with bounded angular ratio. Let $1 < p < \infty$ and for S a* positive *contraction on $L_p(\mathbb{S}, m)$ put $T = T_\mu(S)$. Then $\sup_n n\|T^n - T^{n+1}\| < \infty$. Consequently $T^n f$ converges a.e. for every $f \in L_p(\mathbb{S}, m)$, and $\sup_n |T^n f| \in L_p(\mathbb{S}, m)$.*

Proof. Note that for $p = 2$ the claim follows from Theorems 4.1 and 2.7.

We first look at the right shift R on $\ell_p = \ell_p(\mathbb{N})$, defined by $R(c_1, c_2, \dots) = (0, c_1, c_2, \dots)$. The shift R is a well-defined isometry on all the ℓ_q spaces, $1 \leq q \leq \infty$, and by Theorem

4.1 we have $\sup_n n\|T_\mu^n(R) - T_\mu^{n+1}(R)\|_2 < \infty$. Now we apply Blunck's interpolation (see Theorem 3.1) and obtain that $\sup_n n\|T_\mu^n(R) - T_\mu^{n+1}(R)\|_p = K < \infty$.

Now define $\phi_N(z) = \sum_{k=0}^N a_k z^k$. Then $\|\phi_N(S) - T_\mu(S)\| \to 0$ for every power-bounded S, and also for any fixed n we have $\|\phi_N^n(S) - T_\mu^n(S)\| \to 0$. Now fix S a positive contraction on our L_p. By Coifman and Weiss [17] (see also [15]), for every polynomial $\phi(z)$ we have $\|\phi(S)\|_{L_p} \le \|\phi(R)\|_{\ell_p}$. Hence for fixed n we obtain

$$n\|T_\mu^n(S) - T_\mu^{n+1}(S)\| = \lim_{N\to\infty} n\|\phi_N^n(S) - \phi_N^{n+1}(S)\| \le$$

$$\lim_{N\to\infty} n\|\phi_N^n(R) - \phi_N^{n+1}(R)\|_p = n\|T_\mu^n(R) - T_\mu^{n+1}(R)\|_p < K.$$

This shows that $T = T_\mu(S)$ satisfies $\sup_n n\|T^n - T^{n+1}\| < \infty$. Now Theorem 2.7 yields the a.e. convergence of $T^n f$ and $\sup_n T^n|f| \in L_p$, for every $f \in L_p$. □

When S is a positive invertible isometry of L_p $(1 \le p < \infty)$, then S^{-1} is an isometry $(\|f\| = \|S(S^{-1})f\| = \|S^{-1}f\|)$. We show that it is positive. Let $0 \le f \in L_p$ and write $S^{-1}f = g - h$ with $g, h \in L_p$ non-negative with disjoint supports. Then $f = Sg - Sh$, and since positive isometries preserve disjointness of supports [38, p. 186], $Sh = 0$ and thus $h = 0$, so $S^{-1}f \ge 0$. Hence for a positive invertible isometry S and a probability μ on $\mathbb{Z}$ (not supported in $\mathbb{N}$) the operator $T_\mu(S) := \sum_{k=-\infty}^{\infty} a_k S^k$ is a positive contraction of L_p.

Theorem 4.3. *Let $\mu = \{a_k\}$ be a strictly aperiodic probability on $\mathbb{Z}$. Then the following conditions are equivalent.*

(i) μ has bounded angular ratio.

(ii) For any fixed $1 < p < \infty$ and any positive invertible isometry S on $L_p(\mathbb{S}, m)$, we have $\sup_n n\|T_\mu^n(S) - T_\mu^{n+1}(S)\| < \infty$.

(iii) For any fixed $1 < p < \infty$ and any positive invertible isometry S on $L_p(\mathbb{S}, m)$, for every $f \in L_p$ we have $T_\mu^n(S)f$ converges a.e. and $\sup_n T_\mu^n(S)|f| \in L_p$.

(iv) There is a $1 < p < \infty$, such that for any positive invertible isometry on $L_p(\mathbb{S}, m)$, we have $\sup_n n\|T_\mu^n(S) - T_\mu^{n+1}(S)\| < \infty$.

(v) For every ergodic invertible measure preserving transformation τ on a probability space $(\mathbb{S}, m)$ with $Sf = f \circ \tau$ and every $1 < p < \infty$, the operator $T_\mu(S)$ has the property that for every $f \in L_p$, $T_\mu^n f$ converges a.e. and $\sup_n T_\mu^n|f| \in L_p$.

(vi) For any fixed $1 < p < \infty$ and any invertible operator S on $L_p(\mathbb{S}, m)$ with $\sup_{n\in\mathbb{Z}} \|S^n\| < \infty$, we have $\sup_n n\|T_\mu^n(S) - T_\mu^{n+1}(S)\| < \infty$.

(vii) For any fixed $1 < p < \infty$ and any positive invertible operator S on $L_p(\mathbb{S}, m)$ with $\sup_{n\in\mathbb{Z}} \|S^n\| < \infty$ and S^{-1} positive, for every $f \in L_p$ we have $T_\mu^n(S)f$ converges a.e. and $\sup_n T_\mu^n(S)|f| \in L_p$.

Proof. We first prove that (i) implies (ii). Let R be the right shift on $\ell_2(\mathbb{Z})$, defined for $\vec{c} := \{c_j\}_{j\in\mathbb{Z}}$ by $R\vec{c} = \{c_{j-1}\}$. Then $T_\mu(R)\vec{c} = \mu * \vec{c}$. The spectrum of the convolution opertor $T_\mu(R)$ is the range of the Fourier-Stieltjes transform of μ [37, Example 4.3.22], so it is contained in a Stolz region since μ has bounded angular ratio. By Proposition 2.5 we have

$\sup_n n\|T_\mu^n(R) - T_\mu^{n+1}(R)\|_2 < \infty$. By Blunck's theorem $\sup_n n\|T_\mu^n(R) - T_\mu^{n+1}(R)\|_p = K_p < \infty$ for every $1 < p < \infty$. The proof is now similar to that of Theorem 4.2. Fix $1 < p < \infty$ and let S be a positive invertible isometry on $L_p(\mathbb{S}, m)$. Then S^{-1} is also a positive isometry. Define $\psi_N(z) = \sum_{k=-N}^{N} a_k z^k$. Then $\|\psi_N^n(S) - T_\mu^n(S)\| \to 0$ as $N \to \infty$ for each fixed n. Since S is an isometry and $\phi_N = z^N \psi_N$ is a polynomial, we have by [17]

$$n\|T_\mu^n(S) - T_\mu^{n+1}(S)\| = \lim_{N\to\infty} n\|\psi_N^n(S) - \psi_N^{n+1}(S)\| = \lim_{N\to\infty} n\|S^{(n+1)N}(\psi_N^n(S) - \psi_N^{n+1}(S))\| \le$$

$$\lim_{N\to\infty} n\|R^{(n+1)N}(\psi_N^n(R) - \psi_N^{n+1}(R))\|_p = \lim_{N\to\infty} n\|\psi_N^n(R) - \psi_N^{n+1}(R)\|_p = n\|T_\mu^n(R) - T_\mu^{n+1}(R)\|_p$$

which shows that $\sup_n n\|T_\mu^n(S) - T_\mu^{n+1}(S)\| \le K_p$.

Clearly (iii) implies (v), and (ii) implies (iii) by Theorem 2.7. Obviously (ii) implies (iv).

We show (iv) implies (v). Let $Sf = f \circ \tau$ for τ as in (v). By (iv) $\sup_n n\|T_\mu^n(S) - T_\mu^{n+1}(S)\|_p < \infty$, and by Blunck's interpolation $\sup_n n\|T_\mu^n(S) - T_\mu^{n+1}(S)\|_q < \infty$ for every $1 < q < \infty$. We now apply Theorem 2.7 in each $L_q(\mathbb{S}, m)$.

Assume (v). If μ does not have bounded angular ratio, then by Losert [47, Theorem 2] μ has the strong sweeping out property – there is τ as in (v) for which there is a set A with $\limsup S^n 1_A = 1$ a.e. and $\liminf S^n 1_A = 0$ a.e., contradicting the assumed convergence in (v).

Clearly (vi) implies (ii) and (vii) implies (iii). The proof that (i) implies (vi) and (vii) follows from Theorem 5.6 and Corollary 5.8 below. □

Remark. In [6] it is shown that (i) implies (v).

Proposition 4.4. *The set of strictly aperiodic probabilities on $\mathbb{Z}$ having bounded angular ratio is convex, and is closed under convolutions.*

Proof. Let $\mu = \{a_k\}$ and $\nu = \{b_k\}$ be strictly aperiodic probabilities with bounded angular ratio, and let $\eta = t\mu + (1-t)\nu$, $0 < t < 1$. The support of η is the union of the supports of μ and ν, so η is strictly aperiodic. By definition, $\hat\eta = t\hat\mu + (1-t)\hat\nu$, so $1 - |\hat\eta(\lambda)| \ge t(1 - |\hat\mu(\lambda)|) + (1-t)(1 - |\hat\nu(\lambda)|)$, which yields

$$|1 - \hat\eta(\lambda)| \le t|1 - \hat\mu(\lambda)| + (1-t)|1 - \hat\nu(\lambda)| \le$$

$$C_1 t(1 - |\hat\mu(\lambda)|) + C_2(1-t)(1 - |\hat\nu(\lambda)|) \le \max\{C_1, C_2\}(1 - |\hat\eta(\lambda)|),$$

so η has bounded angular ratio. For the convolution $\mu * \nu$, we have $\widehat{\mu * \nu} = \hat\mu \cdot \hat\nu$, so

$$|1 - \hat\mu(\lambda)\hat\nu(\lambda)| \le |1 - \hat\mu(\lambda)| + |\hat\mu(\lambda)| \cdot |1 - \hat\nu(\lambda)| \le (C_1 + C_2)(1 - |\hat\mu(\lambda)\hat\nu(\lambda)|),$$

so $\mu * \nu$ has bounded angular ratio. □

5. Averages of representations of LCA groups by positive L_p isometries

In this section we look at averages of representations of a locally compact Abelian (LCA) group G, with Haar measure m_G, by positive isometries of an L_p space, extending some of the results of the previous section (where $G = \mathbb{Z}$). We assume G to be σ-compact, with dual group $\hat{G}$ with (properly normalized) Haar measure $\hat{m}_{\hat{G}}$.

Proposition 5.1. *let μ be a probability on G, and denote the operator of convolution by μ on $L_p(G, m_G)$ by $T_{\mu,p}$. Then $T_{\mu,2}$ is a normal operator with*

$$\sigma(T_{\mu,2}) = \overline{\{\hat{\mu}(\gamma) : \gamma \in \hat{G}\}}. \tag{17}$$

Proof. Normality follows from commutativity of G. By Plancherel's theorem [33, p.226] $L_2(G, m_G)$ is isometrically isomorphic to $L_2(\hat{G}, \hat{m}_{\hat{G}})$, and $T_{\mu,2}$ is represented by multiplication by the Fourier transform $\hat{\mu}$. The spectrum of the latter is the right-hand side of (17) by Exercise VII.5.15 of [26]. □

Definitions. Let μ be a probability on G. It is called *strictly aperiodic* if $|\hat{\mu}(\gamma)| < 1$ for any $0 \neq \gamma \in \hat{G}$. A probability μ has *bounded angular ratio* if it is strictly aperiodic and

$$\sup_{0\neq\gamma\in\hat{G}} \frac{|1 - \hat{\mu}(\gamma)|}{1 - |\hat{\mu}(\gamma)|} < \infty.$$

Proposition 5.2. *A probability μ on the LCA group G has bounded angular ratio if and only if*

$$\sup_n n\|T^n_{\mu,p} - T^{n+1}_{\mu,p}\| < \infty \qquad \text{for any } 1 < p < \infty. \tag{18}$$

Proof. Bounded angular ratio means that $\overline{\{\hat{\mu}(\gamma) : \gamma \in \hat{G}\}}$ is in a closed Stolz region. Equivalently, $\sigma(T_{\mu,2})$ is in a Stolz region, by (17), and Proposition 2.5 yields that it is equivalent to $\sup_n n\|T^n_{\mu,2} - T^{n+1}_{\mu,2}\| < \infty$. Since m_G is invariant for the convolution, Blunck's interpolation (Theorem 3.1) yields the equivalence with (18) for every $1 < p < \infty$. □

Definition. Let G be a LCA group. An *operator representation* of G in a Banach space X is a homomorphism $\mathbf{S}$ from G to the group of invertible bounded operators on X, i.e. $\mathbf{S}(t + s) = \mathbf{S}(t)\mathbf{S}(s)$. We assume that the representation is strongly continuous, i.e. $\mathbf{S}(t)x$ is continuous when X has its norm topology. In reflexive spaces this is equivalent to weak continuity – $\langle x^*, \mathbf{S}(t)x\rangle$ is continuous for each $x \in X$ and $x^* \in X^*$ [32, p. 340]. The regular representation by translations will be denoted by $\mathbf{R}$.

Definition. *An action* of a LCA group G in a σ-finite measure space $(\mathbb{S}, m)$ is a family $\{\theta_t : t \in G\}$ of invertible measure preserving transformations of $(\mathbb{S}, m)$, satisfying $\theta_{t+s} = \theta_t\theta_s$ and $\theta_0 = id$, such that $\theta_t\omega$ is measurable on $G \times \mathbb{S}$; if $\lim_{t\to0} \|f - f \circ \theta_t\|_p = 0$ for any $1 < p < \infty$ and $f \in L_p(\mathbb{S}, m)$, the action is called *continuous*. For $1 < p < \infty$, a continuous action induces a continuous representation $\mathbf{S}_p$ in $L_p(\mathbb{S}, m)$, defined by $\mathbf{S}_p(t)f = f \circ \theta_t$; in $L_2(\mathbb{S}, m)$ this is a unitary representation. These representations are by positive invertible isometries.

Let μ be a probability on G and $\mathbf{S}$ a bounded operator representation in X. The operator $T_\mu(\mathbf{S})$, the μ-average of the representation $\mathbf{S}$, is defined as the Bochner integral $T_\mu(\mathbf{S})x := \int \mathbf{S}(t)x\, d\mu(t)$; we then have

$$\langle x^*, T_\mu(\mathbf{S})x\rangle = \int \langle x^*, \mathbf{S}(t)x\rangle d\mu(t) \quad \text{for every} \quad x \in X,\ x^* \in X^*.$$

In reflexive spaces, the above equality can be used as the definition of $T_\mu(\mathbf{S})$ [32, p. 335]. Similarly, we can define $T_\eta(\mathbf{S})$ for every finite signed measure η, and we have $\|T_\eta(\mathbf{S})\| \le \|\eta\| \sup_{t\in G} \|\mathbf{S}(t)\|$. We also have $\mathbf{S}(s)(T_\eta(\mathbf{S})x) = \int \mathbf{S}(s)\mathbf{S}(t)x\, d\eta(t)$.

In this section we are interested in representations of G by (positive) isometries on $L_p(\mathbb{S}, m)$ of a σ-finite space for a fixed $1 < p < \infty$, and properties of their μ-averages when μ has bounded angular ratio. For $p = 2$ we have the following.

Proposition 5.3. *Let μ be a probability on the σ-compact LCA group G and let $\mathbf{S}$ be a continuous bounded representation of G in a Hilbert space. If μ has bounded angular ratio, then $\sup_n n\|T_\mu^n(\mathbf{S}) - T_\mu^{n+1}(\mathbf{S})\| < \infty$.*

Proof. It is well-known [25, Théorème 6] (see also [48, p. 83]) that $\mathbf{S}$ is equivalent to a unitary representation, say $\mathbf{U}$, so it is enough to prove the assertion for $T_\mu(\mathbf{U})$. By the general Stone spectral theorem for unitary representations of LCA groups [55, section 140], we have $\mathbf{U}(t) = \int_{\hat G} \gamma(t)E(d\gamma)$, where $E(\cdot)$ is a spectral measure on the Borel sets of $\hat G$. By the definitions and Fubini's theorem,

$$T_\mu(\mathbf{U}) = \int_G [\int_{\hat G} \gamma(t)E(d\gamma)]d\mu(t) = \int_{\hat G} \hat\mu(\gamma)E(d\gamma).$$

Thus, if $\lambda \notin \overline{\{\hat\mu(\gamma) : \gamma \in \hat G\}}$, then $\frac{1}{\lambda - \hat\mu(\gamma)}$ is a bounded continuous function on $\hat G$, and $\int \frac{1}{\lambda-\hat\mu(\gamma)} E(d\gamma)$ yields the inverse of $\lambda I - T_\mu(\mathbf{U})$. Thus $\sigma(T_\mu(\mathbf{U})) \subset \overline{\{\hat\mu(\gamma) : \gamma \in \hat G\}}$, which is contained in a Stolz region when μ has bounded angular ratio. Proposition 2.5 now yields the result. $\square$

Theorem 5.4. *Let $\{\theta_t : t \in G\}$ be a continuous action of a σ-compact LCA group G, and let μ be a probability on G with bounded angular ratio. Then for every $1 < p < \infty$ we have $\sup_n n\|T_\mu^n(\mathbf{S}_p) - T_\mu^{n+1}(\mathbf{S}_p)\|_p < \infty$, and consequently, $T_\mu^n(\mathbf{S}_p)f$ converges a.e. for every $f \in L_p(\mathbb{S}, m)$.*

Proof. The operator $T_\mu(\mathbf{S})f = \int f \circ \theta_t d\mu(t)$ is a positive Dunford-Schwartz operator. By Proposition 5.3 $\sup_n n\|\mathbb{T}_\mu^n(\mathbf{S}_2) - T_\mu^{n+1}(\mathbf{S}_2)\|_2 < \infty$, so Blunck's interpolation (Theorem 3.1) yields $\sup_n n\|\mathbb{T}_\mu^n(\mathbf{S}_p) - T_\mu^{n+1}(\mathbf{S}_p)\|_p < \infty$ for $1 < p < \infty$. The asserted a.e. convergence follows from Theorem 2.7. $\square$

The following "transfer principle" is essentially due to Coifman and Weiss [16] for absolutely continuous measures supported in compact sets.

Theorem 5.5. *Let η be a finite signed measure on a σ-compact locally compact amenable group G, and let $\mathbf{S}$ be a continuous representation of G in $L_p(\mathbb{S},m)$ ($1<p<\infty$ fixed), with $\sup\{\|\mathbf{S}(t)\| : t \in G\} = C < \infty$. Let $T_\eta(\mathbf{R})$ be the convolution operator $T_\eta(\mathbf{R})\phi(s) = \int \phi(s\cdot t)d\eta(t)$ on $L_p(G,m_G)$, and let $T_\eta(\mathbf{S})f(x) = \int(\mathbf{S}(t)f)(x)d\eta(t)$ on $L_p(\mathbb{S},m)$. Then $\|T_\eta(\mathbf{S})\| \le C^2\|T_\eta(\mathbf{R})\|_p$.*

Proof. We first prove the theorem when η (i.e. its variation $|\eta|$) is supported in a compact set K. We adapt the proof of [16], skipping some of the details. Put $A := \|T_\eta(\mathbf{R})\|_p$. We denote $T_\eta(\mathbf{S})$ by T.

Since G is amenable, by Leptin's condition [53, pp. 62-72], for $\epsilon > 0$ there is an open V with $0 < m_G(V) < \infty$ such that

$$m_G(V\cdot K) < (1+\epsilon)m_G(V). \tag{19}$$

Fix $\epsilon > 0$ and V as in (19). Writing $Tf = \mathbf{S}(s^{-1})\mathbf{S}(s)Tf$ for $s\in G$ we obtain

$$\int_{\mathbb{S}} |Tf(x)|^p dm(x) \le C^p \int_{\mathbb{S}} |\mathbf{S}(s)Tf(x)|^p dm(x), \quad s\in G.$$

Integration of this inequality over V and dividing by $m_G(V)$, inserting the definition of Tf and using the fact that $\mathbf{S}$ is a *continuous* representation yields, after changing order of integration and remembering that η is supported on K:

$$\int_{\mathbb{S}} |Tf(x)|^p dm(x) \le \frac{C^p}{m_G(V)}\int_{\mathbb{S}}\Big\{\int_V \Big|\int_G \mathbf{S}(st)f(x)\cdot \mathbf{1}_{V\cdot K}(st)d\eta(t)\Big|^p dm_G(s)\Big\}dm(x).$$

Note that the inner integral is actually over K. For fixed x put $f_x(t) := \mathbf{S}(t)f(x)$. Then

$$\Big|\int_G \mathbf{S}(st)f(x)\cdot\mathbf{1}_{V\cdot K}(st)d\eta(t)\Big| = |(\eta * (f_x\mathbf{1}_{V\cdot K}))(s)|,$$

which yields, when integrating over G instead of V, using the norm of the convolution operator in $L_p(G,m_G)$ and Fubini's theorem,

$$\int_{\mathbb{S}} |Tf(x)|^p dm(x) \le \frac{C^pA^p}{m_G(V)}\int_G \mathbf{1}_{V\cdot K}(s)\Big\{\int_{\mathbb{S}}|\mathbf{S}(s)f(x)|^p dm(x)\Big\}dm_G(s) \le \frac{C^{2p}A^p}{m_G(V)}\|f\|_p^p m_G(V\cdot K).$$

The choice of V yields $\int_{\mathbb{S}}|Tf(x)|^p dm(x) \le C^{2p}A^p\|f\|_p^p(1+\epsilon)$, and letting $\epsilon\to 0$ yields the desired inequality.

We now obtain the general case. Since G is σ-compact, there is an increasing sequence $\{K_j\}$ of compact sets with union G. Then $\eta_j(\cdot) := \eta(\cdot\cap K_j)$ tends to η in total variation norm, and so $\|T_{\eta_j}(\mathbf{R}) - T_\eta(\mathbf{R})\| \to 0$ and $\|T_{\eta_j}(\mathbf{S}) - T_\eta(\mathbf{S})\| \to 0$; with the inequality $\|T_{\eta_j}(\mathbf{S})\| \le C^2\|T_{\eta_j}(\mathbf{R})\|$ proved above we obtain the result. □

Theorem 5.6. *Let G be a σ-compact LCA group, and μ a probability on G with bounded angular ratio. Fix $1<p<\infty$ and let $\mathbf{S}$ be a bounded continuous representation of G in $L_p(\mathbb{S},m)$. Then $\sup_n n\|T_\mu^n(\mathbf{S}) - T_\mu^{n+1}(\mathbf{S})\| < \infty$.*

Proof. Put $\eta_n = n(\mu^n - \mu^{n+1})$. Then by Theorem 5.5 (LCA groups are amenable),

$$n\|T_\mu^n(\mathbf{S}) - T_\mu^{n+1}(\mathbf{S})\| = \|T_{\eta_n}(\mathbf{S})\| \le \|T_{\eta_n}(\mathbf{R})\|_p = n\|T_{\mu,p}^n - T_{\mu,p}^{n+1}\|.$$

Since μ has bounded angular ratio, Proposition 5.2 yields the assertion. $\square$

Definition. A bounded linear operator on $L_p(\mathbb{S}, m)$ is called a *Lamperti operator* if $(Tf)(Tg) = 0$ whenever $fg = 0$ $(f, g \in L_p)$; this means that T preserves disjointness of supports. For $1 < p < \infty$, isometries of $L_p(\mathbb{S}, m)$, $p \neq 2$, and positive isometries of L_2 are Lamperti operators (see [38, p. 186]).

Theorem 5.7. *Let $1 < p < \infty$, and let $\mathbf{S}$ be a bounded representation of a σ-compact LCA group G by Lamperti operators on $L_p(\mathbb{S}, m)$. If μ is a probability on G with bounded angular ratio, then for every $f \in L_p(\mathbb{S}, m)$ the sequence $\{T_\mu^n(\mathbf{S})f\}$ converges a.e., and $\sup_n |T_\mu^n(\mathbf{S})f| \in L_p(\mathbb{S}, m)$.*

Proof. By Proposition 5.2 we can apply the maximal inequality of Le Merdy and Xu [44, Theorem 4.1] to the operator of convolution by μ on $L_p(G, m_G)$, and we obtain that $\big\| \sup_n \mu^n * |\phi| \big\|_p \leq c\|\phi\|_p$ for every $\phi \in L_p(G, m_G)$. The extension of Calderon's transfer principle to amenable groups by Lin and Wittmann [46, Theorem 3.1] yields that

$$\| \sup_n |T_\mu^n(\mathbf{S})f| \,\|_p \leq c\big(\sup_{t\in G} \|S(t)\|\big)^2 \|f\|_p \quad \text{for } f \in L_p(\mathbb{S}, m).$$

By Theorem 5.6, we have the convergence rate $\sup n\|T_\mu^n(\mathbf{S}) - T_\mu^{n+1}(\mathbf{S})\| < \infty$, which yields the a.e. convergence as in the proof of Theorem 2.7. $\square$

Corollary 5.8. *Let $1 < p < \infty$, and let $\mathbf{S}$ be a bounded representation of a σ-compact LCA group G by* positive *operators on $L_p(\mathbb{S}, m)$. If μ is a probability on G with bounded angular ratio, then for every $f \in L_p(\mathbb{S}, m)$ the sequence $\{T_\mu^n(\mathbf{S})f\}$ converges a.e., and $\sup_n |T_\mu^n(\mathbf{S})f| \in L_p(\mathbb{S}, m)$.*

Proof. Kan [36] proved that an invertible operator T on $L_p(\mathbb{S}, m)$ such that both T and T^{-1} are positive is Lamperti, so Theorem 5.7 applies. $\square$

Remarks. 1. Theorem 2.7 does not yield directly Corollary 5.8, because $T_\mu(\mathbf{S})$ is not necessarily a contraction. Note that the maximal inequality of Le Merdy and Xu is used in the proof of Theorem 5.7.

2. When $p \neq 2$, Theorem 5.7 applies to isometric representations in L_p which are not necessarily positive. In this case Theorem 2.7 does not yield the convergence, because $T_\mu(\mathbf{S})$ is not necessarily positive (though it is a contraction).

3. It is possible to have μ strictly aperiodic with $\mu^n * \phi \to 0$ a.e. on G for every $\phi \in L_P(G, m_G)$, without a maximal inequality. In this case $T_\mu^n(\mathbf{S})f$ need not converge a.e. For example, on $\mathbb{Z}$ let $\mu = \frac{1}{2}(\delta_0 + \delta_1)$; the pointwise convergence of $\mu^n * \phi$ for $\phi \in L_p(G, m_G)$, $1 \leq p < \infty$, follows from norm convergence since we have a discrete group. However, J. Rosenblatt [56, Theorem 10] proved that for every action of $\mathbb{Z}$ in a separable non-atomic probability space $(\mathbb{S}, m)$ there is a set B for which $\limsup T_\mu^n(\mathbf{S})\mathbf{1}_B = 1$ a.e. and $\liminf T_\mu^n(\mathbf{S})\mathbf{1}_B = 0$ a.e. For a dense subspace of $L_p(\mathbb{S}, m)$ a.e. convergence holds by [6, Remark, p. 103].

Jones, Rosenblatt and Tempelman [35] studied the a.e. convergence of $T_\mu^n(\mathbf{S}_p)f$ for the μ-averages of actions of general σ-compact locally compact metric groups. They introduced an analogue, for the non-Abelian case, of the bounded angular ratio property. From their approach we obtain the following.

Theorem 5.9. *Let G be a σ-compact locally compact metric group, and let $\{\mathbf{U}_\gamma : \gamma \in \Lambda\}$ be the set of all irreducible unitary representations of G ($\mathbf{U}_\gamma$ acts on a Hilbert space H_γ). Let μ be a probability on G such that $\|T_\mu(\mathbf{U}_\gamma)\| < 1$ for $\mathbf{U}_\gamma \neq Id$. If*

$$\sup_{Id \neq \gamma \in \Lambda} \frac{\|I_\gamma - T_\mu(\mathbf{U}_\gamma)\|}{1 - \|T_\mu(\mathbf{U}_\gamma)\|} = C < \infty \tag{20}$$

then for every unitary representation $\mathbf{S}$ by positive operators in $L_2(\mathbb{S}, m)$ we have

$$\sup_n n\|T_\mu^n(\mathbf{S}) - T_\mu^{n+1}(\mathbf{S})\| < \infty. \tag{21}$$

Consequently for any $f \in L_2(\mathbb{S}, m)$ the sequence $\{T_\mu^n(\mathbf{S})f\}$ converges a.e., with $\sup_n T_\mu^n(\mathbf{S})|f|$ in $L_2(\mathbb{S}, m)$.

Proof. The assumptions yield, by the proof in [35, p. 548], that (9) holds with $p = 2$, $T = T_\mu(\mathbf{S})$, and $C_2 = C$. By positivity of the representation, T is positive, and (21) holds by Theorem 2.9. The a.e. convergence holds by Theorem 2.7. □

Remark. In particular, under the assumptions of the theorem, (18) holds.

Corollary 5.10. *Let $\{\theta_t : t \in G\}$ be a continuous action of a σ-compact locally compact metric group G, let $\mathbf{S}(t)f = f \circ \theta_{t^{-1}}$, and let μ be a probability on G satisfying (20). Then for every $1 < p < \infty$ we have $\sup_n n\|T_\mu^n(\mathbf{S}_p) - T_\mu^{n+1}(\mathbf{S}_p)\|_p < \infty$, and consequently, $T_\mu^n(\mathbf{S}_p)f$ converges a.e. for every $f \in L_p(\mathbb{S}, m)$.*

The convergence is the result of [35]. The proof is like that of Theorem 5.4 (which is the special case of G Abelian).

Theorem 5.11. *Let $1 < p < \infty$, and let $\mathbf{S}$ be a bounded representation of a σ-compact locally compact amenable group G by* positive *operators on $L_p(\mathbb{S}, m)$. If μ is a probability on G which satisfies $\sup_n n\|T_{\mu,2}^n - T_{\mu,2}^{n+1}\| < \infty$, then for every $f \in L_p(\mathbb{S}, m)$ the sequence $\{T_\mu^n(\mathbf{S})f\}$ converges a.e., and $\sup_n |T_\mu^n(\mathbf{S})f| \in L_p(\mathbb{S}, m)$.*

Proof. By Blunck's interpolation, the assumption on μ is equivalent to (18). In the proofs of Theorems 5.6 and 5.7, replace the assumption that μ has bounded angular ratio by (18), and then the proof of Corollary 5.8 yields the assertion. □

Remarks. 1. Dungey [27, Theorem 1.2] gives sufficient conditions for (18) when μ is absolutely continuous.

2. Example 3.10 of [35] exhibits an amenable (countable) group such that for any μ adapted (20) fails, although for any μ symmetric strictly aperiodic (21) holds for unitary representations by positive operators (see [35, p. 549]).

3. In [35, Theorem 3.15] it is shown that if G is a discrete group with Kazhdan's property (T) (hence not amenable), then (20) holds for every strictly aperiodic μ.

4. If μ is strictly aperiodic on G non-amenable with $e \in supp(\mu)$, then $\|T_{\mu,2}\| < 1$ by Derriennic-Guivarc'h [23] (see also [9], [5]), so (18) obviously holds. However, (21) need not hold, as shown in the next example.

Example 5. *μ strictly aperiodic on G non-amenable satisfying (18) and not (21).*
Let $G = \mathbb{F}_2$ be the free group with two generators a, b and let S be the positive unitary operator on $L_2(\mathbb{S}, m)$ of a separable non-atomic probability space, induced by an ergodic measure preserving transformation on $\mathbb{S}$. For elements of $\mathbb{F}_2$ in their reduced representations, we define

$$\mathbf{S}(\prod_{j=1}^{N} a^{k_j} b^{n_j}) = S^{\sum_{j=1}^{N} k_j}.$$

Then $\mathbf{S}$ is a unitary representation. Since $\mathbf{S}(b^n) = S^0 = I$, the probability $\mu := \frac{1}{4}(\delta_e + 2\delta_a + \delta_b)$ is adapted with $e \in supp(\mu)$, so $\|T_{\mu,2}\| < 1$, hence (18) holds. But $T_\mu(\mathbf{S}) = \frac{1}{2}(I + S)$, so by [56, Theorem 10] the final conclusion of Theorem 5.9 fails, so (21) does not hold.

Remark. Example 5 shows also that Theorem 5.11 need not hold if G is non-amenable.

6. Fractional coboundaries of positive Ritt L_p-contractions

Let T be a power-bounded operator on a Banach space X. Let $0 < \alpha < 1$. Following Derriennic-Lin [24], we define $(I - T)^\alpha := I - \sum_{k\geq 1} a_k^{(\alpha)} T^k$, where $\{a_k^{(\alpha)}\}$ is given by the power-series expansion $(1-t)^\alpha = 1 - \sum_{k\geq 1} a_k^{(\alpha)} t^k$, $|t| \leq 1$, with $a_k^{(\alpha)} > 0$ and $\sum_{k=1}^\infty a_k^{(\alpha)} = 1$ (see Example 2). The elements of $(I-T)^\alpha X$ are called *fractional coboundaries* of T (of order α). It was proved in [24, Theorem 2.11] that $y \in (I-T)^\alpha X$ if and only if $\sum_{k=1}^\infty \frac{T^k y}{k^{1-\alpha}}$ converges in norm. These conditions are equivalent to the norm convergence of $\sum_{k=0}^\infty b_k^{(\alpha)} T^k y$, where $\{b_k^{(\alpha)}\}$ is the sequence of coefficients in the expansion $(1-t)^{-\alpha} = \sum_{k=0}^\infty b_k^{(\alpha)} t^k$ for $|t| < 1$, which are all positive (see [24]).

In this section we study additional properties of fractional coboundaries of a positive L_p-contraction ($1 < p < \infty$) T which satisfies Ritt's condition, i.e. $\sup_n n\|T^n - T^{n+1}\| < \infty$. For some properties of coboundaries see Proposition 2.11.

We start with a general property of fractional coboundaries in Banach spaces. Remember that for T power-bounded on X, we have

$$X_{erg} := \{x \in X : \frac{1}{n}\sum_{k=1}^{n} T^k x \ \text{ converges } \} = F(T) \oplus \overline{(I-T)X}$$

T is *mean ergodic* if $X_{erg} = X$. If X is reflexive, every power-bounded T is mean ergodic.

Proposition 6.1. *Let T be a power-bounded operator on a Banach space X satisfying $\sup_n n\|T^n - T^{n+1}\| = K < \infty$, and let $0 < \alpha < 1$. Then*

(i) $\sup_{n\geq 1} n^\alpha \|T^n (I-T)^\alpha\| < \infty$.

(ii) For any $f \in X_{erg}$ *we have* $n^\alpha \|T^n(I-T)^\alpha f\| \underset{n\to\infty}{\longrightarrow} 0$.

Proof. We prove (i). Let $\alpha \in (0,1)$ and $f \in X$. Using the asymptotics of $a_k^{(\alpha)}$ in (14) and (15), $\sup_n \|T^n\| = C < \infty$ and the assumption, we obtain

$$\|(I-T)^\alpha T^n\| = \|\sum_{k\ge 1} a_k^{(\alpha)}(I-T^k)T^n\| \le$$

$$\|\sum_{k=1}^n a_k^{(\alpha)} \sum_{j=0}^{k-1} T^j(I-T)T^n\| + \|\sum_{k\ge n+1} a_k^{(\alpha)}(I-T^k)T^n\| \le$$

$$\frac{C}{n}\sum_{k=1}^n k a_k^{(\alpha)} \|n(I-T)T^n\| \; + \; (C+1)C \sum_{k\ge n+1} |a_k^{(\alpha)}| \le \tilde{C}/n^\alpha \, .$$

To prove (ii), it is enough to prove it on a dense subspace, by (i). Obviously $(I-T)^\alpha f = f$ for $f \in F(T)$, so it suffices to prove (ii) for $f = (I-T)g$. Since

$$n^\alpha \|T^n(I-T)^\alpha(I-T)\| \le \frac{n^\alpha}{n}\|(I-T)^\alpha\| n\|T^n(I-T)\| \le \frac{C+1}{n^{1-\alpha}}K \to 0.$$

we obtain that $n^\alpha\|T^n(I-T)^\alpha(I-T)g\| \underset{n\to\infty}{\longrightarrow} 0$ for every $g \in X$. □

Remark. Our proof is valid also in real Banach spaces. Part (i) was proved for complex Banach spaces in [3, Proposition 2.8].

For $r > 0$, we define $(I-T)^r = (I-T)^{[r]}(I-T)^{r-[r]}$. It was proved in [24] that $\{(I-T)^r : r \ge 0\}$ is a C_0-semi-group on $\overline{(I-T)X}$; its infinitesimal generator was proved to be $-\sum_{n=1}^\infty T^n/n$ [14], [30].

Corollary 6.2. *Let T be as in Proposition 6.1. Then for every $r > 0$ we have*
(i) $\sup_{n\ge 1} n^r \|T^n(I-T)^r\| < \infty$.
(ii) For any $f \in X_{erg}$ *we have* $n^r\|T^n(I-T)^r f\| \underset{n\to\infty}{\longrightarrow} 0$.

Proof. The condition $\sup_n n\|T^n(I-T)\| < \infty$ implies that for every positive integer k we have $\sup_n n^k\|T^n(I-T)^k\| < \infty$ [60, Lemma 2.1]. Given $r > 0$, let $k = [r]$ and $\alpha = r - k$. Then (i) follows from

$$n^r\|T^n(I-T)^r\| = \|n^k T^{[n/2]}(I-T)^k n^\alpha T^{[(n+1)/2]}(I-T)^\alpha\| \le$$

$$\|n^k T^{[n/2]}(I-T)^k\| \cdot \|n^\alpha T^{[(n+1)/2]}(I-T)^\alpha\|.$$

(ii) follows from the above and Proposition 6.1(ii). □

We now turn to the case of positive L_p-contractions. We start with a consequence of the work of Arhancet-Le Merdy [3]. Note that they use a different definition of $(I-T)^\alpha$, appropriate only for T Ritt, but for Ritt operators both definitions coïncide.

Definition. We say that an operator T on $L_p(\mathbb{S},m)$ is *R-Ritt* if there exists $C>0$ such that, for every sequence of integers $\{n_k\}_{k\geq 1}$ and any sequence $\{f_k\}_{k\geq 1}$ in $L_p(\mathbb{S},m)$,

$$\Big\|\Big(\sum_{k\geq 1}|T^{n_k}f_k|^2\Big)^{1/2}\Big\|_p \leq C\Big\|\Big(\sum_{k\geq 1}|f_k|^2\Big)^{1/2}\Big\|_p,$$

$$\Big\|\Big(\sum_{k\geq 1}|n_k(T^{n_k}-T^{n_k+1})f_k|^2\Big)^{1/2}\Big\|_p \leq C\Big\|\Big(\sum_{k\geq 1}|f_k|^2\Big)^{1/2}\Big\|_p.$$

Proposition 6.3. *Let $1<p<\infty$ and let T be a positive contraction of $L_p(\mathbb{S},m)$ which satisfies (1). For $0<\alpha<1$, there exists $C_{p,\alpha}>0$ such that for every $h\in\overline{(I-T)L_p(\mathbb{S},m)}$,*

$$\frac{\|h\|_p}{C_{p,\alpha}} \leq \Big\|\Big(\sum_{n\geq 1}n^{2\alpha-1}|T^n(I-T)^\alpha h|^2\Big)^{1/2}\Big\|_p \leq C_{p,\alpha}\|h\|_p\,. \tag{22}$$

Proof. By Theorem 2.9(iii), there is a Stolz region for which (10) holds, which in the terminology of [42] means that T has a bounded $H^\infty(B_\gamma)$ calculus. Hence, by [42, Proposition 7.4], T is R-Ritt. Hence, by Theorem 3.3 of [3] (with $\beta=1$), there exists $C'_{p,\alpha}>0$ such that

$$\frac{1}{C'_{p,\alpha}}\Big\|\Big(\sum_{n\geq 1}n|T^n(I-T)h|^2\Big)^{1/2}\Big\|_p \leq \Big\|\Big(\sum_{n\geq 1}n^{2\alpha-1}|T^n(I-T)^\alpha h|^2\Big)^{1/2}\Big\|_p$$
$$\leq C'_{p,\alpha}\Big\|\Big(\sum_{n\geq 1}n|T^n(I-T)h|^2\Big)^{1/2}\Big\|_p.$$

By [44, Theorem 3.3(2)] and [44, Corollary 3.4] (both with $m=1$), the extreme terms are equivalent to $\|h\|_p$ for $h\in\overline{(I-T)L_p(\mathbb{S},m)}$. □

We now give a complement to the above mentioned characterization of fractional coboundaries of [24], in the spirit of the paper [19], when T is a positive Ritt contraction of $L_p(\mathbb{S},m)$.

Proposition 6.4. *Let $1<p<\infty$ and let T be a positive contraction of $L_p(\mathbb{S},m)$ satisfying $\sup_n n\|T^n-T^{n+1}\|<\infty$. For $\alpha\in(0,1)$, the following are equivalent for $f\in L_p(\mathbb{S},m)$:*

(*i*) $f\in(I-T)^\alpha L_p(\mathbb{S},m)$;
(*ii*) $\sum_{k\geq 1}\frac{T^kf}{k^{1-\alpha}}$ *converges in L_p-norm.*
(*iii*) $\Big(\sum_{k\geq 1}k^{2\alpha-1}|T^kf|^2\Big)^{1/2}\in L_p(\mathbb{S},m)$.
(*iv*) $\Big(\sum_{r\geq 0}2^{2r\alpha}\max_{2^r\leq n<2^{r+1}}|T^nf|^2\Big)^{1/2}\in L_p(\mathbb{S},m)$.

In particular, f satisfies all the above conditions if

$$\sum_{n\geq 1}n^{p\alpha-1}\|T^nf\|_p^p<\infty \qquad \text{when } 1<p\leq 2\,; \tag{23}$$

$$\sum_{n\geq 1}n^{2\alpha-1}\|T^nf\|_p^2<\infty \qquad \text{when } p\geq 2. \tag{24}$$

When $p=2$, (i) holds if and only if $\sum_{n\geq 1}n^{2\alpha-1}\|T^nf\|_2^2<\infty$.

Proof. The equivalence of (i) and (ii) is in Theorem 2.11 of [24] (valid for any mean ergodic power-bounded operator).

Let $f \in (I-T)^\alpha L_p(\mathbb{S},m)$. Then there exists $h \in L_p(\mathbb{S},m)$ such that $f = (I-T)^\alpha h$, and since T is mean ergodic, by [24, Theorem 2.11] one may assume that $h \in \overline{(I-T)L_p(\mathbb{S},m)}$. Then (iii) follows from Proposition 6.3.

Assume (iii). By reflexivity, Corollary 2.12 of [24] yields that $f \in (I-T)^\alpha L_p(\mathbb{S},m)$ if $\{\| \sum_{k=0}^N b_k^{(\alpha)} T^k f\|_p\}_{N\ge 1}$ is bounded.

We first show that $f \in \overline{(I-T)L_p}$, by proving that $\frac{1}{n}\sum_{k=1}^n T^k f \to 0$ a.e. and using Akcoglu's theorem. By (iii) the series $\big(\sum_{k\ge 1} k^{2\alpha-1}|T^kf|^2\big)^{1/2}$ converges a.e. When $\alpha \ge 1/2$, this implies $|T^kf|^2 \to 0$ a.e., so $|\frac{1}{n}\sum_{k=1}^n T^kf| \le \frac{1}{n}\sum_{k=1}^n |T^kf| \to 0$ a.e. When $\alpha < \frac{1}{2}$, Cauchy's inequality followed by Kronecker's lemma yield

$$\Big|\frac{1}{n}\sum_{k=1}^n T^kf\Big|^2 \le \frac{1}{n}\sum_{k=1}^n |T^kf|^2 \le \frac{1}{n^{1-2\alpha}}\sum_{k=1}^n |T^kf|^2 \to 0.$$

Since α is fixed, we denote below $a_k = a_k^{(\alpha)}$ and $b_k = b_k^{(\alpha)}$. Put $u_N = \sum_{k=0}^N b_k T^k f$. Then u_N is in $\overline{(I-T)L_p}$, and Proposition 6.3 yields

$$\|u_N\|_p \le C_{p,\alpha}\Big\|\Big(\sum_{n\ge 1} n^{2\alpha-1}|T^n(I-T)^\alpha u_N|^2\Big)^{1/2}\Big\|_p. \tag{25}$$

By Proposition 2.4(i) of [24], we have

$$(I-T)^\alpha u_N = f + \sum_{\ell=N+1}^\infty \big(\sum_{k=0}^N b_k a_{\ell-k}\big)T^\ell f\,.$$

Then by the triangle inequality in ℓ_2 and in L_p, we have

$$\Big\|\Big(\sum_{n\ge 1} n^{2\alpha-1}|T^n(I-T)^\alpha u_N|^2\Big)^{1/2}\Big\|_p \le$$

$$\Big\|\Big(\sum_{n\ge 1} n^{2\alpha-1}|T^nf|^2\Big)^{1/2}\Big\|_p + \sum_{\ell\ge N+1}\big(\sum_{k=0}^N \beta_k\alpha_{\ell-k}\big)\Big\|\Big(\sum_{n\ge 1} n^{2\alpha-1}|T^{n+\ell}f|^2\Big)^{1/2}\Big\|_p. \tag{26}$$

As noted at the beginning of the proof of Proposition 6.3, T is R-Ritt. Hence, by taking in the definition $f_k = k^{\alpha-\frac{1}{2}}T^kf$ and $n_k = \ell$ for each k, we obtain

$$\Big\|\Big(\sum_{k\ge 1} k^{2\alpha-1}|T^{k+\ell}f|^2\Big)^{1/2}\Big\|_p \le C_p\Big\|\Big(\sum_{k\ge 1} k^{2\alpha-1}|T^kf|^2\Big)^{1/2}\Big\|_p.$$

By [24, Lemma 2.3], $\sum_{\ell\ge N+1}(\sum_{k=0}^N \beta_k\alpha_{\ell-k}) = 1$ for every $N \ge 0$. Putting it all in (26) and using (25), we obtain by (iii) that $\sup_N \|u_N\|_p < \infty$, so $f \in (I-T)^\alpha L_p(\mathbb{S},m)$.

Assume (iv). Then

$$\Big(\sum_{n=1}^\infty n^{2\alpha-1}|T^nf|^2\Big)^{1/2} = \Big(\sum_{\ell=0}^\infty \sum_{k=2^\ell}^{2^{\ell+1}-1} k^{2\alpha-1}|T^kf|^2\Big)^{1/2} \le 2^{2\alpha-1}\Big(\sum_{\ell=0}^\infty 2^{2\ell\alpha}\max_{2^\ell\le k<2^{\ell+1}}|T^kf|^2\Big)^{1/2}.$$

Hence (iii) holds.

It remains to prove that (i) implies (iv). Let $f \in (I-T)^\alpha L_p(S,m)$.

Let $r \ge 1$ and $2^r \le n \le 2^{r+1}-1$. Then $n = 2^{r-1}+\ell$ for some $2^{r-1} \le \ell \le 2^{r+1}-2^{r-1}-1$. For every $h \in L_p$, we have, using $(a+b)^2 \le 2a^2+2b^2$ and the Cauchy-Schwarz inequality,

$$|T^\ell h|^2 = \Big|\frac{1}{\ell}\sum_{j=0}^{\ell-1} T^j h + \frac{1}{\ell}\sum_{j=1}^{\ell} j(T^j h - T^{j-1}h)\Big|^2 \le$$

$$2\Big|\frac{1}{\ell}\sum_{j=0}^{\ell-1} T^j h\Big|^2 + 2\Big|\frac{1}{\ell}\sum_{j=1}^{\ell} j(T^j h - T^{j-1}h)\Big|^2 \le 2\frac{1}{\ell}\sum_{j=0}^{\ell-1}|T^j h|^2 + 2\sum_{j=1}^{\ell} j|T^j h - T^{j-1}h|^2.$$

Taking $h := T^{2^{r-1}}f$ and maximizing over a block, we obtain

$$\max_{2^r \le n < 2^{r+1}} |T^n f|^2 = \max_{2^{r-1}\le \ell < 2^{r+1}-2^{r-1}} |T^\ell(T^{2^{r-1}}f)|^2 \le$$

$$\frac{2}{2^{r-1}}\sum_{j=2^{r-1}}^{2^{r+1}-1}|T^j f|^2 + 2\sum_{j=2^{r-1}}^{2^{r+1}-1} j|T^j f - T^{j-1}f|^2 \le 8\sum_{j=2^{r-1}}^{2^{r+1}-1}\frac{|T^j f|^2}{j} + 2\sum_{j=2^{r-1}}^{2^{r+1}-1} j|T^j f - T^{j-1}f|^2\,.$$

We shall multiply this estimate by $2^{2r\alpha} \le 4^\alpha j^{2\alpha}$ for j in the summations. We then take the sum over r. When doing so we notice that our blocks $[2^{r-1}, 2^{r+1}-1]$ overlap when r varies, but any integer $j > 1$ will appear twice. Hence, we obtain

$$\sum_{r\ge 1} 2^{2r\alpha}\max_{2^r\le n<2^{r+1}}|T^n f|^2 \le \tilde{C}\Big(\sum_{j\ge1} j^{2\alpha-1}|T^j f|^2 + \sum_{j\ge 1} j^{2\alpha+1}|T^j f - T^{j-1}f|^2\Big).$$

By assumption, there exists $g \in L_p$ such that $f = (I-T)^\alpha g$. Since (i) and (iii) are equivalent, $\big(\sum_{j\ge1} j^{2\alpha-1}|T^j(I-T)^\alpha g|^2\big)^{1/2} \in L_p$. By [3, Theorem 3.3], with $\beta = \alpha+1$, we have also $\big(\sum_{j\ge1} j^{2\alpha+1}|T^j(I-T)^{1+\alpha} g|^2\big)^{1/2} \in L_p$. Hence (iv) holds.

We now prove that (23) or (24) imply (iii).

Assume that $1 < p \le 2$. Then, estimating on blocks of length 2^ℓ and using the norm inequality $\|\cdot\|_{\ell_2} \le \|\cdot\|_{\ell_p}$, we obtain

$$h^2 := \sum_{n=1}^{\infty} n^{2\alpha-1}|T^n f|^2 \le \sum_{\ell=0}^{\infty}(2^{\ell+1})^{(2\alpha-1)}2^\ell \max_{2^\ell\le k<2^{\ell+1}}|T^k f|^2 =$$

$$2^{2\alpha-1}\sum_{\ell=0}^{\infty}2^{2\alpha\ell}\max_{0\le j<2^\ell}|T^{2^\ell+j}f|^2 \le 2^{2\alpha-1}\sum_{\ell=0}^{\infty}\Big(2^{p\alpha\ell}\max_{0\le j<2^\ell}|T^{2^\ell+j}f|^p\Big)^{2/p}.$$

By the maximal inequality of Le Merdy and Xu [44], $\|\sup_j |T^j f|\,\|_p \le c\|f\|_p$, so

$$\int h^p dm = \int (h^2)^{p/2}dm \le C\Big(\sum_{\ell=0}^{\infty}2^{p\ell\alpha}\|\max_{0\le j<2^\ell}\|T^j(T^{2^\ell}f)\|_p^p\Big) \le C\Big(\sum_{\ell=0}^{\infty}2^{p\ell\alpha}c^p\|T^{2^\ell}f\|_p^p\Big).$$

Since T is a contraction, we estimate on blocks $(2^{\ell-1}+1, 2^\ell]$ to obtain

$$\sum_{n=1}^{\infty} n^{p\alpha-1}\|T^n f\|_p^p \ge \sum_{\ell=1}^{\infty}2^{(\ell-1)(p\alpha-1)}2^{\ell-1}\|T^{2^\ell}f\|_p^p = 2^{-p\alpha}\sum_{\ell=1}^{\infty}2^{\ell p\alpha}\|T^{2^\ell}f\|_p^p\,,$$

which shows that (23) implies (iii).

When $p \geq 2$, we use the identity $\|h\|_{p/2} = \|h^{1/2}\|_p^2$ for $h \geq 0$ and (24) to obtain

$$\Big[\int \Big(\sum_{k=1}^{\infty} k^{2\alpha-1}|T^k f|^2\Big)^{p/2} dm\Big]^{1/p} = \Big\| \sum_{k=1}^{\infty} k^{2\alpha-1}|T^k f|^2 \Big\|_{p/2}^{1/2} \leq$$

$$\Big(\sum_{k=1}^{\infty} k^{2\alpha-1}\| \, |T^k f|^2\|_{p/2}\Big)^{1/2} = \Big(\sum_{k=1}^{\infty} k^{2\alpha-1}\| \, |T^k f| \, \|_p^2\Big)^{1/2} < \infty.$$

Thus in either case, (iii) holds.

Finally, for $p = 2$ (23) is exactly (iii). □

Remark. The fact that (23) implies (i) was proved in [20] when $p = 2$ and T is normal.

Theorem 6.5. *Let $1 < p < \infty$ and let T be a positive contraction of $L_p(\mathbb{S}, m)$ satisfying $\sup_n n\|T^n - T^{n+1}\| < \infty$. Let $\alpha \in (0, 1)$ and $f \in (I-T)^\alpha L_p(\mathbb{S}, m)$. Then:*

(i) The series $\sum_{n\geq 1} n^{\alpha-1}T^n f$ converges m-a.e. and $\sup_{n\geq 1} |\sum_{k=1}^n k^{\alpha-1}T^k f| \in L_p(\mathbb{S}, m)$.

(ii) $n^\alpha T^n f \underset{n\to\infty}{\longrightarrow} 0$ m-a.e. and $\sup_{n\geq 1} n^\alpha |T^n f| \in L_p(\mathbb{S}, m)$.

(iii) The square variation norm $\|\{n^\alpha T^n f\}_{n\geq 1}\|_{L_p(v^2)}$ is finite.

Proof. Let $f \in (I-T)^\alpha L_p(\mathbb{S}, m)$. The proof that the series $\sum_{n\geq 1} n^{\alpha-1}T^n f$ converges m-a.e. and that $\sup_{n\geq 1} |\sum_{k=1}^n k^{\alpha-1}T^k f| \in L_p(\mathbb{S}, m)$ is similar to that of Theorem 3.9 of [24], since by Theorem 2.7, $\sup_n T^n|g| \in L_p$ for every $g \in L_p(\mathbb{S}, m)$. This proves (i).

To prove (ii), we first observe that by Kronecker's lemma, (i) yields $n^{\alpha-1}(f + \cdots + T^{n-1}f) \underset{n\to\infty}{\longrightarrow} 0$ m-a.e. and $\sup_{n\geq 1} n^{\alpha-1}|f + \cdots + T^{n-1}f| \in L_p$.

Recall that for every $n \geq 1$, we have

$$T^n f - \frac{1}{n}(f + \cdots + T^{n-1}f) = \frac{1}{n}\sum_{j=1}^{n} j(T^j f - T^{j-1}f)\,. \tag{27}$$

By the Cauchy-Schwarz inequality,

$$\frac{n^\alpha}{n}\sum_{j=1}^{n} j|T^j f - T^{j-1}f| = \frac{1}{n^{1-\alpha}}\sum_{j=1}^{n} j^{\frac{1}{2}-\alpha}j^{\frac{1}{2}+\alpha}|T^j f - T^{j-1}f| \leq$$

$$\frac{1}{n^{1-\alpha}}\Big(\sum_{j=1}^{n} j^{1-2\alpha}\Big)^{1/2}\Big(\sum_{k=1}^{n} k^{1+2\alpha}|T^k f - T^{k-1}f|^2\Big)^{1/2} \leq C\Big(\sum_{k=1}^{n} k^{1+2\alpha}|T^k f - T^{k-1}f|^2\Big)^{1/2}.$$

Hence

$$\max_{0\leq k\leq n} k^\alpha|T^k f| \leq \max_{0\leq k\leq n} \frac{k^\alpha|f + \cdots + T^{k-1}f|}{k} + C\big(\sum_{k=1}^{\infty} k^{2\alpha+1}|T^k f - T^{k-1}f|^2\big)^{1/2}. \tag{28}$$

To prove $\sup_n n^\alpha |T^n f| \in L_p$, we have to show that the series on the right-hand side of (28) is in L_p. We write the terms of the series as

$$(2k)^{2\alpha+1}|T^{2k}f - T^{2k-1}f|^2 = 2^{2\alpha+1}\Big|k(T^{k+1}-T^k)(k^{\alpha-1/2}T^{k-1}f)\Big|^2,$$
$$(2k+1)^{2\alpha+1}|T^{2k+1}f - T^{2k}f|^2 \le 2^{2\alpha+1}\Big|(k+1)(T^{k+1}-T^k)((k+1)^{\alpha-1/2}T^{k}f)\Big|^2,$$

Since T is R-Ritt, the second inequality of the definition of R-Ritt yields

$$\Big\|\big(\sum_{k\ge 1} k^{2\alpha+1}|T^k f - T^{k-1}f|^2\big)^{1/2}\Big\|_p \le C\Big\|\big(\sum_{k\ge 1} k^{2\alpha-1}|T^k f|^2\big)^{1/2}\Big\|_p, \tag{29}$$

which is finite by (iii) of Proposition 6.4. Hence $\sup_{n\ge 1} n^\alpha |T^n f| \in L_p$.

To show $n^\alpha T^n f \to 0$ a.e., note that by (iv) of Proposition 6.4, for $2^r \le n < 2^{r+1}$ we have

$$n^\alpha |T^n f| \le 2^\alpha \cdot 2^{r\alpha} \max_{2^r \le n < 2^{r+1}} |T^n f| \to 0 \text{ a.e.}$$

We now prove (iii). Denote $u_j := j^\alpha T^j f$. Let $\{t_k\}_{k\ge 0}$ be an increasing sequence of integers. We will use properties (iii) and (iv) of Proposition 6.4 to bound $\big(\sum_{k\ge 0}|u_{t_k} - u_{t_{k+1}}|^2\big)^{1/2}$ by a function of L_p which does not depend on $\{t_k\}$.

We partition the sequence $(t_k)_{k\ge 0}$ into the blocks of dyadic integers, and accordingly partition the indices. Put $\{\ell_n\} := \{\ell : [2^\ell, 2^{\ell+1}) \cap \{t_k\} \ne \emptyset\}$, and let k_n be the first index k with $2^{\ell_n} \le t_k < 2^{\ell_n+1}$, i.e. $2^{\ell_n} \le t_k < 2^{\ell_n+1}$ if and only if $k_n \le k < k_{n+1}$.

We start with the following simple observation

$$\sum_{k\ge 0} |u_{t_k} - u_{t_{k+1}}|^2 = \sum_{n\ge 0}\sum_{k=k_n}^{k_{n+1}-1} |u_{t_k} - u_{t_{k+1}}|^2$$

For $k_n \le k \le k_{n+1}-2$, $t_{k+1} < 2^{\ell_n+1}$, and by the Cauchy-Schwarz inequality we obtain

$$|u_{t_k} - u_{t_{k+1}}|^2 = \Big[\sum_{j=t_k}^{t_{k+1}-1}(u_j - u_{j+1})\Big]^2 \le (t_{k+1}-t_k)\sum_{j=t_k}^{t_{k+1}-1}|u_j - u_{j+1}|^2 \le 2^{\ell_n}\sum_{j=t_k}^{t_{k+1}-1}|u_j - u_{j+1}|^2.$$

With the convention that $\sum_j^{j-1} = 0$ we obtain, after separating the index from a different block,

$$\sum_{k=k_n}^{k_{n+1}-1} |u_{t_k} - u_{t_{k+1}}|^2 \le \sum_{k=k_n}^{k_{n+1}-2} 2^{\ell_n}\sum_{j=t_k}^{t_{k+1}-1}|u_j - u_{j+1}|^2 \,+\, 2|u_{t_{k_{n+1}-1}}|^2 + 2|u_{t_{k_{n+1}}}|^2\ .$$

By the definitions,

$$|u_{t_{k_{n+1}-1}}|^2 \le 4^\alpha 2^{2\ell_n\alpha} \max_{2^{\ell_n}\le j<2^{\ell_n+1}} |T^j f|^2 \quad \text{and} \quad |u_{t_{k_{n+1}}}|^2 \le 4^\alpha 2^{2\ell_{n+1}\alpha} \max_{2^{\ell_{n+1}}\le j<2^{\ell_{n+1}+1}} |T^j f|^2.$$

Moreover, $|u_j - u_{j+1}|^2 \le C\big(j^{2\alpha-2}|T^j f|^2 + j^{2\alpha}|T^j f - T^{j+1} f|^2\big)$. Hence, we infer that

$$\sum_{n\ge 0}\sum_{k=k_n}^{k_{n+1}-1}|u_{t_k} - u_{t_{k+1}}|^2 \le C\sum_{n\ge 0}\Big(\sum_{k=k_n}^{k_{n+1}-2}\sum_{j=t_k}^{t_{k+1}-1}\big(j^{2\alpha+1}|T^j f - T^{j+1} f|^2 + j^{2\alpha-1}|T^j f|^2\big)\Big)$$
$$+C\sum_{n\ge 0}\Big(2^{2\ell_n\alpha}\max_{2^{\ell_n}\le j<2^{\ell_n+1}}|T^j f|^2 + 2^{2\ell_{n+1}\alpha}\max_{2^{\ell_{n+1}}\le j<2^{\ell_{n+1}+1}}|T^j f|^2\Big) =$$
$$C\sum_{j\ge 1}\big(j^{2\alpha+1}|T^j f - T^{j+1} f|^2 + j^{2\alpha-1}|T^j f|^2\big) + 2C\sum_{\ell\ge 0}2^{2\ell\alpha}\max_{2^\ell\le j<2^{\ell+1}}|T^j f|^2.$$

Since $(\sum_{i=1}^3 |b_i|)^{1/2} \le \sum_{i=1}^3 |b_i|^{1/2}$, we obtain

$$\Big(\sum_{k=1}^\infty |u_{t_k} - u_{t_{k+1}}|^2\Big)^{1/2} \le$$
$$C\Big[\big(\sum_{j\ge 1} j^{2\alpha+1}|T^j f - T^{j+1} f|^2\big)^{1/2} + \big(\sum_{j\ge 1} j^{2\alpha-1}|T^j f|^2\big)^{1/2} + \big(2\sum_{\ell\ge 0}2^{2\ell\alpha}\max_{2^\ell\le j<2^{\ell+1}}|T^j f|^2\big)^{1/2}\Big].$$

Since $f \in (I-T)^\alpha L_p$, the second and the third term are in L_p by (iii) and (iv) of Proposition 6.4, and the first term was then proved to be in L_p in (29). □

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