Circuits and Systems Expositions

The Fluorescent Lamp Circuit

Emanuel Gluskin

Abstract—The exposition of a theory of the strongly nonlinear and, in practice, very important fluorescent lamp circuits is presented. After introducing the topic via a simple singular model of the lamp, and calculating the most immediate circuit parameters, a detailed discussion of the nontrivial theoretical aspects of the circuits is given.

Index Terms—High-frequency characteristic, power systems, sensitivity, singularity, zero crossings.

I. INTRODUCTION

A. General

The topic under discussion is very important in practice. According to various reports, the total consumption of fluorescent lighting is close to 20% of all electrical power generated. We present an introduction, based on two simple models of the complicated lamp's voltage-current characteristic, to the field of lamp circuits, focusing on the parameters of the circuits which are the most important in practice. This introduction allows one to understand the lamp circuits, teach them in a basic or advanced course, and be able to think about possible improvements in these circuits and in their operation.

The modeling regards the lamp as a macroscopic circuit element. Regarding the physical processes in the lamp let us note only the most basic of them, which are the excitation by the accelerating electrons of the atoms of mercury which are present in some small amount (the low-pressure discharge) in the gas filling the lamp and the resulting emission, by the excited atoms, of photons of ultraviolet radiation, which causes the emission of visible light by the luminophore powder placed on the internal surface of the tube. That one photon of the ultraviolet radiation usually causes, for basic quantum reasons, the emission of only one photon of visible light, already implies significant energy losses associated with the difference in the energies of the two photons. Nevertheless, the power efficiency of the regular fluorescent lamp (≈24%) is significantly higher than the efficiency of an incandescent lamp (5–8%).

The physical properties of the lamp are responsible for both turning electrical energy into visible light and for the strong nonlinearity of the lamp, which is our main concern. However, contrary to the situation, e.g., regarding electric motors where the basic relevant laws which we use for the derivation of the macroscopic governing equations are known from early courses in physics, an understanding of the processes in the fluorescent lamps [4]–[6] requires knowledge of quantum physics, which are studied much later. This is the reason why fluorescent lamp circuits, which are practically very important, are ignored in the basic standard courses for future electrical engineers, and remain unknown to circuit specialists.

However, when focusing on the role of the lamp as a circuit element, and accepting the nonlinearity of the lamp as an empirical fact, we have a reasonably formulated problem which is theoretically interesting, important in practice, and worth studying at both undergraduate and graduate levels.

B. The Circuit, The First Model, and The Targets

The circuit under investigation is shown, in its two simplest and most common variants, in Fig. 1(a) and (b). Symbol ‘e’ denotes the lamp, whose voltage-current characteristic is schematically shown in Fig. 2(a). See also oscilloscope photographs in Fig. 2(b) and in [1], [5], and [7].

Despite the simplicity of the circuit, the $L - C$ connection and the simplicity of the sinusoidal input voltage function given by the line, this circuit is difficult to analyze precisely because of the strong nonlinearity of the lamp. We can, however, introduce a simple model which will allow us to understand the main circuit features. Namely, we idealize the realistic $v - i$ characteristic to the following relation:

$$v = A \text{sign} \ i$$

using the function signum, which equals 1 for positive argument and −1 for the negative argument and is undefined.
in the limits (-1, 1) for the zero argument, and the positive parameter $A$, which is the only parameter that characterizes the lamp in this model. See Fig. 2(c). This model will allow us many useful calculations. Later, we shall transfer to a more precise model.

We shall name the model $v = A \text{sign } i$ the hardlimiter model.

In order to justify this model we consider that, because of the power character of the circuit, the problem of the approximation of the $v - i$ characteristic is subject to the requirement to well estimate the power consumption of the lamp and not minimize so much a root mean square error between the approximation and the realistic characteristic. The importance of the power forces us to give different weights to the deviations of the approximation from the realistic characteristic in different parts of the curve which we are approximating. The integral nature of the average power

$$P = \frac{1}{T} \int_0^T v(t)i(t) \, dt$$

suggests that it is most important to take the lamp’s voltage correctly at the highest values of the current. Deviations of an approximation for $v(t)$ from the correct function at low values of the current should influence the integral less.

Another argument on behalf of the hardlimiter model is the unconventional situation regarding lamp circuits, associated with the fact that lamps of the same operational specification, but produced by different firms, are not entirely similar, and that the lamp is a “bad” circuit element which changes its parameters relatively strongly during its life time and is very unreliable. This stresses the importance of having a good theory of the first approximation for these circuits, which would allow one to simply estimate their practically most important parameters, while not missing the important nonlinear effects.

These motivating arguments will be greatly supported by the following discussion, and we shall see that the hardlimiter model permits a rather easy introduction of the more precise approximation, shown in Section VII-B, which better fits the details of the realistic $v - i$ characteristic.

It has to be noted that usually, because of not having a better theory, a linear (series $L' - R$, with a small inductance $L'$) equivalent circuit for the lamp is used (e.g., [4], [8]). The disadvantages of the linear model are, however, well recognized. This is seen, for instance, in the pessimistic discussion of a typical empirical connection between $u(t)$ and $v(t)$ in [4, pp. 310–312]. One notes that the linear model cannot even explain why we must use ballasts in the lamp circuits, i.e., why the lamps are not designed directly for the line voltage. At the same time, the linear model correctly notes the inductive features of the lamp associated with the hysteresis of the realistic $v - i$ characteristic. We shall consider these inductive features in Section VII-B.

In Sections II–V, we demonstrate the validity of the hardlimiter model by calculations of some practically important circuit parameters. In Section VI we consider a way in which to arrive at the topic of the lamp circuits in a standard electrical course, which is an important point for the teacher’s attention. This route arises from the basic theory of electrical transformers, showing the lamp as an appropriate load for an autotransformer having a leakage flux. Another way, given in Section VII-A, starts with a comparison of the resistive hysteresis of the realistic $v - i$ characteristic of the lamp with the magnetic and ferroelectric hystereses.

Section VIII considers the basic lamp’s features which are relevant for electronic design, and the topic of electronic ballasts is briefly discussed in Section IX. This consideration is intended to give one a theoretical basis and an interest in further study of the high-frequency operation of the lamps’ circuits. Consideration of the numerous electronic circuits would require a separate presentation in view of the general principles of power electronics, and is outside the scope of
the present work. However, the consideration of the lamp’s features at high frequencies leads us, in Section VIII, to an interesting point concerning the definition of the lamp as a circuit element. This point, as well as the Appendix, which gives a rigorous mathematical foundation for the theory should be appreciated by a professional circuit theoretician.

The calculational targets of Sections II–V are the following.

1) Estimate the maximal voltage on the series capacitor in the circuit shown in Fig. 1(b).
2) Calculate the amplitudes of the harmonic currents in the most common circuit shown in Fig. 1(a).
3) Calculate $L$ in the latter circuit.
4) Understand the difference in the features of the lamp circuits with the $L$ and $LC$ ballasts.

Let us briefly consider these four targets.

A straightforward calculation shows that the maximal value of the voltage on the series capacitor is directly defined by the power consumed by the lamp. The calculation also shows that the electrical charge which passes through the lamp during a half cycle, rather than the average power being consumed by the lamp, is the most fundamental parameter in the theory based on the hardlimiter model.

The high harmonic currents, which are completely lost in the above-mentioned linear model, are the most immediate manifestation of the nonlinearity of fluorescent lamp circuits in operational practice, and these currents are, generally, very unwanted in the electrical power grid; thus, one must know how to calculate them.

The choke’s inductance is the most immediate parameter to calculate for one who wishes to design the circuits. Despite the simplicity of the circuit shown in Fig. 1(a), it is difficult to find in the literature a theoretically justifiable procedure for the calculation of $L$.

The role of the ballast is, in general, very multilateral and the difference between the $L$ and $LC$ ballasts is far from being as simple as it would be if the lamp was to be a linear element. This difference is seen here, first of all, in the very different power behavior of the circuits shown in Fig. 1(a) and (b).

C. Main Notations and Agreements

$t$  Time.

$v(t)$  Electrical current function of the circuit.

$t_1$  A $+/-$ zero crossing of $v(t)$ belonging to a period. In Section III the time origin, $t = 0$, will be chosen at such a zero crossing of $v(t)$.

$v(t)$  The voltage function of the lamp.

$u(t)$  The input voltage function.

$\phi$  The phase shift of the line voltage with respect to the first harmonic of $v(t)$.

$U$  Amplitude of the line voltage.

$A$  The voltage parameter which characterizes the lamp in the hardlimiter model. In Section VII where a different model of the lamp’s $v(t) - \dot{u}(t)$ characteristic is accepted, we shall use the more general definition of $A$ as $v$ taken at $\dot{u}_{max}$.

$\mu = A/U$  This parameter is used only in the Appendix.

$A'$  The parameter which replaces $A$ in the corrected model in Section VII-B.\footnote{The symbols $U$, $A$, and $A'$ are measured in volts, not in volts rms.}

$n$  Integers, except for in the discussion of the transformer in Section VI.

$\dot{i}_n(t)$  Harmonic currents, except for Section VI where $\dot{i}_1(t)$ and $\dot{i}_2(t)$ are, respectively, the input and output currents of a transformer.

$L_n$  Amplitude of $\dot{i}_n(t)$.

$f$  Line frequency (in the numerical calculations, 50 Hz exclusively).

$T = 1/f$  The period of $u(t)$ and of all other time functions involved.

$e$  Fluorescent lamp.

$k$  Coupling factor of a transformer.

$L'$  Equivalent inductance of the fluorescent lamp.

$v_L$  The voltage on the ballast’s inductor.

$u_c$  The voltage on the capacitor.

$q(t)$  The electrical charge function of the capacitor.

$q(t)/2$  The charge which passes through the lamp and the ballast circuit during the time period $(0, T/2)$.

$P = T^{-1} \int_0^T v(t) \dot{u}(t) \, dt$  The average active power consumed by the lamp.

$K = d\ln P/d\ln U$  Power sensitivity.

$hf$  High frequency (both noun and adjective).

$\sim$  Direct proportionality.

For a sinusoidal $u(t)$, the functions $\dot{i}(t), v(t)$, and $q(t)$ are assumed from the very beginning, to possess the property of alternating symmetry $f(t + T/2) = -f(t)$, which, in particular, implies a zero time average, and to possess two zero crossings per period $T$, which is a realistic requirement for these circuits. Regarding $\dot{i}(t)$, we require it to have a zero time average in any case and, except in Section II, to be a zero crossing function, i.e., to be of alternating polarity, having
only isolated (single) zeros. The isolation (singularity) of a zero implies that the derivative of the function is nonzero at the zero point, although we may have to distinguish between the left and the right derivatives, which are both of the same sign. For the zero-crossing function there is no solution for the system of equations $\dot{i}(t) = 0$; $d\dot{v}/dt = 0$ and we can add a small nonzero function with a bounded derivative to the zero-crossing function so that the resulting function also will be zero-crossing, which means some stability of the zero-crossing waveform. The requirement of $\dot{i}(t)$ to be a zero-crossing function is necessary for transferring from the hardlimiter dependence $v = A \text{sign } \dot{i}$ to the dependence between the time functions $v(t) = A \text{sign } \dot{i}(t)$ which gives $v(t)$ as a rectangular wave.

II. The Maximal Voltage on the Series Capacitor

Finding the maximal voltage on the series capacitor [Fig. 1(b)] $C_v v_{\text{max}}$ is associated with obtaining a formula which connects this voltage in the theory based on the equality $v(t) = A \text{sign } \dot{i}(t)$ with the average active power $P$, which is actually consumed by the lamp, in a very simple manner:

$$P = 4fCAv_{\text{max}}$$

where $f$ is the frequency of the line. We can also write this as

$$v_{\text{max}} = (4fCA)^{-1}P.$$ (1a)

This formula is relevant for any linear ballast and for many, not necessarily sinusoidal, input voltage functions. This is seen from the following derivation.

In order to derive (1) we choose the time origin to be at one of the zeros of the current function $\dot{i}(t)$, which is assumed to possess (as does the line voltage) the property of alternating symmetry $\dot{i}(t+T/2) = -\dot{i}(t), T = 1/f$, and to be nonnegative within half the period which follows (modulo $T$) this zero. For the derivation of (1) we need not require that $\dot{i}(t)$ is a purely zero-crossing function, that is, $\dot{i}(t)$ may have pauses where it is identically zero, as a pulsed function. Denoting the half-period interval where $i \geq 0$ as $(0, T/2)$ we obtain

$$P = T^{-1} \int_0^T v(t)\dot{i}(t) \, dt = fA \int_0^{T/2} \dot{i}(t) \text{sign } \dot{i}(t) \, dt$$

$$= 2fA \int_0^{T/2} \dot{i}(t) \, dt.$$

Noting that the integral in the last expression in (2) is the electrical charge which passes through the lamp during the half period $(0, T/2)$, and denoting this charge as $q_{T/2}$, obtaining

$$P = 2fAq_{T/2}.$$ (2)

We observe then that

$$q_{T/2} = q_{\text{max}} - q_{\text{min}}$$

where $q_{\text{max}}$ and $q_{\text{min}}$ are, respectively, the maximum and the minimum value of the charge function $q(t)$ of the capacitor, because the capacitor is connected in series with the lamp and the minimal value of $q(t)$ is at the $-/+$ zero of $\dot{i}(t)$ (i.e., at $t = 0$), and the maximal value of $q(t)$ is at the $+/-$ zero of $\dot{i}(t)$ (i.e., at $t = T/2$), obviously.

Since, further,

$$q_{\text{min}} = -q_{\text{max}}$$

and since

$$q(t) = Cv_c(t)$$

we obtain

$$q_{T/2} = 2q_{\text{max}} = 2Cv_{\text{max}}.$$ (3)

Thus, (2) is turned into (1)

$$P = 4fACv_{\text{max}}.$$ (4)

III. The Harmonic Currents

According to the idealized $v/i$ characteristic and the assumed properties of $\dot{i}(t)$, the lamp’s voltage as a time function
is (see Fig. 3) a square wave which may be written (see e.g., [8]) using its harmonic components as

$$v(t) = A \text{ sign } i(t)$$

$$= 4A\pi^{-1} (\sin \omega t + 3^{-1} \sin 3\omega t + 5^{-1} \sin 5\omega t + \cdots)$$

$$= 4A\pi^{-1} \sum_{n=1}^{\infty} \sin n\omega t, \quad n = 1, 3, \ldots, \quad \omega = 2\pi/T. \tag{3}$$

This expression for \(v(t)\) is associated with the choice of the time origin at the \(-/+/\) zero-crossing of \(i(t)\). In the general case, denoting this zero-crossing as \(t_1\), we would have to write the voltage wave as \(4A\pi^{-1} \sum_{n=1}^{\infty} \sin n\omega(t-t_1)\), \(n = 1, 3, \ldots\) which, in terms of (3), means \(v(t = t_1)\).

After choosing the time origin at the zero-crossing of \(i(t)\), we have to introduce a phase shift in the expression for the sinusoidal input voltage function (the line voltage) \(u(t)\):

$$u(t) = U \sin (\omega t + \phi).$$

That \(\phi\) is unknown is not a difficulty in the following straightforward calculation. The point is that by keeping \(\phi\) in the resulting expressions as an unknown parameter, it is possible to easily find it, after finding an expression for the current function, from the equality \(i(0) = 0\) which directly follows from the choice of time origin at the zero-crossing of \(i(t)\).

Considering the circuit of Fig. 1(a) we find, for the inductor’s voltage function, \(v_L(t)\):

$$v_L(t) = L \frac{di}{dt} = u(t) - v(t)$$

$$= U \sin (\omega t + \phi) - 4A\pi^{-1} \sum_{n=1}^{\infty} \sin n\omega t$$

from which by integration

$$i(t) = -U(\omega L)^{-1} \cos (\omega t + \phi)$$

$$+ 4A(\pi \omega L)^{-1} \sum_{n=1}^{\infty} n^{-2} \cos n\omega t. \tag{4}$$

According to (4), the amplitudes of the high \((n = 3, 5, \cdots)\) harmonic currents are thus

$$I_3 = 4A(3\pi \omega L)^{-1}, \quad I_5 = 4A(5\pi \omega L)^{-1}, \cdots.$$

Regarding the realistic circuit, the expression for \(I_3\) is most precise because the idealization of the realistic \(v(t)\) to a square wave makes the fronts of the blocks of the waveform of \(v(t)\) extremely steep which, of course, more strongly influences the higher harmonics.

It is very well confirmed by an experiment in which a spectrum analyzer is used that, in the range of \(U\) where \(i(t)\) is a zero-crossing function, \(I_3\) is (contrary to \(I_1\)) independent of the amplitude of the input voltage.

In order to find \(I_1\) we have to find \(\phi\). From the condition \(i(0) = 0\) and (4)

$$\cos \phi = 4A(\pi U)^{-1} \sum_{n=1}^{\infty} n^{-2}$$

$$= 4A(\pi U)^{-1}(1 + 1/9 + 1/25 + \cdots)$$

$$= 4A(\pi U)^{-1}(\pi^2/8),$$

i.e.,

$$\cos \phi = (\pi/2) AU^{-1}. \tag{5}$$

From (4) the first harmonic current \(i_1(t)\) is

$$i_1(t) = (\omega L)^{-1} [(-U \cos (\omega t + \phi) + 4A\pi^{-1} \cos \omega t]$$

$$= (\omega L)^{-1} [(-U \cos \phi + 4A\pi^{-1}) \cos \omega t]$$

$$+ U \sin \phi \cdot \sin \omega t].$$

Using this and (5), we can find the amplitude \(I_1\) of \(i_1(t)\)

$$I_1 = (\omega L)^{-1} [(-U \cos \phi + 4A\pi^{-1})^2 + (U \sin \phi)^2]^{1/2}$$

$$= (\omega L)^{-1} [U^2 - (4\pi^{-1} - 10\pi^{-2})A^2]^{1/2}$$

$$\approx (\omega L)^{-1} [U^2 - 2.38 A^2]^{1/2}.$$

It is required here that \(U > (2.38)^{1/2} A \approx 1.54 A\). However, this condition is weaker than the condition \(U > \pi/2A \approx 1.57 A\) which is required by (5) and which is a necessary condition for \(i(t)\) to be a zero-crossing function.

For the ratio \(I_3/I_1\) we obtain

$$I_3/I_1 = 4A(9\pi \omega L)^{-1}(1 - 2.38(A/U)^2)^{-1/2}. \tag{6}$$

For the typical values of \(U = 220\sqrt{2} V \approx 311 V\) and \(A \approx 115 V\) we find this ratio to be close to 6.4%. This may seem to be a small value. However one has to see that the unwanted effects which are caused by high harmonic currents on input transformers (the eddy currents in the cores) depend on the derivative of the current which enhances the effect of the high frequency, and that the dependence of \(I_3/I_1\) on \(U\) given by (6) is rather strong, especially for \(U\) below the rated value. Since a 10–15% decrease in \(U\) with respect to the nominal value is often observed in developing areas, \(I_3/I_1\) may be significantly increased.

For an \(LC\) ballast the harmonic currents may be [10] stronger, and 9–10% value for this ratio is often obtained for standard \(LC\) ballasts at rated parameters.

High harmonic currents are an important problem in the fluorescent lamp circuits grid. More details about such currents in these circuits, and also for the topic of the currents in power systems in general are given in [10] and [11].
IV. THE CALCULATION OF $L$ IN THE $L-e$ CIRCUIT

According to (5), for the circuit with an $L$ ballast, the phase $\phi$ is independent of $L$. As is seen from (4), $L^{-1}$ is a scaling parameter for $i(t)$, which does not influence the waveform of $\tilde{i}(t)$, but influences the amplitude of the current and thus the lamp’s power.

We shall find $L$ to provide the nominal power of the lamp.

For the $L-e$ circuit shown in Fig. 1(a), starting from (2)

$$P = 2fA \int_0^{T/2} i(t) \, dt$$

and requiring that the lamp’s power be of its nominal value, we consider the series (4) for $\tilde{i}(t)$. Noting that for odd $n$

$$\int_0^{T/2} \cos n\omega t \, dt = 0, \quad n = 1, 3, 5, \ldots$$

and thus the sum in (4) will not contribute to the integral in (2), we have

$$P = -2fA U(\omega L)^{-1} \int_0^{T/2} \cos(\omega t + \phi) \, dt$$

$$= 2A U(\pi\omega L)^{-1} \sin \phi = 2A U(\pi\omega L)^{-1}(1 - \cos^2 \phi)^{1/2}.$$

Using (5), we obtain for $P$, finally

$$P = 2A U(\pi\omega L)^{-1}[1 - (\pi/2)^2(A/U)^2]^{1/2}$$

and resolving for $L$

$$L = 2A U(\pi\omega P)^{-1}[1 - (\pi/2)^2(A/U)^2]^{1/2}.$$

For $P = 40$ W, $U = 220\sqrt{2}$ V, $A = 115$ V we obtain $L \approx 1.47$ H which is slightly higher than the average typical (1.2–1.3 H) $L$ for the chokes produced by different firms for such a lamp. This difference is explained by the inductivity of the lamp, which is ignored by the hardlimiter model.


Why is there the need for the series power capacitor if an $L$ ballast can properly limit the current flowing through the lamp?

There are several reasons for the use of the capacitor.

First, such a capacitor shifts the phase of the first harmonic current and, thus, if we simultaneously operate lamps with $L$ and $LC$ ballasts in the same room, then the total light created by the fixtures together will not be stroboscopic. This avoids the stroboscopic effect of seeing a rotating part as stationary in the stroboscopic light at certain speeds, which is very dangerous for users of machines with open rotating parts. Another immediate reason is that such a capacitor makes very strong hysteresis of its $v-i$ characteristic, the topic of the sensitivity of $P$ to variations in $U$ becomes so crucial that such lamps can be operated only with $LC$ ballasts. Another nontrivial reason is that by using $L$ and $LC$ ballasts we can try to reduce the total high harmonic currents in the local electrical grid.

There are different requirements which may affect the choice of the capacitor. Among the useful points for practice there is the possibility (see [7] for the proof) of adding a certain capacitor in series with the standard choke, so that the power consumption of the lamp will not be changed. Another requirement follows from the fact that the rectangular-wave lamp’s voltage includes high odd harmonics. The resonant frequency of the $LC$ ballast should not be close to a high frequency of the nonsinusoidal current in order not to enhance this frequency. In fact, only a circuit which includes many $L$ and $C$ elements allows us satisfy all the requirements of the ballast and, if the use of such a circuit were not associated with additional power losses, cost, space and weight, one could see such a complicated ballast as desirable.

For choice of $L$ and $C$ in the practical circuit, one can state the following requirements (see the calculation examples in [7]), two of which are inequalities.

1) The power consumed by the lamp to be nominal;
2) The logarithmic sensitivity of $P$ on $U$ (below the power sensitivity) defined as

$$K = \frac{d \ln P}{d \ln U} = \frac{dP/P}{dU/U}$$

to be less than the value 2, which would be the value of $K$ if the lamp were to have a linear $v-i$ characteristic, when it would be $P \sim U^2$. (We shall see that for the nonlinear circuit, $K$ may be much higher or much lower than 2: a very high $K$ is unacceptable.)

3) The total rms value of the high harmonic currents to be less than 11–12% (the usual standard requirement [10]) of the total rms. value of the current.

The sensitivity $K$ is the most important operational characteristic of the nonlinear power system and, as a matter of fact, the study of the nonlinear circuit which is presented here began from the study of the unusual sensitivity of the power to changes in $U$ near the nominal value of $U$.

A. The Power Sensitivity Function $K(U)$

Since the logarithmic-type dependence of the function $K = d \ln P/d \ln U = K(U)$ on $U$ is significantly weaker than the direct dependence $P = P(U)$, the graphs of $K(U)$ related to different (by kinds of lamp and ballast) circuits are more readily understood than the steeply increasing corresponding graphs of $P(U)$ and, as shown in detail in [1], [2], and [7], the different roles of the $L$ and $LC$ ballasts in the circuit’s power features are well seen through the forms of the associated...
This makes the function $K(U)$ a suitable analytical tool.

To demonstrate the features of $K(U)$ we use the function $P(U)$ obtained for an $L$ ballast in Section IV

$$P(U) = 2AU(\pi\omega L)^{-1}(1 - \cos^2 \psi)^{1/2}$$

$$= 2AU(\pi\omega L)^{-1}[1 - (\pi/2)^2(A/U)^2]^{1/2}$$

writing $P(U)$ as

$$P(U) = AU\psi(A/U)$$

using the function

$$\psi = 2(\pi\omega L)^{-1}(1 - \cos^2 \psi)^{1/2}$$

$$= 2(\pi\omega L)^{-1}[1 - (\pi/2)^2(A/U)^2]^{1/2}$$

whose dependence on $U$ is obtained only through its dependence on the phase $\phi$.

From these formulae

$$\ln P = \ln A + \ln U + \ln \psi(A/U)$$

and

$$K = 1 + d\ln \psi(A/U)/d\ln U.$$  

It is not difficult to see [1] that for any type of ballast, for the hardlimiter model $d\ln \psi/d\ln U \geq 0$, and that for any ballast $d\ln \psi/d\ln U$ monotonically decreases, tending finally to zero with an increase in $U$ to infinity. That is, in the hardlimiter model $K \geq 1$ for any ballast, and $K \to 1$ as $U \to \infty$.

The minimal value $K = 1$ means that $P \sim AU \sim U$ or that $\psi$, and thus the phase $\phi$, are independent of $U$.

For a certain $LC$ ballast, and never for an $L$ ballast, we can obtain [7] $K = 1$ also for finite practical values of $U$, which is very important for stability of the light output.

Another feature of the case of $K = 1$ is that since in this case $P \sim AU \sim A$, we can add another lamp in series (which means $A \to 2A$) to the circuit with the specific $LC$ ballast, while not changing the input voltage, without changing the power consumption of the already connected lamp, and thus having the nominal power for both lamps (since $A \to 2A$ means $P \to 2P$), which is an interesting point. The transfer $A \to 2A$ is limited by the constraints on the ratio $A/U$ stated in the above, but we can (see the next section) correspondingly raise $U$, using a transformer instead of the choke.

It is also important to note from the expressions for $L$ ballast

$$\psi = 2(\pi\omega L)^{-1}[1 - (\pi/2)^2(A/U)^2]^{1/2},$$

$$P = AU\psi$$

that for small $U$, when the ratio $A/U$ is increased and is close to the critical value $4/\pi^2$ when $\psi$ and $P$ become small, the power sensitivity becomes very large according to the relation

$$K = 1 + d\ln \psi(A/U)/d\ln U \sim d\ln \psi(A/U)/d\ln U \sim 1/\psi,$$

which is unacceptable.

The expressions and conclusions obtained are instructive also in the case of a significant hysteresis of the $v - i$ characteristic of the lamp where we use (see Section VII-B) an improved $v(t) - i(t)$ model. The hysteresis increases the factor before the hardlimiter term, which leads to an increase in $A/U$ and, according to the equations related to the $L$-ballast lamp circuit, this leads to an increase in $K$. The increase in the power sensitivity is so significant that the $L$ ballast is unacceptable for lamps with strong hysteresis, even for the nominal line voltage. However, for an $LC$ ballast, using the improved model of Section VII-B we can obtain, as will be explained, $K$ even smaller than 1, which yields a very stable light output of the lamp. We see that in the nonlinear circuit, the influences of the hysteresis on $K$ in the cases of the $L$ and $LC$ ballasts are opposite.

Table I shows some typical measured values of $K$.

In general, we see from Table I that the power features of the lamp circuits are very far from the power features of linear circuits where $K$ always equals 2.

**VI. THE FLUORESCENT LAMP FED FROM A LOW-VOLTAGE LINE AND THE TRANSFORMER WITH A MAGNETIC SHUNT**

According to the common technology of lamp production, the lamp has to be operated via a series ballast from a 220-V rms line and the fluorescent lamp is an important practical example of a load for which a transformer is needed when there is a lower line voltage. The 110 V rms. U.S.- or Canada-rated line voltage is the most important case. As a point arising from the theory of the transformers [9], [13], [14] there is here an application of a transformer whose core has a magnetic shunt. Using a transformer with a magnetic shunt we can realize $L$ as the leakage inductance and, thus, a transformer can simultaneously perform both the required amplifying and buffering functions, which is very suitable from the production point of view.

The standard (e.g., [12]) equations of a linear transformer which define the input and output voltages $v_1(2)(t)$ by means of the input and output currents $i_1(2)$ are

$$v_1(t) = L_1 di_1/dt + M di_2/dt$$

$$v_2(t) = M di_1/dt + L_2 di_2/dt$$

As an example, the circuit diagram shows a $L$-ballast circuit fed from a 220-V line and a transformer with a magnetic shunt. The transformer has a leakage inductance $L_1$ and a magnetizing inductance $L_2$. The input current $i_1(t)$ flows through the primary winding of the transformer, and the output current $i_2(t)$ flows through the secondary winding. The input voltage $v_1(t)$ is applied to the primary winding, and the output voltage $v_2(t)$ is applied to the secondary winding.
where $L_1$ and $L_2$ are the self inductances of the windings and $M$ is their mutual inductance. As is well known [9], [12], $M^2 \leq L_1 L_2$. The physical essence of the difference between the features of the so-called perfect transformer, i.e., the transformer with the maximal magnetic coupling between the input and output windings when $M^2 = L_1 L_2$, and the case of the imperfect transformer when $M^2 < L_1 L_2$ is some leakage flux which may be the flux in the magnetic shunt of the core.

The simplest configuration of a transformer with a magnetic shunt is shown in Fig. 4(a). This shunt is meant here to be specifically designed in order to provide a certain required magnetic coupling $M$ between the input and output windings. The reason why we wish the coupling factor $M$ to be smaller than 1 will be immediately clear if we substitute in (8) $\mu r_1$ found from (7), obtaining

$$v_2(t) = (M/L_1) v_1(t) + (L_2 - M^2/L_1) \frac{d}{dt} v_2(t)$$

(we assume the direction of the windings to be such that $M > 0$), which may be also written as

$$v_2(t) = k \mu r_1(t) + (1 - k^2) L_2 \frac{d}{dt} v_2(t)$$

where we introduced the notation $n = (L_2/L_1)^{1/2}$. For the perfect transformer, from (9) $v_2(t) = \mu r_1(t)$ and, in this case $v_2(t)$, copies the form of $v_1(t)$, regardless of whether or not $v_2(t)$ is zero, i.e., regardless of whether or not the transformer is loaded.

The property $v_2 \sim v_1$ of the perfect transformer is clearly unwanted in the case when $v_1$ needs to be the line voltage and $v_2$ needs to be the fluorescent lamp’s voltage.

For the imperfect transformer we note that (9) is the Kirchhoff’s voltage law equation of the circuit shown in Fig. 4(b). There is an input perfect transformer whose output voltage is $k \mu r_1(t)$ and the inductor of the leakage inductance of $(1 - k^2) L_2$ in this circuit. Now $v_2(t)$ copies the form of $v_1(t)$ only if there is no load.

The property of the transformer with the magnetic shunt to both raise the voltage and perform voltage buffering is precisely the situation we need when we use, on the 110-V rms line, a fluorescent lamp designed for a 220-V rms line.

It is important to find how the actual leakage flux is connected with the parameter $L$. For the 110-V rms line voltage, which is $v_2(t)$ in (9), we can require that $k \mu r_1 = 2$, i.e., $k^2 L_2 = 4 L_1$ which gives the rated 220 V rms before the leakage inductance and $(1 - k^2) L_2$ to equal the known value $L$ of the standard buffering inductance used for the 220-V rms line. This yields the equation $L_2 = 4 L_1 = L$. Among others, there is the usual requirement that the input current of the transformer not be too low, i.e., the input impedance of the whole circuit must not be too low, which requires $L_1$ not to be too small. Taking for the estimation, $L_1 = 2 L$, we obtain $L_2/L_1 = 4.5$. This gives $k = (8/9)^{1/2} \approx 0.94$.

According to this high value of $k$, the absolutely necessary flux in the magnetic shunt is much weaker than the flux which is common to the windings, and thus the shunt must have a significant reluctance, i.e., be relatively thin or, rather, to have an air gap. The latter is usually preferred for mass production.

In the United States, in addition to the magnetic shunt in the core, such a transformer is also constructed as an autotransformer [13], [14], thus economizing the size of the core and the winding. While having a good technological performance, such leakage reactance autotransformers, intended to feed the lamps from the 110-V rms line, are sometimes even smaller (without increasing the power losses) than the simple chokes for the 220-V rms circuits which include the same lamps.

VII. HYSTERESIS OF THE REALISTIC VOLTAGE-CURRENT CHARACTERISTIC OF THE LAMP AND THE IMPROVED MODEL FOR THE REALISTIC $v(t) - i(t)$ CONNECTION

A. Resistive Hysteresis Compared to Magnetic and Ferroelectric Hysteresis

It is possible and is essentially justified to study the topic of the hysteresis of the $v - i$ characteristic, together with the topics of magnetic [9], [13], [14], and ferroelectric [15]–[17] hystereses, at the very beginning of a basic course when the basic physical features of electrical elements are being studied and the topic of nonlinearity of the elements’ characteristics is introduced.

The three hysteresis loops are schematically shown in Fig. 5(a)–(c). In Fig. 5(b) and (c), the variables of integration in the energetic consideration which is usually applied to the topic of hysteresis were chosen as the abscissas variables. For the capacitor

$$\int v \, dt = \int \frac{v}{dq}$$

for the energy-losses integral, taken over the hysteresis loop,
and for the inductor (using the Faraday and Ampere laws)
\[
\int v i \, dt \approx \int (dB/dt)H \, dt = \int H \, dB
\]
with the magnetic field \( H \) and the induction \( B \).

When \( B \) is the abscissa, the direction of passing through the magnetic hysteresis loop is clockwise, and is the same as the direction of passing through the hysteresis loop in the \( v - q \) plane, shown in Fig. 5(c). This is required by the positivity of the energy-losses integral.

Comparing the three hysteresis loops, we first note their specific duality. While in the case of the magnetic or the ferroelectric material [Fig. 5(b) and (c)] the hysteresis indicates some power losses in the mainly reactive energy-storing passive element, occurring during the time-dependent process, i.e., some resistive features of the element, in the case of the lamp which is basically a resistive element, the hysteresis of the \( v - i \) characteristic [Fig. 5(a)] indicates some reactive features of the element.

While from the very fact that an inductor with a magnetic core or a ferroelectric capacitor is a passive element it follows that the loop is travelled in the clockwise direction, when we do not know \textit{a priori} the physical processes in the resistive nonlinear element we must permit, in principle, either direction of passage through the hysteresis loop. Energy conservation by itself cannot prescribe whether the energy-dissipating element will have some inductive or some capacitive features. The clockwise passing direction of the loop for the lamp at regular line frequencies indicates some inductive features of the lamp, while the counterclockwise direction (observed for some more complicated forms of the loop at higher frequencies, see Fig. 6) would mean that the lamp has some capacitive features. Indeed, the maximum of the voltage function \( v(t) \) in the first case [the usual, see Fig. 3(a)] is obtained before the maximum of the current function \( i(t) \) and, in the second case, after the maximum of the current function.

For the low-frequency case where the lamp has some inductive features we have to add an inductance to the equivalent circuit of the lamp and to consider the stored electrical energy associated with this inductance.

A very interesting point is that these inductive features of the lamp and the associated energy are not (or almost completely not) associated with a magnetic flux, which would be usual, but with separation of the positive and negative charges (in particular with the ambipolar diffusion \cite{4}) in the lamp’s gas, which causes some delay in the current with respect to the voltage: a typical feature of an inductor. \textit{Thus, the energy associated with the inductor is of an electrostatic nature.}

Being quantitatively influenced by the mass of the gas in which the discharge occurs, the hysteresis and the associated inductive feature are always more strongly exhibited in 36-mm diameter (T-12 tube) lamps than in the 28-mm diameter (T-8 tube) lamps, and they are really very strong in the long-tube (240-cm; 100- to 125-W) T-12 lamps.

**B. The Hysteresis and the Improved Model for the Realistic \( v(t) - i(t) \) Connection at Regular Frequencies**

Returning to the equational side, considering the noted inductive features of the lamp in terms of the time functions \( v(t) \) and \( i(t) \), we see that an approximation for the \( v(t) - i(t) \) connection which would be better than \( v(t) = A \, \text{sign} \, i(t) \) is

\[
v(t) = A' \, \text{sign} \, i(t) + L' \, di/dt
\]
where $L'$ is the equivalent lamp’s inductance and $A'$ is a constant.

A delicate point is that we cannot associate $A'$ with only the parameter $A$ which was used in the model $i(t) = \frac{A}{L} e^{-\frac{i(t)}{L}}$. Also, in order to come to the correct $A'$ we have to reconsider the definition of $A$. It is, at any rate, natural to do this since in the new model $v$ is not constant when $|i|$ is increased.

We now define $A$ as the value of $v$ taken at $t_{\text{max}}$. [See Fig. 2(a), considering that actually the 115-V empirical estimation of $A$ in the above was found as $\frac{v(i_{\text{max}})}{L}$ from such a realistic hysteresis curve.] Then, dealing with any experimental $v-i$ curve, we still can uniquely determine $A$, as related to this range, as an easily observable parameter.

Since the new definition of $A$ obviously includes the previous definition as a particular case of an infinitely thin hysteresis [see Fig. 2(c)] with the completely saturated $v(i)$, using the new definition we can speak about $A$ in both the hardlimiter and the hysteresis cases, stressing, however, that the parameter $A'$ which appears in (10) is not $A = v(i_{\text{max}})$.

The new parameter $A'$ which is not directly seen on the $v-i$ characteristic is, however, simply analytically connected with the redefined $A$. It may be shown [2] that the relative difference between $A'$ and $A$

$$
\frac{(A' - A)}{A} \approx \frac{L'}{L}
$$

with the ballast inductance $L$ included, and that we can estimate more precisely

$$
A' \approx (1 + 2L'/L)A = (1 + 2L'/L)v(i_{\text{max}}),
$$

(11)

Since $L'$ and $L$ are positive, $A' > A$, i.e., the hysteresis causes the factor before sign $\frac{v(i)}{L}$ to be larger than $v(i_{\text{max}})$. For a long lamp $L'/L$ may even be as large as 0.3.

C. Dependence of $L'$ on $U$

Because of the physical processes in the lamp, the lamp’s inductance $L'$, and $A$, and thus $A'$, may decrease with an increase in the effective value of the current, or in the input voltage amplitude $U$ (see the experimental curves in [2]) and for many of the common lamps this decrease is rather significant. The decrease in $A'$ with the increase in $U$ contributes to a weaker dependence of the lamp’s power $P$ on $U$, and having $dA'/dU$, and thus $\frac{d}{dU}A'/dU$, negative we can obtain, for an $LC$ ballast, $K < 1$, the power and the light of the fluorescent lamps with such $LC$ ballasts are very stable.

The decrease in $L'$ with the increase in $U$, finally, explains the connection considered in the often-quoted book [4] between the lamps voltage and the effective current in the lamp. In the context of considering the problematicity of the linear model of the fluorescent lamp, this connection is written (i.e., there means the effective current value) in [4, p. 310] as

$$
v = f(i, \frac{i^{-1}}{dt} dt)
$$

with a function $f$ which is said to be unknown. The dependence of $f$ on the first of the arguments is given in (10) by the signum term, and the factor $i^{-1}$ in the second argument means, more or less, the realistic dependence $L' \sim U^{-1}$.

For many lamps, a good model for the $v(t) - i(t)$ connection is

$$
v(t) = A' \frac{\text{sign}(i(t))}{L'} \frac{di}{dt} - R' i(t)
$$

where $R' > 0$ is some small resistive parameter. When ignoring the term $L' \frac{di}{dt}$ associated with the hysteresis, we perform some averaging of $v(t) - i(t)$ dependence, and the term $R' \frac{di}{dt}$ describes some negative slope of the averaged $v-i$ characteristic. Clearly decreasing $dP/di_{\text{max}} \sim dP/dU'$, the negative slope also contributes to stabilizing the power. However, for some common types of lamps the term $-R' i(t)$ is very small.

VIII. THE LAMP’S $v-i$ CHARACTERISTIC AT HIGH FREQUENCIES

Up to now we have dealt with the lamp’s $v-i$ characteristic at regular line frequencies, and with the problems associated with the singular nonlinearity of this characteristic at these frequencies. Fig. 6 shows, schematically, the lamp’s characteristic at an intermediate frequency of about 400 Hz. At higher frequencies, up to hundreds of kilohertz, this characteristic is changed even more—with the increase in frequency it becomes close to that of a linear resistor (with the addition, on average, of a weak positive cubic term to a straight line), but with the resistance (i.e., the slope of $v(i)$) strongly decreasing with increased intensity (amplitude, r.m.s. value) of the current. The decrease in the resistance is associated with a specific amplitude dependence of $v_{\text{max}}$ on $i_{\text{max}}$ which is observed [19] to be very similar to the strongly saturated dependence of $v$ on $i$ at the regular low frequencies, which we have studied.

A. The Hardlimiter Model Applied to the Amplitude Dependence $v_{\text{max}}(i_{\text{max}})$

Following [19] we ignore (Fig. 7) the cubic nonlinearity and the hysteresis of the hf $v-i$ characteristic. Regarding the hysteresis, we only note that the narrow hysteresis loop in Fig. 7(a) is not unidirectional, i.e., the lamp demonstrates both inductive and capacitive features at the high frequencies.

In addition to ignoring the hysteresis and the nonlinearity of $v(i)$, we limit ourselves here to only the hardlimiter idealization of the amplitude dependence $v_{\text{max}}(i_{\text{max}})$, taking $v_{\text{max}} \sim A$, and $v_{\text{min}} \sim -A$, as shown in Fig. 7(b).

Regarding the latter approximation, it should be noted that in [19] a description of the $v_{\text{max}} - i_{\text{max}}$ line is given, taking into account a negative slope of the line. In [20] such a slope is considered with regard to the stability of operation of a relevant electronic circuit. The negative slope is relevant to the power sensitivity and, even for the low frequency supply, we have an immediate topic for a stability analysis, i.e., to try to explain the ripple on the $v-i$ characteristic clearly seen in Fig. 2(b). Keeping to the line of the analysis of the power consumption of the lamp, which is usually quite stable, we shall not enter into such a stability analysis. This is the more justified by the fact that as we assume in the hardlimiter idealization, for some methods of supplying the hf, and (or) for some types of lamps, $v_{\text{max}}$ appears to be independent in a wide interval, of $i_{\text{max}}$ without the negative slope, as is.
B. The Inertial Nature of the HF Nonlinearity and the Causality Principle

In spite of using for \( i_{\text{max}} \) fixed a linear model of the characteristic \( v(i) \), the property that the linear resistance changes with change in \( i_{\text{max}} \) presents, of course, a nonlinearity. That this specific (amplitude) nonlinearity is realized only at sufficiently high frequencies is worth considering. The elementary argument of causality reveals that the nonlinearity is associated with inertia of the physical processes in the lamp.

Indeed, because of causality, the rate of increase in \( v(t) \) which would be obtained with a very slow increase in \( i(t) \) cannot be defined by the yet to be obtained value of \( i_{\text{max}} \) [as (12) requires]. Causality requires that the dependence of \( v_{\text{max}} \) on \( i_{\text{max}} \) can be obtained only for frequencies which are higher than the inverse characteristic period of the charge relaxation processes in the lamp, so that the lamp always “remembers,” via the excited atoms, the maximal value of the current from the previous cycle. This means that (12), which ignores any hysteresis, implies memory, or inertia, without, however, any macroscopic reactive feature of the element, which would be observed (as in the low-frequency case) in the characteristic \( v(i) \).

In general, such a nonlinearity belongs to the class of inertial nonlinearities to which the (seemingly) linear resistor whose resistance is strongly dependent on temperature and, thus, (for any fixed cooling conditions) on the effective value of the current passing relates. However, the relevant time scale in the case of the lamp is very different from that of the temperature-dependent resistor, since charge relaxation is a much quicker process than heating.

The nonlinearity of the lamp in the hf range can be clearly expressed in terms of power sensitivity.

For a sinusoidal current of a certain amplitude, and speaking about the linear resistor, we have \( P = v_{\text{max}} i_{\text{max}} / 2 = A i_{\text{max}}^2 / 2 \). Variating (slowly changing) \( i_{\text{max}} \) gives \( P \sim i_{\text{max}}^2 \) and the sensitivity \( d\ln P / d\ln i_{\text{max}} = 1 \) which is not the scaling property of a truly linear resistor. Note that \( i_{\text{max}} \) is directly proportional to the amplitude of the input voltage of the circuit.

Despite the strong difference between the lamp’s low and high frequency behavior, expression (1) for \( P \) derived in the low-frequency case is changed very little in the hf case. Indeed, if there is a capacitor connected in series with the lamp, then for the sinusoidal current we obtain \( i_{\text{max}} = \omega C u_{\text{max}} \) and thus \( P = (1/2) A \omega C u_{\text{max}}^2 \) or

\[
\frac{P}{A}=\pi f C \omega u_{\text{max}}^2 \tag{13}
\]

The only difference regarding (1) is the numerical factor.

C. On the Definition of the Lamp As a Circuit Element in the HF Range

That the fluorescent lamp is, first of all, a resistor is a requirement directly following from the energy conservation law. Since we wish to obtain a significant energetic light output from the lamp, the lamp must be (at any supply frequency) an element which consumes, on average, significant electrical power, which is the main physical feature of the resistor.
On the other hand, by the circuit-theoretic definition (e.g., [12]), a resistor is an element whose voltage is uniquely defined by the instantaneous value of its current, and no history of the process or memory of the element are required to know the voltage.

In the low-frequency case, knowing that we are moving on the hysteresis curve in the clockwise direction, and assuming the hysteresis to be not very significant, we can say that the lamp is a resistor with a weak inductive feature.

The high-frequency case is more difficult from the definitional point of view. We saw in Section VIII-B that even if we completely ignore the hysteresis, i.e., any macroscopic reactive feature, the analytical role of $i_{\text{max}}$ is associated with some memory without which we cannot know the slope of the line $v(i)$. When not knowing $a$ priori, without the memory, $i_{\text{max}}$ and the slope of $v(i)$, we have, for any certain given $i < i_{\text{max}}$, a continuum of possible values of the $v$ to choose from. Thus, with regard to the classical definition of a resistor, the lamp at the high frequencies is, as it were, even less a resistor than at low frequencies when we have, ignoring the memory, only two possible points on the hysteresis curve to choose.

We see that the classical circuit-theoretic definition of a resistor has to be reconsidered for such a type of element, and that the feature of inertia of the charge relaxation process, or the associated feature of memory, must be introduced in the very definition of the lamp as a circuit element in the hf range. Considering again that there is such an inertial element as a thermistor also dependent on amplitude-type parameters, we can suggest for the lamp in the hf range the title of an hf-inertial current-controllable resistor. This concept also can be relevant, of course, to some other elements whose $v - i$ characteristic possesses some specific inertial features at the hf.

Because of the inertia of the internal processes, it follows that contrary to the low-frequency case, the light emitted by the lamp having the hf supply is very weakly modulated by the instantaneous power, and no stroboscopic effect is obtainable using this light.

IX. ELECTRONIC BALLASTS

Fig. 8 presents a block schematic of the lamp with an electronic ballast which provides the hf feed. The ballast is, essentially, a frequency convertor and there are some $L$ and $C$ elements which adjust, usually by forming a resonant circuit, the convertor's output to the lamp. There are many different topologies. For the design details of electronic ballasts see [19]–[21] and the references therein and general courses on power electronics.

There are in use several industrial types of electronic ballasts which work in the range of 10's or 100's of kHz (e.g., [22], [23]). According to, e.g., [22], the severe problem of input harmonic currents also exists for the hf voltage which is supplied via adjusting elements to the lamp. Nevertheless, the topic of the hf supply of the lamps today attracts much attention from power specialists. This is because of the general importance of the lamp circuits and because there is an easily realizable nonlinear load for power frequency convertors, which is helpful for the laboratory study and development of the convertors.

X. CONCLUSIONS

A discussion of the basic features of the practically very important fluorescent lamp circuits was presented. The strongly nonlinear hardlimiter approximation to a realistic characteristic of the lamp in an electronic circuit. These are, mainly, the low reliability of the lamp (expressed in flickering, etc.) and the possibly rather significant changes in its main parameters during its lifetime, which are especially relevant when the lamp is operated in difficult weather or safety conditions, e.g., in an agricultural environment. These bad features of the lamp as a circuit element present difficulties which, even in power electronics, are not typical for electronic design and lead finally to a relatively complicated electronic device which is much more expensive than regular passive ballasts. Thus, although electronic ballasts have been known since approximately 1960, except for common emergency light devices, small fluorescent lamp lanterns, and small fluorescent lamps with incandescent-type sockets which cause a very strong line noise, they are still unusual, especially outside of the United States.
of an element is the simplest one, and is much more adequate than the often used linear model. This approximation quickly leads to a Fourier representation of all of the functions involved which is a very good and very rare situation in nonlinear problems. The hardlimiter model has been completed in order to include the lamp’s inductivity. For one who wishes to enquire more deeply into the theoretical side of the research, the Appendix is given and works [1]–[3] are recommended. Some additional calculations are given in [7] and [10].

Among the elementarily formulated new concepts, there is the power sensitivity considered in Section V. The characteristic $K(U)$ makes the clear power features of the lamp circuits in both the low- and high-frequency ranges, and from the point of view of the power sensitivity, the hf and lf range nonlinearities are equally strong, despite the fact that for a fixed input amplitude, the lamp at hf is almost a linear resistor. The characteristic $K(U)$ should be considered as a means of detailed classification of the hf circuits and should also be applied to the analysis of other nonlinear power loads.

Knowing how to calculate the lamp circuits at regular line frequencies enables one to optimize (concerning the total harmonic currents, light stability, etc.) a local fluorescent lamp grid using different types of ballasts. Additionally, several nontrivial points for electronic design may be found, e.g., the idea of measuring the lamp’s power by sampling the maximal voltage value on the series capacitor (see Sections II and VIII-B), or the suggestion [5] to feed the lamp from an electronic supply whose supply frequency is in turn low and high. As is argued in [5] this may influence the light-output efficiency of the lamp. Finding a good equivalent electrical circuit model of the lamp, which would be relevant to both the regular and the hf ranges, is an interesting target.

For an introduction to the field of nonlinear circuits in general, the reader is referred to, e.g., [12], [24]–[26]. Among classical works on nonlinear circuits are [27]–[29]. These works, however deal with monotonous nonlinear characteristics which make a strong difference compared to the present case. Work [30] deals with the types of singularities in differential equations which are of some relevance here. The topic of circuits with elements possessing hysteresis is treated, e.g., in [31]–[35] and the references therein. Zerocrossings of time functions, which are important (see the Appendix) for the rigorous theory of the circuits, are used also in signal processing (e.g., [36]–[38]) and appear in methods of optimal control [39]. Minimization of the power sensitivity is considered in connection with variational principles of dynamics in [26, App.].

# Appendix

## Foundation and Generalization of the Hardlimiter Model

### A.1. The Mathematical Approach

In order to justify the calculations of Sections II–V, based on the model $v(t) = A\text{sign } i(t)$, we have to consider the conditions for the lamp’s current $i(t)$ to be, for the periodic input, a uniquely definable periodic zerocrossing function having a known density of zerocrossings, and in the case when the input function possesses a symmetry of the type $f(t + T/2) = -f(t)$ to also possess this symmetry. The symmetry case will follow from the uniqueness of $i(t)$ since, in this case, the Fourier series of $i(t)$ which satisfies the circuit equation includes only terms of odd frequencies, possessing this symmetry.

In order to better see the mathematical essence of the problem we shall generalize the system and the input function. The ballast may be seen now as a complicated $LC$ (or $LCR$ with a small resistance) one-port, or even a two-port, with one of the ports connected to the line and the other loaded by the lamp. Using the latter scheme [1] does not cause any difficulty in what follows. The ballast must be asymptotically inductive as the frequency tends to infinity. The input voltage function $u(t)$ is given now as

$$u(t) = U\xi(t)$$

with the scaling parameter $U$ and a $T$-periodic physically nondimensional function $\xi(t)$ normalized in any desirable way which defines the waveform of $u(t)$. The function $\xi(t)$ is required to provide, together with the ballast circuit, that if the lamp is short-circuited ($A = 0$), then the current passing via the ballast is a zerocrossing function. (For the ballast considered as a two-port, we have to speak about the current flowing through a short-circuited output port.).

This particular current function obtained at $A = 0$ will be denoted as $i_0(t)$. Since the ballast is a linear circuit, $i_0(t)$ includes only the frequencies which are present in $u(t)$, is $T$-periodic, and $\max\{i_0(t)\} \sim U$.

In the case when the lamp is connected ($A > 0$), function $i_0(t)$ is one of the two terms of the total current $i(t)$, which is because of the superposition of the linear ballast’s electrical current responses to the input and the lamp’s voltages. In this case $i_0(t)$ plays a directive role. Correctly limiting $A/U$, we can ensure that the zero-crossing features of $i(t)$ are, qualitatively, those of $i_0(t)$.

Thus, we are presenting the input–output mapping $u(t) \rightarrow i(t)$ realized by the circuit as two sequential mappings $u(t) \rightarrow i_0(t)$ and $i_0(t) \rightarrow i(t)$. The problematic mapping is the latter one.

#### A.2. $i(t)$ As a Function of Its Zerocrossings

Consider first $i(t)$ in the simplest case of an $L$-ballast and a sinusoidal input given by (4) in Section III, obtained due to the superposition property of the ballast’s response, written now for the time origin chosen to be at the zero of the input sinusoidal voltage

$$i(t) = -U(\omega L)^{-1} \cos \omega t + 4A(\pi \omega L)^{-1} \sum_{n=1}^{\infty} n^{-2} \cos n \omega (t - t_1) = i_0(t)$$

The parameter $t_1$ is connected with the phase $\phi$ in Section III as $\omega t_1 = \phi$. This is a jump point of $v(t)$ and a zerocrossing of $i(t)$.

The first term in the right-hand side of (A1) remaining at $A = 0$ is $i_0(t)$. 


We can see that the waveform of $i(t)$ (and thus its zeros) is given here by the function (we divide (A1) by $U_0^2$) 
\[-\cos \omega t + (4/\pi)(A/U) \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\omega (t-t_1),\]
\[n = 1, 3, \ldots.\]

Clearly, $\frac{A}{U} \to 0$ means that the waveform of $i(t)$ tends to that of $i_0(t)$, and, in general, the physically nondimensional parameter $\mu = A/U$ plays a crucial role in the properties of the mapping $i_0(t) \to i(t)$.

Since the coefficients of the infinite series in (A1) which "spoils," for $A > 0$, the waveform given by $i_0(t)$ are of order $1/\mu^2$, the term with $A$ represents a continuous function whose derivative has finite jumps.

Up to now the choices of the time origin either at a zero of $u(t)$, as in (A1), or at a jump point of $u(t)$, as in (4), were equally good, i.e., the uses of $\phi$ or $t_1$ were equally good. However, in the case of a complicated $u(t)$ which causes a larger number of the zero crossings of $i(t)$ period, it is more suitable to choose the time origin at a zero of $u(t)$ since then all the zero crossings of $i(t)$ (which are meant to be taken modulo $T$ belonging to the same period) $t_p, p = 1, 2, \ldots, m$, appear analytically similar in the mathematical expressions.

With this observation we transfer, for a more complicated $u(t)$ and the correspondingly more complicated $i(t)$, from (A1) to

\[i(t) = i_0(t) - A F(t + t_1, t_2, \ldots, t_m) \quad \text{(A2)}\]

with a function $F$ which is an explicit function of $t$ and of the zero crossings of $i(t)$. More precisely, $F$ is a function of the differences $t - t_1, t - t_2, \ldots, t - t_m$. The case of (A1) is the simplest case of (A2) corresponding to $m = 2$ since the square wave can be presented as a sum of two differently oriented sawtooth waves, one with the argument $t - t_1$ and the other with the argument $t - t_2$ where $t_2 = t_1 + T/2$.

Since in (A2) $i(t)$ is presented as an explicit function of $t$ and its own zero crossings, if we find $t_p, p = 1, 2, \ldots, m$, we know $i(t)$ and have the equation circuit solved. The zero crossings may be found from the equations $i(t) = 0$ which, in view of (A2), are [replacing (5)]

\[AF(t_p; t_1, t_2, \ldots, t_m) = i_0(t_p), \quad p = 1, 2, \ldots, m. \quad \text{(A3)}\]

Since for any certain $u(t) = U_0(t), i_0(t_p) \sim U$, the zero crossings defined by (A3) are some functions of the parameter $\mu = A/U$, $t_p = t_p(\mu)$, while $t_p(0)$ are the zero crossings of $i_0(t)$. $\{t_p\}$ may be independent of $\mu$, and it is shown in [1] that this is the case of the minimal power sensitivity, $K = 1$.

A.3. $\{t_p\}$ and the Mapping $i_0(t) \to i(t)$

Considering the role of $\mu$, we interpret (A2) so that $i_0(t)$ is turned into $i(t)$ by a continuous increase in $\mu$ starting from zero and up to a particular value. Since it is physically obvious that for a fixed $U$ we can find $A$ so large that the current through the ballast will be interrupted or will not flow at all (e.g., $A > U$) there is an upper limit for $\mu$ for $i(t)$ to be a zero crossing function which can be represented as in (A2). The increase permitted here in $\mu$ changes the waveform of the current function and, generally, can move the zero crossings on the time axis, but must not be so strong that zeros appear or disappear. In other words, we limit $\mu$ so that $i(t)$ keeps, qualitatively, its zero crossing features which are defined at $\mu = 0$, i.e., those of $i_0(t)$. Using the term stability in the sense of structural stability, we can speak about stability of the zero crossing form of $i(t)$ with respect to some limited changes in $\mu$.

To better treat this point we also introduce, for the part of the ballast's current response which is caused by the lamp's voltage, a compact notation

\[i(t) = i_0(t) + i_e(t)\]

with $i_e(t)$ being caused by $-u(t)$, Parameter $A$ appears only in $i_e(t)$, but not only as a factor because $i_e(t)$ depends on the zero crossings which, in general, depend on $A$.

Clearly, $i_e(t) \to 0$ as $A \to 0$, and it is not difficult to prove [1] that $A^{-1} d i_e(t)/dt$ is limited for any $A$ which is associated with the required asymptotic inductive features of the ballast. These properties of $i_e(t)$ ensure [1] that for any zero crossing $i_0(t)$ for $\mu$ small enough, $i_e(t)$ and $d i_e(t)/dt$ will be so limited that $i_0(t) + i_e(t)$ will also be a zero crossing function having the same average density of zeros as $i_0(t)$. The limitation on $d i_e(t)/dt$ also ensures that the zero crossings of $i(t)$ are continuously moved on the time axes with a continuous increase in $\mu$, and starting from zero means starting the zero crossings' movement from their positions defined by $i_0(t)$.

We shall refer to these statements related to the structural stability of the zero crossing waveform of $i(t)$ as the Lemma.

Using these concepts, we shall prove that for the $T$-periodic $u(t)$ the lamp's functions $v(t)$ and $i(t)$ are $T$-periodic, and we shall present them, using a Fourier series, as explicit functions of the differences $t - t_1, t - t_2, \ldots, t - t_m$.

We turn now to the proofs of the existence and uniqueness of the $T$-periodic $i(t)$ having the zero crossing features of $i_0(t)$. Despite the space limitation here, a reader can see the beautiful side of the mathematical aspect of the nonlinear theory of the circuits in which the preliminary formalities associated with the easily observable zero crossings are almost immediate, the central point of the proof of existence is the use of a very elementary theorem from calculus, and the proof of uniqueness is reduced to a simple physically clear argument.

A.4. The Existence and Uniqueness Theorem

In order to prove that there exists a $T$-periodic $i(t)$ with the claimed zero crossing features, we introduce the following iterative procedure in which we calculate the current’s term $i_0(t)$, each time using the rectangular wave of the lamp’s voltage which is defined by the zeros of the $i(t)$ obtained in the previous iteration step. Iteration number $n$ will be denoted by superscript $(n)$.

Introducing the notation $i^{(n)}(t) = A \text{ sign } \psi^{(n)}(t)$, we write the iterations $i^{(n)}(t)$ as $i^{(n)}(t) = L[-i^{(n-1)}(t)]$, using the notation $L$ of a linear bounded operator which describes the ballast’s steady-state current response and which is a smoothing integral operator.
GLUSKIN: THE FLUORESCENT LAMP CIRCUIT

Setting
\[ i^{(1)}(t) = i_o(t) \]
we obtain for further iterations \((n = 1, 2, 3, \cdots)\) of \(i(t)\)
\[ i^{(n+1)}(t) = i_e(t) + L [-A \text{sign } i^{(n)}(t)] \]
\[ = i_o(t) - AL[\text{sign } i^{(n)}(t)], \quad (A4) \]

Since they originate from \(i_e(t)\), all the functions \(\hat{i}^{(n)}(t)\) and \(\text{sign } \hat{i}^{(n)}(t)\) are \(T\)-periodic, obviously. According to the Lemma, we can, for \(i_e(t)\) known, take \(A\) so small but nonzero that each of the \(\hat{i}^{(n)}(t)\) will be, for this \(A\), a zero-crossing function having, similar to \(i_e(t)\), \(m\) zero-crossings per period.

The sets of the zero-crossings of the functions \(\hat{i}^{(n)}(t)\) taken modulo \(T\) will be denoted as \(\{t_{1}^{(n)}, p = 1, 2, \cdots, m\}\) for \(n = 1, 2, 3, \cdots, \).

The periodic rectangular-wave function \(\text{sign } \hat{i}^{(n)}(t)\) can be presented as a function of the differences \(t - t_{1}^{(n)}, t - t_{2}^{(n)}, \cdots, t - t_{m}^{(n)}\) by means of a trigonometric series which will be given below. This ensures, as is easily seen, e.g., from the Fourier representation of the periodic functions, that \(L[\text{sign } \hat{i}^{(n)}(t)]\) will also be an explicit function of these differences. According to \((A2)\) we denote this function as \(F(t; t_{1}^{(n)}, t_{2}^{(n)}, \cdots, t_{m}^{(n)})\), Thus
\[ \hat{i}^{(n+1)}(t) = i_o(t) - AF(t; t_{1}^{(n)}, t_{2}^{(n)}, \cdots, t_{m}^{(n)}) \]
which presents \(\hat{i}^{(n+1)}(t)\) as an explicit function of time and the zero-crossings of \(\hat{i}^{(n)}(t)\).

As the point of the proof, we shall show that the sequence of the sets of the zero-crossings related to different \(i_e(t)\) converges as \(n \to \infty\) to a certain limiting set which we denote as \(\{t_{p}, p = 1, 2, \cdots, m\}\) and, as a result, the sequence of the functions \(\hat{i}^{(n)}(t)\) converges to a certain periodic zero-crossing function \(i^*(t)\) which has, from the way in which it is obtained, the same period and density of its zero-crossings as those of \(i_e(t)\).

Consider the \(m\) infinite sequences \(\{t_{1}^{(n)}, t_{2}^{(n)}, \cdots, t_{m}^{(n)}\}\), presented by the columns of the table. Recalling that all of the zero-crossings are related to the same interval of duration \(T\), we refer to the known theorem of calculus \([40]\) which says that any infinite sequence, belonging to a finite interval, has points of condensation in this interval. Choosing one such condensing point for each of the sequences given by the columns: \(t_{1}^{(n)}\) for \(\{t_{1}^{(n)}\} \cdots, t_{m}^{(n)}\) for \(\{t_{m}^{(n)}\}\) we obtain a condensing set \(\{t_{p}\}, p = 1, 2, \cdots, m\). This leads to a certain \(T\)-periodic rectangular wave function \(i^*(t)\).

It is also not difficult to prove \([1]\), using the properties of the smoothing operator, that the associated set \(\hat{i}^{(n)}(t) = L[-\hat{i}^{(n-1)}(t)]\) simultaneously converges to \(L[i^*(t)]\). This leads us, finally, to a \(T\)-periodic function \(i(t) = i^*(t) - L[i^*(t)]\) which is condensing for the sequence \(\hat{i}^{(n)}(t)\), and possesses the zero-crossing properties of \(i_e(t)\).

Having proved the existence of the condensing set of the zero-crossings and of the \(T\)-periodic function \(i(t)\), we have to prove that these objects are unique.

In order to prove the uniqueness, we assume that there are two nonidentical functions, denoted as \(\hat{i}_u(t)\) and \(\hat{i}_v(t)\), which both satisfy the circuit equation
\[ \hat{i}_u(t) = i_o(t) - AI[\text{sign } i_e(t)] \quad (A5) \]
\[ \hat{i}_v(t) = i_o(t) - AL[\text{sign } i_e(t)] \quad (A6) \]
and show that this assumption leads to a contradiction.

Subtracting \((A6)\) from \((A5)\) and introducing the notation
\[ s(t) = A[\text{sign } i_o(t) - \text{sign } i_e(t)] \]
we obtain
\[ \hat{i}_u(t) - \hat{i}_v(t) = -L[s(t)], \quad (A7) \]
Multiplying both sides by \(s(t)\) we obtain
\[ A[\hat{i}_u(t)] + [|\hat{i}_u(t)|][1 - \text{sign } i_o(t) \text{ sign } i_e(t)] = -s(t)L[s(t)]. \quad (A8) \]
The contradiction here is that while the left-hand side of \((A8)\) is essentially nonnegative, the right-hand side is of an alternating polarity. Indeed, by the very sense of the operator \(L\), the expression \(L[s(t)]\) presents the current response of the ballast fed by only a (say, specially introduced) generator of the voltage function \(s(t)\). (If we think about the ballast as a two-port, then for a short-circuited input port, the output port, whose current is of interest, is connected to the generator). Then \(s(t)L[s(t)]\) presents the instantaneous power function of the ballast in such a connection. Since the LC ballast is lossless (or has some small power losses), the average power consumed by the ballast must be zero (or small) and thus the integral of \(s(t)L[s(t)]\) over the period must be zero (small) even for \(s(t)\) not small. Obviously, the smallness of the integral requires that \(s(t)L[s(t)]\) be an alternating-polarity function (the zeros of \(s(t)\) do not coincide with those of \(L[s(t)]\)), which is the point for \((A8)\).

There is thus a unique \(T\)-periodic zero-crossing \(i(t)\) having \(m\) zero-crossings per period.

The uniqueness of \(i(t)\) justifies the guessing in Sections II–IV of the square waveform of the hardlimiter’s voltage \(v(t)\). There is no other \(i(t)\) and \(v(t)\) for the case considered there. Simultaneously, the assumption there of the inducing of the alternating symmetry of the type \(f(t + T/2) = -f(t)\) from \(u(t)\) onto \(i(t), v(t), q(t)\), etc, which was important for the guessing, is justified. We have thus completed the rigorous foundation of the results of Sections II–V.
It remains to give the trigonometric series representation of the $T$-periodic rectangular wave $v(t)$, possessing the values $A$ and $-A$, and having $m$ jump points per period. This series, which introduces the differences $\{t - t_p \}$ is

$$\phi(t) = v_0 + (2A/\pi) \sum_{j=1}^{m} (-1)^{j+1} \sum_{n=1}^{\infty} \frac{n^{-1}}{n} \sin \pi n \omega (t - t_p)$$

with an easily found time-constant term.

Series (A9) may be obtained by integration of the Fourier series of the adequate, properly mutually shifted, $T$-periodic sequences of $\delta$-functions (combs), with a separate calculation of $v_0$.

Since it is important for $\phi(t)$ not to have a time constant term, if the input function $u(t)$ does not provide by its symmetry, zero $v_0$, then a series capacitor in each branch which connects the ballast’s input and output is needed.

Using (A9), it is easy to finally obtain $\dot{\phi}(t) = \dot{i}(t) - L \omega(t)$ as an explicit function of $t$ and $\{t - t_p \}$.

ACKNOWLEDGMENT

The author thanks the listeners to a relevant lecture, which was given recently in the course, Energy Conversion, in the College of Judea and Samaria, and in the course, Nonlinear and LTV Networks, which was given at Ben-Gurion University, for their helpful feedback, and he is grateful to the unknown reviewers for their important comments.

REFERENCES


Emanuel Gluskin received the M.Sc. Degree in physical engineering from the Faculty of Radioelectronics, Leningrad Polytechnical Institute, Leningrad, Russia, in 1974, and the Ph.D. Degree from the Electrical Engineering Department, Ben-Gurion University, in 1994.

He has worked in both research and industry. He is currently a lecturer at the Ben-Gurion University, Israel, and at the Center for Technological Education, Holon, Israel. His research interests are in the area of the contacts of physics and system theory which often includes difficult nonlinear problems. He has more than 30 independent scientific publications in physics, mathematics, and electrical engineering journals, as well as pedagogical works.