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Let us teach this generalization of the final-value theorem

Emanuel Gluskin

Holon Academic Institute of Technology, Holon 58102, Israel
and
Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

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Abstract

A suggestion relevant to teaching the use of Laplace transforms in a basic course of engineering mathematics (or circuit theory, automatic control, etc) is made. The useful 'final-value' theorem for a function $f(t)$, $f(\infty) = \lim sF(s)$, $s \to 0$, makes sense only if $f(\infty) = \lim f(t)$, $t \to \infty$, exists. A generalization of this theorem for time functions for which $f(\infty)$ does not exist, but the time average $\langle f \rangle$ exists, states that as $s \to 0$, $\lim sF(s) = \langle f \rangle$. This generalization includes the case of *periodic* or *asymptotically periodic* functions, and almost-periodic functions that can be given by finite sums of periodic functions.

The proofs include the case of $f(t)$ tending to $f_{as}(t)$ exponentially, which is realistic for the main physics and circuit applications.

Extension of the results to discrete sequences, treatable by the z-transform, is briefly considered.

The generalized form of the final-value theorem should be included in courses of engineering mathematics. The teacher can introduce interesting new problems into the lesson, and provide a better connection with the (usually later) study of the Fourier series and Fourier transform.

1. Introduction

This paper discusses a generalization of the known Laplace-transform final-value theorem. Among the widely used textbooks [1–15], a formula of type (5) can be found only in [8]. However, this result and its further generalization (6) are of significant pedagogical value for a basic course.

We begin by recalling the equality of the average of a periodic function in the whole infinite interval to its average in the period. This equality is proved; the *mistake* of interpreting the average over a period as a *definition* of the average for the periodic function is very common, and may confuse one when we start to consider averages of non-periodic functions, using the correct definition.

The features of realistic systems are the background for the discussion, and some relevant system concepts are recalled in the appendix.

For a teacher who is very limited in time, it may be advisable to use at least (5), with the statement of section 7 ($\langle f_{\mathrm{out}}\rangle = H(0)\langle f_{\mathrm{inp}}\rangle$). This is also the *absolute minimum* that should be included in the standard textbooks.

Below, ∞ means $+\infty$, and the point '0' where a limit is taken, means 0^+.

2. The concept of the average

In the following, we shall widely use the concept of *the (integral) time average* of a real-valued function $f(t)$. By definition, the average is taken over the whole range where $f(t)$ is given. Thus, if the function is defined on the interval $(0, \infty)$, the average is

$$\langle f\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\lambda)\,\mathrm{d}\lambda. \tag{1}$$

For a periodic in $(0, \infty)$ function, having period T, we consider the integer N so that $0 < t - NT < T$. Certainly, $N \to \infty$ as $t \to \infty$. Denoting $\Delta(t) = t - NT$, we have $t = NT + \Delta(t)$, and

$$\langle f\rangle = \lim_{\substack{t\to\infty\\(N\to\infty)}} \frac{\int_0^{NT} f(\lambda)\,\mathrm{d}\lambda + \int_{NT}^{NT+\Delta(t)} f(\lambda)\,\mathrm{d}\lambda}{NT + \Delta(t)} = \lim_{\substack{t\to\infty\\(N\to\infty)}} \frac{N\int_0^{T} f(\lambda)\,\mathrm{d}\lambda + \int_0^{\Delta(t)} f(\lambda)\,\mathrm{d}\lambda}{NT + \Delta(t)}$$

because of the periodicity of $f(t)$. We obtain

$$\langle f\rangle = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t = \frac{1}{T}\int_a^{T+a} f(t)\,\mathrm{d}t, \qquad \forall a \geqslant 0, \tag{2}$$

i.e. *for a periodic function the average over the infinite interval equals its average over the period.* It is easy to see from the proof that the average over (b, ∞), $\forall b > 0$, of a function periodic on $(0, \infty)$, is also given by (2).

Below we deal only with the half-infinite interval $(0, \infty)$, and existence of $\langle f\rangle$ for any function f involved is always meant. Observe from (1) that if $\langle f\rangle \neq 0$, $\int_0^\infty f(t)\,\mathrm{d}t$ diverges.

3. Some properties of $\langle f\rangle$

(i) It is easy to prove that if $\langle f\rangle$ exists over $(0, \infty)$, it is independent of the scale of the time axis. For instance, functions $|\sin\omega t|$ and $|\sin 2\omega t|$ have the same average of $2/\pi$.

(ii) Since the integral averaging operation $\langle\cdot\rangle$ is linear, for f_1 and f_2 possessing finite averages, and for a and b constants, $\langle af_1 + bf_2\rangle = a\langle f_1\rangle + b\langle f_2\rangle$.

(iii) If for a continuous and bounded on $(0, \infty)$ 'f'

$$f(\infty) = \lim_{t\to\infty} f(t)$$

exists, then

$$\langle f\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\lambda)\,\mathrm{d}\lambda = f(\infty). \tag{3}$$

This equality may be proved by using L'Hôpital's rule for this limit—'uncertainty' of the type $\frac{\infty}{\infty}$. (Consider $t \to \infty$ in the nominator and the denominator separately.)

Equality (3) means that the functions possessing $f(\infty)$ are a particular case of the functions possessing average over the infinite interval, and $f(\infty)$ is the average.

Considering for a moment $f(\cdot)$ to be a function not of time, but a spatial variable, one sees that the physical sense of (3) is that in a problem where the boundary conditions at infinity can be introduced, in the averaging over the whole space, the infinity is 'almost the whole space', and thus $\langle f\rangle = f(\infty)$. This simple argument is also helpful when considering discrete sequences below.

4. The final-value theorem

As is usual in one-sided Laplace transform applications, $f(t)$ is meant to satisfy the equation $f(t) = f(t)u(t)$ where $u(t)$ is the unit-step function, $u(t < 0) = 0, u(t > 0) = 1$. This simply means that for $t < 0$ $f(t) \equiv 0$. Except for when dealing with time-shifts in section 6, we need not write the factor $u(t)$.

If

$$f(\infty) = \lim_{t\to\infty} f(t)$$

exists, then we have the Laplace transform ($F(s) = L[f(t)]$) *final-value theorem* [1–14],

$$f(\infty) = \lim_{s\to 0} sF(s). \tag{4}$$

For this equality to be true, we assume that f possesses a first derivative which is bounded in $(0, \infty)$, and absolutely integrable in $[0, \infty)$.

For a sinusoidal f, (4) is not valid, since the right-hand side of (4) gives 0, while $f(\infty)$ does not exist, and similarly for any other periodic, non-constant function. However, in this case the right-hand side of (4) turns out to be $\langle f\rangle$ (see (5) below). Below, we shall also prove that the equality with $\langle f\rangle$ still holds for a wider class of functions f, namely those that, as $t \to \infty$, are asymptotically equal to a finite sum of periodic functions. We shall keep the notation $f(t)$ for such a more general function, and the associated asymptotic function will be denoted as $f_{as}(t)$. Regarding $f_{as}(t)$, we note that precisely periodic functions coincide with their $f_{as}(t)$, and that for a function with $f(\infty)$ existing, $f_{as}(t) \equiv f(\infty)$, i.e. is constant.

5. The generalized form of (4)

Let us start from a strictly periodic case. Using (see [1–14]) that for function $f(t)$ having (for $t > 0$) period T

$$F(s) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}},$$

and noting that

$$\lim_{s\to 0} \frac{s}{1 - e^{-sT}} = 1/T,$$

one easily finds that

$$\lim_{s\to 0} sF(s) = \frac{1}{T}\int_0^T f(t)\,dt = \langle f\rangle. \tag{5}$$

Already, the immediate generalization of (4), given by (5), for strictly periodic functions is important and, as teaching experience shows, interesting to students.

Theorem 1. *Let 'f' be a real-valued function, continuous and absolutely integrable in* $[0, \infty)$*, which is asymptotically equal to (a sum of) periodic function(s),* f_{as}*, that is*

$$|f(t) - f_{as}(t)| < \phi(t)$$

with ϕ *absolutely integrable in* $[0, \infty)$ *and vanishing at infinity. Then*

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\lambda)\,d\lambda = \langle f\rangle, \tag{6}$$

where F *is the Laplace transform of* f.

Remark 1. Any *zero-state response* (ZSR) (see the appendix for recalling this circuit-theory concept) of a deterministic *linear time-invariant* (LTI) circuit with losses, which has as its input a periodic (or an asymptotically periodic) function, *tends to* a periodic (or, respectively, asymptotically periodic) function, but is very seldom periodic (or asymptotically periodic) from the very beginning. Thus the *asymptotic features* of $f(t)$ are very important in the applications. The asymptotic inequality $|f(t)–f_{as}(t)| < \phi(t)$ is certainly true for LTI systems for which $\phi(\cdot)$ may be always taken as exponentially decaying.

Proof of theorem 1. In order to prove (6), we assume, at the start, that $f(t)$ tends to an $f_{as}(t)$ having a finite period. Then, according to (5),

$$\langle f_{as}\rangle = \lim_{s\to 0} sF_{as}(s).$$

Using then the given condition

$$|f(t) - f_{as}(t)| < \phi(t)$$

where $\phi(\cdot)$ is integrable in $[0, \infty)$ and vanishes at infinity, we have

$$\left|\frac{1}{t}\int_0^t (f(\lambda) - f_{as}(\lambda))\,\mathrm{d}\lambda\right| \leqslant \frac{1}{t}\int_0^t \phi(\lambda)\,\mathrm{d}\lambda \leqslant \frac{1}{t}\int_0^\infty \phi(\lambda)\,\mathrm{d}\lambda,$$

and taking the limit as $t \to \infty$, obtain zero in the right-hand side, and

$$\langle f\rangle = \langle f_{as}\rangle \tag{7}$$

which is an important generalization of (3).

Since, furthermore, ϕ has a finite Laplace transform for $s \geqslant 0$, and it is easy to prove that

$$\lim_{s\to 0} |s(F(s) - F_{as}(s))| \leqslant \lim_{s\to 0} s\Phi(s) = 0,$$

where $\Phi(s)$ is the Laplace transform of ϕ, we finally obtain

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} sF_{as}(s) = \langle f\rangle$$

which is (6). □

As the next point, we note that because of the linearity of the Laplace-transform operator and of $\langle\cdot\rangle$, the above proof evidently holds when f_{as} is a finite sum of periodic functions. Since a sum of periodic functions may represent an almost-periodic function [15], for instance $\sin(\omega t) + \sin(\sqrt{2}\omega t)$, theorem 1 is valid for f_{as} to be of this particular class of almost-periodic functions.

We shall not try here to prove theorem 1 for the case when f_{as} is a more general almost-periodic function.

6. Examples of functions possessing $f(\infty)$

As a preparation for use of the important equation (8) below, we find in this section averages of some simple functions that may be typical inputs of electrical systems.

Example 1. Consider

$$f_1(t) = A(1 - \mathrm{e}^{-\gamma t}\cos\omega t)\varphi(t) + Bt\mathrm{e}^{-4\gamma t}\sin 3\omega t$$

where A, B, γ and ω are constants, while γ is positive, and $\varphi(t)$ is a *periodic* function, which equals D for the interval $(0, \tau)$, and E for the interval (τ, T), in each period, with D and E constants.

Since, obviously, $(f_1(t))_{as} = A\varphi(t)$, according to (7)

$$\langle f_1\rangle = \langle (f_1)_{as}\rangle = A\langle\varphi\rangle = \frac{A}{T}[D\tau + E(T - \tau)].$$

Example 2. Using $f_1(t)$ from example 1, consider

$$f_2(t) = f_1(t-a)u(t-a) + K\mathrm{e}^{-8\gamma t}u(t)$$

where a and K are constants, and 'a' is positive.

We have

$$\begin{aligned}(f_2)_{\mathrm{as}} &= (f_1(t-a)u(t-a))_{\mathrm{as}} = (f_1(t-a))_{\mathrm{as}}u(t-a)\\ &= A\varphi(t-a)u(t-a).\end{aligned}$$

Function $\varphi(t-a)u(t-a)$ is periodic in the interval (a, ∞), and its average equals the average of $\varphi(t)u(t)$. We thus obtain

$$\langle f_2\rangle = \langle f_1\rangle = \frac{A}{T}[D\tau + E(T-\tau)].$$

Example 3. Using $f_1(t)$ and $f_2(t)$ from the preceding examples, we have for

$$f_3(t) = \mu_1 f_1(t) + \mu_2 f_2(t),$$

where μ_1 and μ_2 are constants, that

$$\langle f_3\rangle = \mu_1\langle f_1\rangle + \mu_2\langle f_2\rangle = (\mu_1+\mu_2)\langle f_1\rangle = (\mu_1+\mu_2)\frac{A}{T}[D\tau + E(T-\tau)].$$

7. An application to a circuit, or a dynamic system

For application of (6) to the LTI periodically driven electrical passive circuits (or mechanical systems, etc), we introduce the transfer function $H(s)$ of the circuit ($H(s) = L[h(t)]$, where $h(t)$ is the shock-response of the circuit), proving the following theorem.

Theorem 2. *If, as $s \to 0$, $\lim H(s) = H(0)$ exists, then the time-average of the response function f_{out} of the circuit (the ZSR) equals H(0) times the time-average of the input function f_{inp}, i.e. $\langle f_{\mathrm{out}}\rangle = H(0)\langle f_{\mathrm{inp}}\rangle$.*

Proof. Considering that for the asymptotic features of the circuit response only ZSR is relevant (and thus we can replace $F_{\mathrm{out}}(s)$ by $H(s)F_{\mathrm{inp}}(s)$), and using (6), first from right to left, and then from left to right, we obtain for $s \to 0$:

$$\langle f_{\mathrm{out}}\rangle = \lim[sF_{\mathrm{out}}(s)] = \lim[sH(s)F_{\mathrm{inp}}(s)] = H(0)\lim[sF_{\mathrm{inp}}(s)] = H(0)\langle f_{\mathrm{inp}}\rangle. \quad (8)$$

□

For instance, for a series R–C circuit, with a voltage source at the input and the capacitor's voltage as the output, for which (the voltage division) $H(s) = (sC)^{-1}[R + (sC)^{-1}]^{-1} = (1 + sRC)^{-1}$, and thus $H(0) = 1$, the average of the capacitor's voltage equals, according to (8), the average of the input voltage. Thus, if, for instance, the input voltage is given by the function $f_3(t)$ from the previous section (example 3), then the average of the capacitor's voltage equals

$$(\mu_1+\mu_2)\frac{A}{T}[D\tau + E(T-\tau)].$$

One notes that for the strictly periodic case, (8) is easily obtained also using Fourier series. However, here the input function has to be periodic (or almost-periodic, relevant to the proof of (6)) *only in the asymptotic sense*.

If $f_{\mathrm{in}}(\infty)$ exists then we can replace the averages in (8) by the asymptotic values, obtaining

$$f_{\mathrm{out}}(\infty) = H(0)f_{\mathrm{inp}}(\infty). \quad (9)$$

Equality (9) also means that as $t\to\infty$, the function $f_{\mathrm{out}}(t)$ *asymptotically approaches $H(0)f_{\mathrm{inp}}(t)$*.

8. Determination of $H(0)$ for a complicated circuit (system)

Regarding applications of (8), it is very important to recall that for any circuit (system) of a finite structure, $H(0)$ is very simply found directly from the circuit, i.e. there is no need, in fact, to calculate (as we did in the above example of the RC circuit) $H(s)$ and then set $s = 0$. Related to only ZSR, the function $H(s)$ may be determined for the system *with zero initial conditions*, when the system's inductors (or masses in a mechanical system) are presented, in the s-domain, only by the impedances having the factor 's', and the capacitors (springs) only by the impedances having the factor $1/s$. Thus $s = 0$ means *short-circuiting the inductors and disconnecting the capacitors*, which directly turns any finite circuit to a very simply calculable purely resistive (or only with dampers) structure whose transfer function is the $H(0)$ of the initially given circuit (system). Delete the capacitor in the above RC circuit; obviously $H(0) = 1$.

9. The z-transform final-value theorem and its possible generalization

Turning from the continuous functions to *discrete sequences*, f_n, one can similarly consider the final-value theorem in terms of the known z-transform [16] that is closely associated (via the discrete Laplace transform [16]) with the Laplace transform;

$$f_\infty = \lim_{z\to 1+} (z-1)F(z) \tag{10}$$

where $f_\infty = \lim f_n$, as $n \to \infty$, and $F(z)$ is the z-transform of f_n. Similarly to the continuous case, (10) requires existence of f_∞. Consider, however, the periodic sequence $f_n = (-1)^n$, for which f_∞ does not exist. For this sequence [16], $F(z) = z/(z+1)$, and (10) gives 0. The *arithmetic average* (i.e. $\langle\cdot\rangle$ *for a discrete function*) of $(-1)^n$, on the set of the integers, exists and also equals 0, obviously. Thus, at least for this example, we have in the discrete case a similar generalization of the final-value theorem to

$$\lim_{z\to 1+} (z-1)F(z) = \langle f_n\rangle. \tag{11}$$

The case of f_∞ existing is automatically included in (11), since then $\langle f_n\rangle$ exists, and $\langle f_n\rangle = f_\infty$ by the same reason (the infinity is 'almost the whole space') as in the continuous case (section 3, equation (3)).

The scheme of the treatment of the continuous case, given in the previous sections, also defines the points for consideration of the discrete case.

10. Conclusions

We have drawn the attention of teachers to (5) and also suggested a generalization of (5) to (6) that is relevant for some *asymptotically almost*-periodic $f(t)$. Instead of the requirement that $f(\infty)$ exists, we state the much weaker requirement that $\langle f\rangle$ exists.

For LTI systems, equality (6) yields (8) for the functions that are usually met in system theory, including functions asymptotically presentable by almost-periodic trigonometric polynomials. Attempting to extend the generalization of the final-value theorem to other non-periodic functions would be interesting.

Using only elementary concepts, the discussion makes the topic of the application of the Laplace transform to circuit theory deeper for a young student. The teaching experience of the author shows that students of an introductory circuit course find examples associated with formulae (5)–(9) to be interesting.

Any possible extension of the results to the discrete sequences treatable by the z-transform, being relevant, in particular, to such courses as 'Signals and Systems', would be no less interesting.

This material unjustifiably does not appear in the basic textbooks.

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Appendix. Zero-input response (ZIR)

The response of any linear system can be presented as a sum of the *zero-input* and the *zero-state responses*, ZIR + ZSR. By its definition, ZIR includes only initial values of the state variables, and can be the whole system's response only if the 'generator' inputs are zero, while ZSR includes only the 'generator' inputs, and is the whole response only if the initial conditions are zero. From the point of view of a differential equation that may be associated with the system, ZSR *is a particular solution of the equation, which satisfies zero initial conditions*.

From the analytical viewpoint, ZSR is the *convolution* of an input function with the shock response of the system, $(h * f_{\text{inp}})(t)$. (This corresponds, in physics problems, to the convolution of a Green function with, say, charge density.) Thus, the Laplace transform of $\text{ZSR}(t)$ is $H(s)F_{\text{inp}}(s)$. The known method of variations of constants, by Lagrange, applicable to the linear equations, leads to a particular solution of type ZSR.

The energetic losses in realistic passive LTI systems lead to *exponentially bounded* tendencies of the ZIR to zero, and to a similar tendency of the ZSR to the asymptotic output steady state. The latter may be seen from the convolution form of the ZSR and the fact that the circuit's shock response is a particular case of ZIR and thus exponentially tends to zero.

Indeed, for an asymptotically periodic (or asymptotically almost periodic) $f_{\text{inp}}(t)$, the transfer, as $t \to \infty$, of the ZSR function

$$\text{ZSR}(t) = \int_0^t h(\lambda) f_{\text{inp}}(t - \lambda)\, \mathrm{d}\lambda \tag{A.1}$$

to $(f_{\text{out}})_{\text{as}}(t)$ means extending the integration to ∞, while keeping t in the integrand. This yields

$$|(\text{ZSR})(t) - (f_{\text{out}})_{\text{as}}(t)| = \left| \int_t^\infty h(\lambda) f_{\text{inp}}(t - \lambda)\, \mathrm{d}\lambda \right| \leqslant \int_t^\infty |h(t)||f_{\text{inp}}(t - \lambda)|\, \mathrm{d}\lambda.$$

Using the boundedness of $f_{\text{inp}}(t)$, and that $h(t)$ decays exponentially, we have that $\text{ZSR}(t) - (f_{\text{out}})_{\text{as}}(t)$ is bounded by an exponentially decaying function of t.

Remark 2. It is useful to consider in (A.1) $f_{\text{inp}} = \sin \omega t$, thus 'observing' *how* $\text{ZSR}(t)$ tends to a sinusoidal $(f_{\text{out}})_{\text{as}}(t)$ whose amplitude is given by two infinite-range integrals.

Note that the often-used term 'forced response' is usually related not to ZSR, but to the simplest particular solution of the system's equation, which represents the *steady-state* system's response ($(f_{\text{out}})_{\text{as}}(t)$) to which ZSR tends as $t \to \infty$.

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