A small theorem on the nonlinearity of switched systems

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Abstract

A criterion for actual examination the nonlinearity of some systems composed of switched linear elements is given. © 2007 Elsevier GmbH. All rights reserved.

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1. Introduction

Switched circuits and systems (SS) are today the most widespread and important class of electrical and electronics systems. It may be even said that the role of SS in the synthesis of complicated electronic systems satisfying different function (operation) requirements, is similar to that of linear systems in classical electrical engineering.

The SS may be linear, or, more precisely – linear time-variant, LTV [1], as, e.g., the well-known linear switched capacitor circuits, SCC, e.g. [2,3], or nonlinear, NL, e.g. [4–6]. The distinction between LTV and NL SS versions is very important; one never obtains chaos in a linear system, and cannot easily control the output by means of the input (think about a 1-port) in a nonlinear system. It is desirable that at least in some cases the designer, or one examining SS, should have simple criteria for checking nonlinearity.

We speak here only about a sufficient criterion of nonlinearity, restricted to a one-port checking, though the more general situation with linear and nonlinear SS is then briefly discussed, with supporting references to the recent conference presentation [7]. This criterion is not just a rejection (in the sense of the registration of a negative result) of the known 1-port test of linearity; it gives direct, specific for SS clear evidence of the nonlinearity of the system.

One sees that for SS the distinction between LTV and NL systems is quite a delicate issue because linear and nonlinear SS can be created from similar components which all may be per se linear (usually, linear time-invariant, LTI), and the point is: which time-functions define the switching instants. The latter depends on the circuit connections [7]. Since we are interested only in nonlinearity caused by the conditions of switching, our system is meant to include only LTI elements and ideal switches. The LTI elements define the system’s dynamics in the finite time-intervals between the switchings.

2. The 1-port formulation of the problem

The scaling change of a function, e.g. for the input (port) function, from $u(t)$ to $ku(t)$, with a positive constant $k$, will be denoted here by a regular arrow, as $u(t) \rightarrow ku(t)$. The latter change may be also written as passing from
"first state" to "second state", as

\[ u_{\text{second state}(t)} \sim u_{\text{first state}(t)} \] (1)

using the sign ‘\( \sim \)' of direct proportionality.

The input–output map, performed by the system under study, will be denoted by the wide arrow, \( u(t) \Rightarrow x(t) \). For a linear system [1], this map means a convolution-type solution of the state-equations; however no detailed description of the map is important here, in either linear or nonlinear cases, and the criterion to be formulated arises solely from the general form of the equations and the singularity of the time-functions involved.

For a 1-port checking of nonlinearity, the immediate idea is to check whether or not the input rescaling \( u(t) \rightarrow ku(t) \) leads to similar rescaling of the state variables, \( x(t) \rightarrow kx(t) \). If one surely sees that the result is negative, one can conclude that the system is nonlinear. The rejection of linearity indeed is the point in the proof below of our criterion, \( k \) means that the time-process under study is

\[ x(t) \rightarrow kx(t). \] (2)

The important comparison of linear and nonlinear versions of an SS is better done using (2), than using the more common form \( dx/dt = F(x, t) \). If the matrices – i.e. the systems elements – are independent of \( x \), then (2) becomes LTV.

In a 1-port, the scalar input function \( u(t) \) is given with a positive scaling (amplitude) factor/parameter \( U \) (think, e.g., about a “variac”, which changes the amplitude of the input voltage wave)

\[ u(t) = U\xi(t). \] (3)

The function \( \xi(t) \) defines the waveform of \( u(t) \) in each particular experiment, and we can let \( \max(\xi(t)) = 1 \).

Changes in \( U \) mean that the time-process under study is repeated/observed for different values of the parameter \( U \), and for simplicity of the argument, we can be limited to a steady-state, periodic process. Such processes are sufficiently common in electronic SS.

It is important to distinguish here between the occurrence of switchings in time for a fixed \( U \) during a process, and their occurrence when \( U \) is increased. For instance, when for a periodic state there are several switchings per period, appearing in time one after another, a certain increase in \( U \) can change the number of the switchings per period, and we observe a new waveform. The role of the changes \( U \), in the “Small Theorem” below (in particular, in Fig. 1 and Eqs. (8) and (9)) should be understood in the latter sense.

Obviously, ‘\( U \)’ in (3) is a case of the parameter ‘\( k \)’ that led to (1), and, generally, \( k \) may be presented as a ratio of two values of \( U \). Namely, \( k > 1 \) means that \( U \) is increased, and \( k < 1 \) means that \( U \) is decreased.

Though we develop the argument for a scalar \( u \), it is desirable for the generality to continue in the general equations to write vector-input \( u(t) \).

3. A sufficient condition for the nonlinearity of a 1-port composed of linear elements, ideal comparators and switches

The concept of singularity of the waveforms of time-functions is basic here, and consideration of the singularity is sufficient for our argument.

Definition. A waveform of a function (or the function itself) is singular if there are points (time instants) at which a derivative (of any order, 0, 1, 2, \ldots) of the function has jumps.

Since one cannot write at the points of singularity any series expansion having coefficients that may be calculated via derivatives, singularity means that the function is not analytical everywhere.

Since differential equations and their solutions are defined on open sets, a dynamic description of singular functions must include “sewing” of the local solutions at the points of singularity. However since coefficients in the Kirchhoff’s equations are instantaneous time-functions, it is possible to present the switched system in the usual state-space equation form [7], as (2) or (4); the matrices including the switched jumpy parameters/functions will define the system’s description in the intervals between the switching (singularity) points.

The internal switchings in the system inevitably cause singularity of the system’s state-variables \( x(t) \equiv \{x_k(t)\} \). Thus, for instance, the addition, or deletion, of a capacitor in parallel to another capacitor in a battery-R-C circuit, sharply changes the slope \( dv/dt \) of the voltage function \( v(t) \) on the terminals of the parallel connection. A similar singular change of \( v(t) \) also occurs when we change (switch) the resistance through which the capacitor is charged. (Consider Fig. 2 given below.) In accordance with such a change in the capacitor voltage, the charging current contains a jump. The waveforms of the voltage (the \( x(t) \)) and the current function are thus influenced by the switching.

It is suitable to formalize this role of switching as follows.

Postulate 1. Any internal switching in the system, of the elements that define the system’s dynamics, cause new point(s) of singularity of at least one of the time-functions \( \{x_k(t)\} \).

Postulate 2. Any change in the singularity of a function \( x_k(t) \) influences its waveform.

Consequence. Each internal switching changes the waveform of at least one of the \( x_k(t) \).

Why the change in the waveform of at least one of \( \{x_k(t)\} \) is important for the point of nonlinearity, is explained by
where $k$ is a constant.

It is analytically suitable to consider $k \in [0, \infty)$, and it is also appropriate to replace (5) by the equivalent form

$$\text{if } u(t) \Rightarrow x(t) \text{ then } ku(t) \Rightarrow kx(t)$$

where $k$ is a constant.

Using that $x(t) \rightarrow kx(t)$ may be written (compare with (1)) as

$$x_{\text{first state}}(t) \sim x_{\text{second state}}(t)$$

we prove the theorem by coming to a direct contradiction with (7). This contradiction appears because of the singularity of $x(t)$ in the SS, and this is the nontrivial point of the theorem.

According to the above Consequence, when we increase $U$, starting from the zero value, then with appearance or disappearance of a switching, i.e. at $U = U_1$, a change in the waveforms of $x(t)$ occurs. Just for simplicity of writing, we divide the interval $[0, \infty)$ for $U$ into only two intervals $[0, U_1)$ and $[U_1, \infty)$. Of course (see Fig. 1) there may be many such intervals, $[U_1, U_2)$, $[U_2, U_3)$, etc., however the simplification of using only $U_1$ does not influence generality of the proof.

Since when $U$ passes on from $[0, U_1)$ to $[U_1, \infty)$, the waveforms of $x_k(t)$ are changed, proportionality of the waveforms of the “second state” and the “first state” is impossible. Formally written (in the spirit of (7)), for any $x$ or $x_k$,

$$x_{0}U \in [U_1, \infty)(t) \sim x_{0}U \in [0, U_1)(t)$$

is impossible, and even (hoping to have at least one proper $k$)

$$x_{3}U \in [U_1, \infty)(t) \sim x_{3}U \in [0, U_1)(t)$$

is impossible for any $x_k$.

In other words, in the whole range $[0, U_1) \cup [U_1, \infty) = [0, \infty)$ for $U$, the replacement $u(t) \rightarrow ku(t)$, which means here a transfer of $U'$ from $[0, U_1)$ to $[U_1, \infty)$, or conversely (depending on whether $k > 1$, or $k < 1$), can not be followed by $x(t) \rightarrow kx(t)$, which is necessary for linearity, and thus such a system is nonlinear. □

For one dealing with a switched 1-port, this is – though only sufficient (i.e. not covering all possible cases) – but simple criterion of nonlinearity. This criterion does not require any detailed description of the system.

4. Discussion of the Small Theorem

Since the point is the change in the singularity of a time function, caused by the switching, the method of observing “one more switching” occurring with increase in $U$, is different from just first checking linearity using (5) or (6), and then – if the measurement insufficiently well confirms the scaling criterion – to conclude that the system is nonlinear. Though we used (6) for the proof, when applying the suggested criterion one need not think about (5) or (6), and the appearance of a new switching with increase in $U$ is finally seen here to be such a clear feature of an NL SS, as, e.g., a stable limit cycle of some oscillations generated in a (switched or “analytical”) nonlinear system.

The suggested criterion seems to be suitable if one would like to make the checking of nonlinearity automatic. Switching may be registered via the resulting pulses of electromagnetic radiation, and thus one can “hear” one more switching “click” in the system.

The influence of the rescaling of the input function on the parameters of the system (here not only for SS) deserves some additional comments as follows

Focusing on $U$, let us first assume a direct description of the influence of the input on the system parameters

$$dx/dt = [A(u, t)]x + [B(u, t)]u(t)$$

or even simpler:

$$dx/dt = [A(U, t)]x + [B(U, t)]u(t)$$

(here in $u$ there is the component $u(t) = U\xi(t)$, or $u(t) = U(t)\xi(t)$ with a sufficiently slowly changed and not necessarily prescribed $U(t)$), and consider the meaning of the...
The scaling test, in regard to which we perform the scaling test, is \( v_c(t) = U \zeta(t) \). In such a test, we can measure, for instance, the ratio max[\( v_c(t) \)]/\( U \), observing whether or not it is constant. The shown connection of the information input ‘1’ of the voltage comparator to the point \( c \) means that the switching of the resistor will be done when \( v_c(t) \) crosses the level \( E_{\text{ref}} \). This is the case of an \([A(x, t)]\)-system, since \( R \) is dependent of \( v_c(t) \). If we connect, instead, the input 1 of the comparator to the point \( a \), then the prescribed voltage ‘\( v_{\text{ref}}(t) \)’ of the auxiliary source defines the switching instants independently on \( v_c(t) \), and the SS is linear (LTV). The third possibility is to connect input 1 to point \( b \). This is the case of (10, 11), i.e. nonlinear with respect to the test. The scaling parameter influences \( R(t) \) and thus \([A]\). The criterion of the Small Theorem is relevant to both cases \( b \) and \( c \).

For the “standard” nonlinearity

\[
\frac{dx}{dt} = [A(x, t)]x + [B(x, t)]u(t),
\]

when it is more common for one to expect any nonlinear effect (e.g. chaos) to be obtained in the system, \( U \) influences the system parameters not directly, but via \( x \). The suggested 1-port test should cover both of the cases of (10), (11) and (2).

All the three possibilities of: \([A(t)]\) – the LTV case of (4), \([A(u, t)]\) – the “parameters-dependent-on-control” nonlinear case of (10, 11), and \([A(x, t)]\) – the “standard nonlinear” case of (2), may be demonstrated by the 1-order dynamic circuit shown in Fig. 2, in which one respectively takes the control for the informational input of the comparator that generates the pulses triggering the switch, from the terminals \( a \), or \( b \), or \( c \).

Experiments with such circuits show that in case ‘\( b \)’ the nonlinearity of the scaling response may be even stronger than in case ‘\( c \)’.

One often uses the concept of “feedback”, meaning feedback coming from the “output” of the circuit which in this case is \( v_c \). However, one has to remember that for a more complicated (of a higher dynamic order) circuit any component of \( x \) may be considered as “output”. Thus, the nonlinearity is much wider seen via form (2) than via the usual use of the concept of feedback. See also [7] where some complicated circuits are suggested for consideration.

A practical comment is also that in the case of an \([A(u, t)]\)-system, it is better to take in \( u(t) = U \zeta(t), \zeta(t) \) as a triangular or sinusoidal wave than as a rectangular wave, because in the latter case the instants of the switching at which the change(s) of the element occur(s) will be poorly influenced by \( U \).

5. The focus on the switching instants

A very important specificity of SS, is also that one actually obtains \([A(t)]\) as \([A(t^*(t), t)]\), \([A(x, t)]\) as \([A(t^*(x), t)]\), and \([A(u, t)]\) as \([A(t^*(u), t)]\), where \( t^*(\cdot) \) is the set of the switching instances. (Here also one can think for simplicity about a periodic process, but not necessarily.) The switching instants are, respectively, either known a priori (the LTV case (4)), or are defined by the initially unknown state variables to be found (the “standard” nonlinear case (2)), or may be known only empirically after it has become clear how (this is defined by some non-prescribed here extended system including the given system) the slow scaling change \( U(t) \) is done, which is the case of (10, 11).

That is, in fact, we see any switched system first of all as a \([A(t^*(\cdot), t)]\)-system”, or, simpler written, as a “\( t^*\)-system”, and only then think what is \( t^*(\cdot) \). The field of SS can be overviewed from these positions, and this outlook also should be applied to elements with “inherent” characteristics having inflection points, as, e.g. hardlimiters and diodes, which also are, in fact, some “switches”. Thus, the nonlinearity of the ideal diode does not result just from the very fact of the inflection of its \( v-i \) characteristic; it is also important that, usually, the diode’s current, which defines by its zeros the diode’s natural “switchings”, is a state variable that has to be found. If switchings of the diode’s states are done at prescribed instants, then the diode is an LTV element. The circuit shown in Fig. 3, in which one can, for instance, check whether or not \( \max[i(t)] \sim E \), illustrates the above.

Assume first that \( S_c \) is in the position ‘1’, i.e. \( C \) is not connected to the part of the circuit with the diode. Then, if the “Control” circuit operates \( S \) in some prescribed manner, independent of \( E \) and the state variable, then the switching of the states of \( D \) is also prescribed a priori, and \( D \) is an LTV element. However, if \( C \) is connected, then the current via the diode need not be interrupted synchronously with the operation of \( S \); this current may be dependent on the capacitor’s voltage and thus \( D \) is, generally, nonlinear.
The diode will be also nonlinear in the case when the action of the control circuit is dependent on the capacitor’s voltage. However this case of the “standard nonlinearity” requires an important comment as follows.

That nonlinearity of SS, composed of per se linear elements, requires dependence of the switches on the state variables is a very important rule. However there is important reservation (exception), demonstrated by the latter circuit, associated with the possibility of decoupling a system, or the set of the system equations, into two parts of which one is an independent “master” for the other. If $S_C$ is open, the capacitor’s voltage is prescribed by the independent subcircuit $E - R - C$, and as regards operation of $S$, it has to be considered as known. Thus, applying the capacitors voltage in some way to the “Control” circuit, we shall still have an LTV system and the LTV diode. Also among the examples in [7] there is one that demonstrates such a decoupling that keeps the linearity.

See also the argumentation related to operation of hardlimiters in oscillatory circuits for which the preceding concept of “zerocrossing nonlinearity” had been introduced in [7] and references there. In fact, for every circuit touched, the point is singularity, and whether or not the instants (points) of singularity are prescribed or not, is the reason for linearity or nonlinearity. In order to see that the most general point of view is on any singular (switched, or with elements having singular characteristics) systems, let us also briefly mention the sampling systems.

Sampling at certain instant $t'$ (or, for a $T$-periodic process, certain instants $t'(\text{mod } T)$)

$$f(t) \rightarrow f(t')$$

is a linear map, since for $a$ and $b$ constant

$$(af_1 + bf_2)(t') = af_1(t') + bf_2(t')$$

obviously. However, if the sampling device (i.e. the sampling instants) start to be dependent on the functions being sampled, or on some functions connected with the functions being sampled and changed together with them, then there are not the same instants—arguments everywhere, (13) is incorrect (rather, senseless), and the sampling system becomes nonlinear.

6. Conclusions

We have formulated – directly in terms of the switching and input scaling – a sufficient condition for nonlinearity of a switched system. Despite the simplicity of the proof, the “Small Theorem” – i.e. the suggested procedure of changing the input scaling parameter in order to check whether or not this causes new switching – is not expected. The author hopes that though the suggested criterion is only sufficient and relates only to the 1-port test, it can be a suitable tool for one actually examining or designing SS.

The work also contributes to the logical side of basic circuit theory. Instead of saying that a system is nonlinear because it is not a linear one, we say that it has a simply detectable property of nonlinearity, noted in the Small Theorem. This more direct “constructive” logic seems to be appropriate, and it is quite in the spirit of characterizing nonlinearity by other clear (though not always so simply defined/detected) features as, e.g., presence of a stable limit cycle of oscillation, or chaos in the system.

Works [7–14] can provide some additional background to NL SS, and [7,15] may be of some interest also for the “philosophical” side.

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References


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His research interests also include autistic vision problem, spectrum estimation (where he introduced the “*ψ-transform*” avoiding in some cases clock-recovery), spatial filtering, percolation theory, and electrical safety.

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