LETTER TO THE EDITOR

An approximation for the input conductivity function of the non-linear resistive grid

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SUMMARY

The 2-term approximations for the input conductivity functions, \( i_{in} = f_{in}(v_{in}) \), of a grid of similar weakly non-linear (parabolic) conductors, and the grid’s symmetric cuts, measured between two close nodes, are derived, using a semi-empirical method; the results of a relevant PSpice simulation are presented. The functions \( f_{in}(v_{in}) \) of the grid’s symmetric cuts possess a common analytical feature. Simulation results show that the error in the calculation of the non-linear terms in the input functions is less than 1 per cent. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: input conductivity function; non-linear resistive grid

1. INTRODUCTION

We refer to the known problem [1–4] of finding the input conductivity of the infinite square grid of similar linear conductors ‘\( g \)’, measured between a pair of two closest nodes (Figure 1). The value \( 2g \) is obtained in References [1–4] for the input conductivity. Replacing the linear elements with non-linear ones having a monotonic conductive characteristic \( f(\cdot) \) [5], we speak, for such a non-linear grid, about its input conductivity function, \( i_{in} = f_{in}(v_{in}) \) (\( i_{in} \) denotes the input current, and \( v_{in} \) the input voltage), measured at the port, using a floating input voltage source.

With ‘\( k \)’ as the element number, the individual conductivity characteristics of the elements is given as

\[
i_k = f(v_k) = gv_k + pv_k^2
\]

Here \( g > 0 \), while ‘\( p \)’ may be of either sign. In the numerical modelling, ‘\( p \)’ will be negative.

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The input voltage $v_{in}$ has to be limited in order to ensure monotonicity of $f(v_k)$, and thus (as the known consequence of Tellegen’s theorem [6]) the uniqueness of the circuit solution. Limiting $v_{in}$, we limit $v_k$ and $i_k$, $\forall k$. The series expansion (in the general case infinite) of a realistic conductivity characteristic, from which the parabolic expression (1) can be taken as the main $O(v^2)$-precise model, also exists only for a limited range of values of the independent variable.

Thus, we are dealing with a weakly non-linear problem, and considering, in view of the above arguments, that it is reasonable to approximate $f_{im}(v_{im})$ with the same precision as $f(v_k)$, we seek an approximation for $f_{im}(v_{im})$ in the form

$$i_{im} = f_{im}(v_{im}) = g_{im}v_{im} + qv_{im}^2$$

(2)

ignoring the higher-degree terms that appear in the calculation of $f_{im}(v_{im})$ for $f(\cdot)$ given by Equation (1). The notation $f_{im}(v_{im})$ and the form (2) are also used for the input conductivity of some finite circuits (grid cuts) which we shall consider.

The linear term in $f_{im}$ is as in the linear problem. This is supported by any calculation performed in the present study, and follows from the clear fact that ‘$q$’ and ‘$p$’ are zero simultaneously. Thus the focus is on the calculation of $q$.

2. THE INFINITE GRID AS THE ‘LARGEST SYMMETRIC CUT’

Figure 2(a) shows the two symmetry (reflection) lines of the grid connected to the source; the ‘vertical’ straight line passing via the input nodes, and the ‘horizontal’ (zero-potential) straight line that is perpendicular to the middle of the segment connecting the input nodes, crossing the voltage source and some of the grid’s elements. These are the only symmetry lines of the grid connected to the input source.

This symmetry is seen, overall, as some ‘central symmetry’ that is well expressed in the currents’ (and voltages’) distribution (Figure 2(b)). ‘Grid cut’ (or ‘cut’) below is an isolated part of the grid in which no internal element is missed; the input of the grid is the input of the cut.
Figure 2. (a) The (only) symmetry lines of the grid connected to the source. (b) The schematic current distribution lines.

Figure 3. Some symmetric cuts of the 2D-grid: (a) the ‘first simplest cut’; its $f_{\text{in}}(v_{\text{in}}) = f(v_{\text{in}})$ is an important part of $f_{\text{in}}(v_{\text{in}})$ of any symmetric cut; (b) the ‘second simplest cut’; (c) another cut to be considered (one ‘wing’ is shown); (d) the infinite symmetric ladder-cut (one ‘wing’ is shown) and (e) a differently lengthened cut (note the $(f/2)$-conductors; this is half of the symmetric cut, precisely).

By ‘symmetric cut’ we mean a cut that is symmetric with respect to both of the lines of the reflection symmetry lines of the conductive grid. Figures 3(a)–(e) show some of the symmetric cuts. Such a cut may be infinite, e.g. an infinite two-directional ‘ladder’. Each of the Figures 3(c) and (d) show only one of the two ‘wings’ of the associated symmetric cut; the finite symmetric cut and the ladder each include two such wings and also the central element. In the case of Figure 3(e) we show half of the circuit, precisely.

The infinite grid is seen as created by sequentially increasing symmetric cuts; ‘convergence’ of the sequence of the cuts to the infinite grid means convergence of the associated sequence of the functions $f_{\text{in}}(v_{\text{in}})$ to the grid’s $f_{\text{in}}(v_{\text{in}})$.
In the numerical simulation we naturally approximate the grid by some large ‘round’ or square symmetric cuts that visually well approximate the grid. However, as revealed by the present study, all the symmetric cuts possess, with high precision, a common analytical feature, and thus the study of relatively small cuts is also very useful.

3. THE PARAMETERS ‘A’ AND ‘B’

The following semi-empirical approach for finding \( f_{in}(v_{in}) \), is based on the symmetry features of the system; the formulae, starting from Equation (4), relate only to symmetric cuts.

Following the idea of Reference [5], but now using more cuts, we introduce two non-dimensional (unitless) positive parameters ‘A’ and ‘B’, rewriting Equation (2) as

\[
f_{in}(v_{in}) = A(gv_{in} + Bpv_{in}^2)
\]

This means that \( g_{in} = Ag \), and \( q = ABp \). For the single central conductor (Figure 3(a)) \( A = B = 1 \). For the infinite grid \( A^{(\infty)} = 2 \), i.e. \( q^{(\infty)} = A^{(\infty)} B^{(\infty)} p = 2B^{(\infty)} p \), and we have to find \( B^{(\infty)} \). For the linear case of \( p = 0 \) we have

\[
f_{in}(v_{in}) = Agv_{in} = Af(v_{in}).
\]

In the method, we consider ‘A’ and ‘B’ as continuous parameters in the interval \( 1 \leq A \leq 2 \) which is the full range for \( A \) for the 2D-grid cuts.

Parameter ‘A’, that may be found from the linear version of the circuit, gives, roughly, the increase in the input conductivity, obtained when transferring from the first simplest to the given cut, or to the whole grid. ‘A’ is some analytical measure of the grid’s ‘size’ in the sense of the size’s influence on \( f_{in}(v_{in}) \).

As for parameter ‘B’, it is obvious from Equation (3) that it compares the non-linearity of the form of \( f_{in}(v_{in}) \) with that of \( f(\cdot) \).

Symmetric cuts of very different forms (some of them infinite) can have very close values of \( A \), and, as we shall see, they have very close functions \( f_{in}(v_{in}) \).

Since, when enlarging the circuit, the added current paths increase the input conductivity, \( A \geq 1 \) and monotonically increases on the set of the cuts (e.g. ‘round’) that approximate the grid. At the same time, since in the added, relatively long, paths the currents and the conductors’ voltages are relatively weak, which causes the non-linearity to be expressed more weakly, \( B \leq 1 \) and monotonically decreases on the set of the cuts. Thus, with the increase in the cut’s size:

1. \( dB/da < 0 \),
2. there is a ‘competition’ in the expression \( q = ABp \) between the increase in \( A \) and decrease in \( B \), which suggests (and this is found to be indeed so) that when transferring from the first cut to the whole grid, we pass via an extreme value of ‘q’.

Observe that it is much more difficult to see that ‘q’ may have an extreme, if directly considering Equation (2).

Based on item ‘1’, let us assume that, for the full range of the variable \( A \), \( 1 \leq A \leq 2 \), i.e. for the symmetric cuts, starting from the central conductor and up to the limiting case of the infinite grid,

\[
B = B_0 - kA \quad (1 \leq A \leq 2)
\]
Approximation for the Input Conductivity Function

Figure 4. (a) The segment of the straight line, $B(A)$. (b) The parabolic function $q(g_{in})$ for symmetric cuts of the 2D-grid. $X \in [1, 2]$ $g_{in}$ may be found from the linear version of the circuit. Actually, only some discrete points of this curve (as well as in part (a)) are relevant to the cuts, but towards the limit $g_{in} = 2g$ the sequence of these points becomes more and more dense.

with some positive constants $B_o$ and $k$. The basic assumption of Reference [5] is that there is a common analytical feature of the $f_{in}(v_{in})$ related to the symmetric cuts obtains thus the form of the dependence of $B$ on $A$ given by a segment of a straight line in the $A$–$B$ plane.

Using $A$ and $B$ of the first two simplest cuts (Figure 3(a) and (b)) and the infinite grid, we obtain from Equation (4)

$$\frac{B^{(\infty)} - B^{(1)}}{A^{(\infty)} - A^{(1)}} = \frac{B^{(\infty)} - B^{(2)}}{A^{(\infty)} - A^{(2)}}$$

(4a)

where the superscripts denote the relation of the parameters to the cuts. Any of the points $(A^{(1)}, B^{(1)})$, $(A^{(2)}, B^{(2)})$ and $(A^{(\infty)}, B^{(\infty)})$, may be replaced in Equation (4a) by the 'variable point' $(A, B)$, $(1 \leq A \leq 2)$, and then Equation (4a) becomes a 'projective' form of Equation (4). In Reference [5] a less precise formula was suggested.

Using that for $A = 1$, $B = 1$ (the first smallest cut), we find from Equation (4) $B_o = 1 + k$, i.e.

$$B = 1 + k - kA = 1 - k(A - 1)$$

(5)

In order to find ‘$k$’ we use the ‘second smallest grid cut’ (Figure 3(b)). For this cut

$$f_{in}(v_{in}) = f(v_{in}) + 2f(v_{in}/3)$$

$$= (5/3)g_{in} + (11/9)p_{in}^2$$

$$= (5/3)[g_{in} + (11/15)p_{in}^2]$$

(6)

i.e. $A = 5/3$ and $B = 11/15$.

Using these values from Equation (5) $k = (1 - B)/(A - 1) = 0.4$. This gives Equation (4), finally, as

$$B = 1.4 - 0.4A \quad (1 \leq A \leq 2)$$

(7)

illustrated by Figure 4(a).
For the infinite 2D-grid we substitute in Equation (7) \( A = A^{(\infty)} = 2 \), finding \( B^{(\infty)} = 0.6 \), and \( q^{(\infty)} = A^{(\infty)} B^{(\infty)} p = 1.2 p \).

\[
    f_{\text{in}}(v_{\text{in}}) = 2g v_{\text{in}} + 1.2 p v_{\text{in}}^2
\]

PSpice numerical simulation (Section 6) of the 2D-grid, using very large ‘round’ cuts, gives a very close result:

\[
    f_{\text{in}}(v_{\text{in}}) = 2g v_{\text{in}} + 1.21 p v_{\text{in}}^2
\]

In view of this experimental result, we can slightly correct Equation (7) for symmetric cuts that are larger than the two simplest cuts, as

\[
    B = 1.01(1.4 - 0.4A) \quad (A \leq 2)
\]

which changes \( B^{(\infty)} \) to 0.606.

Stressing the role of the symmetric cuts here, we observe that for two similar symmetric cuts connected in parallel, we obtain a doubled \( f_{\text{in}}(v_{\text{in}}) \). This means according to Equation (3), that \( A \rightarrow 2A \) and \( B \rightarrow B \). However, if in \( B = B_0 - kA \), \( A \) is increased, \( B \) cannot remain unchanged, and thus if Equation (4) was valued for \( f_{\text{in}}(v_{\text{in}}) \), it is not valued for \( 2f_{\text{in}}(v_{\text{in}}) \). However, connecting two symmetric cuts in parallel, we clearly do not obtain a symmetric grid cut; already the central conductor of the connection has the improper conductivity \( f(\cdot)/2 \).

We conclude that the equality \( B = B_0 - kA \), \( (1 \leq A \leq 2) \), relates only to symmetric cuts, and it is not invariant under linear transformations of the functions \( f_{\text{in}}(v_{\text{in}}) \). In particular, if a parallel connection of two circuits gives a symmetric cut, neither of the circuits can be a symmetric cut. For instance, \( f_{\text{in}}(v_{\text{in}}) \) of the ‘wing’ of any symmetric cut does not satisfy Equation (7).

It is important to distinguish between the absolute ‘\( q \)-non-linearity’ of \( f_{\text{in}}(v_{\text{in}}) \), which may be directly measured, and the relative \( B \)-non-linearity. If e.g. we use a sinusoidal input voltage source for the symmetric cuts, the highest ‘\( dc \)’ and high-harmonic components in the input current, which are associated with ‘\( g \)’, will be obtained for the optimal cut size. However, the respective value of ‘\( B \)’, which characterizes the non-linearity of this cut equals 0.707. Comparing this figure with the general limits \( 1 \geq B \geq B^{(\infty)} = 0.606 \), we see that the cut that is maximally non-linear in the sense of ‘\( q \)’ is not strongly non-linear in the sense of \( B \).

4. TWO EXAMPLES FOR EQUATION (7a)

Equation (7a) has been checked for the symmetric cut and the infinite ladder shown in Figures 3(c) and 3(d). Both of the circuits are calculated in Reference [7] using precise circuit equations.

For the circuit associated with Figure 3(c), a rather difficult calculation in Reference [6] gives

\[
    f_{\text{in}}(v_{\text{in}}) = (19/11)g v_{\text{in}} + 1.2374155 p v_{\text{in}}^2 + \cdots
\]

that is, \( A = 19/11 = 1.727(27) \) and \( B = 1.2374155/(19/11) \approx 0.71642 \). For this \( A \), Equation (7a) gives 0.7091 for \( B \), which is only about 1 per cent smaller than the true value \( B = 0.71642 \).
For the two-directional infinite ladder associated with Figure 3(d), Reference [6] gives $f_{in}(v_{in})$ as

$$[1 + 2(1 + \sqrt{3})^{-1}]gv_{in} + (1 + 2 \times 0.1196)pv_{in}^2$$

i.e. $A = 1 + 2(1 + \sqrt{3})^{-1} \approx 1.73205$ and $B = (1 + 2 \times 0.1196)/1.73205 \approx 0.71545$. For this value of 'A' the right-hand side of Equation (7a) gives 0.7072, which is also about 1 per cent smaller than the true $B = 0.71545$.

These two examples present some ‘horizontally lengthened’ cuts. Calculation of the ‘vertically lengthened’ cut shown in Figure 3(e) (to be given in another work) is also in good agreement with Equation (7a).

5. THE MAXIMAL ‘q’ FOR THE 2D-GRID CUTS

Using Equation (7a), we obtain

$$q(A) = AB(A)p = 1.01(-0.4A^2 + 1.4A)p \quad 1 \leq A \leq 2$$

(8)

The associated dependence of $|q|$ on $g_{in}$ is illustrated by Figure 4(b). For the continuous $A$–$B$ model, the extremal value of $q(A)$ is obtained for $A = 1.75$ (though there is no cut precisely with such an $A$), and $q(1.75) = 1.23725p$.

The direct consequences of the existence of the maximum of $q(g_{in})$ are as follows:

(a) In the full interval $1 \leq A \leq 2$, $q$ is changed from $p$ to 1.237$p$, i.e. very weakly, and one could very simply obtain a quite good approximation for $f_{in}(v_{in})$ of the infinite grid by taking $g_{in} = 2g$ from the linear version, and $q = 1.22p$ from the ‘second smallest grid cut’.

(b) The two smallest symmetric cuts and the infinite grid (or a very large ‘round’ cut) are ‘far’ from the extreme situation regarding $q$, but, for instance, the infinite ladder is close to it.

(c) We can obtain (ignoring the discrete nature of $A$) the non-linear term in $f_{in}(v_{in})$ of a small ‘cut’ to be the same as that in $f_{in}(v_{in})$ of the infinite grid. Indeed, the expression (we are using Equation (7))

$$q/p = AB(A) = -0.4A^2 + 1.4A$$

equals 1.2 not only for $A = 2$, but also for $A = 1.5$.

6. SIMULATION OF THE 2D-GRID USING PSpice

The idea of the PSpice simulation of the grid is to take larger and larger (and as symmetric and round as possible) grid cuts, until convergence of the associated sequence of functions $f_{in}(v_{in})$ is observed.

The numerical model for the individual characteristic was taken as

$$f(v_k) = v_k - 0.2v_k^2$$

(9)

i.e. $g = 1$, and $p = -0.2$. 

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Figure 5. The maximal grid’s PSpice model used. Note the voltage source in the centre. There are 97 ‘internal’ elements of the grid, and 30 border elements that are interconnected, in particular, by means of the ‘grounding’. (The formally needed grounding resistances are so large that their currents may be ignored.)

Realizations of grid cuts used up to about 127 grid elements (Figure 5), and the resulting associated functions $f_{in}(v_{in})$ were plotted (Figure 6(a)), demonstrating good convergence of this set to the limiting function. For $0 < v_{in} < 2.8$ (V) it is impossible to measure on the figure the difference between the curves related to the cuts with 71 and 127 grid’s elements.

Figure 6(b) shows the limiting function from Figure 6(a), which is our experimental $f_{in}(v_{in})$ of the infinite grid for $g = 1$, and $p = -0.2$, and the curve plotted according to the expression
Figure 6. The simulated set of curves for the input conductivities of the grid cuts, including the (most separated) curve related to the ‘first cut’, which is the individual characteristic of the single element, given by Equation (1). The numbers of the elements of the cuts used are, respectively, 1, 31, 71, 127. Good monotonic convergence is observed. (The difference between the curves related to circuits with 71 and 127 elements may be measured in this figure only at $v_{\text{in}} < -3$ (V); the curve marked by squares is the final result) (b) The experimental $f_{\text{in}}(v_{\text{in}})$ (the upper curve here) that is the ‘limiting’ curve in (a), and the $f_{\text{in}}(v_{\text{in}})$ obtained in Reference [5].
$2gv_{in} + 1.35pv_{in}^2$ for $f_{in}(v_{in})$ obtained in Reference [5] for the 2D-grid, for the same $g$ and $q$. We find the experimental $g_{in}$ and $q$ by comparing these two curves and simultaneously estimate the error of the result in Reference [5].

Since the realistic (experimental) $f_{in}(v_{in})$ includes higher degrees of $v_{in}$, we have first to find a relevant range for $v_{in}$. For this we check up to what values of $v_{in}$ the difference between the two curves increases proportionally to $v_{in}^2$. The range $0 < v_{in} < 1.5$ (V) is thus found.

It is obvious from the graph that the linear terms that are dominant for small $v_{in}$, are equal for the two curves.

Subtracting the curves from their common linear extrapolation starting from $v_{in} = 0$, we measure, at a point in the range found, the ratio of the thus obtained differences, which is the ratio of the quadratic terms, i.e. of the associated values of ‘$q$’.

We find in this way the experimental value of ‘$q$’ to be close to 1.21 $p$. The coefficient in the quadratic term obtained in Reference [5] is about 9.5 per cent larger, in absolute value, than the experimental ‘$q$’. (Since the coefficient ‘$q$’ is negative, the theoretical curve passes below the experimental curve.) Contrary to that, formula (7) gives ‘$q$’ with error of less than 1 per cent.

7. CONCLUSIONS

The semi-empirical ‘$A$–$B$ method’, leads to the simple and very useful formula (7), $B = -0.44 + 1.4$, $1 \leq A \leq 2$, where the non-dimensional (unitless) parameters $A$ and $B$ characterize the geometrical form of $f_{in}(v_{in})$ of any symmetric cut, including the infinite grid. As the PSpice simulation shows, this formula has a precision of about 1 per cent for all the symmetric grid cuts, including the whole grid. The ‘$A$–$B$ method’ also simply shows that the ratio $q/p$ obtains a maximal value for some symmetric cuts.

For the 2D-grid we found $f_{in}(v_{in}) = 2gv_{in} + 1.2pv_{in}^2$, and for the 3D-grid the ‘$A$–$B$ method’ similarly gives $f_{in}(v_{in}) = 3gv_{in} + 1.2857pv_{in}^2$.

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