On the complexity of a locally perturbed liquid flow

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1. Introduction

As is well known (see, e.g., refs. [1–13]), the use of precise dynamic equations is insufficient and additional arguments and even hypotheses, mainly of a statistical character, are widely used in the theories of turbulent motion and in the (in general very different) theories of the appearance of such a motion. Justification for these hypotheses lies in the fact that solving the dynamic equations is not only very difficult, but here in some sense redundant, since these equations define the vector velocity and pressure fields which, for the turbulent state, need not be precisely known, and we have to know, actually, only the degree of complexity of the fields. In the present work, which is also thought to contribute to the problem of the occurrence of turbulence, we assume that when considering the (more or less ordered) pre-turbulent state we may also concentrate the attention on the degree of complexity of this state. The giving up the complete description of a state is typical, in fact, for many theories of the transfer to the disordered state [5–7].

Specifically, we shall deal with the increase in the complexity of the velocity vector field of initially laminar, or slightly vorticed liquid flow as the result of a spatially local, time dependent disturbance of the flow. We shall consider the flow as a system whose structure depends on its inputs, which is, generally, a result of the nonlinearity of the system, or of the describing equations. The concepts of “input” and “system” will be made precise below, in the context of an approach which is based on a relatively simple assumption. This approach arises from an investigation in system theory, motivated by the problem of the realization of delayed functional maps [14].

We are concentrating attention on the connection of the degree of complexity of the system with an approximate solution of the Navier–Stokes equation, revealing the role of the physical variables. Certain
models of the structure, directly following from the basic assumption and not associated directly with the NS equation, are presented in a more extended work [15]. Some results of ref. [15] which may be understood in the concepts introduced here, are listed in section 10.

Our argument is based on the facts that the influence of the perturbation propagates (with the flow) with a delay to a point of observation, and that for realization of a prescribed delay, the structure of the realizing system must be dependent on its inputs and inevitably will be very complicated when the input is a not very complicated waveform. For the latter point, we follow, first of all, the explanation in section 3 of ref. [14] of the fact that realization of a prescribed delay for different waveforms by a system forces us, in an analysis, to speak not about a single system, but about an ensemble of systems whose structures must be connected with the particular functions (waveforms) which may actually appear at the inputs of the systems. The single system, even when very complicated, would perform the realization only approximately. We shall consider the pure delay (i.e. the perturbation, as it is, carried by the main flow) as the zero approximation for the velocity field, and the attempt at the realization of the delay necessarily leads to an increase in the complexity, which results in additional terms of a different analytical structure already in the next approximation.

2. The reasons for the complexity and the delay hypothesis

The well known Wiener–Paley criterion (see, e.g., refs. [16,17], we follow ref. [17]) for the physical realizability of passive systems ("filters"), based on the principle of causality, says that

$$\int_0^{\infty} \ln |H(\omega)| \frac{d\omega}{\omega^2 + \omega^2} < \infty,$$

where \(\omega\) is the (cyclic) frequency, \(\omega_0\) is a nonzero constant, and \(H(\omega)\) is the usual transfer function [16,17] of the system, which equals the ratio of the Laplace transform of the output function to such a transform of the input function. Condition (1) does not permit \(H(\omega)\) to be identically zero in a finite interval, or to be exponential (increasing or decreasing) in an infinite interval.

The relevant here realization of a prescribed delay of an arbitrary function, requires, however, the transfer function of a realizing system \((s = i\omega, \text{see section 7 for details})\) to be

$$H(s) \sim e^{-ds},$$

where \(d\) is the delay or "shift". The fact that such a universal shifting transfer function, which does not include parameters of the function to be shifted, is not realizable, is relevant to systems of any kind, here to the hydrodynamic system. From the very non-realizability it follows that the transfer function of the system which would provide (more or less precisely) the prescribed time shift for a particular time function must have an analytic structure defined by the spectrum features of the function to be shifted. This presents the possibility of connecting the structure of a system with its inputs. When we apply the delay operation to a description of the process of the flow, the resulting problem of the functional (one-dimensionally delayed) input/output mapping includes the reasons for the complexity of the structure of the velocity vector field. This is discussed now in detail.

2.1. The role of the frequency spectrum

Consider the problem of delaying different time functions in the terms of spectrum. It is well known that when a sinusoidal function is delayed (in a steady state) by a linear system, the delay depends on the structure of the system. Thus if we consider a prescribed delay for two pure sinusoidal functions of different periods, we already meet the rather serious problem of finding the system which would realize the same delay for the two functions. If we now consider the delay of a nonsinusoidal function or the delay of two such functions having different (or only partly overlapping) frequency spectra, which is more relevant to the hydrodynamic context, then there is the very serious problem of the realization of the prescribed delay, especially for a wide frequency range. In this case the structure of the realizing system is required to be very complicated and even then we can only speak about an approximate realization. We can also refer to the fact, known from filter and
automatic control theories, that the number of elements required for realization of a prescribed delay without changing the waveform of a time function, tends to infinity when the range of the frequency spectrum of the function does so. The latter may be simply understood if one considers the delay of a signal which is localized in time, having thus an infinite frequency range. Considering the time shift as changing the signal at the points of its definition, we find that an infinite number of the points need be treated in this case. For a periodic function we can use the well known Nyquist (Kotelnikov, in Russian literature) sampling theorem which says that such a function may be completely defined by a finite number of points belonging to a period, whose density depends on the upper limit of the frequency range of the function. Changing the function only at these points we can obtain the needed shift. This may require a finite, though complicated, realizing system. The greater the number of the describing points, the more complicated the realizing system needs to be.

Clearly, there is no significant difference between the problems of similar delay of different functions and the delay of one function, without changing its waveform, because of the possibility of representation of the function to be shifted as a sum of some other functions.

The role of delay leads to the following possible scheme of application of electronic modeling. For a given perturbation and also a given velocity of the laminar flow we state the requirement for an electronic system to similarly delay (with a relevant precision) the functions (taken on the DCS) of the given different spectra. The realization (however compact, cheap, and electronically integrated it is) requires a certain minimal number \( N \), defined by the given conditions, of resistors, capacitors and p–n junctions. Concerning the very transfer to turbulence, we can then require the hydrodynamic experience to say what is the critical spatial density of elementary, clearly observed and undoubtedly not random, structural units (of which not necessarily all are vortices), for which the transfer to turbulence occurs.

Leaving the development of this point of view to ref. [15], we shall treat here the idea of the complexity in the context of the Navier–Stokes (NS) equation, formulating first the basic assumption, and then suggesting a specific structure of the solution of this equation, in agreement with this assumption.

2.2. The defining cross-section and the delay hypothesis

In order to connect the fluid system with the input–output model, in a manner which will be suitable both here and in ref. [15], consider the flow at the point of observation, which we name “output”, as deterministically defined by the flow at the points on a cross-section of the flow to the left (against the direction of the stream) of the observation point; this cross-section includes the point where a spatially local, strongly time-dependent, externally made perturbation of the flow occurs. According to the deterministic point of view, we shall name this cross-section (which need not necessarily be a cross-section of the whole flow, because of the relatively local character of the process under consideration) the defining cross-section (DCS). Though the observation point is meant to be not very close to the DCS, and thus the precise form of the cross-section is not very important here, this form can certainly be defined, for instance so that it maximizes the average angle between the cross-section and the vector of the local velocity (ignoring the “singular” point of the perturbation).

Consider now a cross-section of the flow which includes the point of observation and which is everywhere to the right (the chosen direction of the flow) of the DCS. At many points on this cross-section, the influence of the perturbation will be nonzero, though different. Since the dependence of the output state of the flow on the input states, is given, finally, by the velocity vector field between the two cross-sections, the influence of the perturbation on the different points of the right cross-section means a complexity of the vector field. From the physical point of view this is associated with the diffusion of the imposed irregularities in the flow, as due to viscosity.

Taking into account the fact that whatever the analytical form of the influence of the flow at DCS on that at the “output”, it necessarily includes a time delay, we make the basic assumption:

The delay hypothesis. The interval of the values of the delays from different points of the DCS to the
output is not very strongly spread, and (at least) there is no strong dependence of these values on the form of the time functions which describe the flow on the DCS.

This formulation of the delay hypothesis, which considers, optionally, both the input waveform (or spectrum) and the possible different inputs serves our needs both in the case when we consider (as here) one complicated input, and in the case when we consider (as in ref. [15]) several inputs.

To justify the delay hypothesis we can say that:

1. The influence of the perturbation is seen against the background of the, in general, laminar movement of the flow, associated with the average velocity; there is a spatial “modulation” of the low-frequency flow which “carries” the influences of the spatially fixed high-frequency source of perturbation and thus the delays from different “inputs” are more or less similar.

2. The empirical fact of the increase of the complexity of the flow because of the propagation of the perturbation obtains a direct theoretical explanation in the consideration of the similarity of the delays, as we shall see.

Leaving only the concept of structure for the description of the system, we can, in principle, find an equivalent system, composed from lumped, unmoving elements, which provides the input/output mapping with the required delay. Using the delay hypothesis, we now ask the basic question: How complicated needs the structure of this system to be to provide approximately the same delay of the influences of different harmonics on the input, or on the several inputs, on the output? The derivation of the degree of the complexity of the structure, is meant to be a criterion for the appearance of turbulence in the initially nonturbulent flow.

3. The delay and the nonlinearity

Comparing the system-theoretic and the usual hydrodynamic approaches, we have to separately stress that in the system-theoretic approach the mathematically most important nonlinearity of the NS equation is taken into account. Indeed, this nonlinearity (which inherently has the macroscopic-structural meaning) originates only from the movement of the liquid: \( \frac{dv}{dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v \), with the nonlinear term \((v \cdot \nabla) v\) having the velocity \(v(r, t)\) with \(t\) time and \(r\) the coordinate vector, while the same movement of the liquid is directly used in the delay hypothesis because the delay is defined by the average velocity of the flow.

The dependence of the “parameters” (the operator \(v \cdot \nabla\)) of the equation (and thus the structure of the system) on the velocity vector field (i.e. on the process of the flow itself which depends on the input conditions) may be seen as the property of the system to adjust itself to the input conditions. Even though the property of adjustability (or its antithesis) may be seen in any nonlinear system, for the kinetic nonlinearity this property is especially strongly revealed and important.

Despite a linearization below of the NS equation, the parallel system-theoretic and hydrodynamic considerations allow us to complete some details of the dynamic process, which could be seen, at first, as necessarily lost because of the simplifications. The addition of the structural arguments to an analysis of the specifically (see below) linearized differential equation permits conclusions to be drawn which would be naturally expected only for nonlinear considerations. This is because of the use, in the general case, of nonrealizable functions in the linearized equation, and the specific consideration of the equation as the tool to obtain a “more realizable” function. For such functions the test of linearity of a system (e.g. superposition) has no physical sense, because this test requires generation of the output function for a given input function, which in this case is, generally, impossible. This is precisely the point of ref. [14] where it is noted that the mathematically trivial linearity of the map \(f(t) \rightarrow f(t - \Delta)\) with \(\Delta\) the same for any \(f(t)\), has no physical sense. Realization of the map means, in the general case, creation each time a different system, fitted to a specific \(f(t)\). There is no certain system here, in the general case, about which we could inquire whether it is linear or nonlinear. We can agree to create each time a linear system, but the “jump” from one linear system to another linear system, which represents the property of the adjustability of the ensemble of the elements to the input, is a property of nonlinearity.

These are the reasons for us to concentrate now
the attention on the structural concepts, letting the
delay be responsible for the effects of the nonline-
arity of the NS equation.

4. The approximation of the exponentional “transfer
function” and the physical role of the viscosity

Consider the known polynomial expression (n is
meant to be positive, not necessarily integer)

\[(1-a/n)^n\]  

(3)

(for \(a \sim s \sim i\omega\)) to be here the approximation for (see
\((2)\) \(e^{-a} = e^{-at}\) where \(t\) is the time-delay between
the output and the input of the “system”. The condi-
tion of realizability (1) is satisfied for an \(H(s)\) of
type (3). As is discussed in section 10, introducing
\(s\) in the degree of (3) we can easily obtain, for the
same \(n\), a much better approximation of \(e^{-a}\), but such
an approximation is nonrealizable. Thus (3) should
be one of the best realizable approximations of the
exponent.

In accordance with the physical context, the num-
ber \(n\) determines the complexity of the vector field
and is a function of the spatial coordinate (\(x\) below)
which defines the duration of the system under ex-
amination. \(n\) will be found below (section 8) also as
a function of viscosity, but not of time or \(s\).

We shall somewhat change the form of (3), writ-
ing it as

\[(1+a/n)^{-n}\],  

\((3')\)

with the negative degree, approximating thus the
function \(e^{-a}\) by means of a polynomial fraction. The
mathematically trivial transfer to the negative de-
gree and the argument of the opposite polarity, is here
physically very essential – below, considering the
process in a wide frequency spectrum, we shall con-
sider large values of \(|s| (a \sim s)\), for which (3') is de-
creasing, for any fixed \(n\), in its absolute value, con-
trary to (3), even though both (3) and (3') have
the same limiting function as \(n \to \infty\). This decrease
will be found to be associated with the positivity of
the viscosity of the liquid, i.e. with the dissipation of
energy. More precisely, we shall obtain below the es-

timation for the difference between the approxi-
mating expression and the exponent itself, which for
the case of (3') is \((a^2/2n)e^{-a}\), and for the case of
(3) is \(- (a^2/2n)e^{-a}\), with the opposite sign. Since
we shall interpret the error in the realization (the es-

cence of the nonrealizability of the exponent) as a
result of the diffusion of the irregularities of the flow,
caused by the viscosity, the term of order \(1/n\) will be
(see section 8) proportional to the viscosity. Thus
the minus sign, in the error of the realization in the
case of (3), leads to a negative \(n\) which is unac-
ceptable because of the meaning of \(n\), associated with
the complexity of the system. Thus the structural
consideration comes out to be very deeply associated
with physics.

5. The analytical structure of the velocity vector
field

In accordance with the above, we shall assume that
the velocity of the flow may be written as

\[v(r, t) = v_0 + v_i(r, t) + \epsilon(r, t)\]  

(4)

Here \(r\) is the vector of the spatial coordinates (\(x, y, z\)), \(v_0\) is the average velocity of the flow, \(v_i(r, t)\) is
the “inertial” term (the subscript \(i\) means “inertial”;
we do not use subscripts for the notation of com-
ponents) which represents the “carried” perturba-
tion \(v_i(0, t)\) made at the coordinate origin \(r=0\):

\[v_i(r, t) = v_i(r-v_0t)\],

and \(\epsilon(r, t)\) is the necessarily present spread velocity,
caused by \(v_0, v_i\), and the viscosity which is respon-
sible for the diffusion of the irregularities made by
the perturbation.

Since the value of \(v_0\) causes a similar delay of all
the frequency components of \(v_i(0, t)\) in \(v_i(r-v_0t)\),
we have in this term the prescribed delay which was
discussed in section 2.

The inertial part of the velocity, which satisfies the equation

\[\partial v_i/\partial t + (v_0 \cdot \nabla) v_i = 0\],  

(5)

represents a small “carried” vector field of any kind,
including vortices or other local irregularities of the
flow. We are not prescribing this local field, but we
shall see that if \(v_i(0, t)\) is not monochromatic, its
highest harmonics are relatively strongly influencing
the spread velocity $\epsilon(r, t)$.

The velocity $v_0$ defines the axis of a cylindrical
(conical, with the vortex at the point of the perturbation)
symmetry of the problem in a macroscopic scale. We shall choose this axis as the $x$-axis, writing
$v_0 = (v_0, 0, 0)$. Because of the symmetry, the dependence of $v_1(r, t)$ on time is revealed in the macro-
scopice scale only with the change in $x$, thus we shall
consider all the components of $v_1(r, t)$ only along the
$x$-axis, writing below $v_1(r, t)$ as $v_1(x-v_0t)$. Similarly, we shall consider $\epsilon(r, t)$ as $\epsilon(x, t)$. (We can say, for brevity, that an averaging of the velocity is done, however, the term “averaging” implies here not only ignorance of the small details, but also a use of the symmetry.)

$x$ is positive here; $x<0$ is to the left to the point of the perturbation where $v_1(r, t)$ does not exist, and
is absolutely irrelevant to the expressions.

In some cases we shall write the argument of $v_1(x-v_0t)$ differently,

$$v_1(-v_0(t-x/v_0)) = v_1(t-x/v_0),$$

somewhat nonrigorously keeping the same functional notation $v_1$, in order to keep the notations
whose physical sense is clear. Since we shall write the argument explicitly each time when this is essential, the free transfer below from $x-v_0t$ to $t-x/v_0$, or back, will not cause any trouble. The ratio $x/v_0$ will be denoted below as $\tau$, the time delay ($\Delta$ in section 2). Then $v_1 = v_1(t-\tau_v)$. The following equality, relevant to the argument $x-v_0t$, will also be widely used,

$$\frac{\partial v_1}{\partial x} = -\frac{1}{v_0} \frac{\partial v_1}{\partial t}.$$

We wish to obtain now an expression for $\epsilon(x, t)$ which would agree with both the NS equation and the polynomial approximation of the exponent, introduced in section 4, associated with the delay of the “input” of the “output”. We shall find $\epsilon(x, t)$ (and thus $v(x, t)$) in the form which though it includes the “running variable”, in its whole is not a function of such a variable, and in that sense is “more realizable” (i.e. better approximates a realistic function) than $v_1(x-v_0t)$.

6. The equation for $\epsilon(r, t)$

Consider the NS equation

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = - (\nabla p)/\rho + \nu \nabla^2 v, \quad \nu = \eta/\rho,$$

written for an incompressible (div $v=0$) liquid, with material density $\rho$, viscosity $\eta$ and kinematic viscosity $\nu$. Since we are not defining $v_1(0, t)$, and since the idea of the work is to find how the degree of the complexity of the flow needs to be increased for the delay of the perturbation, we are not seeking a complete solution of the NS equation, but are concentrating attention on the propagation of the perturbation, which is the reason for the following simplifications. First of all, instead of (6) we shall consider the equation

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \nu \nabla^2 v,$$

where the term with pressure is omitted. We omit this term because we are considering a volume which is much smaller than the diameter of the pipe through which the liquid is flowing and because our consideration here ignores the small details of the structure of the flow. In such a situation the change in the pressure may be required only in order to provide the energy, for the $v_0$ given, for the heating of the liquid because of the increased internal friction, associated with the time-dependent part of the velocity. This seems to be just a requirement for the average pressure, i.e. the requirement of the pump which generates the flow.

We shall consider only the part of the volume with the spread velocity where we can assume

$$v_0 \gg v_1 \gg \epsilon,$$

for the amplitude values. This inequality will result in a limitation on $x$. Thus we consider an initial part of the volume. According to this, we ignore the small terms, obtained after the substitution of (4) into (7), and using also (5), obtain the linearized equation for $\epsilon(r, t)$,

$$\frac{\partial \epsilon(r, t)}{\partial t} + (v_0 \cdot \nabla)\epsilon(r, t) = \nu \nabla^2 (r-v_0t).$$

The function (4) with the so-defined $\epsilon(r, t)$ will be named the first approximation for the velocity $v(r, t)$. As was said, we are not describing $v_1(r, t)$ in detail, and, considering it to be “carried” by the main flow, we present the right-hand side of (8) as a func-
tion of the scalar argument \( x - v_0 t \); and writing \( \epsilon(r, t) \) also in such a manner, we obtain, finally, the following equation for \( \epsilon(x, t) \),

\[
\frac{\partial \epsilon(x, t)}{\partial t} + v_0 \frac{\partial \epsilon(x, t)}{\partial x} = \nu \frac{\partial^2 \epsilon(x - v_0 t)}{\partial x^2} .
\]

(9)

Equation (9) will be our basic equation from the hydrodynamic point of view. As is shown in the next two sections, its precision corresponds to the precision of order \( O(1/n) \) of the approximation of the purely exponent "transfer function". Using (9) and the structural consideration, we shall simultaneously determine \( n(x, \nu) \) and \( \epsilon(x, t, \nu) \).

7. The analysis from the structural point of view

In order to speak in terms of the transfer function, we make a Laplace transform of the velocity function, using the two-sided Laplace transform \[ 18 \], denoted as operator \( L \),

\[
L[f(t)] = (Lf)(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt .
\]

For the two-sided transform, the equality

\[
L[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0) e^{-st} dt = \exp(-st_0) (Lf)(s) ,
\]

with a constant \( t_0 \), is satisfied precisely. (For the one-sided transform this is precise only if \( f(t) = 0 \) for \( t \leq 0 \), which is an undesirable requirement here since the moment \( t=0 \) is here physically not important. According to the positive and negative values of the time, \( s \) is also “two-directional” and its real part has to be, correspondingly, positive and negative. Thus \( s \) is defined in an infinite vertical band in the complex plane, having the frequency variable \( \omega \) as its imaginary part. See ref. \[ 18 \] for details.) For \( \nu_1(t-t_x) \) with \( t_x = x/v_0 \) we obtain

\[
L[\nu_1(t-t_x)] = \exp(-st_x) V_1(s) ,
\]

(10)

where

\[
V_1(s) = L[\nu_1(t)]
\]
is the Laplace transform of the perturbation itself (i.e. \( \nu_1(x-v_0 t) \) at \( x=0 \), or \( \nu_1(0, t) \)).

In these notations, the basic hypothesis of this work, formulated in sections 1–3, may be written as

\[
L[v(x, t) - v_0] = L[\nu_1(x, t) + \epsilon(x, t)]
\]

\[
= (1 + st_x/n) - s V_1(s) ,
\]

(11)

with the polynomial-fraction (the transfer function) which approximates \( \exp(-st_x) \), expressing the mapping of \( \nu_1(0, t) \) into the time-dependent part of \( v(x, t) \).

Applying the Laplace transform to (4) written as

\[
\epsilon(x, t) = [v(x, t) - v_0] - \nu_1(t-t_x)
\]

and using (11) and (10) we obtain

\[
L[\epsilon(x, t)]
\]

\[
= [(1 + st_x/n) - s - \exp(-st_x)] V_1(s) ,
\]

(12)

which is our basic equation from the system-theoretical point of view. The analysis below is essentially based on the comparison between (12) and (9). Estimating the difference in the square brackets and making the inverse Laplace transform in (12), we obtain \( \epsilon(x, t) \) in a form which will agree with (9) after an easy matching of \( n \).

Denoting again \( st_x \) as \( a \), we estimate, with the \( O(1/n) \)-precision, the difference \((1 + a/n)^n - e^{-a}\) in (12), using the fact that the limit of \((1 + a/n)^n\) as \( 1/n \to 0 \) is precisely \( e^a \), and calculating first the differential of \((1 + a/n)^n\) (with the positive power), taken for the small value \( 1/n \),

\[
\frac{1}{n} \frac{d[(1 + a/n)^n]}{dn} .
\]

Denoting \((1 + a/n)^n\) as \( y \) and calculating first \( d(\ln y)/dn \), one easily finds this differential to be precisely

\[
\frac{1}{n} (n-2)(1+a/n)^n \left( \ln(1+a/n) - \frac{a}{a+n} \right) .
\]

(13)

Using the expansions: \( \ln(1+a/n) = a/n - a^2/2n^2 \pm \ldots \); \( a/(a+n) = (a/n) (1-1/n \pm \ldots) \), we shall ignore in (13) the terms of order \( 1/n^2 \), which allows us to replace here \((1 + a/n)^n\) by \( e^a \). Thus we obtain, finally,

\[
(1 + a/n)^n - e^a \approx - \frac{1}{2} (a^2/n) e^a ,
\]

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or
\[(1 + a/n)^n \approx e^a (1 - a^2/2n)\].

Returning to the negative power, we find
\[(1 + a/n)^{-n} \approx e^{-a} (1 - a^2/2n)^{-1} \approx e^{-a} (1 + a^2/2n),\]
or
\[(1 + a/n)^{-n} - e^{-a} = (a^2/2n) e^{-a},\]
and thus (12) may be written as
\[L[\epsilon(x, t)] \approx (1/2n) s^2 t^2 x \exp(-Stx) \mathcal{V}(s). \quad (12')\]

It is easily seen now that if we were using the approximation \((1 - a/n)^n\), with the positive power \(n\), for approximation of \(e^{-a} = \exp(-s t x)\), then we would obtain a minus sign in right-hand side of \((12')\), which, as was already said in section 3, would lead below (see section 7) to an unacceptable negative value of \(n\). As was said, this is also associated with the physically essential property of the chosen finite-degree transfer function to fall with an increase in \(|s|\), i.e. in the frequency, which physically is because of the positivity of the viscosity and the energy dissipation. It may be thus said that here the very existence of the “structure” is because of the energy dissipation.

Based on \((12')\) and using the inverse Laplace transform and the known convolution theorem, we can now write
\[\epsilon(x, t) = \int_{-\infty}^{\infty} f(t - \tau) \eta_1(\tau) \, d\tau,\]
where \((\sigma > 0)\)
\[f(t) = \frac{t_2^2}{2\pi i} \frac{\delta(t - t_1)}{2n} \exp(-s t_1) \, e^{s t} \, ds\]
\[= \frac{t_2^2}{4\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} s^2 \exp[s(t - t_1)] \, ds\]
\[= \frac{t_2^2 \delta''(t - t_1)}{2n},\]
with the second derivative of the Dirac \(\delta\)-function.

Thus
\[\epsilon(x, t) = \frac{t_2^2}{2n} \int_{-\infty}^{\infty} \delta''(t - \tau) \eta_1(\tau) \, d\tau\]
\[= \frac{x^2}{2n} \frac{d^2 \nu_1(t - t_1)}{dt^2},\]
where we used that \(t_1 = x/v_0\).

Because of the factor \(x^2/n(x)\), which will be found below to be dependent on \(x\), \(\epsilon(x, t)\) is not a function of \(t - x/v_0\). In that sense it is “more realizable” than \(v_1(t - x/v_0)\). The itself nonrealizable factor in \(\epsilon(x, t)\) with the argument \(t - x/v_0\), was obtained because of the simplifying replacement of \((1 + a/n)^n\) by \(e^a\) in (13). It may be assumed that in a precise theory, based on an infinite number of iterations, the expression \(t - x/v_0\) could be excluded.

Thus the structural point of view results, finally, in
\[\nu(x, t) = \nu_0 + \nu_1(t - x/v_0)\]
\[+ \frac{x^2}{2n} \frac{d^2 \nu_1(t - x/v_0)}{dt^2}. \quad (14)\]

8. The agreement with eq. (9) and the determination of \(n\)

Turning to the writing of \(\nu_1\) with the argument \(x - v_0 t\) (see section 5 for the notations), we rewrite \(\epsilon(x, t)\) as
\[\epsilon(x, t) = \frac{x^2}{2n} \frac{d^2 \nu_1(x - v_0 t)}{dx^2},\]
and thus
\[\nu(x, t) = \nu_0 + \nu_1(x - v_0 t) + \frac{x^2}{2n} \frac{d^2 \nu_1(x - v_0 t)}{dx^2}. \quad (14')\]

Substituting the above expression for \(\epsilon(x, t)\) into (9) and taking into account that \(n\) is dependent on \(x\), we obtain
\[- \frac{v_0 x^2}{2n} \nu'' + v_0 \frac{2x}{2n} \nu' - \frac{x^2 n'}{2n^2} \nu'' + \frac{x^2}{2n} \nu'' = \nu \nu'\]

where a prime denotes differentiation with respect to \(x\). The terms with \(\nu''\) are mutually cancelled, and dividing by \(\nu'\), we obtain
\[ \frac{\nu_0 x}{n} - \frac{\nu_0 x^2}{2 n^2} n' = \nu , \]

or

\[ \nu_0 x^2 \frac{dn}{dx} - 2 \nu_0 x n + 2 \nu n^2 = 0 . \]

This differential equation has the solution \( n(x) = Cx \), with a constant \( C \) which is immediately found as \( \nu_0/2 \nu \).

Thus

\[ n(x, \nu) = \frac{\nu_0}{2 \nu} \]

and the positivity of \( n \) is clearly associated with that of \( \nu \).

Returning to \( \epsilon(x, t) \), and using (15), we obtain

\[ \epsilon(x, t) = \frac{\nu x d^2 \epsilon(x - \nu_0 t)}{v_0} , \]

with the factor which is linearly dependent on \( x \). Thus finally

\[ \nu(x, t) = \nu_0 + \nu_1(x - \nu_0 t) + \frac{\nu x d^2 \epsilon(x - \nu_0 t)}{v_0} \frac{d^2 \epsilon(x - \nu_0 t)}{dx^2} . \]

Summing up the above, we can say that the essence of the mathematical derivation here is expressed by the facts that the structural analysis gives the second derivative of \( \epsilon \) by time, while the linearized NS equation includes the second derivative with respect to the coordinate. The argument \( x - \nu_0 t \) of \( \nu_1 \) permits us to connect these derivatives and thus during the derivation of the expressions for \( n \) it was possible to divide both sides of the resulting equation by \( \nu_0 \epsilon' \), indeed finding \( n \) to be independent of time, according to its structural meaning. There is the mathematical role of the introduction of the "inertial" part of the velocity here.

The assumption \( \epsilon \ll \nu_0 \), which was used in the derivation of (9), means now (we return to differentiating with respect to \( t \) in \( \epsilon \) which permits us to introduce the frequency)

\[ \frac{\nu x \omega^2}{v_0} \ll 1 , \]

or

\[ x \ll \frac{v_0^3}{\nu_0 \omega^2} , \]

where \( \omega \) is a typical cyclic frequency of the perturbation.

When fixing \( x \), we can consider the above inequality, written as

\[ \omega \ll \frac{v_0^{3/2}}{(v_0 x)^{1/2}} , \]

as a limitation on the frequency range which is allowed in this approximation. This limitation follows already from the inequality \( a^2/2 \pi \ll 1 \) (which now coincides with (16)) which had been required in the approximate structural consideration.

In accordance with the assumption of the macroscopically symmetrical and homogeneous distribution of the \( \epsilon(x, t) \) in the \((y, z)\)-plane, the linear dependence of \( \epsilon \) on \( x \) suggests that the spread term exists in a macroscopically conical volume having the \( x \)-axis as the axis of symmetry and with the vortex at the spatial origin. This proposition, which is obtained by the structural argument, seems to be natural for the preturbulent state and the limited \( x \), providing a very simple way for experimentally checking the precision of the approximation taken by means of experimental determination of the deviation of the form of the volume with \( \nu_1 \) and \( \epsilon \) from the conical one.

9. The analog of the Reynolds number

The linear dependence of \( n \) on \( x \) given by (15) means that in the approximation taken, each cross-section of the perturbed part of the flow includes the same number of the irregularities of the flow. It also follows from (15) that for the same volume the system will be more complicated if the average velocity is larger and the kinematic viscosity lower. Introducing the viscosity \( \eta = \nu \rho \) instead of \( \nu \), we turn (15) into \( n = x \nu_0 \rho / 2 \eta \) and see that the lighter the liquid, the smaller is \( n \), for the same \( \eta \), i.e. the complexity, which means, reasonably, that the viscous forces better "keep in order" the light liquid.

We can interpret the value \( \frac{1}{2} x \) as the "characteristic distance" for the particular volume between the points of perturbation \((0)\) and observation \((x)\). Then \( n \) may be interpreted as the Reynolds number \([1]\) of the fluid system. Since we are dealing with the situation where there is no constant characteristic dis-
tance (which could be used for creation of the Reynolds number as usual) defined by the geometry of the problem, the parameter to be watched, as having a critical value, should be the density of the irregularities, averaged in the \((y, z)\)-plane, which may be defined as \(\frac{dn}{dx} = \frac{v_0}{2\nu}\).

Considering the analogy between \(n\) and the Reynolds number, we shall denote \(n(x, \nu)\) as \(R_x\). We can then write the approximating transfer function as

\[
H(s) = \left(1 + \frac{st_n}{R_x}\right)^{-R_x}
\]

and, using (15), as

\[
H(s) = \left(1 + \frac{2\nu s}{v_0^2}\right)^{-\frac{v_0^2}{2\nu}}.
\]

(17)

Noting that \(x\) is included here only in the degree; we can denote \((1 + 2\nu s/v_0^2)^{-\frac{v_0^2}{2\nu}}\) by \(A\), writing the transfer function as \(A^n\), noting that \(A = A(\nu, v_0, s)\), and \(d|A|/ds < 0\). For a fixed \(x\) this transfer function falls with an increase in \(|s|\), i.e. in the frequency. However, as was shown, for a fixed \(x\) (and thus \(n\)), the relative difference between the polynomial fraction and the exponent is proportional to \((st_n)^2\) and increases with the increase in \(s\). Thus though the high frequencies are relatively strongly damped, they relatively strongly influence the structure of the system. This agrees with the results of ref. [15], obtained in a different way.

It follows from (17) that

\[
|H(s)| = \left[1 + 2\nu v_0^{-2} \text{Re}(s)\right]^2 + \left[2\nu v_0^{-2} \text{Im}(s)\right]^2 = \frac{v_0^2}{2\nu}.
\]

Since we deal with a preturbulent structure, \(\text{Re}(s)\) is negative. If \(|\text{Re}(s)| \ll |\text{Im}(s)|\), then \(|H|\) is decreasing with an increase in \(x\). If, however, the expression in the curly brackets equals 1, or is less than that, which means that \(\text{Re}(s) = -\nu v_0^{-2} |s|^2\), or \(\text{Im}(\omega) \gg v_0^{-2} |\omega|^2\), then \(|H|\) is constant or increasing with an increase in \(x\), for \(s\) or \(\omega\) fixed.

10. Final remarks

The following additional points, along the main line of the treatment, require special attention. First, let us note, that as \(n \to \infty\),

\[
(1 + 1/n)^{n+1/2} = (1 + 1/n)^n(1 + 1/n)^{1/2} \\
\approx (1 + 1/n)^n(1 + 1/2n)
\]

approximates

\[
e = 2.718281828459045235360...
\]

much more quickly than \((1 + 1/n)^n\). This follows from the results of section 7 for \(a = 1\) and may be easily checked using a simple calculator. The reason lies, of course, in the fact that the error in the approximation by means of \((1 + 1/n)^n\) is of order \(O(1/n)\), while for \((1 + 1/n)^n(1 + 1/2n)\) it is of order \(O(1/n^2)\). Unfortunately, it is not possible to use the expression

\[
(1 + 1/n)^{n+1/2} a^n = (1 + a/p)^{p+a/2},
\]

for the approximation of \(e^a\), because the expression with \(s\) in the power is nonrealizable. (The integral in (1) is diverging.) If, for instance, \(n + \ln s\) were to appear in the power, the realizability would remain, but it is not clear how such a degree might be introduced. The realizable approximation \((1 + a/p)^{p+a^2/(2p)}\) for \((1 + a/p)^{p+a/2}\), is irrelevant for a wide range of \(s\). We can see the polynomial-fractional approximation (3) or (3') as one of the best among those realizable.

One notes, further, that if we had an approximation of a precision better than \(O(1/n)\), then we would obtain in the terms appearing (e.g. in \((st_n)^n(1-n^{-2})\)) higher degrees of \(s\), which would lead in the inverse Laplace transform (see section 7) to higher (e.g. fourth) derivatives of \(\nu\). This requires the introduction of such derivatives also into the simplified NS equation, which indeed appear, as is easy to see, in the successive approximations when we use \(e\) in the term with \(\Delta \nu\).

In ref. [15] the method of the structural consideration is developed with more detail. In continuation of the above analysis of the NS equation, spatial correlations of the velocity function are calculated and the (weak) spatial decay of the inertial term of the velocity, the possibility of which was ignored here, is estimated, using the hypothesis of the conservation of the average square of the time-variable part of the velocity function. This leads to the prediction.
of the saturation of the cross-section of the perturbed flow, and comparison with experiment can lead to checking the hypothesis asserted. The decay is given by the relation

\[ \langle v_i(x, t)^2 \rangle \approx (1 + \nu \omega^2 / \nu_0^2)^{-1/2} \langle v_i(0, t)^2 \rangle , \]

where \( \langle \rangle \) denotes time-averaging, and \( \omega \) is as in (16). The volume where the time-dependent part of the velocity exists is thus expected in the macroscopic scale to be first linearly increasing, and then saturated.

It is shown in ref. [15] that physically meaningful qualitative conclusions may be obtained even without any mathematical treatment, directly using the concepts of input and response. Thus, for instance, it is very simply explained there that if one rotates, with a not very small constant radius and constant angular frequency, the liquid in the plane of the DCS, so that a circle of water in the DCS is made to rotate as a whole, this is ineffective for the occurrence of the turbulence. It is also explained there, by an electrical analogy, that if there is interaction of irregularities of the flow, the energy has the tendency to transfer from a low-frequency, large-size motion to a higher frequency, small-size motion.

A model of the chains of electrical inductor–resistor units, replacing the "chains" of the vortices, is analyzed in ref. [15], and it is shown there that the polynomial approximation to the exponential transfer function may be obtained in such a way. This model leads in ref. [15] to estimations of spectral properties of energy density and some corresponding properties of power dissipation. In a more general system consideration, it is shown there that the expression \( \ln(\sum H_k(s)s^k) \), where \( H_k(s) \) are the transfer functions related to certain different frequency ranges, has a simple macroscopic meaning and thus should be used in any informational or a thermodynamic characterization of the system.

Regarding the criterion of the realizability (1), a change of this criterion, with a transfer to the "cut" upper limit of integration, and writing a large, but finite number in the right-hand side of the inequality, is suggested in ref. [15], in order to adjust this criterion to the use of the nonrealizable transfer functions which appear in the approximations. The "cutting" of the limit of the integration is justified also by the fact that in reality we never obtain an infinite frequency spectrum even in a fully developed turbulent process.

The use of the nonrealizable terms in ref. [15] is somewhat different from that here. Namely, a system with several inputs of very different spectral properties is considered, and it is assumed that each of the input functions is transferred to the output with a pure delay. These delays are assumed to be different, but not strongly. Analytical investigation of the small difference between the delays appears to be a tool in finding the connection of the system's structure with properties of the input in the terms of the transfer function.

Concerning the possibility of explaining the very moment of the transfer to turbulence, in the terms of the theory, a radical step is taken in ref. [15], interpreting this final step as the attempt of the system to seek the parameter \( a \) (i.e. \( \nu \), or \( \omega \) which may be complex because of the power losses), such that for a finite \( n \), the "equality"

\[ (1 + a/n)^{-n} = e^{-a} , \]

with a complex \( a \), would be "realized", which is impossible, of course, but means, first of all, a complete uncertainty in the phase of \( a \). The uncertainty in the phases is obtained in ref. [15] from the most simple features of functions of complex variables. (Note that when turning to the equality we do not see the exponent as a precise expression which is approximated by the polynomial, and we can "recover" \( a \) from any side of the equation.)

In other words, this means, that the transition to turbulence occurs when the fluid system comes to a state, where it is not able, because of the increased complexity and some of its fundamental threshold features, to distinguish between the ordered ("polynomial") structure and the nonrealizable ("exponential") one.

If we assume that a complex system may be considered as a statistical (i.e. "large") system, then we can compare this point of view on the occurrence of turbulence with the conclusions of ref. [19], where it is said that (my translation, E.G.) the existence of a spectrum is an automatic consequence of statistical stationarity, and does not at all necessarily point to a real appearance of the process under study from superposition of purely periodic components. To con-
nect the uncertainty in the phases with a statistical frequency spectrum would not be easy, however, the disordered state of a flow, which could be statistically described, may be, in some sense, a limiting case here. Indeed, since the statistical properties of a fully developed turbulence are similar at each point of the flow, we can speak, in some sense, about the “pure delay” of the everywhere statistically identical process, though we have then to introduce in some way the delay-operator, which now cannot be precisely (2) or even (3), because not s, but some statistical distributional characteristics need to be used in the operator now.

In the scope of the main results, precise “realization” of the exponential operator, understood in any sense, is undoubtedly associated with an infinite complexity.

Thus the point of view on the transition to turbulence here may be as follows. The input frequency spectrum and the delay cause a complexity of the flow. If this complexity is strong enough, we obtain a statistical system with a turbulent spectrum which need not be connected with the input spectrum.

Introduction of the empirical critical density of the irregularities of the flow, associated with its threshold features, seems to be inevitable here.

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**References**