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An investigation related to the optimal speed of motion of a car (in sunny and rainy days, summer and winter) and of a submarine apparatus

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Abstract

The dependence of the minimal amount of energy (and thus of fuel) needed for a given journey (where the engine is never switched off), $W_{\text{min}}$, with the density of the air, $\rho$, is estimated, for high velocities, as $W_{\text{min}} \sim \rho^{1/3}$. Rainy conditions are especially interesting. The modelling uses the concept of average speed of the vehicle, allowing one to think about motion with constant speed. Different kinds of resistive force, appropriate also for motion in liquid, i.e. for submarine apparatuses, are considered. The discussion seems to be methodologically and motivationally appropriate for the classroom.

Keywords: resistive force, turbulent motion, laminar motion, optimization, education

1. Introduction

Using the speed $v$ of the motion of a car, included in the equality

$$\Delta t = \frac{\Delta x}{v}$$

(1)

where $\Delta x$ is the given distance (e.g. between two towns) to be travelled, and $\Delta t$ is the time the journey takes, we are interested in the minimization of the total energy losses required for the journey, assuming that the minimization of the amount of fuel required is associated with the minimization of energy.

One can assume, for simplicity, that the road is strictly horizontal and there is no wind.

The energy losses $W$ are assumed to be composed of two main parts/terms, both written using the average speed:

$$W(v) = W_1(v) + W_2(v).$$

(2)
The term $W_1$ relates to the macroscopic mechanical work that equals the physical force $F$ multiplied by the distance,

$$W_1(v) = F(v) \Delta x,$$

(3)

a term directly proportional to $\Delta x$.

The term $W_2$ is directly proportional to $\Delta t$, and we write it using some constant $B$ as

$$W_2(v) = B \Delta t.$$

(4)

Using (3) and (4), (2) becomes

$$W(v) = F(v) \Delta x + B \Delta t = \left[ F(v) + \frac{B}{v} \right] \Delta x$$

(5)

with the pole-type term in the ‘generalized force’. For $v > 0$, the existence of a single minimum of $W(v)$ is obvious in each particular case of $F(v)$ considered below.

Comment: Our basic equation $W = W_1 + W_2 = F \Delta x + B \Delta t$ is a case of the energy-conservation equation, written, as usual, in differentials

$$d(\text{lost energy}) = d(\text{work done}) + d(\text{generated heat}).$$

The common way (e.g. [1]) is to immediately focus on the statistically defined entropy $S$ (see also [2]) appearing in the presentation

$$d(\text{generated heat}) = T \, dS$$

where $T$ is the absolute temperature. In our case, we just macroscopically consider the generation of heat occurring as time passes,

$$d(\text{generated heat}) = B \, dt,$$

and then use that $dt = dx/v$, for constant velocity. Regarding the most general form of the energy conservation equation [1, 3], it needs to be stressed that since we deal with a steady state, we ignore any internal accumulation of energy in the car (as, e.g., that associated with charging the battery), for which fuel is also needed, of course. Observe also that the relatively small, constant in time and independent of the velocity, power losses, consumed by the headlights can be taken into account by some small increase in $B$. Since $\Delta x$ is fixed, the equation $dW/dv = 0$ for the optimal speed $v_{opt}$ is

$$\frac{dF}{dv} (v = v_{opt}) = \frac{B}{v_{opt}^2}. \quad (6)$$

We shall consider the following cases of $F(v)$, comparing the results each time, and gradually completing the analytical details and the integral view.

1. The turbulent resisting force [3], relevant since it is obvious that at usual car velocities the motion of the air around a car of the usual shape is not laminar. This force is also relevant ([4] and appendix B) to the quick motion of submarine (underwater) apparatuses, for which fuel economy is important for safety reasons in particular.

For the turbulent force [3],

$$F(v) = Av^2, \quad \text{with} \quad A \sim \rho. \quad (7)$$

where $\rho$ is the material density of the medium, either the air, or the liquid. For this force, the viscosity of the medium is much less important than the density.
(2) The rolling friction force $E$ that is independent of $v$, and is proportional to the weight of the car (see [5] for more details). This force, relevant only to the car motion, will be added, in our first model, to the turbulent force, i.e. for the car we shall deal with the force
\[ F(v) = Av^2 + E. \]

Observe also that the constant slope of the road can be taken into account by a change in $E$.

(3) The Stokes force,
\[ F = Dv, \quad D \sim \eta \]
where $\eta$ is the viscosity of the medium. This force relates to the slower motions, when (see [3]) $\eta$ is much more important than $\rho$. When using the force $F = Dv + E$, we again relate the analysis to the car, but we always can set $E = 0$ in the final expression, making the result relevant to the motion in liquid.

(4) The mixed force
\[ F(v) = Av^2 + Dv + E \]
which includes all the previous simpler cases. Since the general structure of this polynomial function is invariant under any constant shift of $v$, the associated analysis (section 4) is relevant also to the case where there is wind with constant velocity. However, this investigation is left for the reader.

2. The model with turbulent force

In this model,
\[ W_1(v) = (Av^2 + E)\Delta x. \]

Equation (5) becomes
\[ W(v) = \left( Av^2 + E + \frac{B}{v} \right) \Delta x. \]

and (6) becomes
\[ 2Av_{\text{opt}} = \frac{B}{v_{\text{opt}}^2} \]
i.e.
\[ v_{\text{opt}} = \left( \frac{B}{2A} \right)^{1/3}. \]

Note that $E$ does not influence $v_{\text{opt}}$.

Substitution of (10) into (9) gives the minimal energy losses $W_{\text{min}}$ as
\[ W_{\text{min}} = \left( A^{1/3}B^{2/3} \left[ \left( \frac{2}{3} \right)^{2/3} + 2^{1/3} \right] + E \right) \Delta x \]
\[ = (3 \cdot 2^{-2/3}A^{1/3}B^{2/3} + E) \Delta x, \]
that is
\[ W_{\text{min}} \approx (1.89A^{1/3}B^{2/3} + E) \Delta x. \]

When ignoring $E$, which is relevant both to the car-motion with high speed and to the motion in liquid, and considering that $A \sim \rho$, we have
\[ W_{\text{min}} \sim \rho^{1/3}. \]
Figure 1. Schematic comparison of the cases of $W(v)$ for larger $\rho$ (rain) and lower $\rho$ (sun), as regards the relative values of: $v_{opt}$, $W_{min}$, and the curvature of $W(v)$ near the smooth parabolic-type minimum. We assume here that rain causes higher $\rho$, but this is a problematic point (see appendix A). However, that the upper curve relates to higher $\rho$ is certain, which is relevant also to the distinction in $\rho$ of the air in winter and summer (the influence of the temperature), and to motion of submarine (underwater) apparatuses in liquid.

For the car, this proportion means that if rain can increase $\rho$, it increases $W_{min}$, despite the fact that it decreases $v_{opt}$ for which we have from (10):

$$v_{opt} \sim \rho^{-1/3}. \quad (13)$$

Since, furthermore, from (9) and (10)

$$\frac{d^2W}{dv^2}(v = v_{opt}) = 2(\Delta x) \left( A + \frac{B}{v^4} \right) (v = v_{opt})$$

$$= 6A \Delta x, \quad (14)$$

i.e.

$$\frac{d^2W}{dv^2}(v = v_{opt}) \sim \rho,$$

we schematically have (assuming that the rain increases $\rho$) the situation shown in figure 1.

Mathematically, $\rho$ appears here as a parameter (that ranges through some of its relevant interval) of a ‘family’ of the curves $\{W(v)\} [6]$.

It has to be stressed that when considering the car’s motion and saying that $\rho$ is larger for rain, we mean water drops, and not the gas of H$_2$O molecules (i.e. vapour) that are lighter than nitrogen (N$_2$) and oxygen (O$_2$) molecules of which dry air is mainly composed. If ‘humidity’ was to be understood in the sense of the gas of H$_2$O molecules, then the curves of figure 1 would be mutually replaced. (This is clear in view of Dalton’s and Clapeyron’s basic laws; see any textbook of molecular physics, considering mixtures of gases.)

By themselves, the drops of water must increase $\rho$. However, according to the estimation of appendix A this effect is not significant, which must mean that the drops occupy a very small part of the volume with the rain. The fact that the car does not receive any frontal shock (blow) when it enters a sharply framed region of rain also does not support the assumption that $\rho$ is
significantly increased by the rain. The fact that any opposite shock is also never felt suggests that the presence of the gas of H₂O molecules, which reduces ρ, is also not significant. It is not felt by the driver that the rain changes ρ in any sense.

At the same time, figure 1 is relevant to the distinction in ρ associated with different average temperatures of the air in winter and summer, because a change of the air temperature by 30° means [7] a change of the density of the air by about 10%. We have here one more reason why winter is more 'expensive' for an individual and the whole nation/country than summer.

3. The case of the Stokes dynamic force (lower velocities)

The case of the relatively low velocities, when the dynamic force is the Stokes force, is most relevant for the motion in liquid, but it can be also relevant to a car. Adding E to the viscose force, we have

\[ W_1 = (Dv + E) \Delta x. \]  

Assuming that there is no other physical force, we obtain for (5)

\[ W(v) = \left( Dv + E + \frac{B}{v} \right) \Delta x \]
\[ = \left[ \left( \sqrt{Dv} - \frac{\sqrt{B}}{v} \right)^2 + 2\sqrt{DB} + E \right] \Delta x. \]

Obviously,

\[ W_{\text{min}} = (2\sqrt{DB} + E) \Delta x, \]

for which, when ignoring E (the case of the motion in liquid) we have

\[ W_{\text{min}} \sim D^{1/2} \sim \eta^{1/2}, \]

obtained at

\[ v = v_{opt} = \frac{\sqrt{B}}{D} \sim \eta^{-1/2}. \]

Calculating also

\[ \frac{d^2W}{dv^2}(v = v_{opt}) \]
\[ = \frac{2B}{v^3} (v = v_{opt}) \Delta x \]
\[ = 2D^{3/2}B^{-1/2} \Delta x \sim \eta^{3/2}, \]

we see that when compared to the case of turbulent force, the role of η comes out (in the sense of figure 1) to be similar to that of ρ there, while the appearing degrees of η are somewhat ‘stronger’.

It is interesting to note that according to our equations, in both cases, for \( E = 0 \), the product \( v_{opt}W_{\text{min}} \) (geometrically, the area in the \((v, W)\)-plane of the rectangle with the vortexes at the points \{(0,0), (0, W_{\text{min}}), (v_{opt}, W_{\text{min}}), (v_{opt}, 0)\}) is independent of the parameter, either ρ, or η. In this sense, \( W_{\text{min}} \sim 1/v_{opt} \), for \( E = 0 \), for these two models.
4. The case of the mixed force

For the mixed force, we have

$$W(v) = \left( A v^2 + D v + E + \frac{B}{v} \right) \Delta x.$$  \hspace{1cm} (17)

Here also, because of $E$, the modelling relates only to a car, but by setting $E = 0$, we can make the results relevant also to the motion of submarine apparatuses for which the flow can be partly laminar and partly turbulent.

The equation for $v_{\text{opt}}$, written in a form that is suitable for graphical solution (see figure 2), is:

$$2 Av + D = \frac{B}{v^2}; \quad v \to v_{\text{opt}}.$$ \hspace{1cm} (18)

The found value of $v_{\text{opt}}$ gives

$$W_{\text{min}} = \left( A v_{\text{opt}}^2 + D v_{\text{opt}} + E + \frac{B}{v_{\text{opt}}} \right) \Delta x.$$ \hspace{1cm} (19)

It is obvious from figure 2 that, in the mixed case, an increase in $A$ or (and) $D$ decreases $v_{\text{opt}}$, but an increase in $B$ increases $v_{\text{opt}}$, i.e. decreases $1/v_{\text{opt}}$.

Whether the terms of (19), including $A$, $D$ and $B$, are increased or decreased with simultaneous increases in $A$, $D$, and $B$, and whether or not $W_{\text{min}}$ as a whole is then increased (as in the particular case of figure 1), is not seen immediately from (19). However, the increase of $W_{\text{min}}$ is seen ‘by contradiction’ directly from (17). Indeed, if, with an increase in the parameters, $W_{\text{min}}$ would be decreased, then at some $v$ placed between the old and the new $v_{\text{opt}}$, $W$ as the continuous function of all its parameters, would also decrease. However, it is obvious from (17) that for any fixed $v$, any increases in the coefficients lead to the increase in $W(v)$. More straightforwardly, since

$$W^{\text{new}}(v) > W^{\text{old}}(v) \geq W^{\text{old}}_{\text{min}}, \quad \forall v,$$ \hspace{1cm} (20)

whatever $v_{\text{opt}}^{\text{new}}$ is, we have

$$W^{\text{new}}_{\text{min}} \equiv W^{\text{new}}(v_{\text{opt}}^{\text{new}}) > W^{\text{old}}_{\text{min}}.$$ \hspace{1cm} (21)
For the curvature of \( W(v) \) near \( v_{\text{opt}} \), we calculate

\[
\frac{d^2W}{dv^2}(v = v_{\text{opt}}) = 2 \left( A + \frac{B}{v_{\text{opt}}^3} \right) \Delta x. \tag{22}
\]

Since from (18),

\[
\frac{B}{v_{\text{opt}}^3} = 2A + \frac{D}{v_{\text{opt}}},
\]

(22) is

\[
\frac{d^2W}{dv^2}(v = v_{\text{opt}}) = \left( 6A + 2 \frac{D}{v_{\text{opt}}} \right) \Delta x. \tag{23}
\]

Since, as was noted, with the increase in \( A \) or (and) \( D \), \( v_{\text{opt}} \) is reduced, both terms of the (23) are increased. Thus, for the mixed model the role of the increase in \( \rho \) or (and) \( \eta \) is as in the particular case of figure 1.

5. Conclusions and final remarks

Some simple models, employing average speed over the whole journey, are used to analyse the fuel economy seen as directly proportional to the energy economy. It is methodologically important that a lot can be understood by considering only the average speed (most simply, assuming that the speed is constant all the time, but its value is not prescribed) and assigning to this speed high and low values. In the case of the low speeds, the fact that the engine works for a long time causes the important pole-type term to appear in the power losses.

As regards the car, assuming that both in sun and rain this speed is sufficiently high for the dynamic component of the model to include the resisting force of turbulent motion (\( \sim v^2 \)), we discuss the change in the density of the air (medium) because of the rain. It appears, as appendix A shows, that the rain does not change the density of the air to any significant degree. However, the argument concerning the role of density is correct, and it is relevant to the fact that the density depends on the temperature, i.e. to driving in winter and summer.

The analysis is relevant also to the no less interesting case of the motion of a submarine apparatus, when fuel economy is even more important. If the motion of such an apparatus is quick, associated with a turbulent flow, then the apparatus should not enter water regions with some heavy material dissolved or suspended in the water. As for slower motion, the viscous friction should be decreased as much as possible (e.g. by secreting a special liquid which could allow the apparatus to increase its speed, or, for a fixed speed, to save energy).

We have considered a continuous-time optimisation problem. However, at present, discrete problems are no less popular. To start with them, consider, for instance, with what velocity should one pass through an orchard in autumn in order to have the least number of incidences of, for a high velocity (our \( W_1 \)), stumbling over tree roots and hitting a tree, and for a low velocity (our \( W_2 \)), any apples falling on one’s head? It is not so easy, for instance, to speak here about the density of the falling apples; contrary to the water drops in the air, this process and its results are individual. It is also hardly possible not to use here (as we did before) the concept of probability [8]. Both regarding the process and its results.

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Appendix A. On the measurement of the density of the rain

For the following argument, knowledge of the speed of falling of the rain drops is needed. This speed is easily estimated when one observes the angle (slope) of the tracks of the drops on the window of the train you ride in, if the wind speed (i.e. the horizontal component of the speed of the drops) is much smaller than that of the train which can be known with a good precision. I thus once calculated about 10 m sec$^{-1}$, and close figures are found, e.g. in [9].

Let us consider the volume through which the rain falls, assuming that it falls vertically (this allows us to use the usual product instead of a scalar product), and also a container placed under this volume, both having the same cross section $S$.

The flow of water, presented as the drops, through the volume, is:

$$\rho_{\text{drops}} v_{\text{drops}} S,$$  \hspace{1cm} (A.1)

and the rate of the change of the amount of the water in the container is

$$\rho_{\text{water}} S \frac{dh}{dt},$$  \hspace{1cm} (A.2)

where $\rho_{\text{water}}$ is the density of the water (liquid), and $h$ is the height of the water in the container.

Equalizing (A.1) to (A.2), and using the realistic

$$\frac{dh}{dt} \approx 10 \text{ cm h}^{-1} \approx 2.8 \cdot 10^{-5} \text{ m s}^{-1}$$

we obtain

$$\rho_{\text{drops}} = \rho_{\text{water}} \frac{v_{\text{drops}}}{v_{\text{drops}}} \frac{dh}{dt} = \left(100 \frac{\text{kg} \cdot \text{s}}{\text{m}^3}\right) \frac{dh}{dt} \approx 2.8 \cdot 10^{-3} \text{ kg m}^{-3}.$$  \hspace{1cm} (A.3)

Comparing this value to [7]

$$\rho_{\text{air}} \approx 1.3 \text{ kg m}^{-3},$$

we see that the addition may be ignored. This obviously means that the drops fill a very small part of the volume considered. (About $2 \cdot 10^{-6}$, i.e. 2 cm$^3$ in 1 m$^3$.) We do not feel this smallness, because the drops up to about 10 m above us reach us within 1 s.

Appendix B. Additional comments regarding the viscous and turbulent forces

As was used by us, the viscous (Stokes) and turbulent forces that can appear while in motion in the air (gas) or in the water (liquid), are associated, respectively, with the dependences $\sim \eta v$ and $\sim \rho v^2$. These forces also depend, of course, on how streamlined the forms of the bodies that move in the medium are. Not every underwater apparatus (vehicle) has the streamlines of a rocket, and even a military submarine always has turbulent movement of the water around its hull, as can be learned, e.g., from [4].

The transfer from the viscous to the turbulent force for uncompressible liquid means that (we use a non-dimensional ’Reynolds number’ [10, 11])

$$R \equiv ud \frac{\rho}{\eta} \gg 1$$  \hspace{1cm} (B.1)

where $u \equiv |\vec{v}|$ and $d$ is some spatial parameter (classically [10], the diameter of a pipe through which the liquid flows, or the diameter of a ball that is moving in the liquid) that has to be associated in some way with the size and the form of the moving vehicle. (Check that (B.1) means that in Navier–Stokes equation [10, 11] the nonlinear term, originally including density, becomes dominant over the term including viscosity.)
According to (B.1), the larger-size or and quicker moving body more easily causes turbulence, while the competing roles of $\eta$ and $\rho$ in the transfer to turbulence (and in the general dynamics of the liquid) is also clear.

Taking in (B.1) [7]:

$$d = 5 \text{ m}, \quad \eta \approx 100 \text{ kg m}^{-1} \text{ s}^{-1}, \quad \rho = 1000 \text{ kg m}^{-3};$$

we obtain

$$u \gg 2 \cdot 10^{-2} \text{ m s}^{-1}.$$  

Thus, say, for $u \approx 1 \text{ m s}^{-1}$ (i.e. about 3.6 km h$^{-1}$) we can expect turbulent resistive force, for a body of such a size. According to [12], for a typical submarine the velocity is/was up to 74 km h$^{-1}$.

Thus, the concern of [4] is understood, and our assumption about relevance of the turbulent force for motion of the submarine apparatus is justified.

References

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http://vixra.org/pdf/1207.0082v1.pdf see the first 10 (introductory) slides