On the equality of the averaged electrical and magnetic energies in electromagnetic wave

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2013 Eur. J. Phys. 34 1543

(http://iopscience.iop.org/0143-0807/34/6/1543)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
This content was downloaded by: gluskin
IP Address: 109.64.127.6
This content was downloaded on 22/10/2013 at 17:37

Please note that terms and conditions apply.
On the equality of the averaged electrical and magnetic energies in electromagnetic wave

Emanuel Gluskin

Kinneret College in the Jordan Valley (on the Sea of Galilee), Israel
E-mail: gluskin@ee.bgu.ac.il

Received 29 July 2013, in final form 17 September 2013
Published 22 October 2013
Online at stacks.iop.org/EJP/34/1543

Abstract
The equality of the averaged densities of the electrical and magnetic energies in the electromagnetic wave is associated, using a very simple argument, with the requirement that the velocity of the wave is \( c = (\varepsilon_0 \mu_0)^{-1/2} \). This may be a heuristically useful point for a discussion in the classroom.

1. Introduction
As is well known, after Maxwell added (in 1865) to the Ampere-law equation a term with a time derivative, and, using also the Faraday-law equation, derived, for either electrical or magnetic field, one equation of second order, i.e. the wave equation, he found that the velocity of the electromagnetic wave in vacuum is

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \]  

(1)

and that the numerical value of this velocity is just that of (already measured at this time) propagating light. Thus, as a triumph of classical electrodynamics, it was understood that light is an electromagnetic wave.

We would like to point out, using a simple direct argument, that \( (1) \) necessarily means that the averaged densities of the electrical and magnetic energies in the wave are equal, which itself is physically interesting and provides, as we discuss, one more angle of vision on the important role of the velocity of light.

We develop this point using Pointing’s vector, in the following.

2. The argument regarding vector pointing
Pointing’s vector is defined (see any standard textbook, for instance [1] or [2]) as

\[ \mathbf{P} = \mathbf{E} \times \mathbf{H} \]  

(2)
where this definition is given together with the derivation of the equation of energy conservation, in which $\vec{E} \times \vec{H}$ appears to be the flow of the density of electromagnetic energy.

We would like, however, to use an ‘additional’ definition of $\vec{\Pi}$, in the spirit of the definition of the flow of any conserved quantity, for instance the flow of electrical charge, i.e. the electrical current density

$$\vec{j} = \rho \vec{v},$$

(3)

where $\rho$ is the density of the charge moving with the (locally averaged) velocity $\vec{v}$. The fact that $\vec{j}$ is employed in the important equation of conservation of charge,

$$\text{div} \vec{j} = -\frac{\partial \rho}{\partial t},$$

(4)

is the matter of a physics law, and, of course, (4) is taught after the definition (3) is introduced. (See [3] regarding the mutual position of physical laws and definitions.)

Putting $\vec{\Pi}$ in the same logical frame, let us consider the equation of energy conservation (written for a vacuum, without a dissipating term $j \vec{E}$, or any other additional term associated with a process in a material medium):

$$\text{div} \vec{\Pi} = -\frac{d (\text{density of the electromagnetic energy})}{dt}$$

together with the equality

$$\vec{\Pi} = \left( \frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2} \right) \vec{v}$$

(5)

which is the direct analogy of (3) and must agree with (2).

In (5), the energy densities

$$W_E = \frac{\varepsilon_0 E^2}{2} \quad \text{and} \quad W_H = \frac{\mu_0 H^2}{2}$$

are assumed to be averaged, that is, ‘a piece of electromagnetic energy’ $\sim W = W_E + W_H$, is moving with some velocity $\vec{v}$. Thus, one can associate $E$ and $H$ with the amplitudes of the sinusoidal waves $\vec{E}(t)$ and $\vec{H}(t)$, as one does when using ‘phasors’.

Comment. The factors $1/2$ in (5) are due to (see, e.g., [1]) $\vec{E}(t) \frac{dE(t)}{dt} = \frac{1}{2} \frac{dE^2(t)}{dt}$ and $\vec{H}(t) \frac{dH(t)}{dt} = \frac{1}{2} \frac{dH^2(t)}{dt}$. Averaging in time (in [4] such an averaging is found in several places) also adds, via $\langle \sin^2 \omega t \rangle$, or $\langle \sin \omega t \sin (\omega t + \alpha) \rangle$, a factor of 1/2. However, since the latter averaging is done for all three values: $\Pi$, $W_E$ and $W_H$; the latter factor is reduced in the following equations.

3. Velocity (1) as an extreme value

Light velocity is so important that one often mentions only its absolute value. Thus, let us take the absolute value of (5), keeping in mind (2) and the orthogonality of the vectors $\vec{E}(t)$ and $\vec{H}(t)$. We obtain

$$EH = \left( \frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2} \right) v$$

i.e.

$$v = \frac{EH}{\varepsilon_0 E^2 + \mu_0 H^2}$$

(6)

It is necessary, of course, to also obtain

$$v = c = (\varepsilon_0 \mu_0)^{-1/2}.$$
The latter comes out if and only if
\[
\frac{\varepsilon_0 E^2}{2} = \frac{\mu_0 H^2}{2}
\] (7)
which itself is an important conclusion, and suggests some useful analogies to the wave.

To understand the formal role of (7), let us recall that for real values, \( a \) and \( b \), \((a - b)^2 \geq 0\), i.e.
\[
\frac{a^2}{2} + \frac{b^2}{2} \geq ab,
\]
which for \(ab > 0\) (as is below) also is
\[
\frac{1}{a^2} + \frac{1}{b^2} \leq \frac{1}{ab}.
\] (8)
while the equality is obtained only when \(a = b\).

If we set
\[
a = \sqrt{\varepsilon_0 E} \quad \text{and} \quad b = \sqrt{\mu_0 H}
\]
(9)
so that \(a = b\), or \(a^2/2 = b^2/2\), is equivalent to (7), then (6) in view of (8) immediately gives the upper limit for \(v\) as
\[
v \leq \frac{EH}{(\sqrt{\varepsilon_0 E}) (\sqrt{\mu_0 H})} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c,
\]
while the equality \(v = c\) is obtained only when (7) is obeyed.

We thus see how important is the equality of the densities of the two kinds of energies in the wave!

One can compare the wave with the usual \(LC\) oscillatory circuit where, with the oscillations, the magnetic energy completely becomes electrical, and then converse energy transfer occurs, etc, from which it is clear that the averaged magnetic and electrical energies are equal.

4. The symmetry argument and the requirement of relativistic invariance—a discussion

We notice, furthermore, that because of eliminating both of the inequalities \(\frac{\varepsilon_0 E^2}{2} > \frac{\mu_0 H^2}{2}\) and \(\frac{\varepsilon_0 E^2}{2} < \frac{\mu_0 H^2}{2}\), \(v = c\) is a kind of symmetry solution. This aspect should not be ignored because symmetry arguments are important in physics and it is always interesting to find one more.

In view of this, let us observe that we can rewrite (6) as
\[
v = \frac{1}{\frac{1}{\scriptstyle \sqrt{\varepsilon_0 E} - \sqrt{\mu_0 H}} + \sqrt{\varepsilon_0 \mu_0}}.
\] (9)
This form immediately shows that (6) does not give a set of physical velocities including \(c\) as the maximal one; \(v = c\) is the only physical solution.

The point is that the term in the denominator:
\[
\frac{1}{\scriptstyle \sqrt{\varepsilon_0 E} - \sqrt{\mu_0 H}}
\]
\[
\frac{1}{\frac{\sqrt{\varepsilon_0 E} - \sqrt{\mu_0 H}}{EH}} + \sqrt{\varepsilon_0 \mu_0}
\]
(10)
cannot be expressed as a function of the quantity
\[
\varepsilon_0 E^2 - \mu_0 H^2,
\] (11)
and the scalar product
\[
\vec{E} \cdot \vec{H},
\] (12)
which are [2] the only relativistic-invariant values for the electromagnetic field, i.e. are unchanged under Lorenz transforms, and thus are the same for any inertial system. This non-invariance of (10) (and thus, generally, of (6)) is one more reason why we must directly require \( \sqrt{\varepsilon_0 E} = \sqrt{\mu_0 H} \), i.e. \( v = c \). Since it is impossible to express (10) as a function of the arguments (11) and (12), we cannot obtain (10) as zero in a limit of the invariants (both becoming zero, for the plane wave).

This non-invariance means that for any \( v \neq c \), the velocity given by (6) would be dependent on the coordinate system in which the fields are measured.

Thus the second postulate of Einstein [1, 2], requiring the invariance of light velocity, can be associated with the analytical symmetry of the simple algebra known to any student. This is one more example showing the usefulness of symmetry considerations in physics.

Of course, formally, the latter discussion of (6) just demonstrates self-consistence of our argument, because the requirement of relativistic invariance for \( v \) means a use of Lorenz transformations in which the velocity finally found, \( c \), is already included as a parameter. Nevertheless, the fact that equation (6) reveals important topics, and the simplicity of the way it is obtained, seem to be of some heuristic value.

5. Conclusions and final remarks

An argument for the physically important equality of the densities of the electrical and magnetic field energies, for the electromagnetic wave in vacuum, is given. Despite the mathematical simplicity of the argument, the topic is sufficiently deep and should be discussed with students.

Advising strong students to read [2] where electrodynamics is preceded by the special theory of relativity, and is essentially logically based on it, would be the reasonable next step.

On the scientific aspect, it would be interesting, in our opinion, to continue the investigation in two directions.

1. To try to apply the same line of thought to particles (and their fields) such as the neutrino, by trying to present their velocity as a physical function of field invariants, of the type

\[
v = \frac{1}{f(\varepsilon_0 E^2 - \mu_0 H^2, E \cdot H) + \sqrt{\varepsilon_0 \mu_0}}.
\]

with some at least partly known \( f(x_1, x_2) \). Then, whether or not such a particle can move faster than electromagnetic wave will be just the matter of sign \( [f] \).

2. Since (6) estimates the velocity using energy, there should be some quantum (‘half-photon’) limit for the precision, which can permit a small (\( \sim \hbar \)) non-invariant part in (6).

References