The Evanescent Field Transform for Estimating the Parameters of Homogeneous Random Fields with Mixed Spectral Distributions

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***Abstract*— Parametric modeling and estimation of complex valued homogeneous random fields with mixed spectral distributions is a fundamental problem in two-dimensional (2-D) signal processing. The parametric model under consideration results from the 2-D Wold-type decomposition of the random field. The same model naturally arises as the physical model in problems of space-time adaptive processing of airborne radar. A computationally efficient algorithm for estimating the parameters of the field components is presented. The algorithm is based on a nonlinear operator that uniquely maps each evanescent component to a single exponential. The exponential's spatial frequency is a function of the spectral support parameters of the evanescent component. Hence, employing this operator, the problem of estimating the spectral support parameters of an evanescent field is replaced by the simpler problem of estimating the spatial frequency of a 2-D exponential. The properties of the operator are analyzed. The algorithm performance is illustrated and investigated using Monte Carlo simulations.**

I. Introduction

We CONSIDER the problem of parametric modeling and estimation of a homogeneous, complex valued, two-dimensional (2-D) random field with mixed spectral distribution. This modeling and estimation problem has fundamental theoretical importance, as well as various applications in wave propagation problems.

It is shown in [1] that any 2-D regular and homogeneous discrete random field can be represented as a sum of two mutually orthogonal components: a *purely indeterministic* field and a *deterministic* one. The deterministic component is further orthogonally decomposed into a *harmonic* field and a countable number of mutually orthogonal *evanescent* fields. This decomposition results in a corresponding decomposition of the spectral measure of the regular random field into a countable sum of mutually singular spectral measures. The purely indeterministic component has an absolutely continuous spectral distribution function. The spectral measure of the deterministic component is singular with respect to the Lebesgue measure, and therefore, it is concentrated on a set of Lebesgue measure zero in the frequency plane. It is shown in [1] that under some mild assumptions (that always hold in practice), each evanescent component can be modeled by a separable model given by the product of a one-dimensional (1-D) purely indeterministic process in one dimension and an exponential in the orthogonal dimension (or a linear combination of such separable random fields). Hence, the spectral supports of the different evanescent components have the form of lines, where the slope of each line is a rational number.

The parametric model that results from these orthogonal decompositions naturally arises as the physical model in problems of space-time signal processing, such as the space-time processing of airborne radar data. In this problem, the target signal is modeled as a random amplitude complex exponential where the exponential is defined by a space-time steering vector that has the target's angle and Doppler. In other words, in the space-time domain, the target model is that of an harmonic component. The purely indeterministic component of the space-time field is the sum of a white noise field due to the internally generated receiver amplifier noise and a colored noise field due to the sky noise contribution. The presence of a jammer results in a barrage of noise localized in angle and distributed over all Doppler frequencies. Hence, in the space-time domain, each jammer is modeled as an evanescent component whose 1-D modulating process is a white noise. Thus, in the angle-Doppler domain, each jammer contributes a 1-D delta function located at a specific angle. (It is therefore parallel to the Doppler axis.) The ground clutter results in an additional evanescent component of the observed 2-D space-time field. The clutter echo from a single ground patch has a Doppler frequency that depends on its aspect with respect to the platform. Hence, clutter from all angles lies in a "clutter ridge" supported on a diagonal line (that generally wraps around) in the angle-Doppler domain.

Due to physical properties of the problem, the different components of the field are assumed to be mutually orthogonal. In the specific application of airborne radar, the evanescent components (the clutter and jamming signals) are considered to be unknown interference. The power of the harmonic component (target) is considerably smaller than that of the interference components. Hence, the objective of the space-time processing is to estimate the unknown evanescent components of the 2-D space-time signal from the available finite dimension observed data. Based on the estimated interference terms, a 2-D filter that represents combined receive beamforming and Doppler filtering is applied in a second stage to the data

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to suppress the interference, thus enabling detection of the harmonic component. (See [2] for a detailed description of this problem.)

The special case of a real valued 2-D random field has many applications in modeling, estimation, and synthesis of textures in images [22], as well as for image coding [23] and restoration problems [24]. In these applications, the texture field is decomposed into a sum of a purely indeterministic component—the structureless, "random looking" component of the texture field, a harmonic component that results in the periodic attributes of the texture, and evanescent components that result in the directional attributes of the observed texture.

In [14], we developed a conditional maximum-likelihood algorithm for jointly estimating the parameters of the harmonic, evanescent, and purely indeterministic components of the field for the case where the slope parameters of the spectral supports of the evanescent fields are *a priori* known. In [15], the algorithm was extended to the case where these parameters are unknown. It is shown that by introducing appropriate parameter transformations, the highly nonlinear least-squares (NLLS) problem that results from maximizing the conditional likelihood function is transformed into a separable NLLS problem. Hence, the computational complexity of the required numerical minimization is reduced significantly. In the transformed problem, only the spectral support parameters of the deterministic components enter nonlinearly into the transformed model equation. Therefore, by first estimating the unknown spectral supports of the harmonic and evanescent components, the problem of solving for the transformed parameters of the field is reduced to linear least squares.

In this paper, we derive a computationally efficient estimation algorithm for the parameters of the evanescent and purely indeterministic components of the field such that no numerical minimization is required. The algorithm is based on a new nonlinear operator derived in this paper. The operator uniquely maps each evanescent component to a single exponential. It is therefore named the evanescent to exponential transform (EET). The exponential's spatial frequency is a function of the spectral support parameters of the evanescent component. Hence, employing this transformation, the problem of estimating the spectral support parameters of an evanescent field (i.e., the slope parameters of its spectral support, and its frequency parameter) is replaced by the simpler problem of estimating the spatial frequency of a 2-D exponential. It should be emphasized that the proposed algorithm is not based on any assumption regarding the probability distribution function of the observed field.

Alternative approaches for estimating the spectral support parameters of the evanescent components can be derived by taking the Radon or Hough transforms [20], [21] of the observed field periodogram. In the presence of evanescent components, the periodogram peaks are concentrated along straight lines. Since on a finite-dimension observed field only a finite number of possible line orientations may exist, any of the foregoing transformations can be applied to estimate the spectral support parameters of the evanescent fields. Since these methods employ the periodogram for obtaining the estimates, their performance is limited by the resolution limits of the discrete Fourier transform (DFT). As we show in this paper, such methods are considerably more sensitive to noise than the method based on the EET. The latter is also computationally more efficient than the methods based on the Radon and Hough transforms. More specifically, periodogram-based estimation of the spectral support parameters using the Radon transform requires the discretization of both the distance from the origin and angle parameters. This step is followed by evaluation of the line integrals (the projections) for each pair of parameters and a search for the projections with highest energy. On the other hand, using the EET, no such search in the parameter space is required.

Once the spectral support parameters of each evanescent component have been estimated, several alternatives for estimating the other parameters of the field are possible. For example, in [18], we have developed a demodulation procedure that provides the estimated 1-D modulating process of each component and its parametric model. (Section IV-D includes a brief summary of the procedure). In the absence of harmonic components, the residual field, after all the evanescent components have been removed, is the purely indeterministic component of the observed field. Its parametric model can now be estimated using existing estimation methods of purely indeterministic random fields (e.g., an AR model [9], [14]). Note that in this case where the observed field has only a purely indeterministic component, the procedure [14] of obtaining a maximum-likelihood estimate of the AR model parameters is reduced to a solution of a linear least squares problem. An alternative method is to employ a two-stage procedure for obtaining a least-squares estimate of the observed field model. In the first stage, the parameter estimation algorithm proposed in this paper is applied to the observed field to estimate the spectral support parameters of the evanescent components. In the second stage, the unknown spectral support parameters are substituted with the estimated ones to reduce a highly nonlinear LS problem (similar to the one in [15]) to a *linear* LS.

The proposed estimation algorithm of the evanescent components opens the way for *parametric* solutions that can simplify and improve existing methods of space-time adaptive processing (STAP). The goal of space-time adaptive filtering is to achieve high gain at the target angle and Doppler and deep nulls along both the jamming and clutter lines. Because the interference covariance matrix is unknown *a priori*, it is typically estimated using sample covariances obtained from averaging over a few range gates. Next, a weight vector is computed from the inverse of the sample covariance matrix. A second common approach of STAP (see, e.g., [2] and [3]), is to employ subspace projections. Algorithms in this class first estimate the subspace spanned by the interference by performing eigenanalysis of the sample covariance matrix. The weight vector is then obtained by projecting the desired response onto the subspace orthogonal to the interference subspace. In this way, the weight vector is forced to null the interference. In this paper, we derive a computationally efficient algorithm that is useful for estimating both the spectral supports and the modulating processes of the jamming and clutter fields. Hence, the contribution of the estimated interference can be subtracted from the observed data. The residual field consists of a purely indeterministic (most often white thermal noise) component and the target signal. The latter is a considerably simpler problem, even when fully adaptive (optimal) STAP is considered [2]. Alternatively, having estimated the interference terms parametric models, their covariance matrix can be evaluated based on the estimated parameters. In [17], we derived expressions of the field covariance matrix in terms of the components parameters. By substituting in these expressions the unknown parameters with the estimated ones, an estimate of the interference covariance matrix is obtained. Such a method may be utilized in the framework of reduced rank adaptive STAP, where the lower computational complexity of the weight application makes this approach much more practical than the computationally prohibitive fully adaptive STAP.

In [6], a matrix enhancement and matrix pencil method for estimating the parameters of 2-D superimposed, complex valued exponential signals was suggested. Assuming the noise field is *white*, the Cramér–Rao lower bound (CRLB) for this problem was derived as well. The performance of the algorithm was analyzed in [7]. The problem of ML estimation of 2-D superimposed, complex valued exponential signals has been recently considered in [8]. However, most of the works on parametric modeling and estimation of 2-D random fields are concerned with the parameter estimation of real-valued 2-D AR fields, (see, e.g., [4], [5], and [9]–[11]), and the statistical inference of Markov random fields (MRF's) (see, e.g., [12], [13], and the references therein). The underlying assumption in the literature is that the random field is purely indeterministic, and hence, it can be fit with a white- or correlated-noise driven linear model. In [17], we derive an *exact* CRLB on the error variance in jointly estimating the parameters of a complex valued homogeneous Gaussian random field with mixed spectral distribution, using the parametric model that results from the orthogonal decomposition of the field [1]. In this paper, we employ the results of [17] to evaluate the performance of the proposed algorithm for estimating the evanescent components' parameters.

The paper is organized as follows. In Section II, we briefly summarize the results of the 2-D Wold-like decomposition and the derivation of the random field model. In Section III, we define the problem considered in this paper and introduce some necessary notations. In Section IV, we introduce the evanescent to exponential transform (EET) and an algorithm for estimating the spectral support parameters of the evanescent components. The proposed algorithm is based on the properties of the EET. These properties are stated and proved. The algorithm presented in Section IV requires knowledge of second-order moments of the observed field or a reliable estimate thereof. However, in many cases, only a finite dimension, single observed realization of the field is available. Hence, in Section V, we further elaborate on the properties of the EET and on the required modifications in its definition so that it can be applied when only a single realization of the field is available. To illustrate the operation of the proposed algorithm and to get further insight into its properties, some numerical examples are presented in Section VI. Finally, in Section VII, we make some concluding remarks.

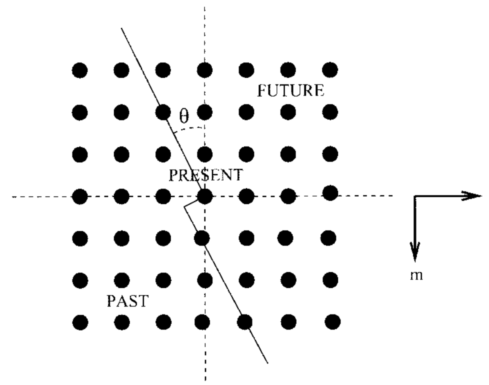

Fig. 1. RNSHP support; example with $\alpha = 2$ and $\beta = 1$.

II. The Homogeneous Random Field Model

The considered random field model is based on the Wold-type decomposition of 2-D regular and homogeneous random fields, presented in [1], and briefly summarized in this section. Let $\{y(n,m), (n,m) \in \mathcal{Z}^2\}$ be a complex-valued, regular, homogeneous random field. Then, $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + v(n,m). \tag{1}$$

The field $\{v(n,m)\}$ is a deterministic random field. The field $\{w(n,m)\}$ is purely indeterministic and has a unique white innovations driven nonsymmetrical half-plane (NSHP) moving average representation given by

$$w(n,m) = \sum_{(0,0)\preceq(k,\ell)} b(k,\ell)u(n-k,m-\ell) \tag{2}$$

where $\sum_{(0,0)\preceq(k,\ell)} |b(k,\ell)|^2 < \infty; b(0,0) = 1$, and $\{u(n,m)\}$ is the innovations field of $\{y(n,m)\}$.

We call a 2-D deterministic random field $\{e_o(n,m)\}$ *evanescent w.r.t. the NSHP total-order* o if it spans a Hilbert space identical to the one spanned by its *column-to-column innovations* at each coordinate (n,m) (w.r.t. the total order o). The deterministic field column-to-column innovation at each coordinate $(n,m) \in \mathcal{Z}^2$ is defined as the difference between the actual value of the field and its projection on the Hilbert space spanned by the deterministic field samples in all previous columns.

It is possible to define [1] a family of NSHP total-order definitions such that the boundary line of the NSHP has a rational slope. Let α and β be two coprime integers such that $\alpha \neq 0$. The angle θ of the slope is given by $\tan\theta = \beta/\alpha$. (See, for example, Fig. 1.) A NSHP of this type is called *rational nonsymmetrical half-plane* (RNSHP). For the case where $\alpha = 0$, the RNSHP is uniquely defined by setting $\beta = 1$. (For the case where $\beta = 0$, the RNSHP is uniquely defined by setting $\alpha = 1$.) We denote by O the set of all possible RNSHP definitions on the 2-D lattice (i.e., the set of all NSHP definitions in which the boundary line of the NSHP has a rational slope). Since it is shown in [1] that interchanging the roles of past and future in any total-order definition results in identical evanescent components, it is sufficient to consider only $0 \leq \theta < \pi$. We therefore assume without limiting the generality of the derivation that $\alpha \geq 0$, whereas β can assume any integer value.

The introduction of the family of RNSHP total-ordering definitions results in the countably infinite orthogonal decomposition of the deterministic component of the random field

$$v(n,m) = p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \quad (3)$$

The random field $\{p(n,m)\}$ is *half-plane deterministic,* i.e., it has no column-to-column innovations w.r.t. any RNSHP total-ordering definition. The field $\{e_{(\alpha,\beta)}(n,m)\}$ is the evanescent component that generates the column-to-column innovations of the deterministic field w.r.t. the RNSHP total-ordering definition $(\alpha,\beta) \in O$.

Hence, if $\{y(n,m)\}$ is a 2-D regular and homogeneous random field, then $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \quad (4)$$

In this paper, all spectral measures are defined on the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$. It is shown in [1] that the spectral measures of the decomposition components in (4) are mutually singular. The spectral distribution function of the purely indeterministic component is absolutely continuous, whereas the spectral measures of the half-plane deterministic component and all the evanescent components are concentrated on a set of Lebesgue measure zero in K. Since for practical applications we can exclude singular-continuous spectral distribution functions from the framework of our treatment, a model for the evanescent field that corresponds to the RNSHP defined by $(\alpha,\beta) \in O$ is given by

$$\begin{aligned} e_{(\alpha,\beta)}(n,m) &= \sum_{i=1}^{I^{(\alpha,\beta)}} e_i^{(\alpha,\beta)}(n,m) \\ &= \sum_{i=1}^{I^{(\alpha,\beta)}} s_i^{(\alpha,\beta)}(n\alpha - m\beta) \\ &\quad \cdot \exp\left(j2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right) \end{aligned} \quad (5)$$

where the 1-D purely indeterministic complex valued processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ and $\{s_j^{(\alpha,\beta)}(n\alpha - m\beta)\}$ are zero-mean and mutually orthogonal for all $i \neq j$. Hence, the "spectral density function" of each evanescent field has the form of a countable sum of 1-D delta functions that are supported on lines of rational slope in the 2-D spectral domain. The slope of the spectral support of each evanescent component is determined by the corresponding (α,β) pair, whereas $\nu_i^{(\alpha,\beta)}$ is the "frequency" parameter of this spectral support.

Define the following parameter transformation as

$$\gamma \triangleq \frac{\alpha}{\alpha^2+\beta^2} \quad (6)$$

$$\delta \triangleq \frac{\beta}{\alpha^2+\beta^2}. \quad (7)$$

We note that the transformed parameters γ, δ are rational numbers. Using (6) and (7), we can rewrite (5) in the form

$$\begin{aligned} e_{(\alpha,\beta)}(n,m) = \sum_{i=1}^{I^{(\alpha,\beta)}} & s_i^{(\alpha,\beta)}(n\alpha - m\beta) \\ & \cdot \exp\big(j2\pi\nu_i^{(\alpha,\beta)}(n\delta + m\gamma)\big). \end{aligned} \quad (8)$$

One of the half-plane-deterministic field components, which is often found in physical problems, is the harmonic random field

$$h(n,m) = \sum_{p=1}^{P} C_p \exp(j2\pi(n\omega_p + m\nu_p)) \quad (9)$$

where the C_p's are mutually orthogonal random variables, and (ω_p, ν_p) are the spatial frequencies of the pth harmonic. In general, P is infinite. The parametric modeling of deterministic random fields whose spectral measures are concentrated on curves other than lines of rational slope, or discrete points in the frequency plane, is still an open question to the best of our knowledge.

III. Estimation of the Evanescent Components Parameters: Problem Definition

The orthogonal decompositions of the previous section imply that if we exclude from the framework of our model those 2-D random fields whose spectral measures are concentrated on curves other than lines of rational slope, $y(n,m)$ is uniquely represented by

$$y(n,m) = w(n,m) + h(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \quad (10)$$

Yet, there are applications, such as in the case of space-time processing of airborne radar data, where the observed field may contain only evanescent random fields embedded in noise. This is the case when no target exists or in secondary data used for estimating and adaptively nulling the interference in the range gate under test. In other applications, such as in that of texture modeling and estimation [22], the deterministic component of the observed field comprises only evanescent components or harmonic components, but not both. In this paper, we concentrate on the problem of estimating the parameters of an observed field given by

$$y(n,m) = w(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m) \quad (11)$$

i.e., when no harmonic component is present. In this framework, the purely indeterministic component can be viewed as an unknown colored noise field.

We next state the assumptions required for proving the results in the next two sections and introduce some necessary notations. Let $\{y(n,m)\}$, $(n,m) \in D$, where $D = \{(i,j) \mid 0 \leq i \leq S-1, 0 \leq j \leq T-1\}$, be the observed random field. Note, however, that the observed field just as well could have any *arbitrary* shape.

Assumption 1: The purely indeterministic component $\{w(n,m)\}$ is a complex-valued field, such that its real and imaginary components are jointly wide sense homogeneous and jointly mean-square ergodic in the first- and second-order moments.

Assumption 2: The number $I = \sum_{(\alpha,\beta)\in O} I^{(\alpha,\beta)}$ of evanescent components in the field is *a priori* known. Note that contrary to [15], here, we do not assume *a priori* knowledge of $I^{(\alpha,\beta)}$ for each $(\alpha,\beta)\in O$.

Assumption 3: For each evanescent field $\{e_i^{(\alpha,\beta)}\}$, the modulating complex valued 1-D purely-indeterministic process $\{s_i^{(\alpha,\beta)}\}$ is a zero-mean process such that its real and imaginary components are jointly wide sense stationary and jointly mean-square ergodic in the first- and second-order moments. It is further assumed that these processes are *not* circular. Let $\mathbf{a}_i^{(\alpha,\beta)}$ denote the parameter vector of $\{s_i^{(\alpha,\beta)}\}$. At the moment, we will not specify the functional dependence of $\mathbf{s}_i^{(\alpha,\beta)}$ on $\mathbf{a}_i^{(\alpha,\beta)}$, but rather leave it implicit. Thus, the parameter vector of each of the evanescent components $\{e_i^{(\alpha,\beta)}\}$ is given by

$$\boldsymbol{\phi}_i^{(\alpha,\beta)} = \left[\alpha,\beta,\nu_i^{(\alpha,\beta)},\left(\mathbf{a}_i^{(\alpha,\beta)}\right)^T\right]^T. \tag{12}$$

Therefore, the parameter vector of the evanescent field $\{e^{(\alpha,\beta)}\}$ is obtained by collecting the vectors $\boldsymbol{\phi}_i^{(\alpha,\beta)}$ into a single-column vector, i.e.,

$$\boldsymbol{\phi}^{(\alpha,\beta)} = \left[\left(\boldsymbol{\phi}_1^{(\alpha,\beta)}\right)^T,\ldots,\left(\boldsymbol{\phi}_{I^{(\alpha,\beta)}}^{(\alpha,\beta)}\right)^T\right]^T. \tag{13}$$

Let $\mathbf{b}$ denote the parameter vector of the purely indeterministic component. At the moment, we will not specify the functional dependence of the purely indeterministic component on $\mathbf{b}$, but rather leave it implicit. Thus, the parameter vector of the observed field $\{y(n,m)\}$ is given by

$$\boldsymbol{\theta} = \left[\mathbf{b}^T,\left\{\left(\boldsymbol{\phi}^{(\alpha,\beta)}\right)^T\right\}_{(\alpha,\beta)\in O}\right]^T. \tag{14}$$

Let

$$\begin{aligned}\mathbf{y} = [&y(0,0),\ldots,y(0,T-1),y(1,0),\ldots\\ &y(1,T-1),\ldots,\ldots,y(S-1,0),\ldots\\ &y(S-1,T-1)]^T\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbf{w} = [&w(0,0),\ldots,w(0,T-1),w(1,0),\ldots\\ &w(1,T-1),\ldots,\ldots,w(S-1,0),\ldots\\ &w(S-1,T-1)]^T\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathbf{e}_i^{(\alpha,\beta)} = \big[&e_i^{(\alpha,\beta)}(0,0),\ldots,e_i^{(\alpha,\beta)}(0,T-1),e_i^{(\alpha,\beta)}(1,0),\ldots\\ &e_i^{(\alpha,\beta)}(1,T-1),\ldots,\ldots\\ &e_i^{(\alpha,\beta)}(S-1,0),\ldots,e_i^{(\alpha,\beta)}(S-1,T-1)\big]^T.\end{aligned} \tag{17}$$

Let

$$\begin{aligned}\boldsymbol{\xi}_i^{(\alpha,\beta)} = \big[&s_i^{(\alpha,\beta)}(0),s_i^{(\alpha,\beta)}(-\beta),\ldots,s_i^{(\alpha,\beta)}(-(T-1)\beta)\\ &s_i^{(\alpha,\beta)}(\alpha),s_i^{(\alpha,\beta)}(\alpha-\beta),\ldots\\ &s_i^{(\alpha,\beta)}(\alpha-(T-1)\beta),\ldots\\ &s_i^{(\alpha,\beta)}((S-1)\alpha),s_i^{(\alpha,\beta)}((S-1)\alpha-\beta),\ldots\\ &s_i^{(\alpha,\beta)}((S-1)\alpha-(T-1)\beta)\big]^T\end{aligned} \tag{18}$$

be the vector whose elements are the observed samples from the 1-D modulating process $\{s_i^{(\alpha,\beta)}\}$. Define the vector of grid points

$$\begin{aligned}\mathbf{v}^{(\alpha,\beta)} = [&0,\alpha,\ldots,(T-1)\alpha,\beta,\beta+\alpha,\ldots\\ &\beta+(T-1)\alpha,\ldots,\ldots,(S-1)\beta\\ &(S-1)\beta+\alpha,\ldots,(S-1)\beta+(T-1)\alpha]^T.\end{aligned} \tag{19}$$

Given a scalar function $f(v)$, we will denote the column vector, consisting of the values of $f(v)$ evaluated for all the elements of $\mathbf{v}$, where $\mathbf{v}$ is a column vector, by $f(\mathbf{v})$. Using this notation, we define

$$\mathbf{d}^{(\alpha,\beta)} = \exp\left(j2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}\mathbf{v}^{(\alpha,\beta)}\right). \tag{20}$$

Thus, using (5), we have that

$$\mathbf{e}_i^{(\alpha,\beta)} = \boldsymbol{\xi}_i^{(\alpha,\beta)}\odot\mathbf{d}^{(\alpha,\beta)} \tag{21}$$

where $\odot$ denotes the Hadamard product of the vectors.

Note that whenever $n\alpha-m\beta=k\alpha-\ell\beta$ for some integers n,m,k,ℓ such that $0\le n,k\le S-1$ and $0\le m,\ell\le T-1$, the same sample of the process $\{s_i^{(\alpha,\beta)}\}$ is repeatedly used in the product form (21). It can be shown that for a rectangular observed field of dimensions $S\times T$, the number of *distinct* samples from the random process $\{s_i^{(\alpha,\beta)}\}$ that are found in the observed field is

$$N_c=(S-1)|\alpha|+(T-1)|\beta|+1-(|\alpha|-1)(|\beta|-1). \tag{22}$$

This is because N_c is the number of different "columns" that can be defined on such a rectangular lattice for an RNSHP defined by (α,β). We therefore define the *concentrated version*, $\mathbf{s}_i^{(\alpha,\beta)}$ of $\boldsymbol{\xi}_i^{(\alpha,\beta)}$ to be an N_c-dimensional column vector of nonrepeating samples of the process $\{s_i^{(\alpha,\beta)}\}$. More specifically, for the case in which $\alpha\ge 0$ and $\beta\ge 0$, $\mathbf{s}_i^{(\alpha,\beta)}$ is given by

$$\mathbf{s}_i^{(\alpha,\beta)} = \left[s_i^{(\alpha,\beta)}(-(T-1)\beta),\ldots,\ldots,s_i^{(\alpha,\beta)}((S-1)\alpha)\right]^T \tag{23}$$

while for the case in which $\alpha\ge 0$ and $\beta<0$, $\mathbf{s}_i^{(\alpha,\beta)}$ is given by

$$\mathbf{s}_i^{(\alpha,\beta)} = \left[s_i^{(\alpha,\beta)}(0),\ldots,\ldots,s_i^{(\alpha,\beta)}((S-1)\alpha-\beta(T-1))\right]^T. \tag{24}$$

Note, however, that due to boundary effects, the vector $\mathbf{s}_i^{(\alpha,\beta)}$ is not composed of consecutive samples from the process $\{s_i^{(\alpha,\beta)}\}$ unless $|\alpha|\le 1$ or $|\beta|\le 1$. In other words, for some arbitrary α and β, there are missing samples in $\mathbf{s}_i^{(\alpha,\beta)}$.

Thus, for any (α,β), we have that

$$\boldsymbol{\xi}_i^{(\alpha,\beta)} = \mathbf{A}_i^{(\alpha,\beta)}\mathbf{s}_i^{(\alpha,\beta)} \tag{25}$$

where $\mathbf{A}_i^{(\alpha,\beta)}$ is rectangular matrix of zeros and ones that replicates rows of $\mathbf{s}_i^{(\alpha,\beta)}$.

IV. The Evanescent to Exponential Transform

In this section, we introduce the evanescent to exponential transform (EET) and an algorithm for estimating the parameters of the evanescent components. The proposed algorithm is based on the EET and its properties. To simplify the presentation, we first describe the EET and the resulting algorithm for the case where the observed field is a sum of evanescent components, i.e., the case where no purely indeterministic component exists in (11). In Section IV-B, we extend the results of Section IV-A to the case of a nonzero purely indeterministic component.

A. Estimation in the Case of a Zero Purely Indeterministic Component

Assume that the observed field is a sum of evanescent components, i.e.,

$$y(n,m) = \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \tag{26}$$

Definition 1: Let τ_m and τ_n be some finite integers. Define

$$\mathcal{M}_2(y(n,m),\tau_n,\tau_m) = E\{y(n+\tau_n, m+\tau_m)y(n,m)\}. \tag{27}$$

The unconjugated correlation operator $\mathcal{M}_2(\cdot,\tau_n,\tau_m)$ is employed as the basic building block of the proposed algorithm for estimating the evanescent components' spectral support parameters. The properties of the mapping induced by applying $\mathcal{M}_2(\cdot,\tau_n,\tau_m)$ to an evanescent field are stated and proved in the next two theorems.

Theorem 1: Let $\{y(n,m)\}$ be given by (26). Then, $\mathcal{M}_2(y(n,m),\tau_n,\tau_m)$ is a sum of constant amplitude 2-D exponentials given by

$$\begin{aligned}\mathcal{M}_2&(y(n,m),\tau_n,\tau_m)\\ &= \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} \mathcal{M}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big)\\ &= \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} \mathcal{C}_i\big(\alpha,\beta,\nu_i^{(\alpha,\beta)},\tau_n,\tau_m\big)\\ &\quad\cdot \exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta+2m\gamma)\big)\end{aligned} \tag{28}$$

where the coefficients $\mathcal{C}_i(\alpha,\beta,\nu_i^{(\alpha,\beta)},\tau_n,\tau_m)$ are neither functions of n nor m.

Proof: Substituting (26) into (27), we have

$$\begin{aligned}\mathcal{M}_2&(y(n,m),\tau_n,\tau_m)\\ &= E\left\{\sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} e_i^{(\alpha,\beta)}(n+\tau_n,m+\tau_m)\right.\\ &\quad\left.\cdot\sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} e_i^{(\alpha,\beta)}(n,m)\right\}\\ &= \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} \big\{E\{s_i^{(\alpha,\beta)}((n+\tau_n)\alpha\\ &\quad -(m+\tau_m)\beta)s_i^{(\alpha,\beta)}(n\alpha-m\beta)\}\\ &\quad\cdot\exp\big(j2\pi\nu_i^{(\alpha,\beta)}(\tau_n\delta+\tau_m\gamma)\big)\big\}\\ &\quad\cdot\exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta+2m\gamma)\big)\\ &= \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} \big\{\mathcal{K}_i^{(\alpha,\beta)}(\tau_n\alpha-\tau_m\beta)\\ &\quad\cdot\exp\big(j2\pi\nu_i^{(\alpha,\beta)}(\tau_n\delta+\tau_m\gamma)\big)\big\}\\ &\quad\cdot\exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta+2m\gamma)\big)\\ &= \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}} \mathcal{C}_i\big(\alpha,\beta,\nu_i^{(\alpha,\beta)},\tau_n,\tau_m\big)\\ &\quad\cdot\exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta+2m\gamma)\big)\end{aligned} \tag{29}$$

where the second equality is due to the mutual orthogonality of the different evanescent components. The third equality is due to the fact that the real and imaginary components of each of the $\{s_i^{(\alpha,\beta)}\}$ processes are jointly wide sense stationary. $\mathcal{K}_i^{(\alpha,\beta)}$ denotes the unconjugated second-order moment of $\{s_i^{(\alpha,\beta)}\}$, and we define

$$\begin{aligned}\mathcal{C}_i&\big(\alpha,\beta,\nu_i^{(\alpha,\beta)},\tau_n,\tau_m\big)\\ &= \mathcal{K}_i^{(\alpha,\beta)}(\tau_n\alpha-\tau_m\beta)\cdot\exp\big(j2\pi\nu_i^{(\alpha,\beta)}(\tau_n\delta+\tau_m\gamma)\big).\end{aligned} \tag{30}$$

□

Thus, if the observed field $y(n,m)$ is a sum of evanescent components, $\mathcal{M}_2(y(n,m),\tau_n,\tau_m)$ is a sum of constant amplitude 2-D exponentials. Hence, we name $\mathcal{M}_2(y(n,m),\tau_n,\tau_m)$ the EET. In the following, we use the notation (ω_q,ϕ_q), $q=1\ldots I$, where $I=\sum_{(\alpha,\beta)\in O} I^{(\alpha,\beta)}$ to denote the spatial frequency of each exponential.

The spectral support parameters of each evanescent component are related to the spatial frequency (ω_q,ϕ_q) of a corresponding exponential through the system

$$\frac{2\nu_i^{(\alpha,\beta)}\beta}{\alpha^2+\beta^2} = \omega_q \tag{31}$$

$$\frac{2\nu_i^{(\alpha,\beta)}\alpha}{\alpha^2+\beta^2} = \phi_q. \tag{32}$$

It is easy to verify using (31) and (32) that $(\omega_q,\phi_q)=(0,0)$ if and only if $\nu_i^{(\alpha,\beta)}=0$. Hence, if $\nu_i^{(\alpha,\beta)}=0$ for more than one component, all these components are mapped to a single exponential whose frequency is $(0,0)$. Let $\tilde{I}$ be the total number of evanescent components in the field such that $\nu_i^{(\alpha,\beta)}\neq 0$. In this and the next subsections, we elaborate on the problem of estimating the parameters of evanescent components with $\nu_i^{(\alpha,\beta)}\neq 0$. The case where for some q, $(\omega_q,\phi_q)=(0,0)$ is treated separately in Section IV-C.

Note that in general, depending on the values of α,β and $\nu_i^{(\alpha,\beta)}$, cyclic frequency folding may lead to a noninvertible mapping to (ω_q,ν_q). A necessary and sufficient condition to avoid this problem is that $(\omega_q,\phi_q)\in K$, which guarantees that no ambiguities exist. In other words, the condition to avoid nonunique mapping of the evanescent components' spectral support parameters is that for all $(\alpha,\beta) \in O$ and for all $i = 1 \ldots I^{(\alpha,\beta)}$

$$\frac{2|\nu_i^{(\alpha,\beta)}| \max(|\alpha|, |\beta|)}{\alpha^2+\beta^2} < 0.5. \tag{33}$$

However, because $|\nu_i^{(\alpha,\beta)}| < 0.5$ while α and β are integers, it is straightforward to verify that except for the cases where $(\alpha,\beta)=(1,0)$ or $(\alpha,\beta)=(0,1)$, (33) is satisfied for all α,β, and $\nu_i^{(\alpha,\beta)}$. If components with $(\alpha,\beta)=(0,1)$ [and similarly for $(\alpha,\beta)=(1,0)$] exist in the observed field, their frequency parameter $\nu_i^{(0,1)}$ can be recovered using the relations in (31) and (32) only up to a ± 0.5 shift. [The term 0.5 is due to the fact that the folding of the exponential frequency is by 1, whereas the estimate of $\nu_i^{(0,1)}$ is obtained by dividing the estimated exponential frequency by a factor of 2.] As a consequence, two different evanescent components with frequency parameters $\nu_i^{(0,1)}$ and $\nu_j^{(0,1)}$, such that $|\nu_i^{(0,1)} - \nu_j^{(0,1)}| = 0.5$, are mapped by the EET to the same exponential. Later in this section, we present an algorithmic solution to this nonuniqueness of the EET.

In the next theorem, we show that if $\nu_i^{(\alpha,\beta)} = 0$ for no more than a single evanescent component, then except for the foregoing two special cases, each evanescent component is mapped to a unique exponential.

Theorem 2: Assume that for all $(\alpha,\beta) \in O$ and $i = 1 \ldots I^{(\alpha,\beta)}$, $\nu_i^{(\alpha,\beta)} \neq 0$. Then, for all $(\alpha,\beta) \neq (1,0)$, $(\alpha,\beta) \neq (0,1)$, the mapping of the spectral support parameters $(\alpha,\beta), \nu_i^{(\alpha,\beta)}$ of each evanescent component in the field to an exponential's frequency (ω_q,ϕ_q) is unique. For $(\alpha,\beta)=(1,0)$ and $(\alpha,\beta)=(0,1)$, the mapping is unique up to a shift of ± 0.5 in $\nu_i^{(1,0)}$ and $\nu_i^{(0,1)}$, respectively.

Proof: Assume that two evanescent components with different spectral support parameters $(\alpha_1,\beta_1),\nu_1$, and $(\alpha_2,\beta_2),\nu_2$, respectively, are mapped by the operator $\mathcal{M}_2(\cdot,\tau_n,\tau_m)$ to two exponentials of identical frequency (ω,ϕ). Let us first consider the case where $\phi \neq 0$ and $\omega \neq 0$. From the assumption and (31) and (32), we have that $\frac{\beta_2}{\alpha_2} = \frac{\beta_1}{\alpha_1}$. Since both (α_1,β_1) and (α_2,β_2) are pairs of coprime integers, we conclude that $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$. Hence, $\nu_1=\nu_2$ as well.

We now consider the case where $\phi = 0$. Since by assumption, $\nu_1 \neq 0$ and $\nu_2 \neq 0$, we have from (32) that $\alpha_1=\alpha_2=0$. Hence, by the RNSHP definition, we have that both $\beta_1 = 1$ and $\beta_2=1$. Substituting these values into (31), the assumption implies that $\nu_1=\nu_2$ or that $\nu_1=\nu_2 \pm 0.5$. A similar argument holds for the case where $\omega = 0$. □

The problem of estimating the spectral support parameters of the evanescent components can now be stated as follows. Given the estimated nonzero spatial frequencies (ω_q,ϕ_q), $q=1\ldots\tilde{I}$, solve for each evanescent component the system (31) and (32) to obtain the spectral support parameters. The system has to be solved under the constraints that α and β are coprime integers, $\alpha \geq 0$, and $0 < |\nu_i^{(\alpha,\beta)}| < \frac{1}{2}$.

If components with $(\alpha,\beta)=(0,1)$ (and similarly for $(\alpha,\beta)=(1,0)$) are detected by an EET-based estimation procedure, their frequency estimate $\nu_i^{(0,1)}$ should be independently verified, using some other procedure. Possible procedures are the solution of the 2-D normal equations, which was used for initializing the ML algorithm in [14], or simply the inspection of the absolute value of the field's 2-D Fourier transform. If a discrepancy close to ± 0.5 between the estimate of $\nu_i^{(0,1)}$ obtained using (31) and the one obtained using the verification procedure is detected, the estimate obtained using (31) is increased/decreased by 0.5. We further note that since the estimate obtained using the verification procedure is required only for the purpose of possible ± 0.5 adjustment of $\nu_i^{(0,1)}$ and the separation of two components with $|\nu_i^{(0,1)} - \nu_j^{(0,1)}| = 0.5$, the verification estimator can be of a low accuracy. To simplify the presentation, we assume in the following that no two evanescent components with $(\alpha,\beta)=(0,1)$ are such that $|\nu_i^{(0,1)} - \nu_j^{(0,1)}| = 0.5$. Next, we give a detailed description of the proposed algorithm. We begin by introducing the definition of a new operator.

Definition 2: Let τ_m and τ_n be some finite integers. Let $\mathcal{FM}_2(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi)$ be defined as the Fourier transform of $\mathcal{M}_2(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m)$, i.e.,

$$\begin{aligned}\mathcal{FM}_2&\big(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi\big)\\ &= \sum_{n=0}^{T-1}\sum_{m=0}^{S-1} \mathcal{M}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big)\\ &\quad \cdot \exp(-j2\pi(n\omega+m\phi)).\end{aligned} \tag{34}$$

In general, $\{y(n,m)\}$ has multiple evanescent components. From Theorems 1 and 2, we have that applying the EET operator to $\{y(n,m)\}$ produces a sum of $\tilde{I}$ constant amplitude complex exponentials, whose frequencies $(\omega_q,\phi_q) \neq (0,0)$, whereas all the evanescent components for which $\nu_i^{(\alpha,\beta)} = 0$ are mapped to $(0,0)$. Hence, $\mathcal{FM}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$ has exactly $\tilde{I}$ spectral peaks at frequencies $(\omega_q,\phi_q) \neq (0,0)$.

Recall that for each evanescent component, the slope parameter of the boundary line of the corresponding RNSHP is defined by the ratio of two coprime integers α and β such that $\tan\theta = \frac{\beta}{\alpha}$. For each component, if $\omega_q = 0$ (in practice, $|\omega_q| < \epsilon$, where ϵ is a small predetermined constant), we decide that $(\hat{\alpha},\hat{\beta}) = (1,0)$, and hence, $\hat{\nu}_i^{(1,0)} = \frac{\phi_q}{2}$. This estimate of $\nu_i^{(1,0)}$ is then verified as explained in the foregoing discussion and adjusted by ± 0.5, if required. An identical procedure is applied if $\phi_q = 0$, where we decide $(\hat{\alpha},\hat{\beta}) = (0,1)$, and $\hat{\nu}_i^{(0,1)} = \frac{\omega_q}{2}$.

If $(\hat{\alpha},\hat{\beta}) \neq (0,1)$, $(\hat{\alpha},\hat{\beta}) \neq (1,0)$ then using (31) and (32), we find that $\frac{\beta}{\alpha} = \frac{\omega_q}{\phi_q}$. We thus search for a coprime integer pair (k,ℓ), $k>0$ such that $\frac{\omega_q}{\phi_q} = \frac{\ell}{k}$ (and, in practice, $|\frac{\omega_q}{\phi_q} - \frac{\ell}{k}| < \epsilon$). These (k,ℓ) pairs, when substituted into (31) and (32) instead of (α,β), should yield a valid $\hat{\nu}_i^{(\alpha,\beta)}$ estimate, i.e., $|\hat{\nu}_i^{(\alpha,\beta)}| < 0.5$. Thus, we consider only (k,ℓ) pairs that satisfy both

$$\left|\frac{\omega_q(k^2+\ell^2)}{2k}\right| < 0.5 \tag{35}$$

and

$$\left|\frac{\phi_q(k^2+\ell^2)}{2\ell}\right| < 0.5. \tag{36}$$

TABLE I
ESTIMATION ALGORITHM OF THE EVANESCENT COMPONENTS SPECTRAL SUPPORT PARAMETERS

0.	Let $\tilde{I}$ denote the total number of evanescent components in the field, such that $\nu_i^{(\alpha,\beta)} \neq 0$.
1.	Find the spatial frequencies (ω_q, ϕ_q) $q = 1 \ldots \tilde{I}$ of the $\tilde{I}$ prominent spectral peaks of $\lvert \mathcal{F}\mathcal{M}_2(y(n,m); \tau_n, \tau_m, \omega, \phi) \rvert$, such that $(\omega_q, \phi_q) \neq (0,0)$.
2.	For each evanescent component such that $\omega_q = 0$, set $(\hat{\alpha}, \hat{\beta}) = (1,0)$ and $\hat{\nu}^{(1,0)} = \frac{\phi_q}{2}$. Add ± 0.5 to $\hat{\nu}^{(1,0)}$, if frequency folding is detected.
3.	For each evanescent component such that $\phi_q = 0$, set $(\hat{\alpha}, \hat{\beta}) = (0,1)$ and $\hat{\nu}^{(0,1)} = \frac{\omega_q}{2}$. Add ± 0.5 to $\hat{\nu}^{(0,1)}$, if frequency folding is detected.
4.	For each one of the remaining evanescent components, find the coprime integer pair (k, ℓ) such that $0 < k$ and $\frac{\omega_q}{\phi_q} = \frac{\ell}{k}$.
5.	For each evanescent component of step 4, set $(\hat{\alpha}, \hat{\beta}) = (k, \ell)$ and $\hat{\nu}^{(\alpha,\beta)} = \frac{1}{2}\left(\frac{\omega_q(k^2+\ell^2)}{2k} + \frac{\phi_q(k^2+\ell^2)}{2\ell}\right)$.

The algorithm is summarized in Table I.

B. Estimation in the Presence of a Purely Indeterministic Component

In Theorem 1, it is proved that in the absence of the purely indeterministic component $\mathcal{M}_2(y(n,m), \tau_n, \tau_m)$ is a sum of constant amplitude 2-D exponentials. Next, we show that a minor modification of the same result holds for the more general case in which the observed signal consists of the sum of multiple evanescent components and a purely indeterministic component.

Theorem 3: Let $\{y(n,m)\}$ be given by (11). Then, $\mathcal{M}_2(y(n,m), \tau_n, \tau_m)$ is a sum of constant amplitude 2-D exponentials given by

$$\begin{aligned} &\mathcal{M}_2(y(n,m), \tau_n, \tau_m) \\ &\quad = \mathcal{C}_0(w(n,m), \tau_n, \tau_m) \\ &\qquad + \sum_{(\alpha,\beta)\in O} \sum_{i=1}^{I^{(\alpha,\beta)}} \mathcal{M}_2\big(e_i^{(\alpha,\beta)}(n,m), \tau_n, \tau_m\big) \\ &\quad = \mathcal{C}_0(w(n,m), \tau_n, \tau_m) \\ &\qquad + \sum_{(\alpha,\beta)\in O} \sum_{i=1}^{I^{(\alpha,\beta)}} \mathcal{C}_i\big(\alpha, \beta, \nu_i^{(\alpha,\beta)}, \tau_n, \tau_m\big) \\ &\qquad \cdot \exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta + 2m\gamma)\big) \end{aligned} \tag{37}$$

where the coefficients $\mathcal{C}_i(\alpha, \beta, \nu_i^{(\alpha,\beta)}, \tau_n, \tau_m)$ are functions of neither n nor m.

Proof: Substituting (11) into (27), we have

$$\begin{aligned} &\mathcal{M}_2(y(n,m), \tau_n, \tau_m) \\ &\quad = E\left\{ \sum_{(\alpha,\beta)\in O} \sum_{i=1}^{I^{(\alpha,\beta)}} e_i^{(\alpha,\beta)}(n+\tau_n, m+\tau_m) e_i^{(\alpha,\beta)}(n,m) \right\} \\ &\qquad + E\{w(n+\tau_n, m+\tau_m)w(n,m)\} \end{aligned} \tag{38}$$

since the purely indeterministic component and the different evanescent components are mutually orthogonal. Because the real and imaginary components of $\{w(n,m)\}$ are jointly wide sense homogeneous, $E[w(n+\tau_n, m+\tau_m)w(n,m)]$ is a function of neither n nor m. Thus, $\mathcal{C}_0(w(n,m), \tau_n, \tau_m) = E[w(n+\tau_n, m+\tau_m)w(n,m)]$ is a constant. Hence, repeating the arguments of the proof of Theorem 1, (37) follows.

C. Estimation of the Spectral Support for Components With Zero Frequency Parameter

In Section IV-A, it is shown that whenever the frequency parameter $\nu_i^{(\alpha,\beta)}$ of an evanescent component is zero, the EET maps this evanescent component to an exponential whose frequency is $(0,0)$, regardless of the values of α and β. This nonuniqueness implies that the EET cannot produce a complete parameter estimate of the spectral support parameters of these evanescent components. Hence, a different operator must be applied to the observed field in order to estimate the (α, β) pairs of the evanescent components whose frequency parameter is zero.

In this section, it is assumed that the spectral support parameters of the $\tilde{I}$ evanescent components, whose frequency parameter is nonzero, have already been estimated using the estimation algorithm summarized in Table I. Hence, the algorithm proposed in this section is designed to estimate the (α, β) pairs of evanescent components for which it is *already known* that their frequency parameter is zero. The spectral measure of these evanescent components is concentrated on lines with rational slope that cross the $(0,0)$ frequency.

The proposed algorithm is a modified version of the Hough transform for detecting straight lines in 2-D arrays [20]. This modification employs the *a priori* knowledge that the frequency parameter of the evanescent components the algorithm is looking for, is zero, as well as of the fact that the spectral support of each one of these components is a line in the frequency plane such that its slope is defined by two coprime integers α and β. Initially, the algorithm identifies the peaks of the field periodogram. In the presence of evanescent components, these peaks are concentrated along lines. On a finite-dimension observed field, only a finite number of (α, β) pairs may be defined. (This is because α and β are integers representing distances between consecutive samples along the "rows" and "columns" defined with respect to the RNSHP total-ordering definition $(\alpha, \beta) \in O$). Therefore, for given dimensions of the observed field, we search among all possible combinations of the spectral support parameters α and β for the $I - \tilde{I}$ pairs of (α, β) that best explain the concentration of peaks along lines that cross the $(0,0)$ frequency.

D. Obtaining a Complete Estimate of the Model Parameters

Using the estimated spectral support parameters of each evanescent component (α, β), $\nu_i^{(\alpha,\beta)}$, several alternatives for estimating the other parameters of the field are possible. We first briefly summarize the method developed in [18]. Note from the evanescent field model (5) that for a fixed $c = n\alpha - m\beta$ (i.e., along a line on the sampling grid), the samples of the evanescent component are nothing but the samples of constant amplitude exponential signal. Multiplying the observed signal $y(n,m)$ by $\exp(-j2\pi \frac{\hat{\nu}_i^{(\hat{\alpha},\hat{\beta})}}{\hat{\alpha}^2+\hat{\beta}^2}(n\hat{\beta}+m\hat{\alpha}))$ and evaluating the arithmetic mean of this signal along a line on the sampling grid such that $c = n\alpha - m\beta$, we have

$$\hat{s}_i^{(\alpha,\beta)}(c) = \frac{1}{N_s} \sum_{n\hat{\alpha}-m\hat{\beta}=c} y(n,m) \cdot \exp\left(-j2\pi \frac{\hat{\nu}_i^{(\alpha,\beta)}}{\hat{\alpha}^2+\hat{\beta}^2}(n\hat{\beta}+m\hat{\alpha})\right) \quad (39)$$

where N_s denotes the number of the observed field samples that satisfy the relation $n\alpha - m\beta = c$. Substituting (11) into the right-hand side of (39), it is easy to verify that (39) indeed provides an estimate of $s_i^{(\alpha,\beta)}(c)$ since the arithmetic mean of each of the terms of $y(n,m)\exp(-j2\pi \frac{\hat{\nu}_i^{(\hat{\alpha},\hat{\beta})}}{\hat{\alpha}^2+\hat{\beta}^2}(n\hat{\beta}+m\hat{\alpha}))$ along the line $n\hat{\alpha} - m\hat{\beta} = c$ tends to zero, except the DC term, which tends to $s_i^{(\alpha,\beta)}(c)$. This procedure provides estimates of the 1-D sequence $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ of the evanescent field.

Having obtained the sequence of estimated samples from the 1-D modulating process $\{s_i^{(\alpha,\beta)}\}$, the problem of estimating its parametric model becomes entirely a 1-D estimation problem. Applying to the sequence any parameter estimation algorithm that corresponds to the model of this complex-valued process (e.g., 1-D AR, MA, ARMA), we obtain estimates of the modulating process parameters as well.

The residual field, after all the evanescent components have been subtracted, is the purely indeterministic component of the observed field. Its parametric model can now be estimated using existing estimation methods of purely indeterministic random fields (e.g., an AR model, [9], [14]). Note that in this case, where the observed field has only a purely indeterministic component, the procedure of obtaining a maximum-likelihood estimate of the AR model parameters [14] is reduced to a solution of a linear least squares problem.

A different approach to obtaining a complete estimate of the observed field model is to obtain a least-squares estimate of its parameters. Substitution of the unknown spectral support parameters of the evanescent components with the estimated ones reduces a highly nonlinear LS problem to a *linear* LS. (See [15] for details).

V. The EET in the Case of a Single Observed Realization of the Field

The EET and the algorithm that employs it (see Table I) are formulated in terms of unconjugated second order moments of $y(n,m)$. Hence, knowledge of these moments, or a reliable estimate thereof, are required. However, in many cases, only a finite-dimension, single-observed realization of the field is available. Hence, in this section, we elaborate on the properties of the EET and on the required modifications in its definition so that it can be applied when only a finite, single-observed realization of the field is available.

From Theorem 3, we have that

$$\begin{aligned}\mathcal{M}_2(y(n,m),\tau_n,\tau_m) &= \mathcal{C}_0(w(n,m),\tau_n,\tau_m) \\ &\quad + \sum_{(\alpha,\beta)\in O} \sum_{i=1}^{I^{(\alpha,\beta)}} \mathcal{M}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big). \end{aligned} \quad (40)$$

To simplify the notations, we first address the problem of estimating $\mathcal{M}_2(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m)$ for a single evanescent component, assuming a zero purely indeterministic component. In that case

$$\begin{aligned}\mathcal{M}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big) &= \big\{\mathcal{K}_i^{(\alpha,\beta)}(\tau_n\alpha - \tau_m\beta)\cdot \exp\big(j2\pi\nu_i^{(\alpha,\beta)}(\tau_n\delta+\tau_m\gamma)\big)\big\} \\ &\quad \cdot \exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta+2m\gamma)\big). \end{aligned} \quad (41)$$

The term $\mathcal{K}_i^{(\alpha,\beta)}(\tau_n\alpha - \tau_m\beta)$ in (41) is the unconjugated second-order moment of a complex-valued process whose real and imaginary components are jointly wide sense stationary and jointly mean-square ergodic in the first- and second-order moments. Hence, it can be consistently estimated by replacing the ensemble average with sample average.

Let $\mathbf{f}$ be an N_c-dimensional vector whose entries are the indices of the samples from $\{s_i^{(\alpha,\beta)}\}$ in (23) [or (24)]. We therefore define the sample unconjugated second-order moment

$$\begin{aligned}\hat{\mathcal{K}}_i^{(\alpha,\beta)}(\tau_n\alpha - \tau_m\beta) &= \frac{1}{N_c} \sum_{\substack{(k\alpha-\ell\beta)\in\mathbf{f} \\ ((k+\tau_n)\alpha-(\ell+\tau_m)\beta)\in\mathbf{f}}} s_i^{(\alpha,\beta)}((k+\tau_n)\alpha - (\ell+\tau_m)\beta) \\ &\quad s_i^{(\alpha,\beta)}(k\alpha - \ell\beta). \end{aligned} \quad (42)$$

Recall that in Section III, we have shown that $\boldsymbol{\xi}_i^{(\alpha,\beta)}$ contains replicated elements. Hence, we can extend the summation in (42) and sum the products of the type $s_i^{(\alpha,\beta)}((k+\tau_n)\alpha - (\ell+\tau_m)\beta)s_i^{(\alpha,\beta)}(k\alpha-\ell\beta)$ over the entire field, and average it appropriately. Hence, (42) gets the form

$$\begin{aligned}\hat{\mathcal{K}}_i^{(\alpha,\beta)}(\tau_n\alpha - \tau_m\beta) &= \frac{1}{ST}\sum_{k=0}^{S-1}\sum_{\ell=0}^{T-1} e_i^{(\alpha,\beta)}(k+\tau_n,\ell+\tau_m)e_i^{(\alpha,\beta)}(k,\ell) \\ &\quad \cdot \exp\big(-j2\pi\nu_i^{(\alpha,\beta)}((k+\tau_n)\delta + (\ell+\tau_m)\gamma)\big) \\ &\quad \cdot \exp\big(-j2\pi\nu_i^{(\alpha,\beta)}(k\delta+\ell\gamma)\big) \end{aligned} \quad (43)$$

where the last equality is due to (5). Thus, using (41) and (43), we conclude that an estimate of $\mathcal{M}_2(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m)$ is obtained by replacing $\mathcal{K}_i^{(\alpha,\beta)}(\tau_n\alpha - \tau_m\beta)$ in (41) with $\hat{\mathcal{K}}_i^{(\alpha,\beta)}(\tau_n\alpha - \tau_m\beta)$. We therefore have

$$\begin{aligned}\hat{\mathcal{M}}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big) &= \frac{1}{ST}\sum_{k=0}^{S-1}\sum_{\ell=0}^{T-1} \big\{e_i^{(\alpha,\beta)}(k+\tau_n,\ell+\tau_m)e_i^{(\alpha,\beta)}(k,\ell) \\ &\quad \cdot \exp\big(-j2\pi\nu_i^{(\alpha,\beta)}(2k\delta+2\ell\gamma)\big)\big\} \\ &\quad \cdot \exp\big(j2\pi\nu_i^{(\alpha,\beta)}(2n\delta+2m\gamma)\big). \end{aligned} \quad (44)$$

Note, that (44) is the 2-D Fourier series expansion of $\hat{\mathcal{M}}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big)$. The series has a single term. The coefficient of this term is the 2-D Fourier transform applied to the product signal $e_i^{(\alpha,\beta)}(n+\tau_n, m+\tau_m)e_i^{(\alpha,\beta)}(n,m)$ evaluated at some frequency $(2\delta\nu_i^{(\alpha,\beta)}, 2\gamma\nu_i^{(\alpha,\beta)})$ and scaled by a constant. Since $2\delta\nu_i^{(\alpha,\beta)}$ and $2\gamma\nu_i^{(\alpha,\beta)}$ are unknown, in principle, this expression has to be evaluated for all frequencies.

Thus, in the estimation algorithm, we replace the EET operator $\mathcal{M}_2(\cdot,\tau_n,\tau_m)$, which uses ensemble moments, with the $\hat{\mathcal{M}}_2(\cdot,\tau_n,\tau_m)$ operator, which uses sample moments. More specifically, the step in which we evaluate $(\hat{\omega}_q,\hat{\phi}_q)$ (step 1 in Table I) is replaced in this single component case by

$$(\hat{\omega}_q,\hat{\phi}_q) = \underset{(\omega,\phi)\in K}{\operatorname{argmax}} \left|\widetilde{\mathcal{FM}}_2\big(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi\big)\right| \quad (45)$$

where

$$\begin{aligned}\widetilde{\mathcal{FM}}_2\big(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi\big) &= \sum_{n=0}^{T-1}\sum_{m=0}^{S-1}\hat{\mathcal{M}}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big)\\ &\quad\cdot\exp(-j2\pi(n\omega+m\phi)).\end{aligned} \quad (46)$$

Using (44) and (46), we have

$$\begin{aligned}&\left|\widetilde{\mathcal{FM}}_2\big(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi\big)\right|\\ &= \frac{1}{ST}\left|\sum_{k=0}^{S-1}\sum_{\ell=0}^{T-1}\left\{e_i^{(\alpha,\beta)}(k+\tau_n,\ell+\tau_m)e_i^{(\alpha,\beta)}(k,\ell)\right.\right.\\ &\quad\left.\left.\cdot\exp\big(-j2\pi\nu_i^{(\alpha,\beta)}(2k\delta+2\ell\gamma)\big)\right\}\right|\\ &\quad\cdot\left|\sum_{n=0}^{T-1}\sum_{m=0}^{S-1}\exp\Big[-j2\pi\Big(\big(\omega-2\nu_i^{(\alpha,\beta)}\delta\big)n\right.\\ &\quad\left.+\big(\phi-2\nu_i^{(\alpha,\beta)}\gamma\big)m\Big)\Big]\right|.\end{aligned} \quad (47)$$

Inspecting (47), it is clear that evaluating the Fourier transform of the product signal $e_i^{(\alpha,\beta)}(n+\tau_n,m+\tau_m)e_i^{(\alpha,\beta)}(n,m)$ for all (ω,ϕ) and setting

$$(\hat{\omega}_q,\hat{\phi}_q) = \underset{(\omega,\phi)\in K}{\operatorname{argmax}} \left|\widehat{\mathcal{FM}}_2\big(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi\big)\right| \quad (48)$$

where

$$\begin{aligned}\widehat{\mathcal{FM}}_2\big(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi\big) &= \sum_{n=0}^{T-1}\sum_{m=0}^{S-1}e_i^{(\alpha,\beta)}(n+\tau_n,m+\tau_m)e_i^{(\alpha,\beta)}(n,m)\\ &\quad\cdot\exp(-j2\pi(n\omega+m\phi))\end{aligned} \quad (49)$$

is equivalent to estimating $(\hat{\omega}_q,\hat{\phi}_q)$ using (45).

Recall that $\{s_i^{(\alpha,\beta)}\}$ is a zero-mean process such that its real and imaginary components are jointly wide sense stationary and jointly mean-square ergodic in the first- and second-order moments. Hence, using the derivation of the $\hat{\mathcal{M}}_2(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m)$ estimator in (43) and (44), it is clear that

$$\begin{aligned}&\lim_{S,T\to\infty}\hat{\mathcal{M}}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big)\\ &\quad= \mathcal{M}_2\big(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m\big)\end{aligned} \quad (50)$$

in the mean square sense. From Theorem 1, we have that $\mathcal{M}_2(e_i^{(\alpha,\beta)}(n,m),\tau_n,\tau_m)$ is a *constant* amplitude exponential with the correct frequency (ω_q,ϕ_q). In other words, the ergodicity of $\{s_i^{(\alpha,\beta)}\}$ guarantees that as $S\to\infty$ and $T\to\infty$, $(\hat{\omega}_q,\hat{\phi}_q)=(\omega_q,\phi_q)$ in the mean square sense.

Alternatively, from the derivation of the estimator, and the proof of Theorem 1, it is clear that the weighting term $\exp(-j2\pi\nu_i^{(\alpha,\beta)}(2k\delta+2\ell\gamma))$ in (44) suppresses the oscillatory behavior of the sample moment. Since in our application we are nonetheless interested in detecting the frequency of this oscillation and not in estimating the moments themselves, we employ the statistic $\widehat{\mathcal{FM}}_2(e_i^{(\alpha,\beta)}(n,m);\tau_n,\tau_m,\omega,\phi)$.

We further note that since (49) is the Fourier transform of the product field $e_i^{(\alpha,\beta)}(n+\tau_n,m+\tau_m)e_i^{(\alpha,\beta)}(n,m)$, it can be evaluated on a discrete 2-D grid of (ω,ϕ) via the 2-D FFT. In fact, due to the properties of the EET, any algorithm for estimating the frequencies of 2-D exponentials can be used as a substitute to the FFT-based implementation we have chosen (see, e.g., [6] and [8]). The reason for our choice is, of course, the simplicity of the implementation.

Let us return now to the general case where the observed field $\{y(n,m)\}$ consists of a purely indeterministic component and multiple evanescent components. Since the purely indeterministic component and all the evanescent components are mutually orthogonal, we compute the statistic

$$\begin{aligned}\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi) &= \sum_{n=0}^{T-1}\sum_{m=0}^{S-1}[y(n+\tau_n,m+\tau_m)y(n,m)-A(\tau_n,\tau_m)]\\ &\quad\cdot\exp(-j2\pi(n\omega+m\phi))\end{aligned} \quad (51)$$

where $A(\tau_n,\tau_m)$ denotes the sample mean of the product field $y(n+\tau_n,m+\tau_m)y(n,m)$, i.e.,

$$A(\tau_n,\tau_m) = \frac{1}{ST}\sum_{n=0}^{T-1}\sum_{m=0}^{S-1}y(n+\tau_n,m+\tau_m)y(n,m). \quad (52)$$

The subtraction of the mean of the product field is meant to eliminate the contribution to $\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$ of the purely indeterministic component through $\mathcal{C}_0(w(n,m),\tau_n,\tau_m)$, as well as the contributions of evanescent components with $\nu_i^{(\alpha,\beta)}=0$.

Applying the foregoing reasoning to the multicomponent case, we conclude that the spectral support parameters of the $\tilde{I}$ evanescent components whose frequency parameter $\nu_i^{(\alpha,\beta)}$ is nonzero are found by estimating the spatial frequencies of the $\tilde{I}$ prominent peaks of $|\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)|$. Let us denote this set of estimates by $\{(\hat{\omega}_q,\hat{\phi}_q)\}_{q=1}^{\tilde{I}}$.

Recall that the purely indeterministic component and the different evanescent components are mutually orthogonal. Since each one of these fields has a zero-mean, while its real and

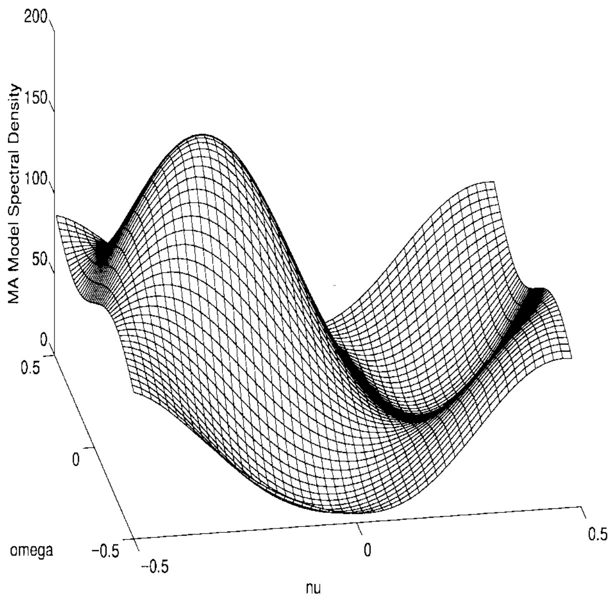

Fig. 2. Spectral density function of the purely indeterministic component.

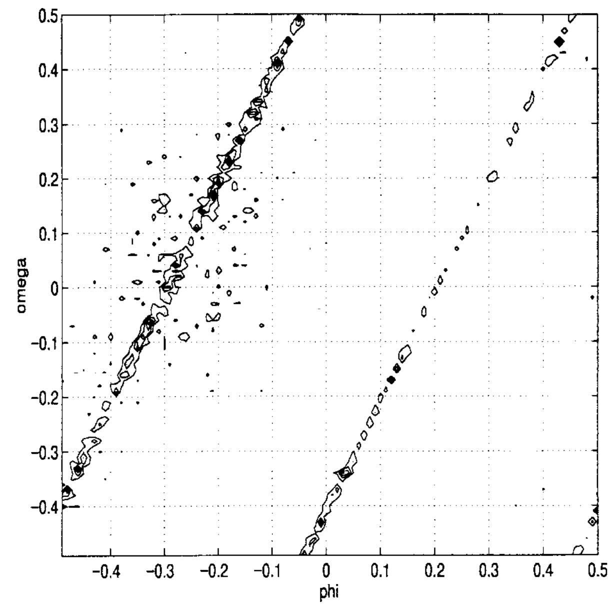

Fig. 3. Absolute value of the Fourier transform of the observed field in Example 1.

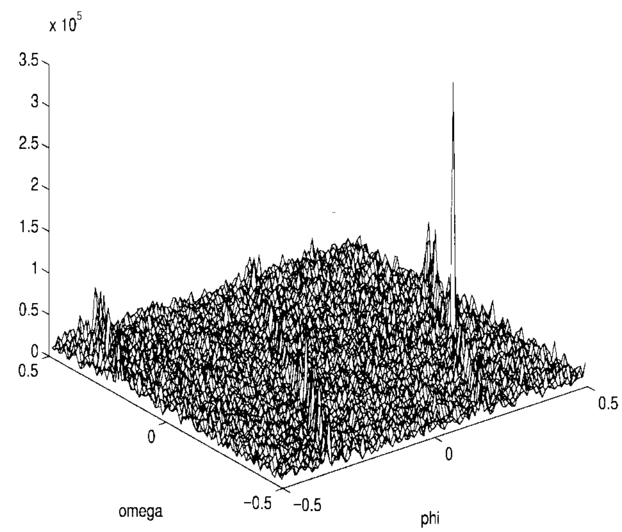

Fig. 4. $\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$—Estimated result of applying the operator $\widehat{\mathcal{FM}}_2(\cdot;\tau_n,\tau_m,\omega,\phi)$ to the observed field in Example 1.

imaginary components are jointly wide sense stationary and jointly mean-square ergodic in the first- and second-order moments, we conclude, using (43), (44) and (50), that

$$\lim_{S,T\to\infty} \hat{\mathcal{M}}_2(y(n,m),\tau_n,\tau_m) = \mathcal{M}_2(y(n,m),\tau_n,\tau_m) \quad (53)$$

in the mean square sense.

In summary, the estimation of the spectral support parameters of the evanescent components of $\{y(n,m)\}$ is performed by applying the operator $|\widehat{\mathcal{FM}}_2(\cdot;\tau_n,\tau_m,\omega,\phi)|$ to the observed field, followed by a search for the $\tilde{I}$ prominent peaks of $|\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)|$. The remaining steps of the algorithm are those developed in the previous section. In other words, if the true second-order moments of the observed field are unknown, the same algorithm derived in the previous section (see Table I) holds when $\mathcal{FM}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$ is replaced by $\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$, and (ω_q,ϕ_q) replaced by $(\hat{\omega}_q,\hat{\phi}_q)$.

VI. Numerical Examples

To illustrate the operation of the proposed algorithm, as well as to gain more insight into its performance, we present several numerical examples.

Example 1: Consider a 2-D homogeneous random field consisting of a sum of a purely indeterministic component and a single evanescent component. The purely indeterministic component is a NSHP MA field with parameters $b(0,1) = -0.9\exp(j0.25\pi)$, $b(1,-1) = 0.1\exp(j0.4\pi)$, $b(1,0) = -0.5\exp(j0.8\pi)$, $b(1,1) = 0.4\exp(-j0.2\pi)$. The driving noise of the MA model is a complex valued white noise field such that its real and imaginary components are independent real Gaussian white noise fields each with zero mean and variance $\sigma^2 = 4$, and $\rho^2 = 25$, respectively. The spectral density function of this purely indeterministic field is depicted in Fig. 2.

The evanescent component spectral support parameters are $(\alpha,\beta) = (2,-1)$, $\nu^{(2,-1)} = 0.4$. The modulating 1-D purely indeterministic process of this evanescent component is a second-order Gaussian MA process such that $a^{(2,-1)}(1) = -0.95\exp(j\pi/4)$, and $a^{(2,-1)}(2) = 0.1\exp(j\pi/4)$. Its driving noise is a complex-valued Gaussian process, whose real and imaginary components are independent real Gaussian white noise processes with zero mean and variances $(\sigma^{(2,-1)})^2 = 9$, and $(\rho^{(2,-1)})^2 = 1$, respectively.

The dimensions of the observed field are 100 $\times$ 100. For illustration purposes, a contour map of the absolute value of the Fourier transform of the observed field is depicted in Fig. 3. Observe that the evanescent component spectral support wraps around the boundary of the spectral domain. In addition, note the presence of the colored background noise, which is due to the purely indeterministic component of the field. Fig. 4 depicts $\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$, i.e., the result of

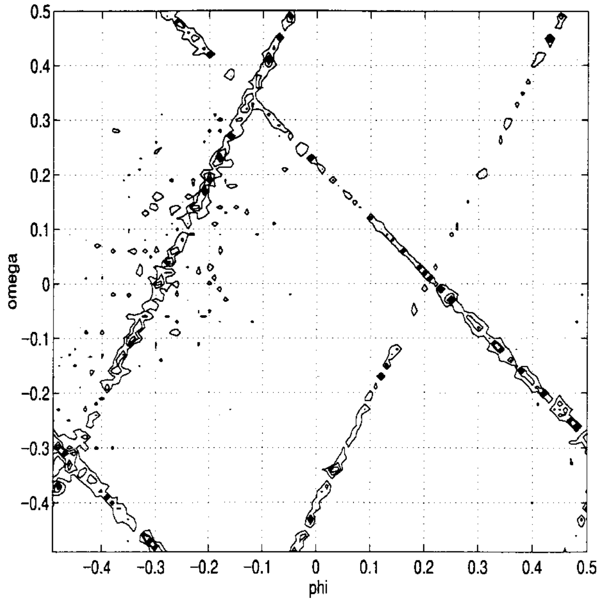

Fig. 5. Absolute value of the Fourier transform of the observed field in Example 2.

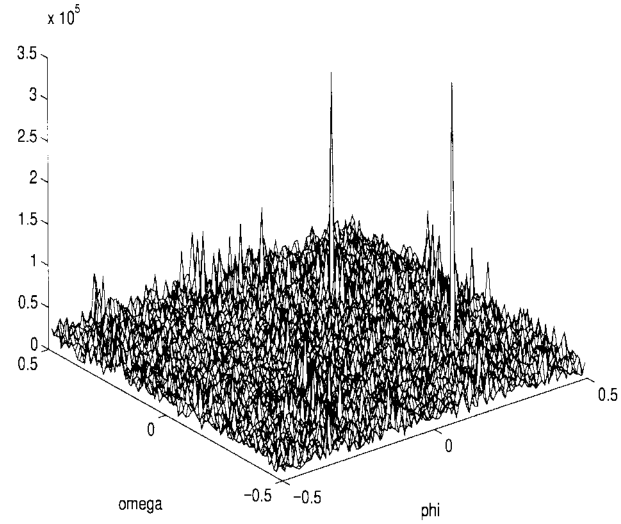

Fig. 6. $\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$—Estimated result of applying the operator $\widehat{\mathcal{FM}}_2(\cdot;\tau_n,\tau_m,\omega,\phi)$ to the observed field in Example 2.

applying the EET to the observed field. The peak due to the exponential produced by the EET operator is clearly visible. Note that the DC component due to the unconjugated second-order moment of the purely indeterministic component has been removed by subtracting the mean of the product field $y(n+\tau_n, m+\tau_m)y(n,m)$. The estimation results for this example are $(\hat{\alpha},\hat{\beta}) = (2,-1)$, and $\hat{\nu}^{(2,-1)} = 0.3999$.

Example 2: Consider a 2-D homogeneous random field consisting of a sum of a purely indeterministic component and two evanescent components. The purely indeterministic component and the first evanescent component are those of Example 1. The spectral support parameters of the second evanescent component are $(\alpha,\beta) = (1,1)$, $\nu^{(1,1)} = 0.2$. The modulating 1-D purely indeterministic process of this evanescent component is a second-order Gaussian MA process, with the same parameters as those of the 1-D modulating process of the first evanescent component. For illustration purposes, a contour map of the absolute value of the Fourier transform of the observed field is depicted in Fig. 5. Note again that the spectral supports of the evanescent components wrap around the boundary of the spectral domain. Fig. 6 depicts $\widehat{\mathcal{FM}}_2(y(n,m);\tau_n,\tau_m,\omega,\phi)$, which results from applying the EET to the observed field. The two peaks are due to the exponentials produced by the EET operator. The estimation results for this example are $(\hat{\alpha}_1,\hat{\beta}_1) = (2,-1)$, $\hat{\nu}_1^{(2,-1)} = 0.4002$, and $(\hat{\alpha}_2,\hat{\beta}_2) = (1,1)$, and $\hat{\nu}_2^{(1,1)} = 0.2000$.

Example 3: In this example, we illustrate the performance of the proposed algorithm using Monte Carlo simulations. The experimental results are based on 100 independent realizations of the observed field for different field sizes. We analyze the bias and the variance of the estimate of the spectral support parameter $\nu^{(\alpha,\beta)}$ obtained by the algorithm and compare the experimental results with the Cramér–Rao lower bound (CRLB), which is derived in [17]. We note here that in the derivation of the CRLB, [17], it is assumed that α and β are *a priori* known. Since in practice α and β are unknown and need to be estimated, the CRLB in this case is not tight and is an "optimistic" lower bound. In this experiment, we also evaluate the probability of correct estimation of the (α,β) pair.

TABLE II
ESTIMATION RESULTS OF THE SPECTRAL SUPPORT PARAMETERS: ERROR RATE IN ESTIMATING (α,β);BIAS AND STANDARD DEVIATION OF $\hat{\nu}^{(\alpha,\beta)}$

data size	*bias*	*std*	(α,β) *error rate*
25	4.1285e-05	4.5972e-03	0.04
50	6.5132e-05	1.3272e-03	0.01
100	9.2803e-06	5.4053e-04	0
200	7.7799e-06	1.9976e-04	0

In this example, the 2-D random field is a sum of a purely indeterministic component and a single evanescent component. The purely indeterministic component is a complex valued white noise field such that its real and imaginary components are independent real Gaussian white noise fields each with zero mean and variance $\sigma^2 = 1$, and $\rho^2 = 1$, respectively. The evanescent component spectral support parameters are $(\alpha,\beta) = (2,-1)$, and $\nu^{(2,-1)} = 0.4$. The modulating 1-D purely indeterministic process of the evanescent component is a second-order Gaussian MA process, such that $a^{(2,-1)}(1) = -0.95\exp(j\pi/4)$, and $a^{(2,-1)}(2) = 0.1\exp(j\pi/4)$. Its driving noise is a complex valued Gaussian process, whose real and imaginary components are independent real Gaussian white noise processes with zero mean and variances $(\sigma^{(2,-1)})^2 = 4$, and $(\rho^{(2,-1)})^2 = 1$, respectively.

In Table II, we present the estimation results of the evanescent component spectral support parameters. It is clear that a wrong estimate of an (α,β) pair would result in wrong estimates of the other parameters of that evanescent component. Since the probability of such event is very small, as indicated by the results in Table II, we consider such events to be outliers. Hence, we ignore the results of these experiments in the computation of the bias and variance in estimating $\nu^{(\alpha,\beta)}$.

The experimental results listed in Table II indicate that the error rate in estimating the (α,β) pair of the evanescent component is low, and rapidly decreases as the dimensions of the observed field are made larger. For moderate size data fields, the error rate in estimating the (α, β) pair becomes zero. Furthermore, the estimates of $\nu^{(2,-1)}$ obtained by the proposed algorithm are essentially unbiased as the experimental bias is much smaller than the standard deviation of the experimental results. The estimation error variance can therefore be compared with the CRLB. (The CRLB is the lower bound on the estimation error variance for any unbiased estimator of the problem parameters). A comparison of the Monte Carlo results with the CRLB for the case where the dimensions of the observed field are relatively small (field size of 25 $\times$ 25) shows that the experimentally computed standard deviation of the $\nu^{(2,-1)}$ estimate is not far from the lower bound, even for such a small data size. The squared root of the exact CRLB on $\nu^{(2,-1)}$ is 6.6096e-04, whereas the experimentally computed standard deviation of the $\nu^{(2,-1)}$ estimate is 4.5972e-03.

Example 4: It is shown in Section IV that whenever the frequency parameter $\nu_i^{(\alpha,\beta)}$, of an evanescent component is zero, the EET maps this evanescent component to an exponential whose frequency is $(0, 0)$, regardless of the values of α and β. This nonuniqueness implies that the EET cannot produce a complete estimate of the spectral support parameters of these evanescent components. Hence, a different algorithm, which is described in Section IV-C, is derived in order to estimate the (α, β) pairs of evanescent components whose frequency parameter is zero. In this example, we illustrate the performance of the proposed algorithm, using Monte Carlo simulations. The experimental results are based on 100 independent realizations of the observed field for different field sizes.

In this example, the 2-D random field is a sum of a purely indeterministic component and two evanescent components. The spectral support parameters of the two evanescent components are $(\alpha, \beta) = (2, -1)$, $\nu^{(2,-1)} = 0$, and $(\alpha, \beta) = (1, 3)$, $\nu^{(1,3)} = 0$, respectively. The purely indeterministic component of the observed field is identical to the purely indeterministic component in Example 1. The modulating 1-D purely indeterministic processes of the two evanescent components are second-order Gaussian MA processes, with the same parameters as those of the evanescent component in Example 1. For illustration purposes, a contour map of the absolute value of the Fourier transform of one realization of the observed field is depicted in Fig. 7.

The experimental results demonstrate that the error rate in estimating the (α, β) pair of the evanescent component is low, as long as the dimensions of the observed field are large. The error rate was zero for 100 $\times$ 100 and 200 $\times$ 200 fields. However, the error rate sharply increases when the dimensions of the observed field are small. For example, for a 50 $\times$ 50 field, the error rate reached 0.49. We thus conclude that the proposed algorithm for estimating the (α, β) pairs of evanescent fields with $\nu_i^{(\alpha,\beta)} = 0$ is useful for relatively large data arrays and is considerably less robust than the EET algorithm for estimating the spectral support parameters of evanescent components with $\nu_i^{(\alpha,\beta)} \neq 0$.

Since, here, the algorithm is searching only for evanescent components with a zero frequency parameter, in principle, it can be applied to the observed field, regardless of the existence in the field of evanescent components with a nonzero frequency parameter. However, our experimental results indicate that improved performance of the algorithm for estimating the (α, β) parameters of evanescent components with $\nu^{(\alpha,\beta)} = 0$ is achieved by first filtering evanescent components with a nonzero frequency parameter out of the observed field. Using this procedure, we first apply the EET-based algorithm, summarized in Table I, to the observed field and obtain the spectral support parameters of the evanescent components whose frequency parameter is nonzero. Having estimated the spectral supports of these components, we apply the procedure described in Section IV-D to eliminate the contributions of these evanescent components to the observed field. The residual field contains only evanescent components with a zero frequency parameter and a purely indeterministic component.

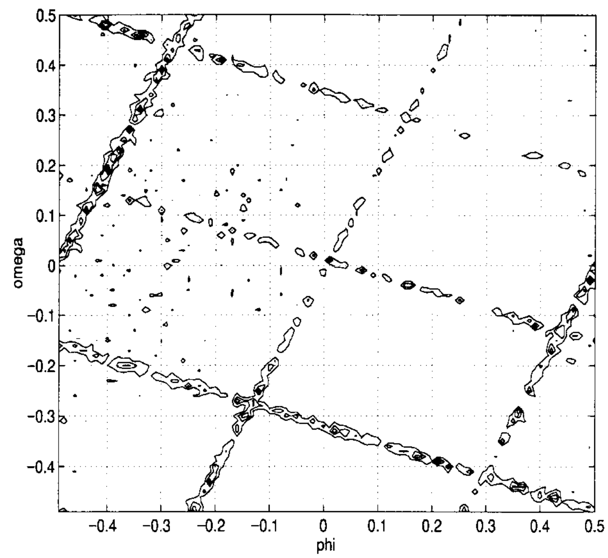

Fig. 7. Absolute value of the Fourier transform of the observed field in Example 4.

The performance gain obtained by adopting this procedure is significant in cases where the dimensions of the observed field are relatively small. The performance gain becomes less significant as the dimensions of the observed field are made larger.

VII. Conclusions

In this paper, we derived a computationally efficient estimation algorithm for the parameters of the evanescent and purely indeterministic components of a homogeneous random field. The algorithm is based on a nonlinear operator derived in this paper. The operator uniquely maps each evanescent component to a single exponential. The exponential's spatial frequency is a function of the spectral support parameters of the evanescent component. Hence, employing this transformation, the problem of estimating the spectral support parameters of an evanescent field is replaced by the simpler problem of estimating the spatial frequency of a 2-D exponential.

The performance of the proposed algorithm was investigated using Monte Carlo simulations. It was found that the error rates in estimating the (α, β) pairs of the evanescent components are low and rapidly decrease as the dimensions of the observed field are made larger. The estimates of the frequency parameters of the evanescent fields were found to be unbiased.

In case the probability density function of the observed field is known, a maximum-likelihood estimate (MLE) of the field parameters can be found by maximizing the log-likelihood function of the observations with respect to the model parameters. Since this objective function is highly nonlinear in the problem parameters, the maximization problem cannot be solved analytically, and we must resort to numerical methods. In order to avoid the enormous computational burden of an exhaustive search, a two-step procedure based on the evanescent to exponential algorithm can be employed. In the first stage, a suboptimal estimate of the parametric models of the field components is obtained using the algorithm derived in this paper. In the second stage, these estimates initialize an iterative numerical maximization of the log likelihood function.

Alternative approaches for estimating the spectral support parameters of the evanescent components can be derived by taking the Radon or Hough transforms of the observed field periodogram. Periodogram-based estimation of the spectral support parameters using the Radon transform requires the evaluation of a line integral for each pair of orientation and distance from the origin parameters and a search for the projections with highest energy. On the other hand, the algorithm based on the evanescent to exponential transform is computationally more efficient as no such search in the parameter space is required. Moreover, using the evanescent-to-exponential transform, we avoid the resolution limitations of the periodogram. It is shown using Monte Carlo simulations that the methods based on the Hough and Radon transforms are considerably more sensitive to noise than the method based on the nonlinear evanescent to exponential mapping.

We therefore conclude that the suggested algorithm provides an attractive solution to the problem of estimating the parameters of a homogeneous random field with a mixed spectral distribution. Since the proposed algorithm does not directly employ the structure of the field covariance matrix, the need to estimate this matrix, whose dimensions can be very large, is avoided. Unlike previously suggested algorithms, the proposed method does not require, in any of its stages, numerical solution of a multidimensional nonlinear minimization problem.

REFERENCES

[1] J. M. Francos, A. Z. Meiri, and B. Porat, "A wold-like decomposition of 2-D discrete homogeneous random fields," *Ann. Appl. Prob.*, vol. 5, pp. 248–260, 1995.

[2] J. Ward, "Space-time adaptive processing for airborne radar," Tech. Rep. 1015, Lincoln Lab., Mass. Inst. Technol., Cambridge, MA, 1994.

[3] Y. L. Gau and I. S. Reed, "An improved reduced-rank CFAR space-time adaptive radar detection algorithm," *IEEE Trans. Signal Processing*, vol. 46, pp. 2139–2146, 1998.

[4] P. Whittle, "On stationary processes in the plane," *Biometrika*, vol. 41, pp. 434–449, 1954.

[5] T. L. Marzetta, "Two-dimensional linear prediction: Autocorrelation arrays, minimum-phase prediction error filters and reflection coefficient arrays, *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 725–733, 1980.

[6] Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil," *IEEE Trans. Signal Processing*, vol. 40, pp. 2267–2280, 1992.

[7] H. Yang and Y. Hua, "Statistical decomposition of an eigendecomposition based method for 2-D frequency estimation," *Automatica*, vol. 30, pp. 157–168, 1994.

[8] C. R. Rao, L. Zhao, and B. Zhou, "Maximum likelihood estimation of 2-D superimposed exponential signals," *IEEE Trans. Signal Processing*, vol. 42, pp. 1795–1802, 1994.

[9] A. K. Jain, "Advances in mathematical models for image processing," *Proc. IEEE*, vol. 69, pp. 502–528, 1981.

[10] S. R. Parker and A. H. Kayran, "Lattice parameter autoregressive modeling of two-dimensional fields—Part I: The quarter plane case," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 872–885, 1984.

[11] C. W. Therrien, T. F. Quatieri, and D. E. Dudgeon, "Statistical model-based algorithms for image analysis," *Proc. IEEE*, vol. 74, pp. 532–551, 1986.

[12] H. Derin and P. A. Kelly, "Discrete-index markov-type random processes," *Proc. IEEE*, vol. 77, pp. 1485–1510, 1989.

[13] N. Balram and J. M. F. Moura, "Noncausal gauss markov random fields: Parameter structure and estimation," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1333–1355, 1993.

[14] J. M. Francos, A. Narasimhan, and J. W. Woods, "Maximum likelihood parameter estimation of the harmonic, evanescent and purely indeterministic components of discrete homogeneous random fields," *IEEE Trans. Inform. Theory*, vol. 42, pp. 916–930, 1996.

[15] J. M. Francos, A. Narasimhan, and J. W. Woods, "Maximum likelihood parameter estimation of discrete homogeneous random fields with mixed spectral distributions," *IEEE Trans. Signal Processing*, vol. 44, pp. 1242–1255, 1996.

[16] A. J. Isaksson, "Analysis of identified 2-D noncausal models," *IEEE Trans. Inform. Theory*, vol. 39, pp. 525–534, 1993.

[17] J. M. Francos, "The cramer-rao bound on the estimation of complex valued deterministic random fields in unknown colored noise," submitted for publication.

[18] G. Cohen and J. M. Francos, "Efficient parameter estimation of evanescent random fields," submitted for publication.

[19] B. Porat, *Digital Processing of Random Signals*. Englewood Cliffs, NJ: Prentice-Hall, 1994.

[20] A. K. Jain, *Fundamentals of Digital Image Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[21] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[22] J. M. Francos, A. Narasimhan, and J. W. Woods, "Maximum likelihood parameter estimation of textures using a wold decomposition based model," *IEEE Trans. Image Processing*, vol. 4, pp. 1655–1666, 1995.

[23] R. Sriram, J. M. Francos, and W. A. Pearlman, "Texture coding using a wold decomposition based model," *IEEE Trans. Image Processing*, vol. 5, pp. 1382–1386, 1996.

[24] R. Krishnamurthy, J. W. Woods, and J. M. Francos, "Adaptive restoration of textured images with mixed spectra using a generalized wiener filter," *IEEE Trans. Image Processing*, vol. 5, pp. 648–652, 1996.

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