Bounds on the Accuracy of Estimating the Parameters of Discrete Homogeneous Random Fields with Mixed Spectral Distributions

Joseph M. Francos, *Member, IEEE*

***Abstract*— This paper considers the achievable accuracy in jointly estimating the parameters of a real-valued two-dimensional (2-D) homogeneous random field with mixed spectral distribution, from a single observed realization of it. On the basis of a 2-D Wold-like decomposition, the field is represented as a sum of mutually orthogonal components of three types: purely indeterministic, harmonic, and evanescent. An exact form of the Cramer–Rao lower bound on the error variance in jointly estimating the parameters of the different components is derived. It is shown that the estimation of the harmonic component is decoupled from that of the purely indeterministic and the evanescent components. Moreover, the bound on the parameters of the purely indeterministic and the evanescent components is independent of the harmonic component. Numerical evaluation of the bounds provides some insight into the effects of various parameters on the achievable estimation accuracy.**

***Index Terms*—Cramer–Rao bound, Fisher information, evanescent fields, harmonic fields, purely indeterministic fields, 2-D Wold decomposition, 2-D mixed spectral distributions.**

I. Introduction

IN this paper, we consider the problem of fitting a parametric model to observations from a single realization of a two-dimensional (2-D) real-valued discrete and homogeneous random field with mixed spectral distribution. This fundamental problem is of great theoretical and practical importance. It arises quite naturally in terms of the texture estimation of images [26], [27], [29], as well as in several areas of radar, sonar, and seismic signal processing.

From the 2-D Wold-like decomposition [1], we have that any 2-D regular and homogeneous discrete random field can be represented as a sum of two mutually orthogonal components: a *purely indeterministic* field and a *deterministic* one. The purely indeterministic component has a unique white innovations driven moving average representation. The deterministic component is further orthogonally decomposed into a *harmonic* field and a countable number of mutually orthogonal *evanescent* fields. This decomposition results in a corresponding decomposition of the spectral measure of the regular random field into a countable sum of mutually singular spectral measures. The spectral distribution function of the purely indeterministic component is absolutely continuous, while the spectral measure of the deterministic component is singular with respect to the Lebesgue measure, and therefore it is concentrated on a set of Lebesgue measure zero in the frequency plane. For practical applications, the "spectral density function" of the regular field's deterministic component can be assumed to have the form of a countable sum of one-dimensional (1-D) and 2-D delta functions. The 1-D delta functions are singular functions which are supported on curves in the 2-D spectral domain. The 2-D delta functions are singular functions which are supported on discrete points in the spectral domain. In [27], [29] the 2-D Wold-like decomposition, and the resulting random field model, are employed for modeling, analysis, and synthesis of natural textures. We refer the interested reader to [27], [29] for examples that demonstrate the identification and parameterization of the decomposition components in images of natural textures. Illustrative synthetic examples can be found in [28].

This paper is devoted to the analysis of the achievable accuracy in estimating the parameters of a regular homogeneous random field, based on the parametric model derived in [1]. In particular, we concentrate here on establishing the lower bound on the error variance in *jointly* estimating the parameters of the purely indeterministic, harmonic, and evanescent components of the field, based on a *finite-dimension, single observed realization* of this field. Assuming that the observed field is a Gaussian random field, we derive closed-form expressions for the lower bound on the error variance of any unbiased estimator of the field parameters. We show that the lower bound on the parameters of the harmonic component is decoupled from the bound on the parameters of the purely indeterministic and the evanescent components. Moreover, the bound on the parameters of the purely indeterministic and the evanescent components is independent of the harmonic component. These results hold regardless of the parametric models of the purely indeterministic and the evanescent components. Next, by assuming a moving average model, or alternatively an autoregressive model, for the modulating purely indeterministic processes of each evanescent field, we find for both cases, closed-form expressions for the Fisher Information Matrix (FIM) entries which correspond to the parameters of the evanescent components. For the case where the 2-D moving average model of the purely indeterministic component is of finite dimensions we derive a closed-form

Manuscript received October 13, 1995; revised June 1996. This work was supported, in part, by The Israel Ministry of Science and The Arts under Grant 6253295.

The author is with the Electrical and Computer Engineering Deptartment, Ben-Gurion University, Beer-Sheva 84105, Israel.

Publisher Item Identifier S 0018-9448(97)02314-6.

exact Cramer–Rao Bound (CRB) on the achievable accuracy in jointly estimating the parameters of the harmonic, evanescent, and purely indeterministic components of the field, from a finite dimension observed realization of it.

The general problem of random fields' parameter estimation has received considerable attention. Many of the works address the problem of the statistical inference of Markov random fields (MRF's), and its applications in image processing (see, e.g., [3], [4], and the references therein). A special class of MRF's is that of Gauss–Markov random fields (GMRF's). It is shown in [6] that a GMRF may be defined in the form of a 2-D autoregressive (AR) field driven by correlated noise. This definition is equivalent to specifying the field joint probability density function, through the Gibbs potentials. A maximum-likelihood (ML) algorithm for estimating the parameters of a GMRF is derived in [5]. A large number of the existing parameter estimation algorithms are concerned with the parameter estimation of 2-D AR fields (see, e.g., [2], [7]–[10]). Parameter estimation of 2-D moving average (MA) random fields is addressed in [25]. Autoregressive, moving average (ARMA) models are introduced in [13]. In general, most of these works implicitly assume the observed random field is purely indeterministic and try to fit it with white- or correlated-noise-driven linear model. A different family of algorithms addresses the problem of estimating the parameters of sinusoidal signals in white noise. Note, however, that in the Gaussian case all of the foregoing problems are only special cases of the general problem which we address in this paper.

The asymptotic Cramer–Rao bound (CRB) on the parameters of a Gaussian purely indeterministic field was derived by Whittle [2]. More recently, this general derivation was specialized for the case of noncausal AR models, and nonsymmetrical half plane (NSHP) AR models in [11]. Parameter estimation techniques of sinusoidal signals in additive white noise include the periodogram-based approximation (applicable for widely spaced sinusoids) to the ML solution [20], extensions to the Pisarenko harmonic decomposition [15], or the singular value decomposition [16]. More recently, a matrix enhancement and matrix pencil method for estimating the parameters of 2-D superimposed, complex-valued exponential signals was suggested in [17], and analyzed in [18]. Assuming the noise field is white, the Cramer–Rao lower bound for this problem was derived as well. The problem of ML estimation of 2-D superimposed, complex-valued exponential signals has been recently considered in [19].

An early discussion on the problem of analyzing 2-D homogeneous random fields with discontinuous spectral distribution functions can be found in [21]. In [22] we have developed a conditional ML algorithm for jointly estimating the parameters of the harmonic, evanescent, and purely indeterministic components of a complex-valued homogeneous random field from a single observed realization of it. In [28], this algorithm is generalized for the case where the random field is real-valued, and has multiple evanescent components of unknown spectral support parameters. In [22] we also derive the conditional Cramer–Rao lower bound on the covariance matrix of the conditional estimates for a complex-valued field with a special type of evanescent component. Here, we derive an *exact* Cramér–Rao bound on the parameters of essentially any real-valued regular and homogeneous Gaussian field, where the field may contain all of the 2-D Wold decomposition components.

The paper is organized as follows. In Section II we briefly summarize the results of the 2-D Wold-like decomposition, which establish the theoretical basis for the suggested solution. In Section III we define the problem considered in this paper and introduce some necessary notations. In Section IV a general form of the CRB for the estimation problem considered here is derived. It is shown that the estimation problem of the harmonic component is decoupled from that of the purely indeterministic and the evanescent components. In Section V we derive closed-form expressions for the lower bound on the achievable estimation accuracy of the field parameters both for the case in which the modulating 1-D purely indeterministic processes of each evanescent field are MA processes, as well as for the case in which the modulating 1-D purely indeterministic processes of each evanescent field are AR processes. In Section VI we present some numerical examples in order to get further insight into the properties of the bound.

II. The Homogeneous Random Field Model

The considered random field model is based on the Wold-type decomposition (of 2-D regular and homogeneous random fields) presented in [1] and briefly summarized in this section. Let $\{y(n, m), (n, m) \in \mathcal{Z}^2\}$, be a real-valued, regular, homogeneous random field. Then $y(n, m)$ can be uniquely represented by the orthogonal decomposition

$$y(n, m) = w(n, m) + v(n, m). \tag{1}$$

The field $\{w(n, m)\}$ is purely indeterministic and has a unique white innovations driven MA representation. The field $\{v(n, m)\}$ is a deterministic random field.

We call a 2-D deterministic random field $\{e_o(n, m)\}$ *evanescent with respect to the NSHP total order* o if it spans a Hilbert space identical to the one spanned by its *column-to-column innovations* at each coordinate (n, m) (with respect to the total order o). The deterministic field column-to-column innovation at each coordinate $(n, m) \in \mathcal{Z}^2$ is defined as the difference between the actual value of the field, and its projection on the Hilbert space spanned by the deterministic field samples in all previous columns.

It is possible to define [1] a family of NSHP total-order definitions such that the boundary line of the NSHP is of rational slope. Let α and β be two coprime integers, such that $\alpha \neq 0$. The angle θ of the slope is given by $\tan\theta = \beta/\alpha$. (See, for example, Fig. 1.) Each of these supports is called *rational nonsymmetrical half-plane* (RNSHP). We denote by O the set of all possible RNSHP definitions on the 2-D lattice (i.e., the set of all NSHP definitions in which the boundary line of the NSHP is of rational slope). The introduction of the family of RNSHP total-ordering definitions results in the following countably infinite orthogonal decomposition of the deterministic component of the random field:

$$v(n, m) = p(n, m) + \sum_{(\alpha, \beta) \in O} e_{(\alpha, \beta)}(n, m). \tag{2}$$

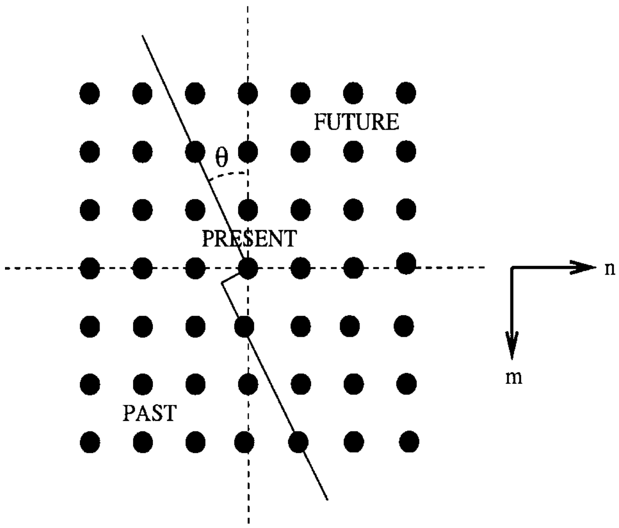

Fig. 1. RNSHP support. Example with $\alpha = 2$ and $\beta = 1$.

The random field $\{p(n, m)\}$ is *half-plane deterministic*, i.e., it has no column-to-column innovations with respect to any RNSHP total-ordering definition. The field $\{e_{(\alpha,\beta)}(n, m)\}$ is the evanescent component which generates the column-to-column innovations of the deterministic field with respect to the RNSHP total-ordering definition $(\alpha, \beta) \in O$.

Hence, if $\{y(n, m)\}$ is a 2-D regular and homogeneous random field, then $y(n, m)$ can be uniquely represented by the orthogonal decomposition

$$y(n, m) = w(n, m) + p(n, m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n, m). \quad (3)$$

In this paper, all spectral measures are defined on the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$. It is shown in [1] that the spectral measures of the decomposition components in (3) are mutually singular. The spectral distribution function of the purely indeterministic component is absolutely continuous, while the spectral measures of the half-plane deterministic component and all the evanescent components are concentrated on a set of Lebesgue measure zero in K. Since for practical applications we can exclude singular-continuous spectral distribution functions from the framework of our treatment, a model for the evanescent field which corresponds to the RNSHP defined by $(\alpha, \beta) \in O$ is given by

$$\begin{aligned} e_{(\alpha,\beta)}&(n, m) \\ &= \sum_{i=1}^{I^{(\alpha,\beta)}} e_i^{(\alpha,\beta)}(n, m) \\ &= \sum_{i=1}^{I^{(\alpha,\beta)}} s_i^{(\alpha,\beta)}(n\alpha - m\beta) \cos\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right) \\ &\quad + t_i^{(\alpha,\beta)}(n\alpha - m\beta) \sin\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right) \quad (4) \end{aligned}$$

where the 1-D purely indeterministic processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$, $\{s_j^{(\alpha,\beta)}(n\alpha - m\beta)\}$, $\{t_k^{(\alpha,\beta)}(n\alpha - m\beta)\}$, $\{t_\ell^{(\alpha,\beta)}(n\alpha - m\beta)\}$ are mutually orthogonal for all i, j, k, ℓ such that $i \neq j$ and $k \neq \ell$; also for all i the processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ and $\{t_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ have an identical autocorrelation function. Hence, the "spectral density function" of each evanescent field has the form of a countable sum of 1-D delta functions which are supported on lines of rational slope in the 2-D spectral domain.

One of the half-plane deterministic field components, which is often found in physical problems, is the harmonic random field

$$\begin{aligned} h(n, m) = \sum_{p=1}^{P} (C_p \cos 2\pi(n\omega_p + m\nu_p) \\ + D_p \sin 2\pi(n\omega_p + m\nu_p)) \end{aligned} \quad (5)$$

where the C_p's and D_p's are mutually orthogonal random variables, $E[C_p]^2 = E[D_p]^2 = \sigma_p^2$, and (ω_p, ν_p) are the spatial frequencies of the pth harmonic. In general, P is infinite. This component generates the 2-D delta functions of the "spectral density." The parametric modeling of deterministic random fields whose spectral measures are concentrated on curves, other than lines of rational slope, or discrete points in the frequency plane, is still an open question to the best of our knowledge.

III. Problem Definition

The orthogonal decompositions of the previous section imply that if we exclude from the framework of our model those 2-D random fields whose spectral measures are concentrated on curves other than lines of rational slope, $y(n, m)$ is uniquely represented by

$$y(n, m) = w(n, m) + h(n, m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n, m). \quad (6)$$

Hence, in this paper, we study the problem of the achievable accuracy in *jointly* estimating the parameters of the harmonic, evanescent, and purely indeterministic components using a finite-size, single observed realization of the field. In this problem the purely indeterministic component can be viewed as an unknown colored noise field.

When expressed in the general form (5), the coefficients $\{C_p, D_p\}$ of the harmonic component are real-valued, mutually orthogonal, random variables. However, since in general, only a single realization of the random field is observed, we cannot infer anything about the variation of these coefficients over different realizations. The best we can do is to estimate the particular values which the C_p's and D_p's take for the given realization; in other words we might just as well treat the C_p's and D_p's as unknown constants, and the harmonic component as the unknown mean of the observed realization.

We next state our assumptions and introduce some necessary notations. Let $\{y(n, m)\}$, $(n, m) \in D$ where

$$D = \{(i, j) | 0 \leq i \leq S-1, 0 \leq j \leq T-1\}$$

be the observed random field. Note, however, that the observed field could just as well have any *arbitrary* shape.

Assumption 1: The purely indeterministic component is a zero-mean, real-valued Gaussian field. Hence the purely indeterministic component is characterized by its covariance matrix which is denoted by $\boldsymbol{\Gamma}_{\mathrm{PI}}$. We assume that the covariance matrix has some known parametric form, where $\boldsymbol{b}$ is the parameter vector. At the moment we will not specify the functional dependence of $\boldsymbol{\Gamma}_{\mathrm{PI}}$ on $\boldsymbol{b}$, but rather leave it implicit.

Assumption 2: The number P of harmonic components in (5) is *a priori* known. The values of the (α, β) pairs, as well as the number $I^{(\alpha, \beta)}$ of evanescent components in (4), are *a priori* known for all the evanescent components.

Assumption 3: The 1-D purely indeterministic processes $\{s_i^{(\alpha, \beta)}\}$ and $\{t_i^{(\alpha, \beta)}\}$ are all assumed to be zero-mean Gaussian processes. Hence each pair $\{s_i^{(\alpha, \beta)}\}$, $\{t_i^{(\alpha, \beta)}\}$ is characterized by its covariance matrix which is denoted by $\boldsymbol{R}_i^{(\alpha, \beta)}$. (Note that $\{s_i^{(\alpha, \beta)}\}$ and $\{t_i^{(\alpha, \beta)}\}$ have the same autocorrelation function.) We assume that the covariance matrix has some known parametric form, where $\boldsymbol{a}_i^{(\alpha, \beta)}$ is the parameter vector. At the moment we will not specify the functional dependence of $\boldsymbol{R}_i^{(\alpha, \beta)}$ on $\boldsymbol{a}_i^{(\alpha, \beta)}$, but rather leave it implicit, as well. Thus the parameter vector of each of the evanescent components $\{e_i^{(\alpha, \beta)}\}$ is given by

$$\boldsymbol{\phi}_i^{(\alpha, \beta)} = [\nu_i^{(\alpha, \beta)}, (\boldsymbol{a}_i^{(\alpha, \beta)})^T]^T.$$

Therefore, the parameter vector of the evanescent field $\{e^{(\alpha, \beta)}\}$ is obtained by collecting the vectors $\boldsymbol{\phi}_i^{(\alpha, \beta)}$ into a single column vector, i.e.,

$$\boldsymbol{\phi}^{(\alpha, \beta)} = [(\boldsymbol{\phi}_1^{(\alpha, \beta)})^T, \cdots, (\boldsymbol{\phi}_{I^{(\alpha, \beta)}}^{(\alpha, \beta)})^T]^T. \tag{7}$$

Let

$$\boldsymbol{c} = [C_1, \cdots, C_P, D_1, \cdots, D_P]^T \tag{8}$$
$$\boldsymbol{\omega} = [\omega_1, \cdots, \omega_P]^T \tag{9}$$
$$\boldsymbol{\nu} = [\nu_1, \cdots, \nu_P]^T. \tag{10}$$

Thus the parameter vector of the observed field $\{y(n, m)\}$ is given by

$$\boldsymbol{\theta} = [\boldsymbol{c}^T \boldsymbol{\omega}^T \boldsymbol{\nu}^T \boldsymbol{b}^T \{(\boldsymbol{\phi}^{(\alpha, \beta)})^T\}_{(\alpha, \beta) \in O}]^T. \tag{11}$$

Let

$$\begin{aligned}\boldsymbol{y} = [&y(0, 0), \cdots, y(0, T-1), y(1, 0), \cdots, \\ &y(1, T-1), \cdots, \cdots, y(S-1, 0), \cdots, \\ &y(S-1, T-1)]^T\end{aligned} \tag{12}$$

$$\begin{aligned}\boldsymbol{h} = [&h(0, 0), \cdots, h(0, T-1), h(1, 0), \cdots, \\ &h(1, T-1), \cdots, \cdots, h(S-1, 0), \cdots, \\ &h(S-1, T-1)]^T\end{aligned} \tag{13}$$

$$\begin{aligned}\boldsymbol{e}_i^{(\alpha, \beta)} = [&e_i^{(\alpha, \beta)}(0, 0), \cdots, e_i^{(\alpha, \beta)}(0, T-1), \\ &e_i^{(\alpha, \beta)}(1, 0), \cdots, e_i^{(\alpha, \beta)}(1, T-1), \cdots, \cdots, \\ &e_i^{(\alpha, \beta)}(S-1, 0), \cdots, e_i^{(\alpha, \beta)}(S-1, T-1)]^T.\end{aligned} \tag{14}$$

Let

$$\begin{aligned}\tilde{\boldsymbol{s}}_i^{(\alpha, \beta)} = [&s_i^{(\alpha, \beta)}(0), s_i^{(\alpha, \beta)}(-\beta), \cdots, s_i^{(\alpha, \beta)}(-(T-1)\beta), \\ &s_i^{(\alpha, \beta)}(\alpha), s_i^{(\alpha, \beta)}(\alpha-\beta), \cdots, \\ &s_i^{(\alpha, \beta)}(\alpha-(T-1)\beta), \cdots, \\ &s_i^{(\alpha, \beta)}((S-1)\alpha), s_i^{(\alpha, \beta)}((S-1)\alpha-\beta), \cdots, \\ &s_i^{(\alpha, \beta)}((S-1)\alpha-(T-1)\beta)]^T\end{aligned} \tag{15}$$

be the vector composed of the observed samples from the 1-D modulating process $\{s_i^{(\alpha, \beta)}\}$ of the evanescent field $\{e_i^{(\alpha, \beta)}\}$. In a similar way we define the vector $\tilde{\boldsymbol{t}}_i^{(\alpha, \beta)}$ of the 1-D modulating process $\{t_i^{(\alpha, \beta)}\}$. Also let

$$\begin{aligned}\boldsymbol{v}^{(\alpha, \beta)} = [&0, \alpha, \cdots, (T-1)\alpha, \beta, \beta+\alpha, \cdots, \\ &\beta+(T-1)\alpha, \cdots, \cdots, (S-1)\beta, \\ &(S-1)\beta+\alpha, \cdots, (S-1)\beta+(T-1)\alpha]^T.\end{aligned} \tag{16}$$

Given a scalar function $f(v)$, we will denote the matrix, or column vector, consisting of the values of $f(v)$ evaluated for all the elements of $\boldsymbol{v}$, where $\boldsymbol{v}$ is a matrix, or a column vector, by $f(\boldsymbol{v})$. Using this notation, we define

$$\begin{aligned}\tilde{\boldsymbol{f}}_i^{(\alpha, \beta)} &= \cos\left(2\pi \frac{\nu_i^{(\alpha, \beta)}}{\alpha^2+\beta^2} \boldsymbol{v}^{(\alpha, \beta)}\right) \\ \tilde{\boldsymbol{g}}_i^{(\alpha, \beta)} &= \sin\left(2\pi \frac{\nu_i^{(\alpha, \beta)}}{\alpha^2+\beta^2} \boldsymbol{v}^{(\alpha, \beta)}\right).\end{aligned} \tag{17}$$

Thus using (4), we can rewrite (14)

$$\boldsymbol{e}_i^{(\alpha, \beta)} = \tilde{\boldsymbol{s}}_i^{(\alpha, \beta)} \odot \tilde{\boldsymbol{f}}_i^{(\alpha, \beta)} + \tilde{\boldsymbol{t}}_i^{(\alpha, \beta)} \odot \tilde{\boldsymbol{g}}_i^{(\alpha, \beta)} \tag{18}$$

where $\odot$ denotes an element by element product of the vectors.

Note that whenever $n\alpha - m\beta = k\alpha - \ell\beta$ for some integers n, m, k, ℓ such that $0 \le n, k \le S-1$ and $0 \le m, \ell \le T-1$, the same element of $\tilde{\boldsymbol{s}}_i^{(\alpha, \beta)}$ (and $\tilde{\boldsymbol{t}}_i^{(\alpha, \beta)}$) appears more than once in the vector. It can be shown that for a rectangular observed field of dimensions $S \times T$ the number of *distinct* samples from the random process $\{s_i^{(\alpha, \beta)}\}$, $(\{t_i^{(\alpha, \beta)}\})$, that are found in the observed field is

$$(S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1).$$

This is because $(S-1)|\alpha|+(T-1)|\beta|+1-(|\alpha|-1)(|\beta|-1)$ is the number of different "columns" one can define on such a rectangular lattice for an RNSHP defined by (α, β). We therefore define the *concentrated version*, $\boldsymbol{s}_i^{(\alpha, \beta)}$ $(\boldsymbol{t}_i^{(\alpha, \beta)})$ of $\tilde{\boldsymbol{s}}_i^{(\alpha, \beta)}$ $(\tilde{\boldsymbol{t}}_i^{(\alpha, \beta)})$ to be an $(S-1)|\alpha|+(T-1)|\beta|+1-(|\alpha|-1)\cdot(|\beta|-1)$ column vector of nonrepeating samples of the process $\{s_i^{(\alpha, \beta)}\}$, $(\{t_i^{(\alpha, \beta)}\})$. More specifically, for the case in which $\alpha > 0$ and $\beta \ge 0$, $\boldsymbol{s}_i^{(\alpha, \beta)}$ $(\boldsymbol{t}_i^{(\alpha, \beta)})$ is given by

$$\boldsymbol{s}_i^{(\alpha, \beta)} = [s_i^{(\alpha, \beta)}(-(T-1)\beta), \cdots, \cdots, s_i^{(\alpha, \beta)}((S-1)\alpha)]^T \tag{19}$$

while for the case in which $\alpha \ge 0$ and $\beta < 0$, $\boldsymbol{s}_i^{(\alpha, \beta)}$ $(\boldsymbol{t}_i^{(\alpha, \beta)})$ is given by

$$\boldsymbol{s}_i^{(\alpha, \beta)} = [s_i^{(\alpha, \beta)}(0), \cdots, \cdots, s_i^{(\alpha, \beta)}((S-1)\alpha-\beta(T-1))]^T. \tag{20}$$

Note, however, that due to boundary effects, the vectors $\boldsymbol{s}_i^{(\alpha,\beta)}$ and $\boldsymbol{t}_i^{(\alpha,\beta)}$ are not composed of consecutive samples from the processes $\{s_i^{(\alpha,\beta)}\}$ and $\{t_i^{(\alpha,\beta)}\}$, respectively, unless $|\alpha| \le 1$ or $|\beta| \le 1$. In other words, for some arbitrary α and β there are missing samples in $\boldsymbol{s}_i^{(\alpha,\beta)}$ and $\boldsymbol{t}_i^{(\alpha,\beta)}$.

Thus for any (α, β) we have that

$$\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)} = \boldsymbol{A}_i^{(\alpha,\beta)} \boldsymbol{s}_i^{(\alpha,\beta)} \tag{21}$$

where $\boldsymbol{A}_i^{(\alpha,\beta)}$ is rectangular matrix of zeros and ones which replicates rows of $\boldsymbol{s}_i^{(\alpha,\beta)}$.

We note that the covariance matrix $\boldsymbol{R}_i^{(\alpha,\beta)}$ which characterizes the pair of processes $\{s_i^{(\alpha,\beta)}\}$, $\{t_i^{(\alpha,\beta)}\}$ is defined in terms of the concentrated version vectors $\boldsymbol{s}_i^{(\alpha,\beta)}$ and $\boldsymbol{t}_i^{(\alpha,\beta)}$, i.e.,

$$\boldsymbol{R}_i^{(\alpha,\beta)} = E[\boldsymbol{s}_i^{(\alpha,\beta)} (\boldsymbol{s}_i^{(\alpha,\beta)})^T] = E[\boldsymbol{t}_i^{(\alpha,\beta)} (\boldsymbol{t}_i^{(\alpha,\beta)})^T] \tag{22}$$

and not in terms of the covariance matrix

$$\tilde{\boldsymbol{R}}_i^{(\alpha,\beta)} = E[\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)})^T] = E[\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)})^T] \tag{23}$$

of the vectors $\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)}$ and $\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)}$. The matrix $\tilde{\boldsymbol{R}}_i^{(\alpha,\beta)}$ is a singular matrix, which is also given by

$$\tilde{\boldsymbol{R}}_i^{(\alpha,\beta)} = \boldsymbol{A}_i^{(\alpha,\beta)} \boldsymbol{R}_i^{(\alpha,\beta)} (\boldsymbol{A}_i^{(\alpha,\beta)})^T.$$

In terms of the Fisher information both $\tilde{\boldsymbol{R}}_i^{(\alpha,\beta)}$ and $\boldsymbol{R}_i^{(\alpha,\beta)}$ represent the same information on the processes $\{s_i^{(\alpha,\beta)}\}$ and $\{t_i^{(\alpha,\beta)}\}$.

Define $\boldsymbol{H}$ (see (24) at the bottom of this page) where the ith column of $\boldsymbol{H}$ consists of the values of the ith-harmonic component evaluated for all $(s, t) \in D$. We therefore have

$$\boldsymbol{h} = \boldsymbol{\rho c} \tag{25}$$

where

$$\boldsymbol{\rho} = [\boldsymbol{H}^R \quad \boldsymbol{H}^I] \tag{26}$$

and $\boldsymbol{H}^R = \mathrm{Re}\,\{\boldsymbol{H}\}$, $\boldsymbol{H}^I = \mathrm{Im}\,\{\boldsymbol{H}\}$.

Since the evanescent components $\{e_i^{(\alpha,\beta)}\}$ are mutually orthogonal, and since all the evanescent components are orthogonal to the purely indeterministic component, we conclude that $\boldsymbol{\Gamma}$, the covariance matrix of $\boldsymbol{y}$, has the form

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\mathrm{PI}} + \sum_{(\alpha,\beta)\in O} \sum_{i=1}^{I^{(\alpha,\beta)}} \boldsymbol{\Gamma}_i^{(\alpha,\beta)} \tag{27}$$

where $\boldsymbol{\Gamma}_i^{(\alpha,\beta)}$ is the covariance matrix of $\boldsymbol{e}_i^{(\alpha,\beta)}$.

Using (4), (18), and the orthogonality of $\{s_i^{(\alpha,\beta)}\}$ and $\{t_i^{(\alpha,\beta)}\}$ we find that

$$\begin{aligned}
\boldsymbol{\Gamma}_i^{(\alpha,\beta)} &= E[\boldsymbol{e}_i^{(\alpha,\beta)} (\boldsymbol{e}_i^{(\alpha,\beta)})^T] \\
&= E[(\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)} \odot \tilde{\boldsymbol{f}}_i^{(\alpha,\beta)})(\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)} \odot \tilde{\boldsymbol{f}}_i^{(\alpha,\beta)})^T] \\
&\quad + E[(\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)} \odot \tilde{\boldsymbol{g}}_i^{(\alpha,\beta)})(\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)} \odot \tilde{\boldsymbol{g}}_i^{(\alpha,\beta)})^T] \\
&= E[(\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)})^T) \odot (\tilde{\boldsymbol{f}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{f}}_i^{(\alpha,\beta)})^T)] \\
&\quad + E[(\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)})^T) \odot (\tilde{\boldsymbol{g}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{g}}_i^{(\alpha,\beta)})^T)] \\
&= \tilde{\boldsymbol{R}}_i^{(\alpha,\beta)} \odot [\tilde{\boldsymbol{f}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{f}}_i^{(\alpha,\beta)})^T + \tilde{\boldsymbol{g}}_i^{(\alpha,\beta)} (\tilde{\boldsymbol{g}}_i^{(\alpha,\beta)})^T] \\
&= \tilde{\boldsymbol{R}}_i^{(\alpha,\beta)} \odot \tilde{\boldsymbol{E}}_i^{(\alpha,\beta)} \\
&= (\boldsymbol{A}_i^{(\alpha,\beta)} \boldsymbol{R}_i^{(\alpha,\beta)} (\boldsymbol{A}_i^{(\alpha,\beta)})^T) \odot \tilde{\boldsymbol{E}}_i^{(\alpha,\beta)}
\end{aligned} \tag{28}$$

where

$$\tilde{\boldsymbol{E}}_i^{(\alpha,\beta)} = \cos\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2} \tilde{\boldsymbol{K}}^{(\alpha,\beta)}\right) \tag{29}$$

and $\tilde{\boldsymbol{K}}^{(\alpha,\beta)}$ is the $ST \times ST$ matrix given by

$$\begin{aligned}
[\tilde{\boldsymbol{K}}^{(\alpha,\beta)}]_{k,\ell} &= \left(\left\lfloor \frac{k-1}{T} \right\rfloor \beta + [((k-1) \bmod T)\alpha]\right) \\
&\quad - \left(\left\lfloor \frac{\ell-1}{T} \right\rfloor \beta + [((\ell-1) \bmod T)\alpha]\right).
\end{aligned} \tag{30}$$

A compact matrix representation of $\boldsymbol{\Gamma}_i^{(\alpha,\beta)}$ for any (α, β) cannot be derived due to the dependence of the matrix structure on (α, β). However, for the case in which $(\alpha, \beta) = (1, 0)$, (and similarly, for $(\alpha, \beta) = (0, 1)$), a more compact representation is possible. More specifically, for this special case (18) can be expressed in the form

$$\begin{aligned}
\boldsymbol{e}_i^{(1,0)} &= \tilde{\boldsymbol{s}}_i^{(1,0)} \odot \tilde{\boldsymbol{f}}_i^{(1,0)} + \tilde{\boldsymbol{t}}_i^{(1,0)} \odot \tilde{\boldsymbol{g}}_i^{(1,0)} \\
&= \boldsymbol{s}_i^{(1,0)} \otimes \boldsymbol{f}_i^{(1,0)} + \boldsymbol{t}_i^{(1,0)} \otimes \boldsymbol{g}_i^{(1,0)}
\end{aligned} \tag{31}$$

where $\otimes$ is the Kronecker product

$$\begin{aligned}
\boldsymbol{f}_i^{(1,0)} &= \mathrm{Re}\,\{[1, \exp\{j2\pi\nu_i^{(1,0)}\}, \cdots, \\
&\qquad \exp\{j2\pi\nu_i^{(1,0)}(T-1)\}]^T\}
\end{aligned} \tag{32}$$

and $\boldsymbol{g}_i^{(\alpha,\beta)}$ is the imaginary part of the same vector. Hence, using the orthogonality of $\{s_i^{(1,0)}\}$ and $\{t_i^{(1,0)}\}$, and the

$$\boldsymbol{H} = \begin{bmatrix}
e^{j2\pi[0\omega_1+0\nu_1]} & e^{j2\pi[0\omega_2+0\nu_2]} & \cdots & e^{j2\pi[0\omega_P+0\nu_P]} \\
e^{j2\pi[0\omega_1+1\nu_1]} & e^{j2\pi[0\omega_2+1\nu_2]} & \cdots & e^{j2\pi[0\omega_P+1\nu_P]} \\
\vdots & \vdots & \vdots & \vdots \\
e^{j2\pi[0\omega_1+(T-1)\nu_1]} & e^{j2\pi[0\omega_2+(T-1)\nu_2]} & \cdots & e^{j2\pi[0\omega_P+(T-1)\nu_P]} \\
e^{j2\pi[1\omega_1+0\nu_1]} & e^{j2\pi[1\omega_2+0\nu_2]} & \cdots & e^{j2\pi[1\omega_P+0\nu_P]} \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
e^{j2\pi[(S-1)\omega_1+(T-1)\nu_1]} & \cdots & \cdots & e^{j2\pi[(S-1)\omega_P+(T-1)\nu_P]}
\end{bmatrix} \tag{24}$$

$$\boldsymbol{R}_i^{(1,0)} = \begin{bmatrix} r_i^{(1,0)}(0) & r_i^{(1,0)}(1) & \cdots & r_i^{(1,0)}(S-1) \\ r_i^{(1,0)}(1) & r_i^{(1,0)}(0) & \cdots & r_i^{(1,0)}(S-2) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & r_i^{(1,0)}(1) \\ r_i^{(1,0)}(S-1) & r_i^{(1,0)}(S-2) & \cdots & r_i^{(1,0)}(0) \end{bmatrix} \tag{34}$$

$$\boldsymbol{E}_i^{(1,0)} = \begin{bmatrix} 1 & \cos(-2\pi\nu_i^{(1,0)}) & \cdots & \cos(-2\pi(T-1)\nu_i^{(1,0)}) \\ \cos(2\pi\nu_i^{(1,0)}) & 1 & \cdots & \cos(-2\pi(T-2)\nu_i^{(1,0)}) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & \cos(-2\pi\nu_i^{(1,0)}) \\ \cos(2\pi(T-1)\nu_i^{(1,0)}) & \cos(2\pi(T-2)\nu_i^{(1,0)}) & \cdots & 1 \end{bmatrix}. \tag{35}$$

properties of the Kronecker product

$$\begin{aligned} \boldsymbol{\Gamma}_i^{(1,0)} &= E[(\boldsymbol{s}_i^{(1,0)} \otimes \boldsymbol{f}_i^{(1,0)})(\boldsymbol{s}_i^{(1,0)} \otimes \boldsymbol{f}_i^{(1,0)})^T] \\ &\quad + E[(\boldsymbol{t}_i^{(1,0)} \otimes \boldsymbol{g}_i^{(1,0)})(\boldsymbol{t}_i^{(1,0)} \otimes \boldsymbol{g}_i^{(1,0)})^T] \\ &= E[(\boldsymbol{s}_i^{(1,0)} \otimes \boldsymbol{f}_i^{(1,0)})((\boldsymbol{s}_i^{(1,0)})^T \otimes (\boldsymbol{f}_i^{(1,0)})^T)] \\ &\quad + E[(\boldsymbol{t}_i^{(1,0)} \otimes \boldsymbol{g}_i^{(1,0)})((\boldsymbol{t}_i^{(1,0)})^T \otimes (\boldsymbol{g}_i^{(1,0)})^T)] \\ &= E[(\boldsymbol{s}_i^{(1,0)}(\boldsymbol{s}_i^{(1,0)})^T) \otimes (\boldsymbol{f}_i^{(1,0)}(\boldsymbol{f}_i^{(1,0)})^T)] \\ &\quad + E[(\boldsymbol{t}_i^{(1,0)}(\boldsymbol{t}_i^{(1,0)})^T) \otimes (\boldsymbol{g}_i^{(1,0)}(\boldsymbol{g}_i^{(1,0)})^T)] \\ &= \boldsymbol{R}_i^{(1,0)} \otimes [\boldsymbol{f}_i^{(1,0)}(\boldsymbol{f}_i^{(1,0)})^T + \boldsymbol{g}_i^{(1,0)}(\boldsymbol{g}_i^{(1,0)})^T] \\ &= \boldsymbol{R}_i^{(1,0)} \otimes \boldsymbol{E}_i^{(1,0)} \end{aligned} \tag{33}$$

where $\boldsymbol{R}_i^{(1,0)}$ and $\boldsymbol{E}_i^{(1,0)}$ are Toeplitz matrices, given by (34) and (35) (see the top of this page).

IV. A General Form of the CRB

The general expression for the Fisher Information Matrix (FIM) of a real Gaussian process is given by (e.g., [23])

$$\begin{aligned} [\boldsymbol{J}(\boldsymbol{\theta})]_{k,\ell} &= \frac{\partial \boldsymbol{\mu}^T}{\partial \boldsymbol{\theta}_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}_\ell} \\ &\quad + \frac{1}{2} \mathrm{tr}\left\{ \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\theta}_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\theta}_\ell} \right\} \end{aligned} \tag{36}$$

where $\boldsymbol{\mu}$ is the mean of the observation vector, $\boldsymbol{\Gamma}$ is the observation vector covariance matrix, and $[\boldsymbol{J}(\boldsymbol{\theta})]_{k,\ell}$ denotes the (k, ℓ) entry of the matrix $\boldsymbol{J}$. In our case $\boldsymbol{\mu} \equiv \boldsymbol{h}$.

Taking the partial derivatives of $\boldsymbol{h}$ we get

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{c}_\ell} = \boldsymbol{\rho}_\ell \tag{37}$$

where $\boldsymbol{\rho}_\ell$ is the ℓth column of $\boldsymbol{\rho}$. Since the evanescent components, as well as the purely indeterministic component, are zero-mean fields, the mean vector is independent of their parameters. Hence

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{b}_k} = 0 \tag{38}$$

$$\frac{\partial \boldsymbol{h}}{\partial [\boldsymbol{\phi}_i^{(\alpha,\beta)}]_k} = 0. \tag{39}$$

Note also that since the field covariance function is independent of the mean

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{c}_k} = 0 \tag{40}$$

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\omega}_k} = 0 \tag{41}$$

and

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\nu}_k} = 0. \tag{42}$$

Hence, the $\frac{1}{2}\mathrm{tr}\{\cdot\}$ term in (36) vanishes for all the FIM entries that correspond to parameters of the harmonic mean. Therefore, $\boldsymbol{J}^{\boldsymbol{b},\boldsymbol{c}} = 0$, $\boldsymbol{J}^{\boldsymbol{b},\boldsymbol{\omega}} = 0$, $\boldsymbol{J}^{\boldsymbol{b},\boldsymbol{\nu}} = 0$, and for the evanescent components we have that for all (α, β) and i, $\boldsymbol{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{c}} = 0$, $\boldsymbol{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{\omega}} = 0$, $\boldsymbol{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{\nu}} = 0$. Hence we conclude that the estimation problem of the harmonic component is decoupled from that of the purely indeterministic and the evanescent components.

Using (37) and (40) we conclude that the FIM elements which correspond to the amplitude parameters of the harmonic component are given by

$$[\boldsymbol{J}^{\boldsymbol{c},\boldsymbol{c}}]_{k,\ell} = \boldsymbol{\rho}_k^T \boldsymbol{\Gamma}^{-1} \boldsymbol{\rho}_\ell. \tag{43}$$

Let

$$\boldsymbol{\tau}_1 = [0, 1, \cdots, (S-1)]^T \otimes \boldsymbol{1}_T \tag{44a}$$

$$\boldsymbol{\tau}_2 = \boldsymbol{1}_S \otimes [0, 1, \cdots, (T-1)]^T \tag{44b}$$

where $\boldsymbol{1}_T$ and $\boldsymbol{1}_S$ are T- and S-dimensional column vectors of ones, respectively. In other words, $\boldsymbol{\tau}_1$ is the vector of the first indices of the elements of $\boldsymbol{h}$ in (13), and $\boldsymbol{\tau}_2$ is the vector of the second indices of the elements of $\boldsymbol{h}$. Taking now the partial derivatives with respect to the harmonic frequencies yields

$$\frac{\partial \boldsymbol{h}}{\partial \omega_p} = 2\pi\, \mathrm{diag}(\boldsymbol{\tau}_1)(D_p \boldsymbol{H}_p^R - C_p \boldsymbol{H}_p^I) \tag{45a}$$

$$\frac{\partial \boldsymbol{h}}{\partial \nu_p} = 2\pi\, \mathrm{diag}(\boldsymbol{\tau}_2)(D_p \boldsymbol{H}_p^R - C_p \boldsymbol{H}_p^I) \tag{45b}$$

where $\mathrm{diag}(\boldsymbol{\tau}_1)$, $(\mathrm{diag}(\boldsymbol{\tau}_2))$, is an $ST \times ST$ matrix whose diagonal is the vector $\boldsymbol{\tau}_1$, $(\boldsymbol{\tau}_2)$, and $\boldsymbol{H}_p^R$, $(\boldsymbol{H}_p^I)$, is the pth column of $\boldsymbol{H}^R$, $(\boldsymbol{H}^I)$.

Substituting (37), (40), (41), and (45a) into (36)

$$[\boldsymbol{J}^{\boldsymbol{c},\boldsymbol{\omega}}]_{k,\ell} = 2\pi \boldsymbol{\rho}_k^T \boldsymbol{\Gamma}^{-1} \operatorname{diag}(\boldsymbol{\tau}_1)(D_\ell \boldsymbol{H}_\ell^R - C_\ell \boldsymbol{H}_\ell^I). \quad (46)$$

In a similar way we obtain

$$[\boldsymbol{J}^{\boldsymbol{c},\boldsymbol{\nu}}]_{k,\ell} = 2\pi \boldsymbol{\rho}_k^T \boldsymbol{\Gamma}^{-1} \operatorname{diag}(\boldsymbol{\tau}_2)(D_\ell \boldsymbol{H}_\ell^R - C_\ell \boldsymbol{H}_\ell^I) \quad (47)$$

$$[\boldsymbol{J}^{\boldsymbol{\omega},\boldsymbol{\omega}}]_{k,\ell} = 4\pi^2 (D_k \boldsymbol{H}_k^R - C_k \boldsymbol{H}_k^I)^T \operatorname{diag}(\boldsymbol{\tau}_1) \boldsymbol{\Gamma}^{-1} \cdot \operatorname{diag}(\boldsymbol{\tau}_1)(D_\ell \boldsymbol{H}_\ell^R - C_\ell \boldsymbol{H}_\ell^I) \quad (48)$$

$$[\boldsymbol{J}^{\boldsymbol{\omega},\boldsymbol{\nu}}]_{k,\ell} = 4\pi^2 (D_k \boldsymbol{H}_k^R - C_k \boldsymbol{H}_k^I)^T \operatorname{diag}(\boldsymbol{\tau}_1) \boldsymbol{\Gamma}^{-1} \cdot \operatorname{diag}(\boldsymbol{\tau}_2)(D_\ell \boldsymbol{H}_\ell^R - C_\ell \boldsymbol{H}_\ell^I) \quad (49)$$

$$[\boldsymbol{J}^{\boldsymbol{\nu},\boldsymbol{\nu}}]_{k,\ell} = 4\pi^2 (D_k \boldsymbol{H}_k^R - C_k \boldsymbol{H}_k^I)^T \operatorname{diag}(\boldsymbol{\tau}_2) \boldsymbol{\Gamma}^{-1} \cdot \operatorname{diag}(\boldsymbol{\tau}_2)(D_\ell \boldsymbol{H}_\ell^R - C_\ell \boldsymbol{H}_\ell^I). \quad (50)$$

Using the orthogonality of the evanescent components, their orthogonality to the purely indeterministic component, and (27), we find that

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{b}_k} = \frac{\partial \boldsymbol{\Gamma}_{\mathrm{PI}}}{\partial \boldsymbol{b}_k} \quad (51)$$

and for all $(\alpha, \beta) \in O$ and i

$$\frac{\partial \boldsymbol{\Gamma}}{\partial [\boldsymbol{\phi}_i^{(\alpha,\beta)}]_k} = \frac{\partial \boldsymbol{\Gamma}_i^{(\alpha,\beta)}}{\partial [\boldsymbol{\phi}_i^{(\alpha,\beta)}]_k}. \quad (52)$$

Substituting (38) and (39) into (36), we conclude that, for all the FIM entries that correspond to parameters of the purely indeterministic and the evanescent components, the mean dependent term of (36) vanishes. In particular

$$\left[\boldsymbol{J}^{\boldsymbol{b},\boldsymbol{b}}\right]_{k,\ell} = \frac{1}{2} \operatorname{tr}\left\{\boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}_{\mathrm{PI}}}{\partial \boldsymbol{b}_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}_{\mathrm{PI}}}{\partial \boldsymbol{b}_\ell}\right\} \quad (53)$$

$$\left[\boldsymbol{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{b}}\right]_{k,\ell} = \frac{1}{2} \operatorname{tr}\left\{\boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}_i^{(\alpha,\beta)}}{\partial [\boldsymbol{\phi}_i^{(\alpha,\beta)}]_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}_{\mathrm{PI}}}{\partial \boldsymbol{b}_\ell}\right\} \quad (54)$$

and

$$\left[\boldsymbol{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{\phi}_j^{(\gamma,\delta)}}\right]_{k,\ell} = \frac{1}{2} \operatorname{tr}\left\{\boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}_i^{(\alpha,\beta)}}{\partial [\boldsymbol{\phi}_i^{(\alpha,\beta)}]_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}_j^{(\gamma,\delta)}}{\partial [\boldsymbol{\phi}_j^{(\gamma,\delta)}]_\ell}\right\} \quad (55)$$

where $(\gamma, \delta) \in O$, and $1 \le j \le I^{(\gamma,\delta)}$.

Using (29) and the separability of (28) we find that

$$\begin{aligned} \frac{\partial \boldsymbol{\Gamma}_i^{(\alpha,\beta)}}{\partial \nu_i^{(\alpha,\beta)}} &= \tilde{\boldsymbol{R}}_i^{(\alpha,\beta)} \odot \frac{\partial \tilde{\boldsymbol{E}}_i^{(\alpha,\beta)}}{\partial \nu_i^{(\alpha,\beta)}} \\ &= -\frac{2\pi}{\alpha^2+\beta^2} (\boldsymbol{A}_i^{(\alpha,\beta)} \boldsymbol{R}_i^{(\alpha,\beta)} (\boldsymbol{A}_i^{(\alpha,\beta)})^T) \\ &\quad \odot \tilde{\boldsymbol{K}}^{(\alpha,\beta)} \odot \sin\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2} \tilde{\boldsymbol{K}}^{(\alpha,\beta)}\right) \end{aligned} \quad (56)$$

while

$$\begin{aligned} \frac{\partial \boldsymbol{\Gamma}_i^{(\alpha,\beta)}}{\partial [\boldsymbol{a}_i^{(\alpha,\beta)}]_k} &= \frac{\partial \tilde{\boldsymbol{R}}_i^{(\alpha,\beta)}}{\partial [\boldsymbol{a}_i^{(\alpha,\beta)}]_k} \odot \tilde{\boldsymbol{E}}_i^{(\alpha,\beta)} \\ &= \left(\boldsymbol{A}_i^{(\alpha,\beta)} \frac{\partial \boldsymbol{R}_i^{(\alpha,\beta)}}{\partial [\boldsymbol{a}_i^{(\alpha,\beta)}]_k} (\boldsymbol{A}_i^{(\alpha,\beta)})^T\right) \odot \tilde{\boldsymbol{E}}_i^{(\alpha,\beta)}. \end{aligned} \quad (57)$$

We have previously concluded that the estimation problem of the harmonic component is decoupled from the estimation problem of the purely indeterministic and the evanescent components. Using (53)–(55) we find that the bound on the purely indeterministic and the evanescent components is found by inverting the FIM block which corresponds to the parameters of the purely indeterministic and the evanescent components, and it is independent of the harmonic component parameters. Therefore, this bound is identical to the one obtained for the case in which no harmonic component exists.

From the Wold-type decomposition (1), it is known that the purely indeterministic component of the field has a unique white innovations-driven MA representation. In [25] we consider the representation of the covariance matrix of a 2-D MA random field in terms of the MA model parameters, for finite-order MA models. The derivatives of the covariance matrix, $\boldsymbol{\Gamma}_{\mathrm{PI}}$, with respect to the MA model parameters are derived as well. Hence, in this paper we consider only a simple special case of the general derivation, and assume that the purely indeterministic component is a zero-mean, white Gaussian field with variance σ^2. Therefore, $\boldsymbol{b} = [\sigma^2]$, and

$$\boldsymbol{\Gamma}_{\mathrm{PI}} = \sigma^2 \boldsymbol{I}_{ST \times ST} \quad (58)$$

where $\boldsymbol{I}_{ST\times ST}$ is an $ST \times ST$ identity matrix. Also,

$$\frac{\partial \boldsymbol{\Gamma}_{\mathrm{PI}}}{\partial \boldsymbol{b}_1} = \boldsymbol{I}_{ST \times ST}. \quad (59)$$

Thus for the case in which the purely indeterministic component is a Gaussian white noise field, substitution of (27) and (59) into (53), followed by substitution of (27) into (46)–(50), (54), (55), provides an expression of the exact CRB on the parameters of the observed homogeneous random field. Similar substitution of the expressions for $\boldsymbol{\Gamma}_{\mathrm{PI}}$ and $\partial \boldsymbol{\Gamma}_{\mathrm{PI}} / \partial \boldsymbol{b}_k$ of an MA modeled purely indeterministic component [25] provides an expression of the exact CRB on the parameters of essentially any Gaussian homogeneous random field.

A. The Case of a Nil Purely Indeterministic Component

In this section we specialize the foregoing general results for the special case of a homogeneous random field with a nil purely indeterministic component. In particular, we concentrate on the case of an observed field which is composed of only a single evanescent component $\{e^{(\alpha,\beta)}\}$. In this special case $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(\alpha,\beta)}$.

Recall that whenever $n\alpha - m\beta = k\alpha - \ell\beta$, for some integers n, m, k, ℓ such that $0 \le n, k \le S-1$ and $0 \le m, \ell \le T-1$, the same element of $\tilde{\boldsymbol{s}}_i^{(\alpha,\beta)}$ (and $\tilde{\boldsymbol{t}}_i^{(\alpha,\beta)}$) appears more than once in the vector. For a rectangular observed field of dimensions $S \times T$ the number of distinct samples from the random process $\{s_i^{(\alpha,\beta)}\}$, $(\{t_i^{(\alpha,\beta)}\})$, that are found in the observed field is

$$(S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1).$$

Using (18) we have that the elements of the observed vector $\boldsymbol{e}^{(\alpha,\beta)}$ are linear combinations of the elements of $\boldsymbol{s}^{(\alpha,\beta)}$ and $\boldsymbol{t}^{(\alpha,\beta)}$ weighted by deterministic cosinusoidal functions of a single parameter $\nu^{(\alpha,\beta)}$.

Since the 1-D modulating processes $\{s^{(\alpha,\beta)}\}$ and $\{t^{(\alpha,\beta)}\}$ are independent, the elements of the vector $\boldsymbol{e}^{(\alpha,\beta)}$ in (18) form a linear space whose dimension is at most

$$2[(S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1)].$$

In other words, all other elements of $\boldsymbol{e}^{(\alpha,\beta)}$ can be expressed as *deterministic* linear combinations of these elements. Hence, in this present case, $\boldsymbol{\Gamma}$, the covariance matrix of the observed field, is a singular matrix. Therefore, a straightforward substitution into the general expression for the FIM of a real Gaussian process (36) is impossible.

In terms of the Fisher information, the information contents of a vector is equivalent to that of any linear transformation thereof. Hence we define the *concentrated version*, $\overline{\boldsymbol{e}}^{(\alpha,\beta)}$ of $\boldsymbol{e}^{(\alpha,\beta)}$ as follows: From every column of the observed field, where "column" is defined with respect to the NSHP total ordering denoted by (α, β), we arbitrarily choose two samples of $\{e^{(\alpha,\beta)}\}$. Note that it is possible that for some combinations of lattice dimensions and an NSHP total ordering definition (α, β), some columns will contain only one sample of the field. Hence the dimension of the constructed column vector $\overline{\boldsymbol{e}}^{(\alpha,\beta)}$ is less or equal to $2[(S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1)]$. The elements of $\overline{\boldsymbol{e}}^{(\alpha,\beta)}$ are guaranteed to be linearly independent, and hence its covariance matrix is invertible. Let us denote this covariance matrix by $\overline{\boldsymbol{\Gamma}}^{(\alpha,\beta)}$, i.e.,

$$\overline{\boldsymbol{\Gamma}}^{(\alpha,\beta)} = E[\overline{\boldsymbol{e}}^{(\alpha,\beta)}(\overline{\boldsymbol{e}}^{(\alpha,\beta)})^T].$$

Thus the FIM for the case in which the observed field is composed of a single evanescent component is given by

$$[\boldsymbol{J}(\boldsymbol{\phi}^{(\alpha,\beta)})]_{k,\ell} = \frac{1}{2}\,\mathrm{tr}\left\{(\overline{\boldsymbol{\Gamma}}^{(\alpha,\beta)})^{-1}\frac{\partial \overline{\boldsymbol{\Gamma}}^{(\alpha,\beta)}}{\partial[\boldsymbol{\phi}^{(\alpha,\beta)}]_k} \cdot (\overline{\boldsymbol{\Gamma}}^{(\alpha,\beta)})^{-1}\frac{\partial \overline{\boldsymbol{\Gamma}}^{(\alpha,\beta)}}{\partial[\boldsymbol{\phi}^{(\alpha,\beta)}]_\ell}\right\}. \quad (60)$$

The derivation of the derivatives of the covariance matrix with respect to the evanescent component parameters is similar to that which leads to (56), (57), and hence is omitted.

V. The FIM for Evanescent Components with Gaussian MA and AR Modulating Processes

In the previous sections we have derived an expression for the exact CRB on the parameters of a homogeneous random field with mixed spectral distribution. In this derivation it was assumed that for each evanescent field the 1-D purely indeterministic processes $\{s_i^{(\alpha,\beta)}\}$ and $\{t_i^{(\alpha,\beta)}\}$ are zero-mean Gaussian processes whose covariance matrix has some known, but unspecified parametric form, where $\boldsymbol{a}_i^{(\alpha,\beta)}$ is the parameter vector. In this section we specialize the results of the previous section. We consider two different parametric models for the modulating 1-D purely indeterministic processes of the evanescent field. First we consider the case in which the modulating 1-D processes are moving average processes. Next we consider the case in which these processes are autoregressive. Using this derivation we finally obtain a closed-form exact expression of the CRB on the error variance in estimating the parameters of the homogeneous random field.

A. Evanescent Components with Gaussian MA Modulating Processes

Let $n^{(\alpha,\beta)} = n\alpha - m\beta$. Assume that the modulating 1-D processes $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ of each evanescent field can be modeled by a finite-order MA model, i.e.,

$$s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = \sum_{\tau=0}^{Q_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-\tau) \quad (61)$$

$$t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = \sum_{\tau=0}^{Q_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-\tau) \quad (62)$$

where

$$n^{(\alpha,\beta)} = \begin{cases} -(T-1)\beta, \cdots, (S-1)\alpha, & \alpha > 0 \text{ and } \beta \geq 0 \\ 0, \cdots, (S-1)\alpha - (T-1)\beta, & \alpha \geq 0 \text{ and } \beta < 0 \end{cases} \quad (63)$$

and $a_i^{(\alpha,\beta)}(0) = 1$. The driving noise processes $\{\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ are independent, zero-mean, white, and Gaussian, and each has variance $(\sigma_i^{(\alpha,\beta)})^2$. We further assume that the MA processes are of known orders, $Q_i^{(\alpha,\beta)}$ where

$$Q_i^{(\alpha,\beta)} \leq (S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1).$$

For the case in which $\alpha > 0$ and $\beta \geq 0$, define the $(S-1)|\alpha| + (T-1)|\beta| + 1 + Q_i^{(\alpha,\beta)}$-dimensional vector of *consecutive* samples

$$\boldsymbol{\xi}_i^{(\alpha,\beta)} = [\xi_i^{(\alpha,\beta)}(-(T-1)\beta - Q_i^{(\alpha,\beta)}), \xi_i^{(\alpha,\beta)}(-(T-1)\beta - Q_i^{(\alpha,\beta)}+1), \cdots, \cdots, \xi_i^{(\alpha,\beta)}((S-1)\alpha)]^T \quad (64)$$

while for the case in which $\alpha \geq 0$ and $\beta < 0$

$$\boldsymbol{\xi}_i^{(\alpha,\beta)} = [\xi_i^{(\alpha,\beta)}(-Q_i^{(\alpha,\beta)}), \xi_i^{(\alpha,\beta)}(-Q_i^{(\alpha,\beta)}+1), \cdots, \cdots, \xi_i^{(\alpha,\beta)}((S-1)\alpha - \beta(T-1))]^T. \quad (65)$$

Hence for both cases we have

$$\boldsymbol{s}_i^{(\alpha,\beta)} = (\boldsymbol{W}_i^{(\alpha,\beta)}\boldsymbol{D}_i^{(\alpha,\beta)})\boldsymbol{\xi}_i^{(\alpha,\beta)} \quad (66)$$

where $\boldsymbol{D}_i^{(\alpha,\beta)}$ is the

$$((S-1)|\alpha| + (T-1)|\beta| + 1) \times ((S-1)|\alpha| + (T-1)|\beta| + 1 + Q_i^{(\alpha,\beta)})$$

Toeplitz matrix as seen in (67) (see the top of the following page) and $\boldsymbol{W}_i^{(\alpha,\beta)}$ is a rectangular matrix of zeros and ones that eliminates rows which correspond to the $(|\alpha|-1)(|\beta|-1)$ samples that are missing from $\boldsymbol{s}_i^{(\alpha,\beta)}$ due to the edge effects. These missing samples result in $\boldsymbol{s}_i^{(\alpha,\beta)}$ being composed of nonconsecutive samples in its top and bottom.

Thus the covariance matrix $\boldsymbol{R}_i^{(\alpha,\beta)}$ of the $Q_i^{(\alpha,\beta)}$-order MA process is given by

$$\boldsymbol{R}_i^{(\alpha,\beta)} = (\sigma_i^{(\alpha,\beta)})^2\boldsymbol{W}_i^{(\alpha,\beta)}\boldsymbol{D}_i^{(\alpha,\beta)}(\boldsymbol{D}_i^{(\alpha,\beta)})^T(\boldsymbol{W}_i^{(\alpha,\beta)})^T. \quad (68)$$

The parameter vector of the 1-D purely indeterministic modulating MA processes $\boldsymbol{a}_i^{(\alpha,\beta)}$ is the $Q_i^{(\alpha,\beta)}+1$-dimensional

$$
\boldsymbol{D}_i^{(\alpha,\beta)} = \begin{bmatrix} a_i^{(\alpha,\beta)}(Q_i^{(\alpha,\beta)}) & \cdots & a_i^{(\alpha,\beta)}(1) & 1 & & & \\ & a_i^{(\alpha,\beta)}(Q_i^{(\alpha,\beta)}) & \cdots & a_i^{(\alpha,\beta)}(1) & 1 & \mathbf{0} & \\ & \mathbf{0} & \ddots & & \ddots & & \\ & & a_i^{(\alpha,\beta)}(Q_i^{(\alpha,\beta)}) & \cdots & a_i^{(\alpha,\beta)}(1) & 1 & \end{bmatrix} \tag{67}
$$

vector

$$
\boldsymbol{a}_i^{(\alpha,\beta)} = [(\sigma_i^{(\alpha,\beta)})^2, a_i^{(\alpha,\beta)}(1), a_i^{(\alpha,\beta)}(2), \cdots, a_i^{(\alpha,\beta)}(Q_i^{(\alpha,\beta)})]^T.
$$

Taking the partial derivatives of $\boldsymbol{R}_i^{(\alpha,\beta)}$ using (68) we have

$$
\frac{\partial \boldsymbol{R}_i^{(\alpha,\beta)}}{\partial(\sigma_i^{(\alpha,\beta)})^2} = \frac{1}{(\sigma_i^{(\alpha,\beta)})^2}\boldsymbol{R}_i^{(\alpha,\beta)} \tag{69}
$$

$$
\frac{\partial \boldsymbol{R}_i^{(\alpha,\beta)}}{\partial a_i^{(\alpha,\beta)}(n)} = (\sigma_i^{(\alpha,\beta)})^2[\boldsymbol{W}_i^{(\alpha,\beta)}\boldsymbol{U}_n(\boldsymbol{D}_i^{(\alpha,\beta)})^T(\boldsymbol{W}_i^{(\alpha,\beta)})^T + \boldsymbol{W}_i^{(\alpha,\beta)}\boldsymbol{D}_i^{(\alpha,\beta)}\boldsymbol{U}_n^T(\boldsymbol{W}_i^{(\alpha,\beta)})^T], \quad n=1,\cdots,Q_i^{(\alpha,\beta)} \tag{70}
$$

where $\boldsymbol{U}_n$ is the up-shift matrix

$$
[\boldsymbol{U}_n]_{k,\ell} = \begin{cases} 1, & \ell - k = Q_i^{(\alpha,\beta)} - n \\ 0, & \text{otherwise.} \end{cases} \tag{71}
$$

Substituting (68)–(71) into (57) we obtain a closed-form expression for $\partial\boldsymbol{\Gamma}_i^{(\alpha,\beta)}/\partial[\boldsymbol{a}_i^{(\alpha,\beta)}]_k$. Using (68), a closed-form expression for (28) is obtained. Thus for the case in which the modulating 1-D processes of the evanescent fields are MA processes, we have obtained a closed-form exact expression for the CRB on the error variance in estimating the parameters of the homogeneous random field.

B. Evanescent Components with Gaussian AR Modulating Processes

In the following, we assume that the modulating 1-D processes $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ of each evanescent field can be modeled by a finite-order AR model, i.e.,

$$
s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = -\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-\tau) + \xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \tag{72}
$$

and

$$
t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = -\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-\tau) + \zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \tag{73}
$$

where the range of $n^{(\alpha,\beta)}$ is given by (63). The driving noise processes $\{\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ are independent, zero-mean, white, and Gaussian, and each has variance $(\sigma_i^{(\alpha,\beta)})^2$. In this section it is assumed that $\boldsymbol{s}_i^{(\alpha,\beta)}$ and $\boldsymbol{t}_i^{(\alpha,\beta)}$ are composed of $(S-1)|\alpha|+(T-1)|\beta|+1$ consecutive samples of the corresponding processes. Note, however, that in general $\boldsymbol{s}_i^{(\alpha,\beta)}$ and $\boldsymbol{t}_i^{(\alpha,\beta)}$ are of a lower dimension $(S-1)|\alpha|+(T-1)|\beta|+1-(|\alpha|-1)(|\beta|-1)$ due to the boundary effects that result is missing data samples. Thus the exact bound on the parameters of the observed homogeneous random field, for the case in which $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ are finite-order AR processes, is tight only for those cases in which $|\alpha| \leq 1$ or $|\beta| \leq 1$. We further assume that the AR processes are of known orders, $V_i^{(\alpha,\beta)}$ where

$$
V_i^{(\alpha,\beta)} \leq (S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1).
$$

It can be shown [24] that the inverse covariance matrix $(\boldsymbol{R}_i^{(\alpha,\beta)})^{-1}$ of a $V_i^{(\alpha,\beta)}$ order AR process is given by

$$
(\boldsymbol{R}_i^{(\alpha,\beta)})^{-1} = \frac{1}{(\sigma_i^{(\alpha,\beta)})^2}(\boldsymbol{C}_i^{(\alpha,\beta)}(\boldsymbol{C}_i^{(\alpha,\beta)})^T - \boldsymbol{B}_i^{(\alpha,\beta)}(\boldsymbol{B}_i^{(\alpha,\beta)})^T) \tag{74}
$$

where $\boldsymbol{C}_i^{(\alpha,\beta)}$ and $\boldsymbol{B}_i^{(\alpha,\beta)}$ are lower triangular Toeplitz matrices such that

$$
[\boldsymbol{C}_i^{(\alpha,\beta)}]_{k,\ell} = \begin{cases} 1, & k=\ell \\ a_i^{(\alpha,\beta)}(k-\ell), & k>\ell \\ 0, & k<\ell \end{cases} \tag{75}
$$

$$
[\boldsymbol{B}_i^{(\alpha,\beta)}]_{k,\ell} = \begin{cases} a_i^{(\alpha,\beta)}((S-1)\alpha + (T-1)|\beta| + 1 - k + \ell), & k \geq \ell \\ 0, & k<\ell \end{cases} \tag{76}
$$

and $a_i^{(\alpha,\beta)}(k) = 0$ for $k<0$ and $k > V_i^{(\alpha,\beta)}$.

The parameter vector of the 1-D purely indeterministic modulating processes $\boldsymbol{a}_i^{(\alpha,\beta)}$ is the $V_i^{(\alpha,\beta)}+1$-dimensional vector

$$
\boldsymbol{a}_i^{(\alpha,\beta)} = [(\sigma_i^{(\alpha,\beta)})^2, a_i^{(\alpha,\beta)}(1), a_i^{(\alpha,\beta)}(2), \cdots, a_i^{(\alpha,\beta)}(V_i^{(\alpha,\beta)})]^T.
$$

Taking the partial derivatives of $(\boldsymbol{R}_i^{(\alpha,\beta)})^{-1}$ using (74) we have

$$
\frac{\partial(\boldsymbol{R}_i^{(\alpha,\beta)})^{-1}}{\partial(\sigma_i^{(\alpha,\beta)})^2} = -\frac{1}{(\sigma_i^{(\alpha,\beta)})^4}(\boldsymbol{C}_i^{(\alpha,\beta)}(\boldsymbol{C}_i^{(\alpha,\beta)})^T - \boldsymbol{B}_i^{(\alpha,\beta)}(\boldsymbol{B}_i^{(\alpha,\beta)})^T) = -\frac{1}{(\sigma_i^{(\alpha,\beta)})^2}(\boldsymbol{R}_i^{(\alpha,\beta)})^{-1} \tag{77}
$$

$$
\frac{\partial(\boldsymbol{R}_i^{(\alpha,\beta)})^{-1}}{\partial a_i^{(\alpha,\beta)}(n)} = \frac{1}{(\sigma_i^{(\alpha,\beta)})^2}[\boldsymbol{Z}_n(\boldsymbol{C}_i^{(\alpha,\beta)})^T + \boldsymbol{C}_i^{(\alpha,\beta)}\boldsymbol{Z}_n^T - \boldsymbol{Z}_{(S-1)\alpha+(T-1)|\beta|+1-n}(\boldsymbol{B}_i^{(\alpha,\beta)})^T - \boldsymbol{B}_i^{(\alpha,\beta)}\boldsymbol{Z}_{(S-1)\alpha+(T-1)|\beta|+1-n}^T], \quad n=1,\cdots,V_i^{(\alpha,\beta)} \tag{78}
$$

where $\boldsymbol{Z}_n$ is the down-shift matrix

$$[\boldsymbol{Z}_n]_{k,\ell} = \begin{cases} 1, & k-\ell = n \\ 0, & \text{otherwise.} \end{cases} \tag{79}$$

Using the identity

$$\begin{aligned} &\partial \boldsymbol{R}_i^{(\alpha,\beta)} / \partial [\boldsymbol{a}_i^{(\alpha,\beta)}]_k \\ &\quad = -\boldsymbol{R}_i^{(\alpha,\beta)} \left[(\partial ((\boldsymbol{R}_i^{(\alpha,\beta)})^{-1}) / \partial [\boldsymbol{a}_i^{(\alpha,\beta)}]_k \right] \boldsymbol{R}_i^{(\alpha,\beta)} \end{aligned}$$

we can now substitute (74)–(79) into (57) in order to obtain a closed-form expression for $\partial \boldsymbol{\Gamma}_i^{(\alpha,\beta)} / \partial [\boldsymbol{a}_i^{(\alpha,\beta)}]_k$. Using (74), closed-form expressions for (28) and (56) are obtained.

Finally, using the well-known relation between the CRB of some parameter vector and any differentiable function of it (see, e.g., [31] or [32]), we have that the CRB of the spectral density function $S_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})$ of the evanescent field modulating AR process is given by

$$\mathrm{CRB}\,(S_i^{(\alpha\,\beta)}(e^{j\omega})) = (\boldsymbol{Y}_i^{(\alpha,\beta)})^T\,\mathrm{CRB}\,(\boldsymbol{a}_i^{(\alpha,\beta)})\boldsymbol{Y}_i^{(\alpha,\beta)} \tag{80}$$

where

$$\begin{aligned} \boldsymbol{Y}_i^{(\alpha,\beta)} &= \left[\frac{\partial S_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})}{\partial (\sigma_i^{(\alpha,\beta)})^2}, \frac{\partial S_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})}{\partial a_i^{(\alpha,\beta)}(1)}, \ldots, \right. \\ &\qquad \left. \frac{\partial S_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})}{\partial a_i^{(\alpha,\beta)}(V_i^{(\alpha,\beta)})} \right]^T \\ &= 2S_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega}) \left[\frac{1}{2(\sigma_i^{(\alpha,\beta)})^2}, -\mathrm{Re}\left\{ \frac{\mathrm{e}^{j\omega(V_i^{(\alpha,\beta)}-1)}}{\mathcal{A}_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})} \right\}, \right. \\ &\qquad \left. -\mathrm{Re}\left\{ \frac{\mathrm{e}^{j\omega(V_i^{(\alpha,\beta)}-2)}}{\mathcal{A}_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})} \right\}, \ldots, -\mathrm{Re}\left\{ \frac{1}{\mathcal{A}_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})} \right\} \right]^T \end{aligned} \tag{81}$$

$$\begin{aligned} \mathcal{A}_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega}) &= \mathrm{e}^{j\omega V_i^{(\alpha,\beta)}} + a_i^{(\alpha,\beta)}(1)\mathrm{e}^{j\omega(V_i^{(\alpha,\beta)}-1)} \\ &\quad + \cdots + a_i^{(\alpha,\beta)}(V_i^{(\alpha,\beta)}) \end{aligned} \tag{82}$$

and $S_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega}) = (\sigma_i^{(\alpha,\beta)})^2)/|\mathcal{A}_i^{(\alpha,\beta)}(\mathrm{e}^{j\omega})|^2$.

C. The FIM for Homogeneous Fields with Evanescent Components of $(\alpha, \beta) = (1, 0)$

As noted in Section III, $\boldsymbol{\Gamma}_i^{(\alpha,\beta)}$ has a different structure for any value of α and β; only for $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, can the matrix $\boldsymbol{\Gamma}_i^{(\alpha,\beta)}$ be compactly represented as the Kronecker product of two smaller Toeplitz matrices. Hence, a more compact closed-form expression for the exact CRB can be derived.

For $(\alpha, \beta) = (1, 0)$, (56) and (57) can be rewritten using (33) in the form

$$\frac{\partial \boldsymbol{\Gamma}_i^{(1,0)}}{\partial \nu_i^{(1,0)}} = \boldsymbol{R}_i^{(1,0)} \otimes \frac{\partial \boldsymbol{E}_i^{(1,0)}}{\partial \nu_i^{(1,0)}} \tag{83}$$

while

$$\frac{\partial \boldsymbol{\Gamma}_i^{(1,0)}}{\partial [\boldsymbol{a}_i^{(1,0)}]_k} = \frac{\partial \boldsymbol{R}_i^{(1,0)}}{\partial [\boldsymbol{a}_i^{(1,0)}]_k} \otimes \boldsymbol{E}_i^{(1,0)}. \tag{84}$$

Note that

$$\frac{\partial \boldsymbol{E}_i^{(1,0)}}{\partial \nu_i^{(1,0)}} = -2\pi \boldsymbol{K}^{(1,0)} \odot \boldsymbol{N}_i^{(1,0)} \tag{85}$$

where

$$\boldsymbol{K}^{(1,0)} = \begin{bmatrix} 0 & 1 & \cdots & T-1 \\ 1 & 0 & \cdots & T-2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & 1 \\ T-1 & T-2 & \cdots & 0 \end{bmatrix} \tag{86}$$

and

$$\boldsymbol{N}_i^{(1,0)} = \sin(2\pi \nu_i^{(1,0)} \boldsymbol{K}^{(1,0)}). \tag{87}$$

Using (68)–(71) for the case of MA modulating processes, or alternatively (74)–(79) for AR modulating processes, compact closed-form expressions for the CRB are obtained.

VI. Numerical Examples

To gain more insight into the behavior of the bound on the different components, we resort to numerical evaluation of some specific examples. In this section, we present several such examples which illustrate the dependence of the bound on various parameters of the field.

Example 1: Consider a 2-D homogeneous random field consisting of a sum of two harmonic components, a single evanescent field $e_{(1,0)}(n, m)$, and a zero-mean, unit variance, white Gaussian purely indeterministic component. The frequencies of the two harmonic components are $(\omega_1, \nu_1) = (0.15, 0.25)$ and $(\omega_2, \nu_2) = (0.16, 0.26)$. The evanescent field frequency parameter is $\nu^{(1,0)} = 0.1$. The evanescent component modulating 1-D purely indeterministic processes are narrowband second-order AR processes whose parameters are $a^{(1,0)}(1) = -1.378$, $a^{(1,0)}(2) = 0.95$. In this example, we investigate the bounds as a function of the variance of the AR model driving noise.

The results indicate that varying the variance of the AR model driving noise from 0.5 to 5 has almost no effect on the CRB for the parameters of the harmonic components, as well as on the bound on the noise variance of the purely indeterministic component. For example, the bound on ω_1 has risen from $4.4022 \cdot 10^{-7}$ to $4.4024 \cdot 10^{-7}$, and similarly the bound on the noise variance of the purely indeterministic component has risen from $1.312 \cdot 10^{-3}$ to $1.316 \cdot 10^{-3}$. This slight increase is due to the presence of the evanescent component whose energy increases from experiment to experiment.

On the other hand, varying the variance of the AR model driving noise has a significant influence on the CRB for the evanescent component frequency parameter, $\nu^{(1,0)}$, as illustrated in Fig. 2. We also note that the bounds on estimating the parameters of the modulating AR processes decrease with the increase in the evanescent component energy.

Example 2: Consider a 2-D homogeneous random field consisting of a sum a single-harmonic component, a single evanescent field $e_{(1,0)}(n, m)$, and a zero-mean, unit variance, white Gaussian purely indeterministic component. The harmonic component frequency is given by $(\omega_1, \nu_1) =$

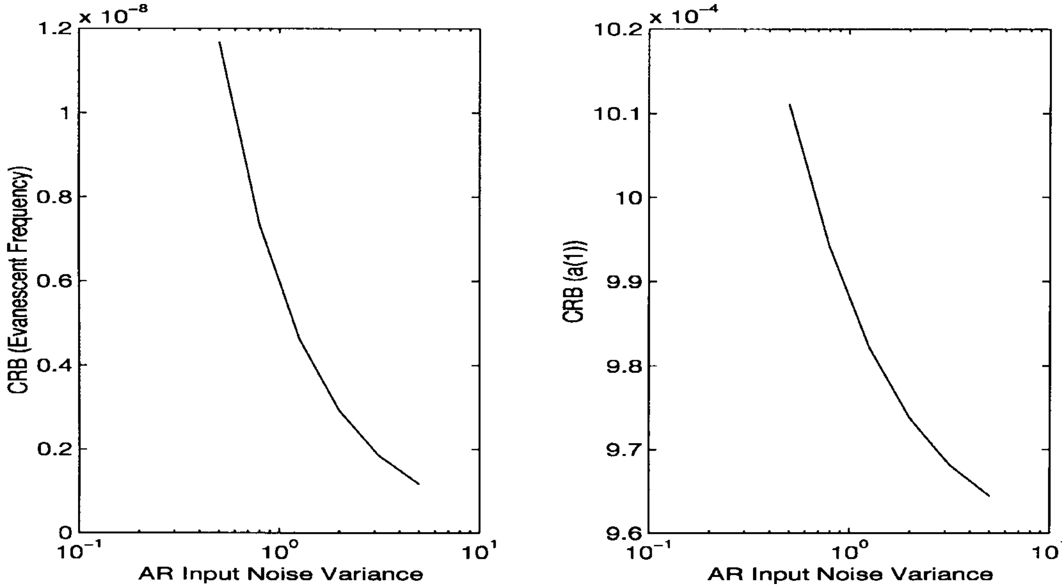

Fig. 2. CRB's on the evanescent field frequency parameter $\nu^{(1,0)}$, and on the evanescent field modulating 1-D AR processes parameter $a^{(1,0)}(1)$, as a function of $(\sigma^{(1,0)})^2$.

$(0.15, 0.25)$, while its amplitude parameters are $C_1 = D_1 = 1$. The evanescent component modulating 1-D purely indeterministic processes are medium-bandwidth second-order AR processes whose parameters are $a^{(1,0)}(1) = -1.183$, $a^{(1,0)}(2) = 0.7$. The peak of the spectral density function of these processes is at $\omega = \pi/4$. In this example, the frequency parameter of the evanescent field $\nu^{(1,0)}$ is changed from experiment to experiment, and we investigate the bounds on the error variance in estimating the parameters of the harmonic component.

The results, Fig. 3, indicate that as the frequency parameter of the evanescent field gets closer to 0.25, which is the ν-axis spatial frequency of the harmonic component, the estimation of the harmonic component parameters becomes more difficult. Note that when $\nu^{(1,0)} = 0.25$, i.e., $\nu^{(1,0)} = \nu_1$, the error variance on ω_1, C_1, D_1, becomes maximal, while the error variance on estimating ν_1, is getting smaller due to the fact that both the harmonic and the evanescent components have their energies concentrated at the same ν-axis frequency. Note also that as long as the harmonic and evanescent components are well-separated, the bounds on the harmonic component parameters remain almost constant.

Example 3: Consider a 2-D homogeneous random field consisting of a sum of two closely spaced harmonic components, two evanescent components of parallel spectral supports, $e_1^{(1,0)}(n,m)$, $e_2^{(1,0)}(n,m)$, and a zero-mean, unit variance, white Gaussian purely indeterministic component. The first-harmonic component frequency is $(\omega_1, \nu_1) = (0.15, 0.25)$, while its amplitude parameters are $C_1 = D_1 = 1$. The second-harmonic component frequency is $(\omega_2, \nu_2) = (0.16, 0.26)$, while its amplitude parameters are $C_2 = D_2 = 1$. The frequency parameter of the first evanescent component is $\nu_1^{(1,0)} = 0.2$. The modulating 1-D purely indeterministic processes of this evanescent component are narrowband second-order AR processes whose parameters are $a_1^{(1,0)}(1) = -1.378$, $a_1^{(1,0)}(2) = 0.95$. The peak of the spectral density function of these processes is at $\omega = \pi/4$. The modulating 1-D purely indeterministic processes of the second evanescent component are medium-bandwidth second-order AR processes whose parameters are $a_2^{(1,0)}(1) = -1.183$, $a_2^{(1,0)}(2) = 0.7$. The peak of the spectral density function of these processes is also at $\omega = \pi/4$. In this example, the frequency parameter, $\nu_2^{(1,0)}$, of the second evanescent component is changed from experiment to experiment, and we investigate the bounds on the error variance in estimating the parameters of the harmonic components, and the first evanescent component.

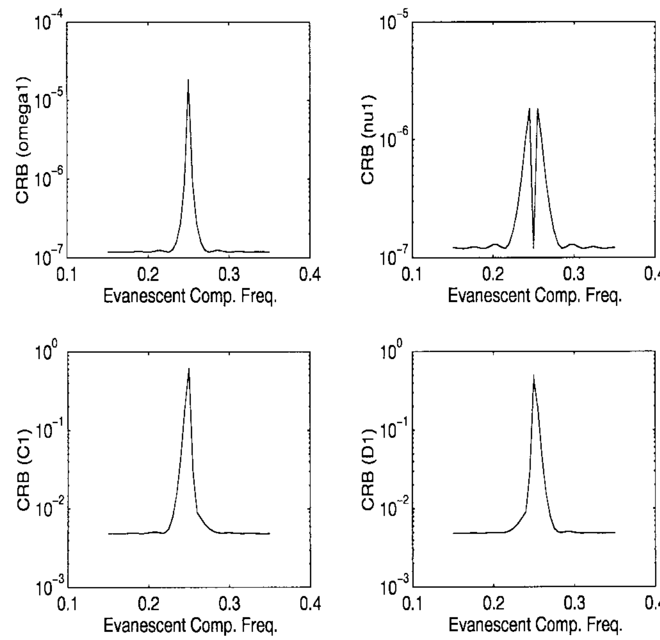

Fig. 3. CRB's on the parameters of the harmonic component, as a function of the evanescent component spectral support parameter, $\nu^{(1,0)}$.

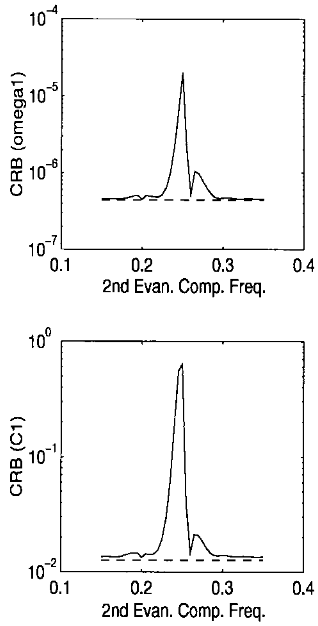

Fig. 4. CRB's on the parameters of the first-harmonic component, as a function of the second evanescent component spectral support parameter, $\nu_2^{(1,0)}$. The dashed line denotes the value of the bound when no evanescent component exists in the observed field.

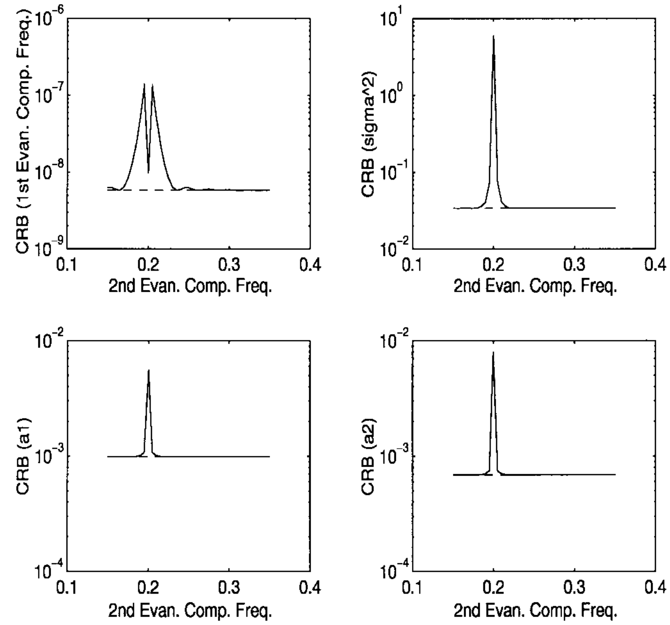

Fig. 5. CRB's on the parameters of the first evanescent component, as a function of the second evanescent component spectral support parameter, $\nu_2^{(1,0)}$. The dashed line denotes the value of the bound when the second evanescent component does not exist.

The results in Fig. 4 indicate that as the frequency parameter of the second evanescent component gets closer to 0.25, which is the ν-axis spatial frequency of the first-harmonic component, the estimation of the harmonic component parameters becomes more difficult. When $\nu_2^{(1,0)} = 0.25$, i.e., $\nu_2^{(1,0)} = \nu_1$, the error variance on ω_1, C_1, D_1, becomes maximal, while the error variance on estimating ν_1 is getting smaller due to the fact that both the harmonic component and the evanescent component have their energies concentrated at the same ν-axis frequency. Note that when $\nu_2^{(1,0)} = \nu_2 = 0.26$, the bounds on all the parameters of the first-harmonic component are slightly lower than their corresponding values for $\nu_2^{(1,0)} = 0.265$. This phenomenon is due to the fact that when $\nu_2^{(1,0)} = 0.26$, $e_2^{(1,0)}(n, m)$ "masks" the second-harmonic component, and thus the error variance in estimating the parameters of the first-harmonic component becomes smaller.

On the other hand, when the frequency parameter of the second evanescent component is far from the frequencies of the harmonic components, the bounds on the harmonic components parameters remain almost constant, and are almost identical to their values when no evanescent component exists in the observed field (dashed lines).

The results in Fig. 5 (solid lines) illustrate the behavior of the CRB's on the parameters of the first evanescent component as a function of the second evanescent component spectral support parameter, $\nu_2^{(1,0)}$. It is clear that as long as the spectral supports of two evanescent components are not very close, the bounds on both $\nu_1^{(1,0)}$, and the parameters of the modulating 1-D AR processes of the evanescent component, are essentially constant. Moreover, the bounds are identical to their values for the case in which no second evanescent component is present (dashed line). When the spectral supports of the two components coincide, the bound on $\nu_1^{(1,0)}$ drops sharply due to the fact that both evanescent components have their energies concentrated on the same spectral support. On the other hand, it is clear that when the spectral supports of the two evanescent components coincide, the problem of estimating the parameters of the modulating 1-D AR processes of the two evanescent components is much more difficult since it requires the separation of these 1-D processes from their sum.

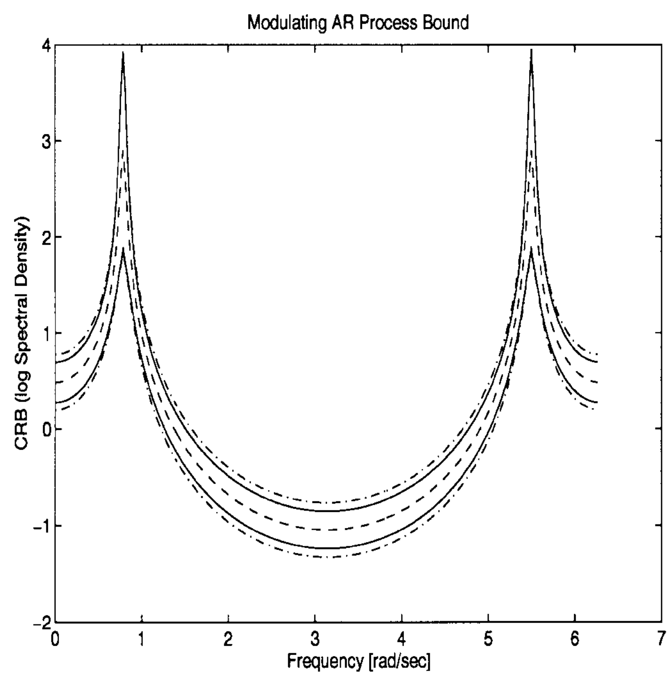

Fig. 6. CRB's on the log of the spectral density function of the first evanescent component modulating 1-D AR processes.

TABLE I
THE CRB FOR A FIELD WITH EVANESCENT COMPONENTS OF INTERSECTING SPECTRAL SUPPORTS

			CRB: Case I	CRB: Case II	CRB: Case III
First harmonic component	ω_1	0.15	4.8289741e-07	4.8289741e-07	4.8289741e-07
	ν_1	0.25	2.9088584e-06	2.9088584e-06	2.9088584e-06
	C_1	1	7.3123960e-02	7.3123960e-02	7.3123960e-02
	D_1	1	1.4620095e-01	1.4620095e-01	1.4620095e-01
Second harmonic component	ω_2	0.16	1.0354122e-06	1.0354122e-06	1.0354122e-06
	ν_2	0.26	4.9685502e-07	4.9685502e-07	4.9685502e-07
	C_2	1	9.1014727e-02	9.1014727e-02	9.1014727e-02
	D_2	1	1.3842889e-02	1.3842889e-02	1.3842889e-02
First evanescent component $e^{(1,0)}(n,m)$	$\nu^{(1,0)}$	0.2	6.2839028e-09	6.2839019e-09	-
	$(\sigma^{(1,0)})^2$	1	3.4523854e-02	3.4521763e-02	-
	$a^{(1,0)}(1)$	-1.378	1.0040566e-03	1.0039962e-03	-
	$a^{(1,0)}(2)$	0.95	6.9080041e-04	6.9080036e-04	-
Second evanescent component $e^{(0,1)}(n,m)$	$\nu^{(0,1)}$	0.15	3.2693599e-08	-	3.2693589e-08
	$(\sigma^{(0,1)})^2$	1	3.4749677e-02	-	3.4746507e-02
	$a^{(0,1)}(1)$	-1.183	7.2832486e-03	-	7.2829150e-03
	$a^{(0,1)}(2)$	0.7	6.7680943e-03	-	6.7680872e-03
Purely indeterministic component	σ^2	1	1.3827854e-03	1.3816976e-03	1.3817262e-03

Example 4: Next, we investigate the bound on the spectral density function of the modulating 1-D AR processes of the first evanescent component, in two cases. In the first case, the observed field has only a single evanescent component, with $\nu_1^{(1,0)} = 0.2$. In the second case, there are two evanescent components in the observed field such that the spectral support of the second evanescent component is parallel and very close to that of the first component. Here, $\nu_2^{(1,0)} = 0.205$. The harmonic and purely indeterministic components are the same as in Example 3, for both cases. The parameters of the modulating 1-D AR processes of both evanescent components are also the same as in Example 3. Fig. 6 depicts the log spectral density function of the first evanescent component modulating AR processes. The mean value of the log spectrum (dashed line) and the mean plus and minus the standard deviation computed from the CRB, are shown. The solid line denotes the bound on the log spectrum for the case in which the observed field has only one evanescent component, while the dashed–dotted line denotes the bound for the case in which the observed field has two evanescent components. As the distance between the parallel spectral supports of the two evanescent components increases, the bound on the spectral density of the first component modulating AR processes becomes identical to its value for the case in which no second evanescent component exists. In fact, in the present example the bounds for the single evanescent component case and the two-components case become essentially identical for relatively small differences of the components' spectral support parameters, $(|\nu_2^{(1,0)} - \nu_1^{(1,0)}| \geq 0.01)$.

We therefore conclude that as the distance between the parallel spectral supports of the two evanescent components becomes larger the FIM tends to a block-diagonal structure, in which separate diagonal blocks correspond to the parameters of the different components. Furthermore, the CRB's for each component become almost the same as in the single-component case.

On the other hand, as the distance between the parallel spectral supports of the two evanescent components decreases, or when the spectral support of any of the evanescent components is close to the frequency of a harmonic component, the bounds become much higher than in the corresponding single-component cases.

Example 5: Here, we consider a case in which the observed random field has two evanescent components whose spectral supports intersect. We compare the CRB on the accuracy of estimating the field parameters, with the CRB for the same field, when only one of the two evanescent components is present. More specifically, we consider a 2-D homogeneous random field consisting of a sum of two closely spaced harmonic components, two evanescent components $e^{(1,0)}(n,m)$, $e^{(0,1)}(n,m)$, and a zero-mean, unit variance, white Gaussian purely indeterministic component. The parameters of the different components of the field are listed in Table I. The modulating 1-D purely indeterministic processes of $e^{(1,0)}(n,m)$ are narrowband second-order AR processes. The modulating 1-D purely indeterministic processes of $e^{(0,1)}(n,m)$ are medium-bandwidth second-order AR processes.

For the case of $(\alpha, \beta) = (0, 1)$, (28) can be expressed in the form

$$\mathbf{\Gamma}_i^{(0,1)} = \boldsymbol{E}_i^{(0,1)} \otimes \boldsymbol{R}_i^{(0,1)} \tag{88}$$

where $\boldsymbol{R}_i^{(0,1)}$ and $\boldsymbol{E}_i^{(0,1)}$ are Toeplitz matrices, given by (89) and (90) (see the top of the following page). Hence,

$$\frac{\partial \mathbf{\Gamma}_i^{(0,1)}}{\partial \nu_i^{(0,1)}} = \frac{\partial \boldsymbol{E}_i^{(0,1)}}{\partial \nu_i^{(0,1)}} \otimes \boldsymbol{R}_i^{(0,1)} \tag{91}$$

$$\boldsymbol{R}_i^{(0,1)} = \begin{bmatrix} r_i^{(0,1)}(0) & r_i^{(0,1)}(1) & \cdots & r_i^{(0,1)}(T-1) \\ r_i^{(0,1)}(1) & r_i^{(0,1)}(0) & \cdots & r_i^{(0,1)}(T-2) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & r_i^{(0,1)}(1) \\ r_i^{(0,1)}(T-1) & r_i^{(0,1)}(T-2) & \cdots & r_i^{(0,1)}(0) \end{bmatrix} \tag{89}$$

$$\boldsymbol{E}_i^{(0,1)} = \begin{bmatrix} 1 & \cos(-2\pi\nu_i^{(0,1)}) & \cdots & \cos(-2\pi(S-1)\nu_i^{(0,1)}) \\ \cos(2\pi\nu_i^{(0,1)}) & 1 & \cdots & \cos(-2\pi(S-2)\nu_i^{(0,1)}) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & \cos(-2\pi\nu_i^{(0,1)}) \\ \cos(2\pi(S-1)\nu_i^{(0,1)}) & \cos(2\pi(S-2)\nu_i^{(0,1)}) & \cdots & 1 \end{bmatrix}. \tag{90}$$

while

$$\frac{\partial \boldsymbol{\Gamma}_i^{(0,1)}}{\partial [\boldsymbol{a}_i^{(0,1)}]_k} = \boldsymbol{E}_i^{(0,1)} \otimes \frac{\partial \boldsymbol{R}_i^{(0,1)}}{\partial [\boldsymbol{a}_i^{(0,1)}]_k}. \tag{92}$$

Here

$$\frac{\partial \boldsymbol{E}_i^{(0,1)}}{\partial \nu_i^{(0,1)}} = -2\pi \boldsymbol{K}^{(0,1)} \odot \boldsymbol{N}_i^{(0,1)} \tag{93}$$

where $\boldsymbol{K}^{(0,1)}$ and $\boldsymbol{N}_i^{(0,1)}$ are given by (86) and (87) with all T's replaced by S's.

We consider three cases: Case I is the case in which both evanescent components exist in the observed field, Case II is the case in which the only evanescent component of the observed field is $e^{(1,0)}(n, m)$, Case III is the case in which the only evanescent component of the observed field is $e^{(0,1)}(n, m)$. In Table I we list the bounds on the error variance in estimating the parameters of the observed field for the three cases. The results indicate that the lower bounds on the error variance in estimating the parameters of the different components are essentially unaffected by the presence of multiple evanescent components with intersecting spectral supports.

Using this example we conclude that in general, the presence of evanescent random fields with intersecting spectral supports has only a negligible effect on the CRB of each component parameters, compared with the case in which this component is the only evanescent component of the field. Using the conclusions of Examples 3 and 4, we finally conclude that the presence of an evanescent component in the field has essentially no effect on the lower bound on the accuracy of estimating the parameters of the other components of the field, unless the spectral support of the evanescent component is parallel and very close to that of another evanescent component, or if the spectral support of the evanescent component is very close to the spectral support of a harmonic component.

VII. Conclusions

In this paper we have investigated the achievable accuracy in jointly estimating the parameters of a real-valued 2-D homogeneous random field with mixed spectral distribution, from a single observed realization of it. An exact form of the Cramer–Rao lower bound on the accuracy of jointly estimating the parameters of the different components was derived. It was shown that the estimation of the harmonic component is decoupled from that of the purely indeterministic and the evanescent components; furthermore, the bound on the purely indeterministic and the evanescent components is independent of the harmonic component.

We have specialized this derivation and derived closed-form expressions of the CRB for the case where the modulating 1-D purely indeterministic processes of each evanescent field are moving average or autoregressive processes, and the purely indeterministic component of the field is a white noise field. In [25], a derivation of a closed-form exact CRB on the parameters of 2-D moving average random fields, and hence of essentially any purely indeterministic random field, is presented. Thus together with the derivation of the closed-form exact CRB on the parameters of 2-D moving average random fields, the derivation in this paper provides a closed-form exact CRB on the parameters of essentially any homogeneous random field.

Using numerical evaluation of specific examples we have found that as the distance between the spectral supports of any two evanescent components with parallel spectral supports is large enough, or when the spectral supports of the evanescent components intersect, the FIM block that corresponds to the evanescent components' parameters, tends to a block-diagonal structure. In this structure, separate diagonal blocks correspond to the parameters of the different evanescent components. Furthermore, the values of the CRB's on the parameters of each component are essentially unaffected by the presence of other evanescent components. On the other hand, as the distance between the spectral supports of any two evanescent components with parallel supports decreases, or when the spectral support of any of the evanescent components is close to the frequency of a harmonic component, the bounds become much higher than in the corresponding single-component cases.

References

[1] J. M. Francos, A. Z. Meiri, and B. Porat, "A Wold-like decomposition of 2-D discrete homogeneous random fields," *Ann. Appl. Prob.*, vol. 5, pp. 248–260, 1995.

[2] P. Whittle, "On stationary processes in the plane," *Biometrika*, vol. 41, pp. 434–449, 1954.

[3] H. Derin and P. A. Kelly, "Discrete-index Markov-type random processes," *Proc. IEEE*, vol. 77, pp. 1485–1510, 1989.

[4] F. Comets and B. Gidas, "Parameter estimation of Gibbs distributions from partially observed data," *Ann. Appl. Prob.*, vol. 2, pp. 142–170, 1992.

[5] N. Balram and J. M. F. Moura, "Noncausal Gauss Markov random fields: Parameter structure and estimation," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1333–1355, 1993.

[6] J. W. Woods, "Two-dimensional discrete Markovian fields," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 232–240, 1972.

[7] T. L. Marzetta, "Two-dimensional linear prediction: Autocorrelation arrays, minimum-phase prediction error filters and reflection coefficient arrays," *IEEE Trans. Acoust., Speech Signal Processing*, vol. ASSP-28, pp. 725–733, 1980.

[8] A. K. Jain, "Advances in mathematical models for image processing," *Proc. IEEE*, vol. 69, pp. 502–528, 1981.

[9] S. R. Parker and A. H. Kayran, "Lattice parameter autoregressive modeling of two-dimensional fields—Part I: The quarter plane case," *IEEE Trans. Acoust., Speech Signal Processing*, vol. ASSP-32, pp. 872–885, 1984.

[10] C. W. Therrien, T. F. Quatieri, and D. E. Dudgeon, "Statistical model-based algorithms for image analysis," *Proc. IEEE*, vol. 74, pp. 532–551, 1986.

[11] A. J. Isaksson, "Analysis of identified 2-D noncausal models," *IEEE Trans. Inform. Theory*, vol. 39, pp. 525–534, 1993.

[12] R. R. Hansen and R. Chellappa, "Two-dimensional robust spectrum estimation," *IEEE Trans. Acoust., Speech Signal Processing*, vol. 36, pp. 1051–1066, 1988.

[13] R. L. Kashyap, "Characterization and estimation of two-dimensional ARMA models," *IEEE Trans. Inform. Theory*, vol. IT-30, pp. 736–745, 1984.

[14] X.-D. Zhang, "Two dimensional harmonic retrieval and its time-domain analysis technique," *IEEE Trans. Inform. Theory*, vol. 37, pp. 1185–1188, 1991.

[15] S. W. Lang and J. H. McClellan, "The extension of Pisarenko's method to multiple dimensions," in *Proc. Int. Conf. on Acoust.ics, Speech, and Signal Processing*, 1982, pp. 125–128.

[16] R. Kumaresan and D. W. Tufts, "A two-dimensional technique for frequency-wavenumber estimation," *Proc. IEEE*, vol. 69, pp. 1515–1517, 1981.

[17] Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil," *IEEE Trans. Signal Processing*, vol. 40, pp. 2267–2280, 1992.

[18] H. Yang and Y. Hua, "Statistical decomposition of an eigendecomposition based method for 2-D frequency estimation," *Automatica*, vol. 30, pp. 157–168, 1994.

[19] C. R. Rao, L. Zhao, and B. Zhou, "Maximum likelihood estimation of 2-D superimposed exponential signals," *IEEE Trans. Signal Processing*, vol. 42, pp. 1795–1802, 1994.

[20] A. M. Walker, "On the estimation of a harmonic component in a time series with stationary independent residuals," *Biometrika*, vol. 58, pp. 21–36, 1971.

[21] M. B. Priestley, "The analysis of two-dimensional stationary processes with discontinuous spectra," *Biometrika*, vol. 51, pp. 195–217, 1964.

[22] J. M. Francos, A. Narasimhan, and J. W. Woods, "Maximum-likelihood parameter estimation of the harmonic, evanescent and purely indeterministic components of discrete homogeneous random fields," *IEEE Trans. Inform. Theory*, vol. 42, pp. 916–930, 1996.

[23] B. Porat and B. Friedlander, "Computation of the exact information matrix of Gaussian time series with stationary random components," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 118–130, 1986.

[24] T. Kailath, A. Vieira, and M. Morf, "Inverses of Toeplitz operators, innovations, and orthogonal polynomials," *SIAM Rev.*, vol. 20, pp. 118–130, 1978.

[25] J. M. Francos and B. Friedlander, "Parameter estimation of two-dimensional moving average random fields," submitted for publication.

[26] J. M. Francos, "Parametric model for textures in 2-D images," D.Sc. dissertation, Dep. Elec. Eng., Technion–Israel Institute of Technology, Haifa, Israel, 1990.

[27] J. M. Francos, A. Z. Meiri, and B. Porat, "A unified texture model based on a 2-D wold like ldecomposition," *IEEE Trans. Signal Processing*, vol. 41, pp. 2665–2678, 1993.

[28] J. M. Francos, A. Narasimhan, and J. W. Woods, "Maximum likelihood parameter estimation of discrete homogeneous random fields with mixed spectral distributions," *IEEE Trans. Signal Processing*, vol. 44, pp. 1242–1255, 1996.

[29] ——, "Maximum likelihood parameter estimation of textures using a wold decomposition based model," *IEEE Trans. Image Processing*, vol. 4, pp. 1655–1666, 1995.

[30] M. B. Priestley, *Spectral Analysis and Time Series.* New York: Academic, 1981.

[31] C. R. Rao, *Linear Statistical Inference and Its Applications.* New York: Wiley, 1965.

[32] B. Porat, *Digital Processing of Random Signals.* Englewood Cliffs, NJ: Prentice-Hall, 1994.