Mesoscopic quantum effect of symmetry breaking for magnetic-dipolar oscillating modes

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Abstract

In quasi-two-dimensional systems the dipolar interaction can play an essential role in determine the magnetic properties. In a case of magnetic-dipolar modes in a normally magnetized thin-film ferrite disk, the oscillations can be considered as the motion process of certain quasiparticles—the light magnons—having quantization of energy and characterizing by effective masses depending on the energy levels. One of the features of magnetic-dipolar oscillations in a normally magnetized ferrite disk resonator is the presence of helicoidal surface magnetic currents. These currents lead to the parity violation effects in magnetic-dipolar oscillations and appearance of anapole moments. Recent experiments show that magnetic-dipolar oscillations in a normally magnetized ferrite disk are strongly affected by a normal component of the external RF electric field. The anapole-moment model gives very convincing arguments for explaining these experimental data.

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1. Introduction. Mesoscopic quantum effects for magnetic-dipolar oscillating modes: Theory and experimental evidence

Quantum effects are rarely observed through macroscopic measurements because statistical averaging over many states usually masks all evidence of discreteness. One of the notable exceptions is a quantum-mechanical effect in the magnetization of a macroscopic ferrite sample: quantum coherence of magnetic-dipolar-mode (MDM) oscillations in thin-film ferrite disks.

In quasi-two-dimensional systems, the dipolar interaction can play an essential role in determining the magnetic properties. In these systems the short-range exchange interactions alone are not necessarily sufficient to establish a ferromagnetically ordered ground state. The dipolar interaction is important in stabilizing long-range magnetic
order in two-dimensional systems, as well in determining the nature and morphology of the ordered states. The long-range nature of the dipolar interaction manifests itself in the low-energy excitations of the system [1]. The MDM wave processes are characterized by negligibly small variation of the exchange energy and negligibly small variation of the electric energy. The feature of disk-film ferrite samples is lack of in-plane translational symmetry for the long-range dipolar interaction. In this case, it is difficult to use the summation technique and perform the Holstein–Primakoff transformation. For films with thickness about tens of micrometers one uses the continuum approach. The continuum analog of the equation of motion of a single spin is the equation of motion of the magnetization vector (the Landau–Lifshitz equation).

In microwaves, ferrite resonators with multiresonance MDM [or magnetostatic (MS)] oscillations may have sizes two–four orders less than the free-space EM wavelength at the same frequency. These “microscopic” oscillating objects—the particles—may interact with the external EM fields by a very specific way, forbidden for the classical description. The character of the experimental multiresonance spectra (obtained with respect to a DC bias magnetic field \( H_0 \)) leads to a clear conclusion that the energy of a source of a DC magnetic field is absorbing “by portions” or discretely, in other words. There are the multiresonance regular experimental spectra with properties similar to an atomic-like \( d \)-function density of states. One can estimate the bias-field energies necessary for transitions from the main level to upper levels. For example, to have a transition from the first level to the second level we need the energy surplus: \( \Delta U_{12} = 4\pi M_0 (H_0^{(2)} - H_0^{(1)}) \), where \( 4\pi M_0 \) is saturation magnetization [2]. The theory of the MDM spectra is based on the notion of magnetostatic-potential wave functions \( \psi \). These functions appear through representation of the RF magnetic fields in a ferrite sample as \( \mathbf{H} = -\nabla \psi \) and usually are considered just as formal quantities convenient for calculations. Really, from the classical electrodynamics it follows that just only the \( \mathbf{E} \) and \( \mathbf{H} \) fields, but not potentials, are regarded as the basic physical quantities. Nevertheless, the fact that in the MDM wave processes one has negligibly small variation of the electric energy raises the questions about the nature of the fields in magnetic-dipole oscillations. In particular, the question about the electromagnetic power flow for MS-wave modes arises: There are no physical mechanisms describing the effect of transformation of the curl electric field to the magnetostatic-potential magnetic field.

MS-potential wave functions may acquire, however, a special physical meaning. This is clearly demonstrated in a case of MDM oscillations in a normally magnetized ferrite disk. As it has been shown recently [3,4], in such a sample the confinement effect leads to the quantum properties described by the Schrödinger-like equation. The oscillations can be considered as the motion process of certain quasiparticles—the light magnons—having quantization of energy and characterizing by effective masses depending on the energy levels. “The energy eigenstates” is the quantum mechanical notion. A ferrite disk with the MDM oscillating spectra is a mesoscopic system in a sense that such a system is sufficiently big compared to atomic and lattice scales, but sufficiently small that quantum mechanical phase coherence is preserved around the whole sample. For a case of a MDM ferrite disk one has the quantized-like oscillating system which preserves the coherence. Following an idea that external interactions should increase the probability of transition of a quantum system to other states leading to line broadening, it was shown in Ref. [2] how the environment may cause decoherence for magnetic oscillations.

To a certain extent, for the MDM oscillating spectra one has situation resembling the dipole-interaction “flat” quasiparticles (electron–hole pairs—excitons) in disk-form semiconductor dots [5]. Together with such similarity with semiconductor quantum dots as discrete energy levels due to confinement phenomena, ferrite particles show other, very unique, properties attributed to the quantized-like systems. Recent experiments show that MS oscillations in a normally magnetized ferrite disk are strongly affected by a normal component of the external RF electric field [2]. The observed multiresonance process cannot be
characterized as the induced electric polarization effect in a particle. There are the eigen-electric-moment oscillations caused by the motion processes in a ferrite resonator. Since the RF electric field does not change sign under time inversion, the eigen electric moment should also be characterized by the time-reversal-even properties. This is the case of an anapole moment, considered, for the first time, by Zel’dovich [6]. As it was theoretically predicted, MDM oscillations in a ferrite disk are accompanied with specific surface magnetic currents, which should cause the parity-violating perturbation and, as a result, should lead to appearance of the eigen electric moments (the anapole moments) in disk-form MS-wave ferrite particles [7]. There are also other examples of special symmetry properties of microwave MDM oscillations. These properties are correlated with the microwave magnetoelectric effect, which was observed in small ferrite disks with metal electrodes [8,9].

Surface magnetic currents (resulting in the anapole moments) in a ferrite disk appear due to symmetry breaking for magnetic-dipolar oscillating modes. As we will show in this paper, in a ferrite disk resonator there exist four helical harmonics for the MS-potential function $\psi$. So the wave function $\psi$ must have four components, which can be combined to form a single-column matrix. Our analysis results in the Dirac-like quasiparticle spectrum. At present, in different 2D condensed matter systems “relativistic” Dirac-like spectrum of quasiparticle excitations becomes a subject of a special attention. Understanding the nature of the quasiparticle energy states and symmetry properties is of considerable importance. In that sense, MDM oscillations are ideal since they have very long wavelength and are easily investigated by experimental techniques.

2. Circular surface magnetic currents for MDM oscillations

For magnetostatic potential functions $\psi$, in a normally magnetized ferrite-disk resonator one has the eigenvalue differential equation [3]:

$$\mu \nabla^2_{\perp} \tilde{\psi}_q = \beta_q^2 \tilde{\psi}_q,$$

where $\tilde{\psi}_q (r, \theta)$ are “in-plane” MS modes with radial $r$ and azimuth $\theta$ distribution of MS potential, $\nabla^2_{\perp}$ is the two-dimensional, “in-plane”. Laplace operator, $\beta_q$ is the propagation constant along $z$-axis, and $\mu$ is the diagonal component of the permeability tensor. A double integration by parts (the Green theorem) on $S$—a square of an “in-plane” cross section of an open ferrite disk—of the integral $\int (\mu \nabla^2_{\perp} \tilde{\psi}) dS$, gives the following boundary condition for the energy orthonormality:

$$\mu \left( \frac{\partial \tilde{\psi}}{\partial r} \right)_{r=R^-} - \left( \frac{\partial \tilde{\psi}}{\partial r} \right)_{r=R^+} = 0,$$

where $R^-$ and $R^+$ designate, respectively, the inner (ferrite) and outer (dielectric) regions of a disk resonator with radius $R$. The boundary conditions (1) are the so-called essential boundary conditions. In accordance with the Ritz method it is sufficient to use basic functions from the energetic functional space with application of the essential boundary conditions [10]. When such boundary conditions are used, the MS-potential eigen functions form a complete functional basis. The essential boundary conditions differ from the homogeneous electrodynamics boundary conditions at $r = R$, which demand continuity for the radial component of the magnetic flux density (together with continuity for potential $\tilde{\psi}$). The last ones are called as natural boundary conditions [10]. In a cylindrical coordinate system, continuity for a radial component of the magnetic flux density (the natural boundary condition) at $r = R$ is described as

$$\mu (H_r)_{r=R^-} - (H_r)_{r=R^+} = -i \mu_a (H_\theta)_{r=R^-},$$

where $\mu_a$ is the off-diagonal component of the permeability tensor, $H_r = -\partial \psi / \partial r$ and $H_\theta = -(1/r)(\partial \psi / \partial \theta)$ are radial and azimuth component of the RF magnetic field, respectively. Supposing that $\tilde{\phi} \sim e^{-i\nu \theta}$, one can rewrite (2) as

$$\mu \left( \frac{\partial \tilde{\phi}}{\partial r} \right)_{r=R^-} - \left( \frac{\partial \tilde{\phi}}{\partial r} \right)_{r=R^+} = -\frac{\mu_a}{R} \psi (\tilde{\phi})_{r=R^-}.$$

One can see that “in-plane” functions $\tilde{\phi}$, being determined by two second-order differential equations (the Bessel equations for functions $\phi$ inside and outside the ferrite region) and one first-order differential equation (3), are dependent on both
a quantity and a sign of \( v \). So the functions \( \tilde{\phi} \) cannot be single-valued functions for angle \( \theta \) varying from 0 to 2\( \pi \). In other words, we have different results for positive and negative directions of an angle coordinate when 0 \( \leq \theta \leq 2\pi \).

Let us consider a circulation of vector \( \mathbf{H} \) along a circular contour \( L = 2\pi R \). This circulation, being expressed as \( C = v \int_0^{2\pi} \tilde{\phi}(L) \, d\theta \), has a non-zero quantity. The solution depends on both a modulus and a sign of \( v \). We have a sequence of angular eigenvalues restricted from above and below by values equal in a modulus and different in a sign, which we denote as \( \pm s^e \). The difference \( 2s^e \) between the largest and smallest values is an integer or zero.

So \( s^e \) can have values 0, \( \pm \frac{1}{2} \), \( \pm 1 \), \( \pm \frac{3}{2} \), \ldots. At a full-angle “in-plane” rotation (at an angle equal to 2\( \pi \)) of a system of coordinates, the “in-plane” functions \( \tilde{\phi} \) with integer values \( s^e \) return to their initial states (single-valued functions) and “in-plane” functions \( \tilde{\phi} \) with the half-integer values \( s^e \) will have an opposite sign (double-valued functions). The only possibility in our case is to suggest that \( s^e \) are the half-integer quantities. Because of the double-valuedness properties of MS-potential functions on a lateral surface of a ferrite disk resonator, we can talk about the “spinning-type rotation” along a border contour \( L \). Along with the well-known notion of the “magnetic spin” as a quantity correlated with the eigen magnetic moment of a particle, we introduce the notion of the “electric spin” as a quantity correlated with the eigen electric moment. For integer quantities \( s^e \) the eigen electric moment is equal to zero, but it is non-zero for half-integer values \( s^e \).

The main feature of boundary condition (2) arises from the quantity of an azimuth magnetic field in the right-hand side. One can see that this is a singular field, which exists only in an infinitesimally narrow cylindrical layer abutting (from a ferrite side), to the ferrite-dielectric border. One does not have any special conditions connecting radial and azimuth components of magnetic fields on other (inner or outer) circular contours, except contour \( L \). With a formal introduction of a quantity of a magnetic current:

\[
\mathbf{J}^m(z) = \frac{1}{4\pi} i\omega \mu_a \mathbf{H}_\theta(z),
\]

one can rewrite the boundary condition (2) as follows:

\[
\delta(r - R) \left[ \frac{1}{4\pi} \omega \mu_a (H_r)_{r=R^-} - \frac{1}{4\pi} \omega \mu_a (H_r)_{r=R^+} \right] = -i^m,
\]

where \( i^m \) is a density of an effective boundary magnetic current defined as

\[
i^m(z) \equiv \delta(r - R) \frac{1}{4\pi} i\omega \mu_a (H_\theta(z))_{\rho=R^-} = \delta(r - R) i^m(z).
\]

In a supposition that (with use of the essential boundary conditions) “flat” functions \( \tilde{\phi} \) form a complete basis in the energy functional space [3,4] it becomes evident that the effective boundary magnetic current slips from the main properties of this functional space. This current cannot be considered as a single-valued function. In the description of the MS-potential functions in a ferrite disk, taking into account the effective surface magnetic current, certain additional coordinates (additional eigenvalues and eigenfunctions) should appear on boundary contour \( L \). These singular functions describing the “spin states” we will denote as \( \tilde{\phi} \). For \( k \)th “border” eigenfunction having amplitude \( B_k \), we can write:

\[
\tilde{\phi}_k = B_k e^{-i\omega \theta},
\]

where \( w_k = k_l \) and \( k \) is an integer (odd (positive or negative) quantity). For a certain “thickness” mode \( \tilde{\zeta}(z) \) and with representation:

\[
(H_\theta(z))_{r=R^-} = -A \tilde{\zeta}(z) \nabla \phi \tilde{\phi},
\]

we have for a certain “flat” mode \( \tilde{\phi} \):

\[
\left( \mathbf{J}^m(z) \right)_k = -A \tilde{\zeta}(z) \frac{i\omega \mu_a}{4\pi R^2} \frac{\partial \phi}{\partial \theta} \bigg|_{r=R^-} = -A \tilde{\zeta}(z) \frac{i\omega \mu_a}{4\pi R^2} w_k B_k e^{-i\omega \theta},
\]

where \( A \) is an amplitude coefficient. The circular surface magnetic current does not exist due to only precession of magnetization. It appears because of the combined effect of precession in a ferrite material and “spinning rotation” caused by the special-type boundary conditions. The “border” MS-potential functions \( \tilde{\phi} \), being characterized by the “spin coordinates”, are antisymmetrical functions. At the same time, as it follows from Eq. (6), the effective magnetic currents are described by
symmetrical functions with respect to the “spin coordinates”. In other words, the effective magnetic current has the same direction for the “right” and “left” spinning states. The signs of magnetic current \( i^m \) are different for different signs of \( \mu_a \). However, the “positive” \( (\mu_a>0) \) and “negative” \( (\mu_a<0) \) magnetic currents do not mutually compensate each other since for different signs of \( \mu_a \) we have structures with different symmetries. Circulation of magnetic current along border contour: \( D_k(z) = \oint_L (i^m)_k \, dl \) gives a non-zero quantity when \( w_k \) is a number divisible by \( \frac{1}{2} \). For non-zero circulation \( D_k(z) \), one can find an electric moment of a whole ferrite disk resonator (in a region far away from a disk) as

\[
\alpha_k = -\frac{i}{2c} \int_0^h dz \oint_L (r \times i^m) \cdot e_z \, dl = A \frac{\partial \mu_a}{4\pi c} \Re B_k \int_0^h \xi(z) \, dz, \tag{7}
\]

where \( e_z \) is the unit vector along \( z \)-axis. The off-diagonal component of the permeability tensor, \( \mu_a \), can be correlated with a magnetic vector of gyration: \( g^m \equiv (\mu_a/4\pi)e_z \). A sign of \( g^m \) corresponds to a sign of \( \mu_a \). A sign of amplitude \( B_k \) depends on orientation of vector \( s^e (s^e = s^e e_z) \) with respect to \( z \)-axis. So one can distinguish two cases: \( s^e \cdot g^m > 0 \) and \( s^e \cdot g^m < 0 \).

The property of helicity is well-known in elementary particle physics. In our case, the spin orientation \( s^e \) is not separated from orientation of a linear momentum \( a^e \), but taking into account also orientation of vector \( g^m \). For both cases: \( s^e \cdot g^m > 0 \) and \( s^e \cdot g^m < 0 \), a sign of an electric moment \( a^e \) is invariant with respect to the time reversal (both vectors \( s^e \) and \( g^m \) are the axial vectors). Following Zel’dovich [6], the so-called toroidal (or anapole) moment is an odd parity magnetic field distribution of rank 1 (dipole). One can trace a clear analogy between the above consideration and electromagnetic properties of a toroidal solenoid. If we picture an element of magnetic current \( i^m \) as a small electric-current loop, the combination of all elements along contour \( L \) can be viewed (for a ferrite disk with a small thickness to diameter ratio) as a toroidal current winding producing an anapole moment.

Now the question about a nature of a source of magnetic current \( i^m \) arises. There is also a question what are the solutions giving different signs of the product \( s^e \cdot g^m \). With consideration of non-zero circulation along contour \( L \) we refer, in fact, to the concept of non-integrable, i.e. path-dependent, phase factor defined by an integral taken around an unshrincable loop. The physics of this topological effect becomes evident from the handedness properties of MDMs [11].

3. Magnetostatic modes in a helical coordinate system and symmetry breaking

Let us consider the MDM propagation in a helical coordinate system. Such an analysis in an infinite ferrite rod bears a formal character, but acquires physical meaning in a case of restricted waveguide sections, such as ferrite disks. For MS-wave waveguide structures the power flow is expressed as a quadratic form of MS potential \( \psi \) and magnetic flux density \( B \) [3]. The boundary conditions demand continuity for \( \psi \) and normal components of \( B \). In Waldron’s helical coordinate system \((r, \phi, \zeta)\) [12], for MS modes in an infinite axially magnetized ferrite rod one has four solutions for components \( B \) (\( B = -\mu \cdot \nabla \psi \)):

\[
B^{(1,2)}_r = -\left[ \frac{\mu}{c} \frac{\partial \psi}{\partial r} \pm i\mu_a \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} - \frac{\tan \zeta}{\zeta} \frac{\partial \psi}{\partial \zeta} \right) \right],
\]

\[
B^{(1,2)}_\phi = -\frac{1}{\cos \zeta} \left[ \frac{\mu}{c} \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} - \frac{\tan \zeta}{\zeta} \frac{\partial \psi}{\partial \zeta} \right) \pm i\mu_a \frac{\partial \psi}{\partial \phi} \right],
\]

\[
B^{(1,2)}_\zeta = -\tan \zeta \left[ \frac{2}{\sin 2\zeta} \frac{\partial \psi}{\partial \zeta} - \frac{1}{r} \frac{\partial \psi}{\partial \phi} + (1 - \mu) \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} - \frac{\tan \zeta}{\zeta} \frac{\partial \psi}{\partial \zeta} \right) \pm i\mu_a \frac{\partial \psi}{\partial \phi} \right],
\]

and

\[
B^{(3, 4)}_r = -\left[ \frac{\mu}{c} \frac{\partial \psi}{\partial r} \pm i\mu_a \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} + \frac{\tan \zeta}{\zeta} \frac{\partial \psi}{\partial \zeta} \right) \right],
\]

\[
B^{(3, 4)}_\phi = -\frac{1}{\cos \zeta} \left[ \frac{\mu}{c} \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} + \frac{\tan \zeta}{\zeta} \frac{\partial \psi}{\partial \zeta} \right) \pm i\mu_a \frac{\partial \psi}{\partial \phi} \right].
\]
\[ B^{(3,4)}_z = \tan z_0 \left[ -\frac{2}{\sin 2x_0} \frac{\partial \psi}{\partial \zeta} - \frac{1}{r} \frac{\partial \psi}{\partial \varphi} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\sin 2x_0} \frac{\partial \psi}{\partial \zeta} \right) + i\mu a \frac{\partial \psi}{\partial r} \right], \tag{9} \]

where \( \tan z_0 = \beta/r, \) and \( \beta = p/2\pi, \) \( p \) is the helical pitch. These solutions are distinguished by directions of the wave propagation in a ferrite rod and different signs of off-diagonal components \( \mu_a \) (in correspondence with different types of helical coordinates: left-handed or right-handed). With the use of Waldron’s equation for the divergence \([12]\) we have:

\[ \nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial}{\partial r} (r \mathbf{B}_r) + \frac{\cos x_0}{r} \frac{\partial \mathbf{B}_\varphi}{\partial \varphi} + \frac{\partial \mathbf{B}_\zeta}{\partial \zeta} = 0. \]

After some transformations we obtain the Walker equation in helical coordinates:

\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \left( \frac{1}{\mu} + \tan^2 z_0 \right) \frac{\partial^2 \psi}{\partial \zeta^2} - 2 \frac{1}{r} \tan z_0 \frac{\partial^2 \psi}{\partial \varphi \partial \zeta} = 0. \tag{10} \]

Outside a ferrite region (where \( \mu = 1 \)) Eq. (10) reduces to the Laplace equation in helical coordinates \([12,13]\). Following Overfelt’s approach \([13]\), we assume that solutions of the Laplace and Walker equations are found as

\[ \psi(r, \varphi, \zeta) = R(r)P(\varphi)Z(\zeta), \tag{11} \]

where

\[ P(\varphi) = A \exp(\pm i \omega \varphi), \]
\[ Z(\zeta) = B \exp(\pm \beta \zeta). \tag{12} \]

Functions \( \psi(r) \) are described by the Bessel equations. Because of the bidirectional MS-wave representation, there are two sets of equations. For a ferrite rod with radius \( \Re \), the first set is:

\[ \frac{\partial^2 \psi(r)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi(r)}{\partial r} - \left[ \frac{\beta^2}{\mu} + \frac{1}{r^2} (w - \bar{\beta})^2 \right] \psi(r) = 0. \tag{13} \]

inside a ferrite rod \((r \leq \Re)\) and

\[ \frac{\partial^2 \psi(r)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi(r)}{\partial r} - \left[ \frac{\beta^2}{\mu} + \frac{1}{r^2} (w + \bar{\beta})^2 \right] \psi(r) = 0. \tag{14} \]

outside a ferrite rod \((r \geq \Re)\). A physically acceptable solution for Eq. (13) is possible only for a negative quantity \( \mu \). This solution is expressed by Bessel function of a real argument:

\[ \psi(r)_{r<\Re} = c_1 J_{(w-\bar{\beta})/\beta}(\mu^{1/2}r). \tag{15} \]

A solution of Eq. (14) is expressed by Bessel function of an imaginary argument:

\[ \psi(r)_{r>\Re} = d_1 K_{(w-\bar{\beta})/\beta}(\beta r). \tag{16} \]

For the second set we have

\[ \frac{\partial^2 \psi(r)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi(r)}{\partial r} - \left[ \frac{\beta^2}{\mu} + \frac{1}{r^2} (w + \bar{\beta})^2 \right] \psi(r) = 0. \tag{17} \]

inside a ferrite rod \((r \leq \Re)\) and

\[ \frac{\partial^2 \psi(r)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi(r)}{\partial r} - \left[ \frac{\beta^2}{\mu} + \frac{1}{r^2} (w - \bar{\beta})^2 \right] \psi(r) = 0. \tag{18} \]

outside a ferrite rod \((r \geq \Re)\). The solutions are expressed as:

\[ \psi(r)_{r<\Re} = c_2 J_{(w+\bar{\beta})/\beta}(\beta^{1/2}r) \tag{19} \]

inside a ferrite rod \((r \leq \Re)\) and

\[ \psi(r)_{r>\Re} = d_2 K_{(w+\bar{\beta})/\beta}(\beta r) \tag{20} \]

outside a ferrite rod \((r \geq \Re)\). Coefficients \( c_{1,2}, d_{1,2} \) are amplitude coefficients.

On a cylindrical surface of a ferrite rod we have the boundary conditions:

\[ \psi(r)_{r=\Re} = (\psi)_{r=\Re^+} \tag{21} \]

and

\[ (B_r)_{r=\Re} = (B_r)_{r=\Re^+}. \tag{22} \]

Based on the above Bessel equations and boundary conditions one obtains characteristic equations for helical MS waves in a ferrite rod.
There are the following two sets of equations:

\[
(-\mu)^{1/2} \frac{J'_w(\omega + \beta \phi)}{J_w(\omega + \beta \phi)} + \frac{K'_w(\omega + \beta \phi)}{K_w(\omega + \beta \phi)} \pm \mu_a \frac{|\omega| - \beta |\phi|}{|\beta | |\phi|} = \frac{1}{0}
\]

(23)

and

\[
(-\mu)^{1/2} \frac{J'_w(\omega + \beta \phi)}{J_w(\omega + \beta \phi)} + \frac{K'_w(\omega + \beta \phi)}{K_w(\omega + \beta \phi)} \pm \mu_a \frac{|\omega| + \beta |\phi|}{|\beta | |\phi|} = \frac{1}{0},
\]

(24)

where the prime denotes differentiation with respect to the argument. Different signs in Eqs. (23) and (24) correspond to different combinations of signs of quantities \(w\) and \(\beta\).

The above analysis of helical MS waves in a ferrite rod gives evidence that in such coordinates we can formally distinguish the waves with clockwise and counter clockwise rotation. Also we can distinguish the forward and backward waves. To characterize the entire properties of the process we should suppose that MS-potential function is a four-component function:

\[
[\psi] = \left( \begin{array}{c}
\psi^{(1)} \\
\psi^{(2)} \\
\psi^{(3)} \\
\psi^{(4)}
\end{array} \right).
\]

(25)

Inside a ferrite region \((r \leq R)\) these components are described as

\[
\psi^{(1)} = a_1 J_w(\omega - \beta \phi) \left(-\mu \right)^{1/2} \beta r e^{-i\omega \phi} e^{-i\beta \phi},
\]

\[
\psi^{(2)} = a_2 J_w(\omega + \beta \phi) \left(-\mu \right)^{1/2} \beta r e^{+i\omega \phi} e^{-i\beta \phi},
\]

\[
\psi^{(3)} = a_3 J_w(\omega - \beta \phi) \left(-\mu \right)^{1/2} \beta r e^{+i\omega \phi} e^{+i\beta \phi},
\]

\[
\psi^{(4)} = a_4 J_w(\omega + \beta \phi) \left(-\mu \right)^{1/2} \beta r e^{-i\omega \phi} e^{+i\beta \phi}.
\]

(26)

For an outside region \((r \geq R)\) one has

\[
\psi^{(1)} = b_1 K_w(\omega - \beta \phi) e^{-i\omega \phi} e^{-i\beta \phi},
\]

\[
\psi^{(2)} = b_2 K_w(\omega + \beta \phi) e^{+i\omega \phi} e^{-i\beta \phi},
\]

\[
\psi^{(3)} = b_3 K_w(\omega - \beta \phi) e^{+i\omega \phi} e^{+i\beta \phi},
\]

\[
\psi^{(4)} = b_4 K_w(\omega + \beta \phi) e^{-i\omega \phi} e^{+i\beta \phi}.
\]

(27)

Coefficients \(a_{1,2,3,4}\) and \(b_{1,2,3,4}\) are amplitude coefficients.

Different components of MS-potential function \([\psi]\) are correlated with different wave processes along \(z\)-axis of a ferrite rod. The main differences between the above four components of function \(\psi\) are introduced by factors combining different signs of the quantities \(\beta\) and \(\mu_a\). Function \(\psi^{(1)}\) describes the right-hand-helix MS wave propagating in a ferrite rod along \(+z\)-axis. Function \(\psi^{(2)}\) describes the left-hand-helix MS wave propagating along \(+z\) axis. Function \(\psi^{(3)}\) describes the right-hand-helix MS wave propagating in a ferrite rod along \(-z\)-axis. Function \(\psi^{(4)}\) is the left-hand-helix MS wave propagating along \(-z\)-axis.

Since we have four types of helical MS waves one should consider every space component of a magnetic flux density as a four-component function:

\[
[B_\phi] = \left( \begin{array}{c}
B^{(1)}_\phi \\
B^{(2)}_\phi \\
B^{(3)}_\phi \\
B^{(4)}_\phi
\end{array} \right),
\]

(28)

These components are expressed by Eqs. (8) and (9).

From the above characteristic equations it becomes clear that we have the same moduli of the propagation constants for helical modes \(\psi^{(1)}\) and \(\psi^{(4)}\) as well as the same moduli of the propagation constants for helical modes \(\psi^{(2)}\) and \(\psi^{(3)}\). At the same time, propagation constants of modes \(\psi^{(1)}\) and \(\psi^{(4)}\) are different from the propagation constants of modes \(\psi^{(2)}\) and \(\psi^{(3)}\). So
we can write
\[ |w^{(1)}| = |w^{(4)}| \neq |w^{(2)}| = |w^{(3)}| \]  
(29)
and
\[ |\beta^{(1)}| = |\beta^{(4)}| \neq |\beta^{(2)}| = |\beta^{(3)}|. \]  
(30)

Taking these relations into account, one can see that for functions \( \psi^{(1)} \) and \( \psi^{(4)} \) there are the same 
Bessel equations:
\[
\frac{\partial^2 \psi^{(1,4)}}{\partial r^2} + \frac{1}{r} \frac{\partial \psi^{(1,4)}}{\partial r} 
- \left[ \left( \frac{\beta^{(1,4)}}{\mu} \right)^2 + \frac{1}{r^2} \left( w^{(1,4)} - \bar{p}\beta^{(1,4)} \right)^2 \right] \psi^{(1,4)} = 0
\]
(31)
inside a ferrite and
\[
\frac{\partial^2 \psi^{(1,4)}}{\partial r^2} + \frac{1}{r} \frac{\partial \psi^{(1,4)}}{\partial r} 
- \left[ \left( \frac{\beta^{(1,4)}}{\mu} \right)^2 + \frac{1}{r^2} \left( w^{(1,4)} + \bar{p}\beta^{(1,4)} \right)^2 \right] \psi^{(1,4)} = 0
\]
(32)
outside a ferrite. Similarly, for functions \( \psi^{(2)} \) and \( \psi^{(3)} \) we have
\[
\frac{\partial^2 \psi^{(2,3)}}{\partial r^2} + \frac{1}{r} \frac{\partial \psi^{(2,3)}}{\partial r} 
- \left[ \left( \frac{\beta^{(2,3)}}{\mu} \right)^2 + \frac{1}{r^2} \left( w^{(2,3)} + \bar{p}\beta^{(2,3)} \right)^2 \right] \psi^{(2,3)} = 0
\]
(33)
inside a ferrite and
\[
\frac{\partial^2 \psi^{(2,3)}}{\partial r^2} + \frac{1}{r} \frac{\partial \psi^{(2,3)}}{\partial r} 
- \left[ \left( \frac{\beta^{(2,3)}}{\mu} \right)^2 + \frac{1}{r^2} \left( w^{(2,3)} - \bar{p}\beta^{(2,3)} \right)^2 \right] \psi^{(2,3)} = 0
\]
(34)
outside a ferrite.

Let us consider a normally magnetized ferrite disk as a section of a ferrite rod restricted by 
planes \( z = 0, d \). On planes \( z = 0, d \) reflections of 
helical MS wave propagating along \( z \)-axis inside a 
ferrite disk take place. For these helical waves one 
has the failure of the law of reflection symmetry. 
It means that any helical wave “incident” on a 
reflection plane cannot be transformed to itself 
and should be transformed to another-type helical 
wave. In other words, on a reflection plane one has 
coupling between different helical waves (between 
waves with different types of symmetry). Suppose, 
for example, that we have a helical MS mode \( \psi^{(1)} \) 
incident (from a ferrite region) on a plane \( z = d \). 
On the plane \( z = d \) this ascending helical wave 
can be coupled, in general, with descending helical MS 
modes \( \psi^{(3)} \) and \( \psi^{(4)} \). At the same time a helical 
MS mode \( \psi^{(2)} \) incident on a plane \( z = d \) can be 
coupled with helical MS modes \( \psi^{(3)} \) and \( \psi^{(4)} \). 
Similarly, due to reflections on a plane \( z = 0 \) 
descending mode \( \psi^{(3)} \) becomes coupled with 
ascenting modes \( \psi^{(1)} \) and \( \psi^{(2)} \). Also mode 
\( \psi^{(4)} \) becomes coupled with modes \( \psi^{(1)} \) and \( \psi^{(2)} \). So in a general consideration there should be 
a system of four linear equations for four 
variables: \( \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \) and \( \psi^{(4)} \). The above 
analysis shows, however, that there could be two 
special cases of the resonance interactions. 
The first resonant case one has due to ascending helical 
wave \( \psi^{(1)} \) and descending helical wave \( \psi^{(4)} \). 
The second resonant case is due to ascending helical 
wave \( \psi^{(2)} \) and descending helical wave \( \psi^{(3)} \). These 
resonant cases we will denote conventionally as the 
“right” and “left” resonances.

The quantities of pitch \( \bar{p} \) one has from the 
evident resonance conditions in a ferrite disk:
\[ p = \frac{d}{2n}, n = 1, 2, 3, \ldots \]  
(35)
So
\[ \tan x_0 = \frac{p}{2\pi n} = \frac{d}{4\pi n} \]  
(36)

We have the “right” and “left” resonances, 
corresponding to the interactions \( \psi^{(1)} \leftrightarrow \psi^{(4)} \) 
and \( \psi^{(2)} \leftrightarrow \psi^{(3)} \), when integer numbers \( n \) in Eq. (35) 
are even quantities. In this case a total period of \( 2\pi \) 
rotation in a cylindrical coordinate system corre-
sponds (for the same time) to the \( 4\pi \) rotation in a 
helical coordinate system. This is the model of 
spinning rotation. Considering the \( \psi^{(1)} \leftrightarrow \psi^{(4)} \) 
resonance in cylindrical coordinates \((r, \theta, z)\), one 
sees that time variation of function \( \psi^{(1)} \) is due to 
interaction with function \( \psi^{(4)} \) via mutual spinning.
rotation. Similarly, time variation of function \( \psi_1^{(2)} \) is due to interaction with function \( \psi_2^{(3)} \) via mutual spinning rotation. These interactions we can describe as two systems of two equations:

\[
iX \frac{\partial \psi_1^{(1)}}{\partial t} = Y e^{-i\theta} \left( \frac{\partial}{\partial r} - i \frac{1}{r} \frac{\partial}{\partial \theta} \right) \psi_1^{(1)},
\]

\[
iX \frac{\partial \psi_1^{(4)}}{\partial t} = Y e^{+i\theta} \left( \frac{\partial}{\partial r} + i \frac{1}{r} \frac{\partial}{\partial \theta} \right) \psi_1^{(4)},
\]

(37)

and

\[
iX \frac{\partial \psi_2^{(2)}}{\partial t} = Y e^{+i\theta} \left( \frac{\partial}{\partial r} + i \frac{1}{r} \frac{\partial}{\partial \theta} \right) \psi_2^{(2)},
\]

\[
iX \frac{\partial \psi_2^{(3)}}{\partial t} = Y e^{-i\theta} \left( \frac{\partial}{\partial r} - i \frac{1}{r} \frac{\partial}{\partial \theta} \right) \psi_2^{(3)},
\]

(38)

where \( X \) and \( Y \) are constant coefficients. The above equations can be transformed in the Cartesian coordinate system with use of the notation

\[
\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} = e^{\pm i\theta} \left( \frac{\partial}{\partial r} \pm i \frac{1}{r} \frac{\partial}{\partial \theta} \right)
\]

(39)

Eqs. (37) and (38) are the Dirac-like equations for massless fermions written in a cylindrical coordinate system. Operator \( iX \frac{\partial}{\partial t} \) is the energy operator \( E \) (like it was for function \( \psi \) described by the Schrödinger-like equation [3,4]). Coefficient \( X \) has the meaning of the “effective Plank constant” and coefficient \( Y \) is the “effective speed of light”. The values \( X \) and \( Y \) can be found from the dispersion characteristics of MS modes. For a monochromatic process Eqs. (37) and (38) describes a certain state. For every given state there are certain coefficients \( X \) and \( Y \).

The importance of the above identification with the Dirac equation is that it immediately permits to construct the azimuthal parts of the spinor wave functions. Based on coordinate transformation between helical and cylindrical systems [12], one can see that in a cylindrical coordinate system, Eqs. (23) and (24) take a form for the “right” and “left” resonances, respectively, as

\[
(-\mu)^{1/2} \left( \frac{J_{\nu_L}}{J_{\nu_R}} \right)_{r=0} + \left( \frac{K'_{\nu_L}}{K'_{\nu_R}} \right)_{r=0} - \frac{\mu a v_R}{|\beta_R|} = 0
\]

(40)

and

\[
(-\mu)^{1/2} \left( \frac{J_{\nu_R}}{J_{\nu_L}} \right)_{r=0} + \left( \frac{K'_{\nu_R}}{K'_{\nu_L}} \right)_{r=0} - \frac{\mu a v_L}{|\beta_L|} = 0
\]

(41)

Quantities \( v_R \) and \( v_L \) are integer numbers. The eigenvalues of the “total angular moment” in the Dirac coordinate system are half-integer and doubly degenerate with the eigenspinors of \( j = \nu + \frac{1}{2} \) given by

\[
\begin{pmatrix}
\delta^{ij} \\
0
\end{pmatrix}
\]

(42)

and

\[
\begin{pmatrix}
0 \\
\delta^{i(j+1)}
\end{pmatrix}
\]

(43)

These states have opposite angular momenta. In this case an analysis can be reduced to consideration of a system of coupled ordinary differential equations [14].

The handedness properties of MDM oscillations show the physics of topological effects and explain the nature of surface magnetic currents in ferrite disks.

4. Conclusion

The MDMs in a ferrite disk resonator are neither electromagnetic nor exchange-interaction waves. We showed that MDM waves in a normally magnetized ferrite disk are spinor wave functions. The “electric spin” (pseudospin) of a MDM is tied to the anapole linear momentum \( a^e \). This is completely analogous to the physical spin of a massless neutrino which points along the direction of propagation. In our case one can distinguish the “particles” (when the pseudospin is antiparallel to the anapole moment) and “anti-particles” (when the pseudospin is parallel to the anapole moment).

References