Diversity-Multiplexing Tradeoff for the Interference Channel With a Relay

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Abstract

We study the diversity-multiplexing tradeoff (DMT) for the slow fading interference channel with a relay (ICR). We first derive an outer bound on the DMT based on the cut-set bound. Then, we derive two inner bounds on the DMT: One is based on the compress-and-forward relaying scheme and the other is based on the decode-and-forward relaying scheme. These results lead to informative insights. We first find conditions on the channel parameters and the multiplexing gains, under which the proposed inner bounds achieve the optimal DMT. Then, we identify cases in which the DMT of the ICR is the same as two parallel fading relay channels, implying that interference does not decrease the DMT for each pair, and that a single relay can be DMT-optimal for two pairs simultaneously. Lastly, we identify conditions under which adding a relay strictly improves the DMT, relative to the interference channel without a relay.

I. INTRODUCTION

The interference channel with a relay (ICR) models the scenario in which a relay helps several independent transmitters in sending messages to their corresponding receivers, simultaneously over a shared channel. The ICR was first studied in [1] and has since gained considerable interest as an extension of the canonical relay and interference channel models. Inner and outer bounds on the capacity region of the two-pair ICR with additive white Gaussian noise (AWGN) were characterized in [1], [2], and [3]. Specifically, in [1] an achievable rate region was obtained by employing a rate splitting scheme at the transmitters, decode-and-forward (DF) strategy at the relay, and a backward decoding scheme at the receivers. The achievable rate region in [2] was obtained using the compress-and-forward (CF) strategy at the relay. Outer bounds were obtained by applying the cut-set bound [2], [3], and by using a potent relay [2]. Furthermore, for the ergodic phase fading and the ergodic Rayleigh fading channels, the capacity region for the strong interference regime was characterized in [4] for the case in which the links from the sources to the relay are good. In this paper we study the diversity-multiplexing tradeoff (DMT) for...
the slow Rayleigh fading ICR. We derive inner and outer bounds on the DMT and identify situations, in which the optimal DMT can be characterized.

In [7], DMT characteristics of several cooperation strategies were obtained. The work [7] showed that the DF scheme with receive channel state information (Rx-CSI) is DMT-optimal for full duplex single-antenna single-relay channels; but it is suboptimal for multiple-antenna relay channels at high multiplexing gains. CF with Rx and Tx CSI, on the other hand, was shown to be DMT-optimal for the multiple-antenna relay channel over all multiplexing gains. In [8] it was shown that quantize-map-and-forward (QMF) achieves the optimal DMT of certain configurations of the half-duplex relay channel without CSI at the relay node. The DMT of single-antenna block Rayleigh fading interference channels (ICs) was studied in [9] for the scenario in which there is Rx-CSI but no Tx-CSI, and studied in [10] for the case in which both Rx-CSI and Tx-CSI are available. It was shown that in the very strong interference regime, successive decoding with interference cancelation is DMT optimal. Note that for the ergodic case, using the same approach achieves the capacity region of the IC. Hence, the same strategy is optimal from both DMT and capacity perspectives. For a general interference regime, [9] proposed a transmission scheme using Han-Kobayashi (HK) type superposition encoding where each receiver jointly decodes both the common messages (from both transmitters) and the private message from its intended transmitter. This scheme was shown to be DMT optimal under some conditions on the strength of the interference and over a certain range of multiplexing gains. The DMT of the Gaussian MIMO ICR was studied in [11], for the case where all links have the same exponential behaviour over the signal-to-noise ratio (SNR). In [11], an outer bound on the DMT was derived using the cut-set theorem, and an achievable DMT was characterized by employing CF at the relay node.

Main Contributions

In this paper we study the DMT of the single-antenna ICR with a full-duplex relay. All links are subject to slow Rayleigh fading. We consider the scenario in which the receivers have perfect Rx-CSI, but there is no Tx-CSI. We allow the direct links gains, interfering links gains, and the relay-to-destination links gains to scale differently as exponential functions of SNR, while the channel is symmetric in the sense that the scaling of the corresponding links is identical for both pairs. The main contributions of this work are:

1) An outer bound on the DMT is presented using the cut-set bound. Note that while [11] studied the scenario in which all links scale identically over the SNR, our outer bound allows different scalings.
2) Two achievable DMT regions are derived based on the CF and DF schemes. We analyze how the gains of the cross-links and of the relay-to-destination links affect the achievable DMT.
3) We provide sufficient conditions under which the optimal DMT can be achieved by either the CF or the DF scheme. We then present sufficient conditions under which the ICR has the same DMT as two parallel single-relay channels. Thus, the relay assistance to one pair does not degrade the DMT performance at the other pair, and in-fact a single relay is simultaneously DMT-optimal for two separate pairs.
4) We compare the DMT of the ICR with that of the IC, and provide sufficient conditions under which adding a relay improves the DMT.
These results give a strong motivation for employing relay nodes in multiuser wireless networks that have to cope with interference.

The rest of the paper is organized as follows: The channel model and notations used in this paper are presented in Section II. The case in which the relay has Tx-CSI and Rx-CSI is studied in Section III in which an outer bound on the achievable DMT is obtained and a CF-based DMT is derived. The case in which the relay has only Rx-CSI is studied in Section IV where an outer bound and a DF-based achievable DMT are presented. Lastly, concluding remarks are presented in Section V.

II. SYSTEM MODEL AND NOTATIONS

We denote random variables (RVs) with capital letters, e.g., $X$, $Y$, and their realizations with lower case letters, e.g., $x$, $y$. $\mathbb{E}\{X\}$ denotes the stochastic expectation of $X$. Bold-face letters, e.g., $\mathbf{x}$, denote column vectors (unless otherwise specified), and the $i$’th element of a vector $\mathbf{x}$ is denoted by $x_i$. We use $x^j$ to denote the vector $(x_1, x_2, \ldots, x_{j-1}, x_j)$, and $X^*$ to denote the conjugate of $X$. We denote the circularly symmetric, complex Normal distribution with mean $\mu$ and variance $\sigma^2$ as $CN(\mu, \sigma^2)$. All logarithms are of base 2. We also define $(x)^+ \triangleq \max\{x, 0\}$, and denote $f(SNR) \triangleq \text{SNR}_c$ if $\lim_{\text{SNR} \to \infty} \frac{\log f(\text{SNR})}{\log \text{SNR}} = c$.

The ICR consists of two transmitters and two receivers, as shown in Fig. 1. $\text{Tx}_k$ sends messages to $\text{Rx}_k$, $k \in \{1,2\}$. The full-duplex relay node assists communication from the transmitters to their respective receivers. The received signals at $\text{Rx}_1$, $\text{Rx}_2$ and at the relay at time $i$ are denoted by $Y_{1,i}$, $Y_{2,i}$, and $Y_{3,i}$ respectively. The channel inputs from $\text{Tx}_1$, $\text{Tx}_2$ and the relay are denoted by $X_{1,i}$, $X_{2,i}$, and $X_{3,i}$ respectively.

In this model, the relationship between the channel inputs and outputs is given by:

$$Y_1 = \sqrt{\text{SNR}} H_{11} X_1 + \sqrt{\text{SNR}}^\alpha H_{21} X_2 + \sqrt{\text{SNR}}^\beta H_{31} X_3 + Z_1$$
$$Y_2 = \sqrt{\text{SNR}}^\alpha H_{12} X_1 + \sqrt{\text{SNR}} H_{22} X_2 + \sqrt{\text{SNR}}^\beta H_{32} X_3 + Z_2$$
$$Y_3 = \sqrt{\text{SNR}}^\beta H_{13} X_1 + \sqrt{\text{SNR}}^\gamma H_{23} X_2 + Z_3,$$

Here, $Z_1$, $Z_2$, and $Z_3$ are mutually independent RVs, distributed according to $CN(0,1)$, independent over time and independent of the channel inputs and channel coefficients. The channel inputs have unit power constraint, i.e.,
\( \mathbb{E} \{|X_k|^2\} \leq 1, k \in \{1, 2, 3\} \). Note that in the above equations, SNR denotes the average received signal-to-noise ratio over the direct link for both pairs. We assume that the cross-link gains scale as \( \sqrt{\text{SNR}}^\alpha \), and the relay-to-destination link gains scale as \( \sqrt{\text{SNR}}^\beta \). The remaining link gains scale as \( \text{SNR} \). Thus, this work will study the impact of interference and relay-destination link scalings on the DMT. In addition, \( H_{kl} \) denotes the channel coefficient from node \( k \) to node \( l \). The different SNR exponents (see also [9]) represent different pathloss scaling behaviours due to different propagation scenarios. For example, when the receiver is located closer to the opposite transmitter, the fading pathloss exponent from the interferer may be smaller than the pathloss exponent from the desired transmitter, as there is less scattering on the path from the interferer to the receiver than on the path from the desired transmitter to the receiver. This is translated into a greater SNR exponent on the interfering link than that on the direct link.

The channel coefficients are i.i.d., each distributed according to \( \mathcal{CN}(0, 1) \), where they change only between messages, corresponding to the slow fading scenario. It is assumed that each receiver \( k \) has Rx-CSI represented by \( \tilde{H}_k = (H_{1k}, H_{2k}, H_{3k}) \in \mathbb{C}^3 \triangleq \mathcal{S}_k, k \in \{1, 2\} \), and that the relay has Rx-CSI represented by \( \tilde{H}_{3,R} = (H_{31}, H_{32}) \in \mathbb{C}^2 \triangleq \mathcal{S}_3 \). In some scenarios, the relay also has Tx-CSI represented by \( \tilde{H}_{3,T} = (H_{31}, H_{32}) \in \mathcal{S}_3 \). However, we do not assume Tx-CSI at Tx1 or Tx2. Let \( \tilde{H} = (\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) \) be the vector of all channel coefficients.

**Definition 1.** An \((R_1, R_2, n)\) code for the ICR consists of two message sets \( \mathcal{M}_k \triangleq \{1, 2, \ldots, 2^{nR_k}\}, k = 1, 2 \), two encoders at the sources, \( e_k : \mathcal{M}_k \mapsto \mathcal{C}^n, k = 1, 2 \), and two decoders at the destinations, \( g_k : \mathcal{C}^n \mapsto \mathcal{M}_k, k = 1, 2 \). The transmitted signal at the relay at time \( i \) is \( x_{3,i} = t_i(\tilde{g}_3^{-1}, \tilde{H}_{3,R}) \in \mathcal{C}, i = 1, 2, \ldots, n \). When Tx-CSI is available at the relay then \( x_{3,i} = t_i(\tilde{g}_3^{-1}, \tilde{H}_{3,R}, \tilde{H}_{3,T}) \in \mathcal{C} \). Let us denote a coding scheme by \( S_c \).

**Definition 2.** The average probability of error is defined as \( P_e^{(n)} \triangleq \Pr(g_1(\tilde{H}_1, Y^n_1) \neq M_1 \text{ or } g_2(\tilde{H}_2, Y^n_2) \neq M_2) \), and each source message is selected independently and uniformly from its message set.

**Definition 3.** A rate pair \((R_1, R_2)\) is called achievable if for any \( \epsilon > 0 \) and \( \delta > 0 \) there exists some blocklength \( n_0(\epsilon, \delta) \) such that for every \( n > n_0(\epsilon, \delta) \) there exists an \((R_1 - \delta, R_2 - \delta, n)\) code with \( P_e^{(n)} < \epsilon \). Let \( \mathcal{R}(\tilde{H}, S_c, \text{SNR}) \) denote the maximum achievable rate region for the ICR whose channel coefficients are \( \tilde{H} \), and the direct-link signal to noise ratio is SNR, achieved by a coding scheme \( S_c \).

**Definition 4.** The probability of an outage event in the ICR, for the scheme \( S_c \) and target rates \( R_{T,1}, R_{T,2} \) for pairs 1 and 2, respectively, at SNR, is defined as: \( P_O(R_{T,1}, R_{T,2}, \text{SNR}, S_c) \triangleq \Pr((R_{T,1}, R_{T,2}) \notin \mathcal{R}(\tilde{H}, S_c, \text{SNR})) \).

**Definition 5.** We say that a coding scheme \( S_c \) for the ICR achieves multiplexing gains (MGs) of \((r_1, r_2)\), if for high SNR there exist rates \((R_1(\text{SNR}), R_2(\text{SNR})) \in \mathcal{R}(\tilde{H}, S_c, \text{SNR})\) that scale as

\[
\lim_{\text{SNR} \to \infty} \frac{R_1(\text{SNR})}{\log(\text{SNR})} = r_1, \quad \lim_{\text{SNR} \to \infty} \frac{R_2(\text{SNR})}{\log(\text{SNR})} = r_2.
\]

**Definition 6.** We say that a scheme \( S_c \) achieves a diversity gain of \( d(r_1, r_2) \) for multiplexing gains \( r_1, r_2 \), if

\[
- \lim_{\text{SNR} \to \infty} \frac{\log P_O(r_1 \log(\text{SNR}), r_2 \log(\text{SNR}))}{\log(\text{SNR})} = d(r_1, r_2).
\]
Note that while the diversity gain is a function of $P_e$, [6], we can follow the arguments in [7, Section III] and characterize the diversity gain via the outage probability. This can be done since

$$
\lim_{\text{SNR} \to \infty} \frac{\log P_0}{\log(\text{SNR})} = \lim_{\text{SNR} \to \infty} \frac{\log P_e}{\log(\text{SNR})}.
$$

III. DMT WITH RX-CSI AND TX-CSI AT THE RELAY

A. DMT Outer Bound

We now present an outer bound on the achievable DMT of the ICR, based on the cut-set theorem:

**Theorem 1.** For the symmetric ICR with Rx-CSI at the receivers and Rx/Tx-CSI at the relay, as defined in Section II, the outer bound on the DMT is given by:

$$
d^+(r_1, r_2) = \min_{k \in \{1, 2, \ldots, 10\}} \{d^+_k(r_1, r_2)\},
$$

where

$$
\begin{align*}
&d^+_1(r_1, r_2) = 2(1 - r_1)^+ \\
&d^+_2(r_1, r_2) = (1 - r_1)^+ + (\beta - r_1)^+ \\
&d^+_3(r_1, r_2) = 2(1 - r_2)^+ \\
&d^+_4(r_1, r_2) = (1 - r_2)^+ + (\beta - r_2)^+ \\
&d^+_5(r_1, r_2) = 2(1 - r_1 - r_2)^+ + 2(\alpha - r_1 - r_2)^+ + 2(\beta - r_1 - r_2)^+ \\
&d^+_6(r_1, r_2) = (2 - r_1 - r_2)^+ + 2(\alpha - r_1 - r_2)^+ \\
&d^+_7(r_1, r_2) = (2\alpha - r_1 - r_2)^+ + 2(1 + \beta - r_1 - r_2)^+ \\
&d^+_8(r_1, r_2) = 4(1 - r_1 - r_2)^+ + 2(\alpha - r_1 - r_2)^+ \\
&d^+_9(r_1, r_2) = 2(2 - r_1 - r_2)^+ + (2\alpha - r_1 - r_2)^+ \\
&d^+_{10}(r_1, r_2) = (2 - r_1 - r_2)^+ + 2(1 + \alpha - r_1 - r_2)^+.
\end{align*}
$$

**Proof:** The cut-set bound for the ICR was evaluated in [11, Proof of Thm. 1] for the case of $\alpha = \beta = 1$. The proof of Thm. [1] is obtained by following similar arguments as in that proof while taking into account the exponents $\alpha$ and $\beta$. A detailed proof can be found in Appendix [A].

**Remark 1.** Observe that (2a) and (2b) are the DMT upper bound for the single-relay channel with Tx_1 as the source and Rx_1 as the destination, while (2c) and (2d) are the DMT upper bound for the Tx_2-relay-Rx_2 relay channel. The remaining DMT bounds correspond to the cut-set bounds for the IC, see [11, Proof of Thm. 1].

B. An Achievable DMT Region via CF

We now derive an achievable DMT region with the CF scheme. In the next section we shall provide sufficient conditions under which the CF achievable DMT region coincides with the DMT outer bound of [1], leading to the
characterization of the optimal DMT of the ICR. The achievable DMT with the CF scheme is characterized in the following theorem:

**Theorem 2.** For the symmetric ICR with Rx-CSI at the receivers and Rx/Tx-CSI at the relay as defined in Section II, an achievable DMT region is given by:

\[ d_{\text{CF}}(r_1, r_2) = \min_{k \in \{1, 2, 3\}} \left\{ d_{k, \text{CF}}(r_1, r_2) \right\}, \]

where

\[ d_{1, \text{CF}}(r_1, r_2) = \begin{cases} (1 - r_1)^+ + (1 - (1 + \alpha - \beta)^+ - r_1)^+ & \alpha > 1 \\ (1 - r_1)^+ + (1 - (2 - \beta)^+ - r_1)^+ & \alpha \leq 1 \end{cases} \]  

(4a)

\[ d_{2, \text{CF}}(r_1, r_2) = \begin{cases} (1 - r_2)^+ + (1 - (1 + \alpha - \beta)^+ - r_2)^+ & \alpha > 1 \\ (1 - r_2)^+ + (1 - (2 - \beta)^+ - r_2)^+ & \alpha \leq 1 \end{cases} \]  

(4b)

\[ d_{3, \text{CF}}(r_1, r_2) = \begin{cases} (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ + 2(1 - (1 + \alpha - \beta)^+ - r_1 - r_2)^+ & \alpha > 1 \\ (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ + 2(1 - (2 - \beta)^+ - r_1 - r_2)^+ & \alpha \leq 1 \end{cases} \]  

(4c)

**Proof:** A detailed proof can be found in Appendix [B](#)

### C. Discussion

**Corollary 1.** Consider the Gaussian ICR defined in Section II. If \( \max\{1, \alpha\} \leq \beta - 1 \) and

\[ \min \left\{ 2(1 - r_1)^+, 2(1 - r_2)^+ \right\} \leq 3(1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+, \]

then the optimal DMT for the Gaussian ICR is

\[ d_{\text{Opt-CF}}(r_1, r_2) = \min \left\{ 2(1 - r_1)^+, 2(1 - r_2)^+ \right\}, \]

and it is achieved with CF at the relay.

**Proof:** The proof is based on Theorem [2](#). Note that if \( \max\{1, \alpha\} \leq \beta - 1 \) and \( \min \left\{ 2(1 - r_1)^+, 2(1 - r_2)^+ \right\} \leq 3(1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+, \) the achievable DMT of the CF scheme is \( \min \left\{ 2(1 - r_1)^+, 2(1 - r_2)^+ \right\}, \) which coincides with the DMT outer bound derived in Theorem [1](#) characterizing the optimal DMT for the ICR.

**Remark 2.** Observe that the optimality holds for large multiplexing gains.

**Remark 3.** Note that if \( \beta < \max\{\alpha, 1\} + 1 \), CF may not be DMT-optimal anymore. Hence, the achievable DMT of CF does not coincide with the DMT of the cut-set bound over all ranges of channel coefficients, contrary to the situation in the single-relay channel observed in [7](#). In such scenarios, the assistance of the relay is not strong enough to support decoding both messages at each receiver.
Remark 4. Note that under the conditions of Corollary 1, (5) corresponds to the optimal DMT of two interference-free parallel relay channels. This can be seen by inspecting the cut-set bound for the relay channel. Consider the relay channel in which the relationship between the channel inputs and its outputs is given by:

\[ Y_1 = \sqrt{\text{SNR}} H_{11} X_1 + \sqrt{\text{SNR}}^{\beta} H_{31} X_3 + Z_1, \]
\[ Y_3 = \sqrt{\text{SNR}} H_{13} X_1 + Z_3, \]

where \( X_1 \) and \( X_3 \) denote the transmitted signals of the source and the relay, respectively, and \( Y_1 \) and \( Y_3 \) denote the received signal at the destination and at the relay node respectively. The rest of the definitions remain the same as given in Section II. Let \( H_R = (H_{11}, H_{13}, H_{31}) \). From [7, Eqns. (4), (5)], for a given realization \( h_r \) the capacity of the relay channel is upper-bounded by

\[
C_{\text{Relay}} \leq \max_{f(x_1, x_2)} \min \{ I(X_1, X_3; Y_1|h_r), I(X_1; Y_1, Y_3|X_3, h_r) \}. 
\]

Following derivations similar to [11, Proof of Thm. 1], we further bound each expression:

\[
I(X_1, X_3; Y_1|h_r) \leq \log \left( \text{SNR}^{1-\theta_{11}} + 2\text{SNR}^{-\theta_{11} + \beta - \theta_{31}} + \text{SNR}^{\beta - \theta_{31}} + 1 \right) 
\]
\[
I(X_1; Y_1, Y_3|X_3, h_r) \leq \log \left( 1 + \text{SNR}|h_{11}|^2 + \text{SNR}|h_{13}|^2 \right) 
= \log \left( 1 + \text{SNR}^{1-\theta_{11}} + \text{SNR}^{1-\theta_{13}} \right). 
\]

Hence, the DMT of the relay channel is upper-bounded by

\[
d_{\text{Relay}}^+(r) = \min \{ 2(1-r)^+, (1-r)^+ + (\beta - r) \}. 
\]

Observe that \( d_{\text{Relay}}^+(r) \leq d_{\text{Opt-CF}}(r, r) \). Recall that in Corollary 1, we have \( \beta \geq 2 \). We conclude that for \( \beta \geq 2 \), \( d_{\text{Relay}}^+(r) = d_{\text{Opt-CF}}(r, r) \). Thus, under the conditions of Corollary 1, the optimal DMT of the ICR coincides with the outer bound on the optimal DMT of two interference-free parallel relay channels. Hence, one relay employing the CF strategy in this situation is DMT optimal for both communicating pairs simultaneously. Furthermore, interference does not degrade the performance in this case.

Remark 5. From Theorem 2, it follows that when \( \alpha \leq 1 \) the DMT achievable with CF is an increasing function of \( \alpha \), that is, increasing the interference between the communicating pairs improves the DMT. On the other hand, if \( \alpha > 1 \), there are two cases:

- If \( \beta \geq \alpha + 1 \), (4a) and (4b) do not depend on \( \alpha \), while (4c) increases with respect to \( \alpha \). We conclude that the DMT performance of the CF improves as the interference becomes stronger.
- If \( \beta < \alpha + 1 \) and \( r_1 + r_2 \leq \beta - \alpha \), (4a) and (4c) decrease when \( \alpha \) increases. Thus, increasing the interference decreases the achievable DMT of the CF strategy.

Note that for the regimes where \( \alpha \leq 1 \) or \( 1 \leq \alpha \leq \beta - 1 \), increasing \( \alpha \) improves decoding the interference at the receivers and, hence, enhances the DMT performance. Similarly, it can be observed that the achievable DMT of CF is a non-decreasing function of \( \beta \), which represents the strength of the links from the relay and the destinations. Thus, better relay-to-destinations links improve the DMT performance for the ICR.
Remark 6. Consider the Gaussian ICR defined in Section II. The maximum achievable diversity gain with CF relaying is:

\[
D_{\text{CF}} = \begin{cases} 
1 + \min \{ (\beta - \alpha)^+, 1 \} & \alpha > 1 \\
1 + \min \{ (\beta - 1)^+, 1 \} & \alpha \leq 1.
\end{cases}
\]

Note that if \( \max \{1, \alpha\} \leq \beta - 1 \), we have \( D_{\text{CF}} = 2 \). In \cite{7} it is shown that if \( \beta = 1 \), the maximum achievable diversity gain of the single-relay channel is 2. Note that in the ICR the same diversity gain is achieved if \( \max \{2, \alpha + 1\} \leq \beta \). This is due to the fact that the achievable DMT in the ICR is affected by the interfering links, while in the single-relay channel there is no interference. Therefore, in order to achieve the same diversity gain, the relay-destination links in the ICR should be stronger than that in the single-relay channel.

Remark 7. The DMT of the IC without a relay is studied in \cite{9}, where it is shown that an outer bound on the achievable DMT of the IC is given by \( \min \{ (1 - r_1)^+, (1 - r_2)^+ \} \). The optimal DMT can be achieved in certain regimes, e.g., in the very strong interference regime, characterized by \( \alpha \geq 2 \). Note that in the scenario considered in Corollary \cite{1} the achievable DMT of ICR is twice the maximum achievable DMT for the IC. This observation gives a strong motivation for employing relay nodes in wireless networks. Additionally, from Theorem \cite{2} it follows that the DMT performance of the ICR is better than that of the IC also in the scenarios where the CF is not DMT optimal: for example, if \( \max \{1, \alpha\} \leq \beta - 1 \) and \( \min \{ (1 - r_1)^+, (1 - r_2)^+ \} \leq 3(1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ \leq \min \{ (1 - r_1)^+, (1 - r_2)^+ \} \). This follows from the fact that, under these conditions, the optimal DMT of the IC is \( d_{\text{Opt-IC}}(r_1, r_2) = \min \{ (1 - r_1)^+, (1 - r_2)^+ \} \), while for the ICR \( d_{\text{CF}}^{-}(r_1, r_2) = 3(1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ \), and hence, \( d_{\text{Opt-IC}}(r_1, r_2) \leq d_{\text{CF}}^{-}(r_1, r_2) \).

IV. DMT WITH ONLY RX-CSI AT THE RELAY

A. DMT Outer Bound

Proposition 1. For the ICR with only Rx-CSI at the relay, as defined in Section II, the region \( d^+(r_1, r_2) \) defined in Eqns. \cite{2} is an outer bound on the DMT region.

Proof: By inspecting the derivation of the outer bound in Thm. \cite{1} we observe that the lack of Tx-CSI at the relay does not affect the final set of expressions. \( \blacksquare \)

B. An Achievable DMT Region via DF

We now derive an achievable DMT for the ICR based on the DF strategy at the relay:

Theorem 3. Consider the ICR with only Rx-CSI at the relay, as defined in Section II. An achievable DMT region is given by

\[
d_{\text{DF}}^{-}(r_1, r_2) = \begin{cases} 
\min \{ d^E(r_1, r_2) + d^{\text{Relay}}(r_1, r_2) \}, & r_1 + r_2 < 1 \\
\min \{ d^E(r_1, r_2) \}, & r_1 + r_2 \geq 1,
\end{cases}
\] (9)
where
\[ d^{\text{Relay}} = \min \left\{ (1-r_1)^+, (1-r_2)^+, 2(1-r_1-r_2)^+ \right\} \]
\[ d^{\text{F}} (r_1, r_2) = \min \left\{ (1-r_1)^+, (1-r_2)^+, (1-r_1-r_2)^+ + (\alpha - r_1 - r_2)^+ \right\} \]
\[ d^{\text{F}} (r_1, r_2) = \min \left\{ (1-r_1)^+ + (\beta - r_1)^+, (1-r_2)^+ + (\beta - r_2)^+, (1-r_1-r_2)^+ + (\alpha - r_1 - r_2)^+ + (\beta - r_1 - r_2)^+ \right\}. \]

**Proof:** A detailed proof can be found in Appendix C. \(\blacksquare\)

C. Discussion

**Corollary 2.** Consider the Gaussian ICR as defined in Section II. If the following inequalities are satisfied:

\[ r_1 + r_2 \leq 1 \] (10a)
\[ \min \left\{ (1-r_1), (1-r_2) \right\} \leq \min \left\{ 2(1-r_1-r_2), (1-r_1-r_2) + (\alpha - r_1 - r_2)^+ \right\} \] (10b)
\[ \min \left\{ (1-r_1) + (\beta - r_1)^+, (1-r_2) + (\beta - r_2)^+ \right\} \leq (1-r_1-r_2) + (\alpha - r_1 - r_2)^+ + (\beta - r_1 - r_2)^+ \] (10c)

the optimal DMT for the Gaussian ICR is

\[ d^{\text{Opt-DF}} (r_1, r_2) = \min \left\{ 2(1-r_1)^+, 2(1-r_2)^+, (1-r_1)^+ + (\beta - r_1)^+, (1-r_2)^+ + (\beta - r_2)^+ \right\}, \] (11)

and it is achieved with DF at the relay.

**Proof:** The proof is based on Theorem 3. Note that if (10) is satisfied, (9) coincides with (1), characterizing the optimal DMT for the Gaussian ICR. \(\blacksquare\)

**Remark 8.** From Theorem 3 we observe that the achievable DMT of DF increases as \(\alpha\) and \(\beta\) increase, i.e., the DMT performance of the DF scheme improves as the relay-destination links and the interference become stronger. This is contrary to CF for which there are regimes of \(\alpha\) and \(\beta\) in which increasing the interference decreases the rate.

**Remark 9.** Recall the upper bound on the DMT of the relay channel given in (8). Setting \(r_1 = r_2 = r\), we note that under conditions (10), the optimal DMT (11) corresponds to the optimal DMT of two interference-free parallel relay channels, as each pair achieves a DMT of \(\min \left\{ 2(1-r)^+, (1-r)^+ + (\beta - r)^+ \right\}\), corresponding to the upper bound (8). We conclude that the DF strategy can also be DMT optimal for both communicating pairs simultaneously. Observe that optimality holds also for large multiplexing gains. Note that using CF this optimality was shown only for \(\beta \geq \max\{1, \alpha\} + 1\); but with DF it applies for any value of \(\beta \geq 0\) as long as (10) is satisfied. It is important to note that for DF and CF this optimality is achieved over different multiplexing gains.

**Remark 10.** The maximum diversity gain achieved by the DF scheme is \(D_{\text{DF}} = \min\{2, 1 + \beta\}\). Compared with the relay channel whose maximum diversity gain is 2, we conclude that the DF scheme achieves for each pair the
maximum diversity gain of the relay channel as long as $\beta \geq 1$. Observe that this diversity gain is obtained for both pairs simultaneously, using only a single relay.

**Remark 11.** When DF is optimal, its DMT outer bounds the optimal DMT of the IC (for the same set of multiplexing gains). Note that this conclusion holds for any value of $\beta > 0$. Moreover, there are scenarios in which the DF achievable DMT for the ICR outperforms the optimal DMT of the IC even when DF is not optimal. One such an example is when (11a), (11c) are satisfied, while (11b) is not satisfied. For example, if we set $r_1 = r_2 = 0.4$, $\alpha = 1.8$, and $\beta = 1$, for the ICR we achieve a diversity gain of 1, while for the IC the achievable diversity gain is upper-bounded by 0.6.

**Remark 12.** Fig. 2 demonstrates the achievable DMT of the CF and the DF strategies for a symmetric scenario with $r_1 = r_2 = r$. The achievable DMT of the CF strategy is presented for the scenario in which the conditions of Corollary 1 are satisfied, i.e., when CF is optimal. Observe that indeed the DMT of CF coincides with the outer bound on the DMT of Theorem 1. The achievable DMT of DF is presented for both the case where the conditions for Corollary 2 are satisfied ($r \leq \frac{1}{3}$), and the case in which DF is suboptimal ($r > \frac{1}{3}$). Note that for the values of $r > 0.5$ the outage event at the relay dominates the outage probability. For these values of $r$, we have $d_{DF}(r) = (1 - r)^+$, corresponding to the optimal DMT of the symmetric IC.

**V. Summary**

We have studied the DMT of single-antenna Gaussian ICRs. We derived an outer bound on the DMT of the ICR using the cut-set theorem, and two achievable DMT regions based on DF and CF. Additionally, we derived conditions on the channel coefficients to achieve optimal DMTs with CF and DF. We also identified situations in which the maximum diversity gain of the ICR obtained using either CF or DF, is equal to the diversity gain of two parallel interference-free relay channels. We showed that for both the CF and DF strategies, the achievable DMT improves as the relay-destination link improves. These results give a strong motivation for employing relay nodes in multi-user wireless networks that have to deal with interference.
APPENDIX A

PROOF OF THEOREM [1]

For the ICR, the cut-set theorem [12, Theorem 15.10.1] provides the following rate constraints:

\[
C_{\text{cut-set}} \triangleq \bigcup_{f(x_1, x_2, x_3)} \left\{ (R_1, R_2) \in \mathbb{R}^2 : \begin{align*}
R_1 &\leq I(X_1; Y_1, Y_3 | X_2, X_3) \\
R_1 &\leq I(X_1, X_3; Y_1 | X_2) \\
R_2 &\leq I(X_2; Y_2, Y_3 | X_1, X_3) \\
R_2 &\leq I(X_2, X_3; Y_2 | X_1) \\
R_1 + R_2 &\leq I(X_1, X_2, X_3; Y_1, Y_2) \\
R_1 + R_2 &\leq I(X_1, X_2, X_3; Y_1, Y_2 | X_3) \end{align*} \right\}
\]

(A.1)

where the joint distribution is chosen among all \( f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3 | x_1 x_2) \). Note that since Rx-CSI is available at the receivers, and since the channel coefficients are independent of the channel inputs and outputs, then we should add the corresponding channel coefficients in the conditioning of each expression in (A.1). For simplicity, however, these channel coefficients are omitted. The DMT corresponding to the cut-set bound \( C_{\text{cut-set}} \) provides an outer bound on the optimal DMT region for the ICR. This follows since the outage probability corresponding to the cut-set bound is a lower bound on the outage probability corresponding to the capacity region of the ICR \( C \):

\[
\Pr( (R_1, R_2) \notin C_{\text{cut-set}}(\tilde{H}) ) \leq \Pr( (R_1, R_2) \notin C(\tilde{H}) ),
\]

which follows since \( C \subseteq C_{\text{cut-set}} \). In order to characterize the DMT corresponding to (A.1), we follow similar steps as those used in [10]. Define \( |h_{kl}|^2 = \text{SNR}^{-\theta_{kl}}, k, l \in \{1, 2, 3\} \), and let \( R_{T,k} = r_k \log \text{SNR}, k \in \{1, 2\} \). Define \( R(\tilde{h}) = R(\tilde{\theta}) \) as the achievable rate region when the channel coefficients are \( h_{kl}, |h_{kl}|^2 = \text{SNR}^{-\theta_{kl}} \), and \( \tilde{\theta} = \{0 \leq \theta_{kl}, k, l \in \{1, 2, 3\}, (k, l) \neq (3, 3)\} \). Then, following the same arguments as in [10] we obtain

\[
P_C(r_1, r_2) \doteq \int_{\mathcal{D}(\tilde{\theta})} \text{SNR}^{-\sum_{l,k \in \{1,2,3\}} \theta_{kl}} d\tilde{\theta} = \int_{\mathcal{D}(\tilde{\theta})} \text{SNR}^{-|\tilde{\theta}|_1} d\tilde{\theta},
\]

where \( \mathcal{D}(\tilde{\theta}) \triangleq \{\theta_{kl} \geq 0, (r_1 \log \text{SNR}, r_2 \log \text{SNR}) \notin R(\tilde{\theta}), k, l \in \{1, 2, 3\}, (k, l) \neq (3, 3)\} \), and \( |\tilde{\theta}|_1 = \sum \theta_{kl}, k, l \in \{1, 2, 3\}, (k, l) \neq (3, 3) \) denotes the unit norm of the vector \( \tilde{\theta} \). Using the Laplace integration method, we obtain

\[
P_C(r_1, r_2) = \text{SNR}^{-\min_{\tilde{\theta} \in \mathcal{D}(\tilde{\theta})} |\tilde{\theta}|_1},
\]

and hence, the DMT is calculated as follows:

\[
d(r_1, r_2) = \min_{\tilde{\theta} \in \mathcal{D}(\tilde{\theta})} |\tilde{\theta}|_1.
\]
A. The DMT Corresponding to the Individual Rates

In this section we characterize the DMT outer bound corresponding to the individual rates, i.e., an outage occurs when the inequalities (A.1a)-(A.1d) are not satisfied. Consider first (A.1a). The outage probability corresponding to (A.1a) is defined as \( \text{Pr}(O^+_1) = \text{Pr}(I(X_1; Y_1, Y_3|X_2, X_3) < r_1 \log \text{SNR}) \). Note that similar to [5, Appendix A], \( I(X_1; Y_1, Y_3|X_2, X_3) \) can be upper bounded as follows:

\[
I(X_1; Y_1, Y_3|X_2, X_3) \leq \log \left(1 + \text{SNR}|h_{11}|^2 + \text{SNR}|h_{13}|^2\right) = \log \left(1 + \text{SNR}^{1-\theta_{11}} + \text{SNR}^{1-\theta_{13}}\right). \tag{A.2}
\]

Thus, \( \text{Pr}(O^+_1) \geq \text{Pr}(1 + \text{SNR}^{1-\theta_{11}} + \text{SNR}^{1-\theta_{13}} < \text{SNR}^{r_1}) \)

Hence, the DMT corresponding to (A.1a) can be obtained by solving the following optimization problem:

\[
\min_{0 \leq \theta_{11}, \theta_{13}} \theta_{11} + \theta_{13} \\
\text{s.t.} (1 - \theta_{11})^+ \leq r_1 \\
(1 - \theta_{13})^+ \leq r_1
\]

Let \( \theta^*_{11} \) and \( \theta^*_{13} \) denote the optimal solution to the above problem, then we obtain

\[
d^+_1(r_1) = \min_{0 \leq \theta_{11}, \theta_{13}} \theta_{11} + \theta_{13} \triangleq \theta^*_{11} + \theta^*_{13} = 2(1 - r_1)^+,
\]

which results in (2a). Next, consider (A.1b). Define \( \text{Pr}(O^+_2) = \text{Pr}(I(X_1, X_3; Y_1|X_2) < r_1 \log \text{SNR}) \). From [5, Eq. (A9)] we have

\[
I(X_1, X_3; Y_1|X_2) \leq \log \left(\text{SNR}|h_{11}|^2 + \text{SNR}^{\frac{1+\beta}{2}} h_{11}^* h_{31}^* + \text{SNR}^{\frac{1+\beta}{2}} h_{31} h_{11} + \text{SNR}^{\beta} |h_{31}|^2 + 1\right) \\
\leq \log \left(\text{SNR}|h_{11}|^2 + 2 \text{SNR}^{\frac{1+\beta}{2}} |h_{11}^* h_{31}^*| + \text{SNR}^{\beta} |h_{31}|^2 + 1\right) \\
\leq \log \left(\text{SNR}|h_{11}|^2 + 2 \text{SNR}^{\frac{1+\beta}{2}} |h_{11}||h_{31}| + \text{SNR}^{\beta} |h_{31}|^2 + 1\right) \\
= \log \left(\text{SNR}^{1-\theta_{11}} + 2 \text{SNR}^{\frac{1-\theta_{11} + \beta - \theta_{31}}{2}} + \text{SNR}^{\beta - \theta_{31}} + 1\right) \tag{A.3}
\]

Thus, \( \text{Pr}(O^+_2) \geq \text{Pr}(\text{SNR}^{1-\theta_{11}} + 2 \text{SNR}^{\frac{1-\theta_{11} + \beta - \theta_{31}}{2}} + \text{SNR}^{\beta - \theta_{31}} + 1 < \text{SNR}^{r_1}) \), hence, the DMT corresponding to (A.1b) can be obtained by solving the following optimization problem:

\[
\min_{0 \leq \theta_{11}, \theta_{31}} \theta_{11} + \theta_{31} \\
\text{s.t.} (1 - \theta_{11})^+ \leq r_1 \\
(\beta - \theta_{31})^+ \leq r_1,
\]

which can be solved to obtain the following DMT bound

\[
d^+_2(r_1) = (1 - r_1)^+ + (\beta - r_1)^+,
\]
which results in (2b). Following similar steps, we obtain the DMT upper bound corresponding to (A.1c) and (A.1d) as follows:

\[ d_3^+(r_2) = 2(1 - r_2)^+ \]
\[ d_4^+(r_2) = (1 - r_2)^+ + (\beta - r_2)^+ \]

resulting in (2c) and (2d), respectively.

**B. The DMT Corresponding to the Sum-Rate**

Next, we derive the outage probability corresponding to the case that (A.1e) and (A.1f) are not satisfied. First, we evaluate the right-hand side (r.h.s.) of (A.1e). Define \( C, H, X, \) and \( Z \) as

\[
C \triangleq \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_1^*, X_2^*, X_3^* \end{bmatrix}, \quad H \triangleq \begin{bmatrix} \sqrt{\text{SNR}} H_{11} & \sqrt{\alpha \text{SNR}} H_{21} & \sqrt{\beta \text{SNR}} H_{31} \\ \sqrt{\alpha \text{SNR}} H_{12} & \sqrt{\text{SNR}} H_{22} & \sqrt{\beta \text{SNR}} H_{32} \end{bmatrix}, \quad X \triangleq \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad Z \triangleq \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}
\]

Using the above definitions, we obtain

\[
I(X_1, X_2, X_3; Y_1, Y_2) = h(Y_1, Y_2) - h(Y_1, Y_2|X_1, X_2, X_3)
\]
\[
= h(Y_1, Y_2) - h(Z_1, Z_2)
\]
\[
= h(H \cdot X + Z) - h(Z)
\]
\[
\leq \log \left( |\text{cov}(H \cdot X + Z)| \right) - \log \left( |\text{cov}(Z)| \right)
\]
\[
\leq \log \left( |\text{cov}(H \cdot X) + \text{cov}(Z)| \right)
\]
\[
= \log \left( |I + H \cdot C \cdot H^H| \right)
\]

where (a) follows from [5, Lemma 2], and (b) follows from the fact that \( Z \) is independent of the channel inputs and the channel coefficients, and since \( |\text{cov}(Z)| = 1 \). Next, note that for a random channel matrix \( H \) we have [7, Eqn. (1)]:

\[
\sup_{0 \leq Q, \text{tr}(Q) \leq M} \log \left( |I + H \cdot Q \cdot H^H| \right) \leq \log \left( |I + M \cdot H \cdot H^H| \right). \tag{A.4}
\]
Hence, as $\mathbb{E}\{|X_k|^2\} \leq 1, k \in \{1, 2, 3\}$, we have that $\text{tr}\{C\} \leq 3$ which results in the following

$$
I(X_1, X_2, X_3; Y_1, Y_2) \leq \log \left( |\mathbf{I} + \mathbf{H} \cdot \mathbf{C} \cdot \mathbf{H}^H| \right)
$$

$$
\leq \log \left( |\mathbf{I} + 3 \cdot \mathbf{H} \cdot \mathbf{H}^H| \right)
$$

$$
= \log \left( 1 + 3(\text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2 + \text{SNR}^\beta|h_{31}|^2
+ \text{SNR}|h_{22}|^2 + \text{SNR}^\alpha|h_{12}|^2 + \text{SNR}^\beta|h_{32}|^2)
+ 9(\text{SNR}|h_{11}|^2|h_{22}|^2 + \text{SNR}^\alpha|h_{12}|^2|h_{21}|^2
+ \text{SNR}^\alpha|h_{11}|^2|h_{32}|^2 + \text{SNR}^\alpha|h_{12}|^2|h_{31}|^2
+ \text{SNR}^\alpha|h_{22}|^2|h_{31}|^2 + \text{SNR}^\alpha|h_{21}|^2|h_{32}|^2) < \text{SNR}^{r_1+r_2}. \right) \tag{A.5}
$$

Denote with $\mathcal{O}_5^+$ the event that (A.16) is not satisfied. Then, $\text{Pr}(\mathcal{O}_5^+)$ can be lower bounded by ignoring the negative terms in (A.5):

$$
\text{Pr}(\mathcal{O}_5^+)=\text{Pr} \left( I(X_1, X_2, X_3; Y_1, Y_2) < (r_1 + r_2) \log \text{SNR} \right)
$$

$$
\geq \text{Pr} \left( 1 + 3(\text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2 + \text{SNR}^\beta|h_{31}|^2
+ \text{SNR}|h_{22}|^2 + \text{SNR}^\alpha|h_{12}|^2 + \text{SNR}^\beta|h_{32}|^2)
+ 9(\text{SNR}|h_{11}|^2|h_{22}|^2 + \text{SNR}^\alpha|h_{12}|^2|h_{21}|^2
+ \text{SNR}^\alpha|h_{11}|^2|h_{32}|^2 + \text{SNR}^\alpha|h_{12}|^2|h_{31}|^2
+ \text{SNR}^\alpha|h_{22}|^2|h_{31}|^2 + \text{SNR}^\alpha|h_{21}|^2|h_{32}|^2) < \text{SNR}^{r_1+r_2} \right).
$$

Thus,

$$
\text{Pr}(\mathcal{O}_5^+)=\text{Pr} \left( I(X_1, X_2, X_3; Y_1, Y_2) < (r_1 + r_2) \log \text{SNR} \right)
$$

$$
\geq \text{Pr} \left( 1 + 3(\text{SNR}^{1-\theta_{11}} + \text{SNR}^{\alpha-\theta_{21}} + \text{SNR}^{\beta-\theta_{31}}
+ \text{SNR}^{1-\theta_{22}} + \text{SNR}^{\alpha-\theta_{12}} + \text{SNR}^{\beta-\theta_{32}})
+ 9(\text{SNR}^{2-\theta_{11}-\theta_{22}} + \text{SNR}^{\alpha-\theta_{12}-\theta_{21}}
+ \text{SNR}^{1+\beta-\theta_{11}-\theta_{32}} + \text{SNR}^{\alpha+\beta-\theta_{12}-\theta_{31}}
+ \text{SNR}^{1+\beta-\theta_{22}-\theta_{31}} + \text{SNR}^{\alpha+\beta-\theta_{21}-\theta_{32}}) < \text{SNR}^{r_1+r_2} \right),
$$

The DMT upper bound corresponding to $\text{Pr}(\mathcal{O}_5^+)$ can be obtained by solving the following optimization problem:

$$
\min_{\theta_{11} \geq 0, k \in \{1, 2, 3\}, \ell \in \{1, 2\}} \theta_{11} + \theta_{21} + \theta_{31} + \theta_{12} + \theta_{22} + \theta_{32}
$$

s.t. \max \left\{ (1 - \theta_{11})^+, (\alpha - \theta_{21})^+, (\beta - \theta_{31})^+, (1 - \theta_{22})^+, (\alpha - \theta_{12})^+, (\beta - \theta_{32})^+ \right\} < r_1 + r_2 \tag{A.6a}

max \left\{ (2 - \theta_{11} - \theta_{22})^+, (\alpha + \beta - \theta_{12} - \theta_{31})^+, (\alpha + \beta - \theta_{21} - \theta_{32})^+ \right\} < r_1 + r_2 \tag{A.6b}

max \left\{ (2\alpha - \theta_{21} - \theta_{12})^+, (1 + \beta - \theta_{11} - \theta_{32})^+, (1 + \beta - \theta_{22} - \theta_{31})^+ \right\} < r_1 + r_2. \tag{A.6c}
The following three bounds on the objective function can be obtained by considering (A.6a), (A.6b), and (A.6c), respectively:

\[
\begin{align*}
\theta_{11} + \theta_{21} + \theta_{31} + \theta_{12} + \theta_{22} + \theta_{32} &\geq 2(1 - r_1 - r_2)^+ + 2(\alpha - r_1 - r_2)^+ + 2(\beta - r_1 - r_2)^+ \\
\theta_{11} + \theta_{21} + \theta_{31} + \theta_{12} + \theta_{22} + \theta_{32} &\geq (2 - r_1 - r_2)^+ + 2(\alpha - r_1 - r_2)^+ + 2(1 - \beta - r_1 - r_2)^+ \\
\theta_{11} + \theta_{21} + \theta_{31} + \theta_{12} + \theta_{22} + \theta_{32} &\geq (2\alpha - r_1 - r_2)^+ + 2(1 - \beta - r_1 - r_2)^+,
\end{align*}
\]

leading to \( d_5^+(r_1, r_2), d_6^+(r_1, r_2) \) and \( d_7^+(r_1, r_2) \) in (2e), (2f), and (2g), respectively. Next, we focus on the outage probability corresponding to (A.11). Define \( T, H, X, \) and \( Z \) as

\[
T \triangleq \mathbb{E} \left\{ \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \mid X_1^*, X_2^* \right\}, \quad H \triangleq \begin{bmatrix} \sqrt{\text{SNR}}H_{11} & \sqrt{\text{SNR}}H_{12} \\ \sqrt{\text{SNR}}H_{13} & \sqrt{\text{SNR}}H_{21} \end{bmatrix}, \quad X \triangleq \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad Z \triangleq \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}
\]

Using the above definitions, we obtain

\[
I(X_1, X_2; Y_1, Y_2, Y_3|X_3) = h(Y_1, Y_2, Y_3|X_3) - h(Y_1, Y_2, Y_3|X_1, X_2, X_3)
\]

\[\leq \log \left( |I + H \cdot T \cdot H^T| \right)\]  

\[\leq \log \left( |I + 2 \cdot H \cdot H^T| \right)\]

\[= \log \left( 1 + 2(\text{SNR}h_{11})^2 + \text{SNR}^\alpha h_{12}^2 + \text{SNR}h_{13}^2 \\
+ \text{SNR}h_{22}^2 + \text{SNR}^\alpha h_{21}^2 + \text{SNR}h_{23}^2 \\
+ 4\text{SNR}h_{11}h_{22} - \text{SNR}^\alpha h_{12}h_{21} \\
+ 4\text{SNR}^\frac{1-\alpha}{2} h_{12}h_{23} - \text{SNR}h_{22}h_{23}^2 \\
+ 4\text{SNR}^\frac{1-\alpha}{2} h_{13}h_{21} - \text{SNR}h_{11}h_{23}^2 \right)\]

where (a) follows from [5] Eq. (A14), and (b) follows from (A.4). Hence, the following lower bound on \( \text{Pr}(O_0^+) \) is obtained:

\[
\text{Pr}(O_0^+) = \text{Pr} \left( I(X_1, X_2; Y_1, Y_2, Y_3|X_3) < (r_1 + r_2) \log \text{SNR} \right)
\]

\[
\geq \text{Pr} \left( 1 + 2(\text{SNR}h_{11})^2 + \text{SNR}^\alpha h_{12}^2 + \text{SNR}h_{13}^2 \\
+ \text{SNR}h_{22}^2 + \text{SNR}^\alpha h_{21}^2 + \text{SNR}h_{23}^2 \\
+ 4(\text{SNR}^2h_{11}^2h_{22}^2 + \text{SNR}^2h_{12}^2h_{21}^2 \\
+ \text{SNR}^{1+\alpha}h_{12}h_{23}^2 + \text{SNR}^2h_{22}h_{23}^2 \\
+ \text{SNR}^{1+\alpha}h_{13}^2h_{21}^2 + \text{SNR}^2h_{11}h_{23}^2 \right) < \text{SNR}^{r_1+r_2} \right).
\]
The DMT upper bound corresponding to \( \Pr(O_+^b) \) can be obtained by solving the following optimization problem:

\[
\begin{align*}
\min_{0 \leq \theta_i \leq 1} & \quad \theta_{11} + \theta_{21} + \theta_{13} + \theta_{22} + \theta_{12} + \theta_{23} \\
{\text{s.t.}} & \quad \max \left\{ (1 - \theta_{11})^+, (\alpha - \theta_{12})^+, (1 - \theta_{13})^+, (\alpha - \theta_{21})^+, (1 - \theta_{23})^+ \right\} < r_1 + r_2 \quad \text{(A.7a)} \\
& \quad \max \left\{ (2\alpha - \theta_{12} - \theta_{21})^+, (2 - \theta_{11} - \theta_{23})^+, (2 - \theta_{22} - \theta_{13})^+ \right\} < r_1 + r_2 \quad \text{(A.7b)} \\
& \quad \max \left\{ (2 - \theta_{11} - \theta_{22})^+, (1 + \alpha - \theta_{13} - \theta_{21})^+, (1 + \alpha - \theta_{12} - \theta_{23})^+ \right\} < r_1 + r_2 \quad \text{(A.7c)}
\end{align*}
\]

The following two bounds on the objective function can be obtained by considering (A.7a) and (A.7b), respectively:

\[
\begin{align*}
\theta_{11} + \theta_{21} + \theta_{13} + \theta_{22} + \theta_{12} + \theta_{23} & \geq 4(1 - r_1 - r_2)^+ + 2(\alpha - r_1 - r_2)^+ \\
\theta_{11} + \theta_{21} + \theta_{13} + \theta_{22} + \theta_{12} + \theta_{23} & \geq 2(2 - r_1 - r_2)^+ + (2\alpha - r_1 - r_2)^+ \quad (\text{B.1a}) \\
\theta_{11} + \theta_{21} + \theta_{13} + \theta_{22} + \theta_{12} + \theta_{23} & \geq (2 - r_1 - r_2)^+ + 2(1 + \alpha - r_1 - r_2)^+, \quad (\text{B.1b})
\end{align*}
\]

leading to \( d_k^+(r_1, r_2) \), \( d_3^+(r_1, r_2) \) and \( d_{10}^+(r_1, r_2) \) in (2h), (2i) and (2j), respectively. By combining all the DMT upper bounds associated with (A.7a) to (A.7b), we obtain the upper bound on the DMT in (1). This completes the proof.

**APPENDIX B**

**PROOF OF THEOREM 2**

The ICR with CF strategy at the relay node is studied in [2]. The achievable rate region for the scenario in which the transmitters send common messages only is given by [2] Theorem 1:

\[
\begin{align*}
R_1 & \leq I(X_1; Y_1, \tilde{Y}_3|X_2, X_3) \quad \text{(B.1a)} \\
R_2 & \leq I(X_2; Y_3|X_1, X_3) \quad \text{(B.1b)} \\
R_1 + R_2 & \leq I(X_1, X_2; Y_1, \tilde{Y}_3|X_3) \quad \text{(B.1c)} \\
R_1 + R_2 & \leq I(X_1, X_2; Y_2, \tilde{Y}_3|X_3) \quad \text{(B.1d)}
\end{align*}
\]

subject to the constraints:

\[
\begin{align*}
I(X_3; Y_1) & \geq I(Y_3; \tilde{Y}_3|X_3, Y_1) \quad \text{(B.2a)} \\
I(X_3; Y_2) & \geq I(Y_3; \tilde{Y}_3|X_3, Y_2) \quad \text{(B.2b)}
\end{align*}
\]

where \( f(x_1)f(x_2)f(x_3)f(y_1, y_2, y_3|x_1, x_2, x_3)f(\tilde{y}_3|y_3, x_3), X_k \sim \mathcal{CN}(0, 1), k \in \{1, 2, 3\}, \) and \( \tilde{Y}_3 = Y_3 + Z_Q, Z_Q \sim \mathcal{CN}(0, N_Q). \) Using the relationships \( I(X_3; Y_k) = h(Y_k) - h(Y_k|X_3) \) and \( I(Y_3; \tilde{Y}_3|X_3, Y_k) = h(Y_k, \tilde{Y}_3|X_3) - h(Y_k|X_3) - \log((\pi e)N_Q), k \in \{1, 2\}, \) we can rewrite the two constraints in (B.2a) and (B.2b) as:

\[
h(Y_k) - h(Y_k|X_3) \geq h(Y_k, \tilde{Y}_3|X_3) - h(Y_k|X_3) - \log((\pi e)N_Q), k \in \{1, 2\}. \quad \text{(B.3)}
\]

Equivalently,

\[
\log((\pi e)N_Q) \geq h(Y_k, \tilde{Y}_3|X_3) - h(Y_k), k \in \{1, 2\} \quad \text{(B.4)}
\]
Hence, we obtain that
\[
\log ((\pi e)N_Q) \geq h(Y_1, \hat{Y}_3|X_3) - h(Y_1) \tag{B.5a}
\]
\[
\log ((\pi e)N_Q) \geq h(Y_2, \hat{Y}_3|X_3) - h(Y_2). \tag{B.5b}
\]

Next, we evaluate the required exponential behavior of \(N_Q\) s.t. \(B.5\) is satisfied. We start with \(B.5a\). Note that as the channel inputs are generated according to the mutually independent complex Normal distribution, then we have
\[
h(Y_1) = \log \left( (\pi e)(1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2 + \text{SNR}^\beta|h_{31}|^2) \right). \tag{B.6}
\]

Define
\[
H \triangleq \begin{bmatrix} \sqrt{\text{SNR}h_{11}} & \sqrt{\text{SNR}^\alpha h_{21}} \\ \sqrt{\text{SNR}h_{13}} & \sqrt{\text{SNR}h_{23}} \end{bmatrix}, \quad X \triangleq \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad Z \triangleq \begin{bmatrix} Z_1 \\ Z_3 + Z_Q \end{bmatrix}
\]

Due Rx-CSI at the receivers, we obtain
\[
h(Y_1, \hat{Y}_3|X_3) = h(\sqrt{\text{SNR}}h_{11}X_1 + \sqrt{\text{SNR}^\alpha}h_{21}X_2 + Z_1, \sqrt{\text{SNR}}h_{13}X_1 + \sqrt{\text{SNR}}h_{23}X_2 + Z_3 + Z_Q) = h(H \cdot X + Z)
\]
\[
= \log \left( (\pi e)^2 |\text{cov}(H \cdot X + Z)| \right)
\]
\[
= \log \left( (\pi e)^2 |H \cdot \text{cov}(X) \cdot H^H + \text{cov}(Z)| \right)
\]
\[
= \log \left( (\pi e)^2 ((1 + N_Q)(1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2)
\]
\[
+ \text{SNR}|h_{13}|^2 + \text{SNR}|h_{23}|^2 + |\text{SNR}^{1+\alpha}h_{13}h_{21} - \text{SNR}h_{11}h_{21}|^2) \right)
\]
\[
\leq \log \left( (\pi e)^2 ((1 + N_Q)(1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2)
\]
\[
+ \text{SNR}|h_{13}|^2 + \text{SNR}|h_{23}|^2 + \text{SNR}^{1+\alpha}|h_{13}|^2|h_{21}|^2 + \text{SNR}^2|h_{11}|^2|h_{23}|^2) \right). \tag{B.7}
\]

Combining \(B.6\) and \(B.7\) we conclude that \(B.5a\) is satisfied if
\[
N_Q \geq \left( \frac{1 + N_Q(1 + \text{SNR}^{1-\theta_{11}} + \text{SNR}^\alpha-\theta_{21}) + \text{SNR}^{1-\theta_{13}} + \text{SNR}^{1-\theta_{23}} + \text{SNR}^{1+\alpha-\theta_{13}-\theta_{21}} + \text{SNR}^{2-\theta_{11}-\theta_{23}}}{1 + \text{SNR}^{1-\theta_{13}} + \text{SNR}^\alpha-\theta_{21} + \text{SNR}^{\beta-\theta_{31}}} \right),
\]
equivalently,
\[
N_Q \geq \frac{1 + \text{SNR}^{1-\theta_{11}} + \text{SNR}^\alpha-\theta_{21} + \text{SNR}^{1-\theta_{13}} + \text{SNR}^{1-\theta_{23}} + \text{SNR}^{1+\alpha-\theta_{13}-\theta_{21}} + \text{SNR}^{2-\theta_{11}-\theta_{23}}}{\text{SNR}^{\beta-\theta_{31}}}
\]

For high SNR regime, this is satisfied if
\[
N_Q \geq \frac{\text{SNR}^{1+\alpha-\theta_{13}-\theta_{21}} + \text{SNR}^{2-\theta_{11}-\theta_{23}}}{\text{SNR}^{\beta-\theta_{31}}}
\]

Since \(0 \leq \theta_{kl}, k, l \in \{1, 2, 3\}\), the above inequality is guaranteed if
\[
N_Q \geq \max\{\text{SNR}^{1+\alpha-\beta+\theta_{31}}, \text{SNR}^{2-\beta+\theta_{31}}\}
\]

Following the same steps for evaluating \(B.5b\), we conclude that
\[
N_Q \geq \max\{\text{SNR}^{1+\alpha-\beta+\theta_{31}}, \text{SNR}^{2-\beta+\theta_{31}}, \text{SNR}^{1+\alpha-\beta+\theta_{32}}, \text{SNR}^{2-\beta+\theta_{32}}\},
\]
Denote $O^{CF}_k$ as the event that inequality $k$ in (B.1a)-(B.1d) is not satisfied. We first calculate $\Pr(O^{CF}_k)$ as follows:

$$
\Pr(O^{CF}_k) = \Pr \left( I(X_1; Y_1, \hat{Y}_3 | X_2, X_3) < r_1 \log \text{SNR} \right)
$$

$$
= \Pr \left( h(Y_1, \hat{Y}_3 | X_2, X_3) - h(Z_1, Z_3 + Z_Q) < r_1 \log \text{SNR} \right)
$$

$$
= \Pr \left( h(\sqrt{\text{SNR}h_{11}}X_1 + Z_1, \sqrt{\text{SNR}h_{13}}X_1 + Z_3 + Z_Q) - \log ((\pi e)^2(1 + N_Q)) < r_1 \log \text{SNR} \right)
$$

Define

$$
H \triangleq \begin{bmatrix} \sqrt{\text{SNR}h_{11}} \\ \sqrt{\text{SNR}h_{13}} \end{bmatrix}, Z \triangleq \begin{bmatrix} Z_1 \\ Z_3 + Z_Q \end{bmatrix}.
$$

Then, we obtain that

$$
\Pr(O^{CF}_1) = \Pr \left( I(X_1; Y_1, \hat{Y}_3 | X_2, X_3) < r_1 \log \text{SNR} \right)
$$

$$
= \Pr \left( h(H \cdot X + Z) - \log ((\pi e)^2(1 + N_Q)) < r_1 \log \text{SNR} \right)
$$

$$
= \Pr \left( \log \left( (\pi e)^2 |HH^H + \text{cov}(Z)| \right) - \log ((\pi e)^2(1 + N_Q)) < r_1 \log \text{SNR} \right)
$$

$$
= \Pr \left( 1 + \text{SNR}|h_{11}|^2 + \frac{\text{SNR}|h_{13}|^2}{1 + N_Q} < \text{SNR}^{r_1} \right).
$$

Note that $1 \leq 1 + N_Q$. Thus,

$$
1 + N_Q = \max \{\text{SNR}^{1+\alpha-\beta+\theta_{31}}, \text{SNR}^{2-\beta+\theta_{31}}, \text{SNR}^{1+\alpha-\beta+\theta_{32}}, \text{SNR}^{2-\beta+\theta_{32}}, \text{SNR}^{0} \}
$$

$$
= \max \{\text{SNR}^{(1+\alpha-\beta+\theta_{31})^+}, \text{SNR}^{(2-\beta+\theta_{31})^+}, \text{SNR}^{(1+\alpha-\beta+\theta_{32})^+}, \text{SNR}^{(2-\beta+\theta_{32})^+} \},
$$

Hence, the DMT corresponding to the event $O^{CF}_1$ can be calculated by solving the following optimization problem:

$$
\min_{0 \leq \theta_{11}, \theta_{13}, \theta_{31}, \theta_{32}} \theta_{11} + \theta_{13} + \theta_{31} + \theta_{32}
$$

$$
\text{s.t.} \left( 1 - \theta_{11} \right)^+ \leq r_1,
$$

$$
\left( 1 - \theta_{13} - \max \left\{ \left( 1 + \alpha - \beta + \theta_{31} \right)^+ + \left( 2 - \beta + \theta_{31} \right)^+, \left( 1 + \alpha - \beta + \theta_{32} \right)^+ + \left( 2 - \beta + \theta_{32} \right)^+ \right\} \right)^+ \leq r_1.
$$

Observe that as $0 \leq \theta_{31}$, and $0 \leq \theta_{32}$, and since there are no other constraints on $\theta_{31}$ and $\theta_{32}$ in the optimization problem above, then the minimum value of the objective function is achieved by $\theta_{31} = \theta_{32} = 0$:

$$
d_{11,CF} = \begin{cases} 
(1 - r_1)^+ + (1 - (1 + \alpha - \beta)^+ - r_1)^+ & \alpha > 1 \\
(1 - r_1)^+ + (1 - (2 - \beta)^+ - r_1)^+ & \alpha \leq 1,
\end{cases}
$$

(B.8)

Similarly, we obtain the achievable DMT corresponding to $R_2$, stated in (45).

We next focus on the outage probability for the sum-rate, i.e., $\Pr(O^{CF}_5)$. Define

$$
H \triangleq \begin{bmatrix} \sqrt{\text{SNR}h_{11}} & \sqrt{\text{SNR}h_{21}} \\ \sqrt{\text{SNR}h_{13}} & \sqrt{\text{SNR}h_{23}} \end{bmatrix}, X \triangleq \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, Z \triangleq \begin{bmatrix} Z_1 \\ Z_3 + Z_Q \end{bmatrix}.
$$
Thus, we obtain
\[
\Pr(O_5^{CF}) = \Pr\left( I(X_1, X_2; Y_3|X_3) < (r_1 + r_2) \log \text{SNR} \right) = \Pr\left( h(EH \cdot X + Z) - \log ((\pi e)^2(1 + N_0)) < (r_1 + r_2) \log \text{SNR} \right) = \Pr\left( \log \left( (\pi e)^2 |HH^H + \text{cov}(Z)| \right) - \log ((\pi e)^2(1 + N_0)) < (r_1 + r_2) \log \text{SNR} \right)
\]
\[
= \Pr\left( 1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2 + \frac{\text{SNR}|h_{13}|^2 + \text{SNR}|h_{23}|^2}{1 + N_0^*} + \frac{\text{SNR}|h_{13}|^2 + \text{SNR}|h_{23}|^2}{1 + N_0^*} < \text{SNR}^r_1 + r_2 \right)
\]
\[
\leq \Pr\left( 1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2 + \frac{\text{SNR}|h_{13}|^2 + \text{SNR}|h_{23}|^2}{1 + N_0^*} < \text{SNR}^r_1 + r_2 \right).
\]

Thus, a lower bound on the DMT can be obtained by considering the following optimization problem:
\[
\min_{0 \leq \theta_{1k,l}, \theta_{2k,l}, \theta_{3k,l} \in \{1, 2, 3\}} (1 - \theta_{11})^+ \leq r_1 + r_2
\]
\[
(\alpha - \theta_{12})^+ \leq r_1 + r_2
\]
\[
(1 - \theta_{13} - \max \left\{ (1 + \alpha - \beta + \theta_{31})^+, (2 - \beta + \theta_{31})^+, (1 + \alpha - \beta + \theta_{32})^+, (2 - \beta + \theta_{32})^+ \right\})^+ \leq r_1 + r_2
\]
\[
(1 - \theta_{23} - \max \left\{ (1 + \alpha - \beta + \theta_{31})^+, (2 - \beta + \theta_{31})^+, (1 + \alpha - \beta + \theta_{32})^+, (2 - \beta + \theta_{32})^+ \right\})^+ \leq r_1 + r_2,
\]

Similar to the previous case, the optimal solution can be found by setting \( \theta_{31} = \theta_{32} = 0 \). This solution is stated as \( d_{3,CF}^-(r_1, r_2) \) in (4c). The same result can also be obtained by bounding \( \Pr(O_5^{CF}) \). This completes the proof.

APPENDIX C

PROOF OF THEOREM 3

The proposed transmission scheme is based on employing DF at the relay and using codebooks generated according to mutually independent zero-mean complex Normal channel inputs. Let \( \mathcal{E} \) denote the outage event at the relay, i.e., the event that the relay fails to decode. Then, the outage probability can be evaluated as follows:
\[
\Pr(\text{outage}) = \Pr(\text{outage}|\mathcal{E}) \Pr(\mathcal{E}) + \Pr(\text{outage}|\overline{\mathcal{E}}) \Pr(\overline{\mathcal{E}}), \tag{C.1}
\]

Similarly to (4), an achievable rate region for decoding at the relay is given by (recall \( \gamma = 1 \))
\[
R_1 \leq I(X_1; X_2, X_3) = \log(1 + \text{SNR}|h_{13}|^2)
\]
\[
R_2 \leq I(X_2; X_3|X_1, X_3) = \log(1 + \text{SNR}|h_{23}|^2)
\]
\[
R_1 + R_2 \leq I(X_1, X_2; Y_3|X_3) = \log(1 + \text{SNR}|h_{13}|^2 + \text{SNR}|h_{23}|^2).
\]

The probability of outage at the relay, corresponds to the event that at least one of the above inequalities is not satisfied. Using similar techniques as in Appendix 3, this probability can be evaluated using union bound as:
\[
\Pr(\mathcal{E}) \leq \text{SNR}^{- \min\{ (1-r_1)^+, (1-r_2)^+, 2(1-r_1-r_2)^+ \}} \triangleq \text{SNR}^{-d_{\text{relay}}(r_1, r_2)}
\]
Hence, following similar steps as in [2] Appendix II, it can be shown that at asymptotically large SNR

\[
\Pr(\mathcal{E}) = \begin{cases} 
\frac{\text{SNR}^{-\min\{(1-r_1)^+, (1-r_2)^+, 2(1-r_1 - r_2)^+\}}}{1} & r_1 + r_2 < 1 \\
1 & r_1 + r_2 \geq 1,
\end{cases}
\]  
(C.2a)

\[
\Pr(\mathcal{E}) = 1 - \Pr(\mathcal{E}) = \begin{cases} 
1 & r_1 + r_2 < 1 \\
0 & r_1 + r_2 \geq 1.
\end{cases}
\]  
(C.2b)

First, consider the case where the relay fails to decode. In this case, the ICR behaves as an IC (recall that receivers have Rx-CSI). Hence, an achievable rate region for the ICR in this scenario can be given as the intersection of the capacity regions of two multiple-access channels (MACs) derived from the IC. This conclusion follows from the fact that in each MAC, the receiver decodes both messages. Hence, the intersection of the capacity regions of two MACs is an inner bound to the capacity region of the IC. This rate region is given by

\[
R_1 \leq I(X_1; Y_1 | X_2) = \log (1 + \text{SNR}|h_{11}|^2) \\
R_2 \leq I(X_2; Y_2 | X_1) = \log (1 + \text{SNR}|h_{22}|^2) \\
R_1 + R_2 \leq I(X_1, X_2; Y_1) = \log (1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2) \\
R_1 + R_2 \leq I(X_1, X_2; Y_2) = \log (1 + \text{SNR}^\alpha|h_{12}|^2 + \text{SNR}|h_{22}|^2).
\]

Denote the target rates \( R_1 = r_1 \log \text{SNR} \) and \( R_2 = r_2 \log \text{SNR} \). An outage event occurs if at least one the inequalities above is not satisfied. Hence, for the scenario where the relay fails to decode we have

\[
\Pr(\text{outage}|\mathcal{E}) = \text{SNR}^{-d^\mathcal{E}(r_1, r_2)},
\]  
(C.3)

where

\[
d^\mathcal{E}(r_1, r_2) = \min \{(1 - r_1)^+, (1 - r_2)^+, (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+\}.
\]

Next, we consider on the case where the relay decodes both messages successfully. Using codebooks similar to those used in [4], an achievable rate region of the ICR is obtained as the intersection of the achievable regions of two multiple-access relay channels (MARCs). This rate region is characterized in [4]:

\[
R_1 \leq I(X_1, X_3; Y_1 | X_2) = \log (1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\beta|h_{31}|^2) \\
R_2 \leq I(X_2, X_3; Y_2 | X_1) = \log (1 + \text{SNR}|h_{22}|^2 + \text{SNR}^\beta|h_{32}|^2) \\
R_1 + R_2 \leq I(X_1, X_2, X_3; Y_1) = \log (1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha|h_{21}|^2 + \text{SNR}^\beta|h_{31}|^2) \\
R_1 + R_2 \leq I(X_1, X_2, X_3; Y_2) = \log (1 + \text{SNR}|h_{22}|^2 + \text{SNR}^\alpha|h_{12}|^2 + \text{SNR}^\beta|h_{32}|^2).
\]

Hence, for the scenario where the relay decodes both messages successfully, we have

\[
\Pr(\text{outage}|\bar{\mathcal{E}}) = \text{SNR}^{-d^\mathcal{E}(r_1, r_2)},
\]  
(C.4)
where
\[ d_{\bar{E}}(r_1, r_2) = \min \left\{ (1 - r_1)^+ + (\beta - r_1)^+, (1 - r_2)^+ + (\beta - r_2)^+, (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ + (\beta - r_1 - r_2)^+ \right\}. \]

Finally, by substituting (C.2), (C.3), and (C.4) in (C.1), we obtain
\[
\Pr(\text{outage}) = \Pr(\text{outage}|E) \Pr(E) + \Pr(\text{outage}|\bar{E}) \Pr(\bar{E})
\]
\[
\equiv \begin{cases} 
\frac{\text{SNR}^{-d_{\bar{E}}(r_1, r_2)} \text{SNR}^{-d_{\text{relay}}}}{\text{SNR}^{-d_{\bar{E}}(r_1, r_2)}} & r_1 + r_2 < 1 \\
\text{SNR}^{-d_{\bar{E}}(r_1, r_2)} & r_1 + r_2 \geq 1,
\end{cases}
\]
This completes the proof.

REFERENCES