Waveguide and Cavity Oscillations in the Presence of Nonlinear Media

DAN CENSOR

Abstract—This paper deals with the problem of waves in metallic structures containing nonlinear media. Problems of this kind are encountered in the analysis of microwave devices operated at high power levels, or when the constitutive parameters of nonlinear materials are investigated by means of microwave measurements.

The Volterra series are the functional analog of the well-known Taylor series for functions. This mathematical tool is adequate for a description of constitutive relations in dispersive nonlinear media. For practical purposes, we deal with weak nonlinearity, such that the series can be truncated. Weak nonlinearity also denotes the absence of shock waves, such that all spectral components of a wave are phase matched (i.e., propagate with the same phase velocity). The main effect of nonlinearity are the production of harmonics, and the dependence of the dispersion equation on the field amplitudes. These are incorporated into the present model.

The development of the present model involves some heuristic assumptions which facilitate the derivation of an algebraic dispersion equation. Therefore, the range of validity of the present model will have to be determined by experimental results, when these are available.

In waveguides, and cavities in particular, the question of the effect of the geometry and boundary conditions arises, too. It is shown here that nonlinearity induces harmonic modes in rectangular structures. In cylindrical and spherical structures, the geometry affects the budget of harmonics and produces mode coupling.

II. GENERAL THEORY OF SELF-INTERACTION

The problem is considered in the frame of electrodynamics in sourceless domains, governed by Maxwell's equations

$$\nabla \times E + \partial B/\partial t = 0 \quad \nabla \times H - \partial D/\partial t = 0$$

$$\nabla \cdot D = 0 \quad \nabla \cdot B = 0.$$  (1)
For reasons explained below, periodic (as opposed to harmonic) plane-wave solutions are considered

\[ E_i = \sum_{q=\infty}^{\infty} E_{q} e^{iq\theta}, \quad \theta = k \cdot x - \omega t \]  

and similar expression for \( B_i, H_i, D_i \), where \( i = 1, 2, 3 \), denotes Cartesian components, \( q \) is the order of the harmonic, \( \theta \) is the phase of the plane wave, \( E_{q} \) the amplitude of the \( i \) component of the \( q \)th harmonic, \( k \) is the propagation vector, and \( x \) the position vector. Substitution of (2) etc. in (1) yields

\[ k \times E_q - \omega B_q = 0 \quad k \times H_q + \omega D_q = 0 \]  

and \( \nabla \cdot D_q = 0, \nabla \cdot B_q = 0 \) are identically satisfied. Before going on, a few observations are in order. The periodic solution, as in (2), has been stipulated in order to include harmonic production due to distortion by the nonlinear medium. The amplitudes \( E_{q}, D_{q}, H_{q}, B_{q} \) are not arbitrary but determined by (3) and the complicated constitutive relations to be introduced below. It is noted that solutions of the kind in (2) already assume phase matching, i.e., all harmonics have identical phase velocities \( \omega / |k| \). We are dealing with weakly nonlinear media, in which the creation of shock waves is excluded. This is adequately described by (2), because shock formation requires that different spectral components propagate with different phase velocities. Phase matching also means that harmonics are produced in a coherent manner, i.e., local nonlinear interactions, although they might be weak, produce waves which interfere constructively, producing significant amplitudes of harmonic waves.

In general, nonlinear constitutive relations are given by \( D(E), B(H) \), but for weak nonlinearity, a hierarchy is assumed

\[ D = D^{(1)} + D^{(2)} + \ldots + D^{(n)} + \ldots \]  

and similarly for \( B \). In (4), it is assumed that the leading terms are predominant, and for practical purposes the sum (4) can be truncated. The term \( D^{(n)} \) in (4) is defined as the \( n \)th term of a Volterra series

\[ D^{(n)}(x, t) = \int d^3x_1 dt_1 \cdots \int d^3x_n dt_n \left[ \epsilon^{(n)}_{i,j,r,s}(x_1, t_1, \ldots, x_n, t_n) E_i(x - x_n, t - t_n) \right] \]  

where indices \( i, j, \ldots, r \) denote Cartesian components and \( \epsilon^{(n)}_{i,j,r,s} \) is a tensor (Einstein’s summation convention is assumed). Thus, for \( n = 1 \), we obtain the linear case, which on substitution of (2) yields

\[ D^{(1)}(qk, q\omega) = \epsilon^{(1)}_{i,j}(qk, q\omega) E_j(qk, q\omega) \]  

for the \( q \)th harmonic, where \( \epsilon^{(1)}_{i,j}(qk, q\omega) \) is the four-dimensional transform according to

\[ (2\pi)^{-4} \int d^3x_1 dt_1 \epsilon^{(1)}_{i,j}(x_1, t_1) e^{-iq\theta(x_1 - t_1)} \]  

between infinite limits of integration. The parameters \( \epsilon^{(n)}_{i,j,r,s} \) of (5) are constants characterizing the system. The first nonlinear term \( D^{(2)} \) upon substitution of (2) yields

\[ D^{(2)}(x, t) = \sum_{q,q'} e^{-i(q+q')\theta(x,t)} \epsilon^{(2)}_{i,j,k}(qk, q\omega, q'k, q'\omega) E_{qj} E_{q'k} \]  

and \( D^{(n)}(x, t) \) involves \( n \) summations [6]. If periodic \( D^{(2)} = \sum_{q} \gamma_{q} e^{-i\gamma \theta} \) is stipulated and substituted in (8), and terms satisfying \( \gamma = q + q' \) are regrouped, then we have

\[ D^{(2)}_{ij} = \epsilon^{(2)}_{ij,k}(\gamma k, \gamma \omega) E_{ij} E_{kk} \]  

where we define

\[ \epsilon^{(2)}_{ij,k}(\gamma k, \gamma \omega) = \sum_{q,q'} \epsilon^{(1)}_{ij,q} E_{qj} E_{q'^k}. \]  

Originally \( \epsilon^{(n)} \) (in (5) hence, also the \( 4 \times n \)-dimensional transform in (8)) are defined as parameters of the medium, independent of field amplitudes. Consequently, (10) etc. display \( \epsilon^{(n)} \) as amplitude dependent; hence, the notation (9) seems to be useless. What is the point of defining characteristic parameters if it turns out that they depend on the variable fields? What we are trying to achieve, and this is a crucial step which has to be tested experimentally, is the following. A plane wave injected into a nonlinear medium will be distorted, i.e., it will undergo self interaction, until a periodic wave is present satisfying Maxwell’s equations (3) and the constitutive relations (4) and (5), and the corresponding relations \( B(H) \). The budget of amplitudes of various harmonics depends on the original excitation and the properties of the medium. In other words, if a periodic plane wave is injected into the medium with exactly the right amplitudes and relative phases of harmonics, this wave will propagate in the medium without modification. The assumption implied in (9) is that the ratios of amplitudes are insensitive to incremental variation, i.e., if all the amplitudes are increased by a small factor, the ratio in (10) becomes

\[ \frac{E_{qj} E_{q'^k}}{E_{ij} E_{kk}} \left( 1 + \frac{\Delta E_{qj}}{E_{qj}} + \frac{\Delta E_{q'^k}}{E_{q'^k}} - \frac{\Delta E_{ij}}{E_{ij}} - \frac{\Delta E_{kk}}{E_{kk}} \right) \]  

and ideally the expression in parentheses in (11) equals 1. As long as the increments in (11) are small enough to justify this approximation, (9) is valid as an approximation. According to (4) and (9), we have for each harmonic \( q \)

\[ D_i = \epsilon^{(1)}_{i,j} E_j + \epsilon^{(2)}_{i,j,k} E_j E_k + \ldots + \epsilon^{(n)}_{i,j,r,s} (E_j \cdots E_r) + \ldots \]  

and if considered as an expansion of \( D(E) \) about \( E = 0 \), then \( \epsilon^{(n)}_{i,j,r,s} \) are the values of the derivatives in a Taylor expansion

\[ \epsilon^{(n)}_{i,j,r,s} = \frac{1}{n!} \frac{\partial^n D_i}{\partial E_j \cdots \partial E_r}. \]
Many studies heuristically start with (12), which is not as systematic as the present approach. The systematic approach through the tool of Volterra’s series clearly displays $\epsilon^{(n)}$ as only approximately constant, as opposed to $\epsilon^{(n)}$.

At this point, (9) etc. is substituted in (3), and for $q = 1$, this yields
\begin{align*}
(k \times E)_q - \omega \epsilon^{(1)}_{ij} H_j - \omega \epsilon^{(2)}_{ijk} H_k H_i - \cdots &= 0 \\
(k \times H)_q + \omega \epsilon^{(1)}_{ij} E_j + \omega \epsilon^{(2)}_{ijk} E_k E_i + \cdots &= 0
\end{align*}
(14)
where the index $q = 1$ has been suppressed, and similar expressions exist for higher harmonic waves. The six homogeneous, algebraic, nonlinear scalar equations (14) define the physical problem at hand. The system (14) can be used to deliver an amplitude dependent dispersion equation, as explained previously [6]. To clarify this, we shall handle the situation in a somewhat primitive way. The second equation (14) can be manipulated to derive expressions for $H_1, H_2, H_3$ in terms of $E_1, E_2, E_3$, and these are substituted in the first line of (14), resulting in three scalar equations on $E_1, E_2, E_3$. In each of these equations, there are terms that do not involve $E_1$, say; hence, the set of equations may be written in the form
\begin{align*}
E_1 f_1(E_1, E_2, E_3) &= g_1(E_2, E_3) \\
E_1 f_2(E_1, E_2, E_3) &= g_2(E_2, E_3) \\
E_1 f_3(E_1, E_2, E_3) &= g_3(E_2, E_3)
\end{align*}
(15)
where $f$ and $g$ are arbitrary functions of the arguments, and in some degenerate cases not all the arguments will be present. By elimination of the factors $E_1$ on the left, in (15), we obtain two scalar equations which have the general form
\begin{align*}
E_2 h_1(E_1, E_2, E_3) &= l_1(E_1, E_3) \\
E_2 h_2(E_1, E_2, E_3) &= l_2(E_1, E_3)
\end{align*}
(16)
and finally, by dividing the equations in (16), one equation of the form
\begin{equation}
E_2 u(E_1, E_2, E_3) = 0.
\end{equation}
(17)
Hence, for the nontrivial solution $E_1 \neq 0$ in (17) prescribes a dispersion relation $u = 0$, which involves $k, \omega$, and field amplitudes. This procedure is equivalent to writing (14) in matrix form $G_\epsilon = F_\epsilon A = 0$, $r, s = 1, \cdots, 6$, where $A = (A_\epsilon) = (E_1, E_2, E_3, H_1, H_2, H_3)$ is a six component vector, and by imposing the condition of solubility det($F_\epsilon$) = 0, the dispersion equation
\begin{equation}
F(k, \omega, A) = 0
\end{equation}
(18)
is obtained. The fact that (18) involves amplitudes is a characteristic feature of the nonlinear problem. The corresponding equations (14) and (18) for higher harmonics are not independent systems of equations, because (14) and (18) already establish a relation between $k$ and $\omega$. Hence, the equations for higher harmonics can only serve to determine the (complex) amplitudes of the harmonic waves. The details are not very important to the main line of our subject.

At this point, we have sufficiently summarized the general theory in order to discuss metallic guides and resonators. We begin, in the next section, with the relatively simple problem of rectangular guides and cavities.

### III. Rectangular Waveguides and Resonators

In order to demonstrate the feasibility of using the above theory for rectangular waveguides and cavities, a simple isotropic constitutive relation is used. This class of problems is still general enough to display typical aspects of the nonlinear class of problems. Accordingly, we define a dielectric medium with scalar constant $\mu$ and $\epsilon$
\begin{align*}
D_i &= \epsilon^{(1)}_{ij} E_j + \epsilon^{(2)}_{ijk} E_j E_k + \cdots \\
&= \epsilon^{(1)} E_i + \epsilon^{(2)} (E_i)^2 + \epsilon^{(3)} (E_i)^3 + \cdots
\end{align*}
(19)
where for expressions containing scalar $\epsilon^{(n)}$, the summation convention is applicable. This means that $\epsilon^{(1)}_{ij}$ is diagonalized by multiplying by a Kronecker $\delta_{ij}$, and the diagonal elements made identical, and a corresponding treatment for the higher order tensors. Equation (19) is compacted in the form
\begin{equation}
D = \epsilon_{\text{eff}} (E) E
\end{equation}
(20)
where it is understood that the field $E$ in $\epsilon_{\text{eff}} (E)$ is the field $E_i$ related to $D_i$. Manipulating Maxwell’s equations (3) yields ($q = 1$)
\begin{equation}
k \times E + \omega^2 \mu \epsilon_{\text{eff}} (E) E = 0.
\end{equation}
This medium admits transversal waves for which the wave equation becomes
\begin{equation}
[k^2 - \omega^2 \mu \epsilon_{\text{eff}} (E)] E = 0.
\end{equation}
(21)
Hence, the expression in brackets is the dispersion equation (18).

The main difficulty in proceeding to analyze the present problem is that, unlike the linear case, the representation of the total field as a superposition of plane waves requires justification. In general, superposition is not valid in nonlinear media. However, in weakly nonlinear systems as discussed here, it appears plausible to assume that only phase-matched nonlinear induced harmonics will be produced with significant efficiency. This implies that in isotropic media as considered here, the interaction of noncollinear waves will be negligible. This heuristic assumption, still requiring experimental support, salvages the linear method of superposition to the extent that new solutions may be constructed from sums of plane waves which are not phase matched.

Accordingly, we take the formulas for rectangular waveguides, e.g., as given by Collin [21], recast them in terms of plane waves, and replace $k, \omega$ with $g_k, q_\omega$, respectively, to obtain the harmonics. Thus, the fields are given by

<table>
<thead>
<tr>
<th>Field</th>
<th>TE</th>
<th>TM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_z$</td>
<td>$C_x C_y \epsilon$</td>
<td>0</td>
</tr>
<tr>
<td>$E_z$</td>
<td>0</td>
<td>$S_x S_y$</td>
</tr>
<tr>
<td>$E_x$</td>
<td>$Z_{h,nm} H_y$</td>
<td>$A C_x S_y \epsilon$</td>
</tr>
<tr>
<td>$E_y$</td>
<td>$-Z_{h,nm} H_x$</td>
<td>$B S_x C_y \epsilon$</td>
</tr>
<tr>
<td>$H_x$</td>
<td>$A S_x C_y \epsilon$</td>
<td>$E_x / Z_{c,nm}$</td>
</tr>
<tr>
<td>$H_y$</td>
<td>$-B C_x S_y \epsilon$</td>
<td>$E_y / Z_{c,nm}$</td>
</tr>
</tbody>
</table>
where in the above shorthand notation \( C_x = \cos(qm \pi x/a) \), and \( S_x \) implies sine of the same argument, and \( C_y = \cos(qm \pi y/b) \), and \( S_y \) follows; \( a \) and \( b \) are the dimensions of the guide in the \( x \) and \( y \) directions, respectively, \( q \) is the number of the harmonics, \( e \) denotes \( e^{i \omega t} \), where \( q_B_{nm} \) is the pertinent component of \( q_k \) in the \( z \)-direction, \( A \) denotes \( i \beta_{nm} \alpha \pi / ak_{c,nm} \) and for \( B \) replace \( \alpha \pi / ak_{c,nm} \) with \( m/b, k_{c,nm}^2 = (n \pi/a)^2 + (m \pi/b)^2 \), and \( \beta_{nm} = k^2 - k_{c,nm}^2 \). The impedances are given by \( Z_{h,nm} = k Z_0 / \beta_{nm} \), \( Z_{e,nm} = \beta_{nm} Z_0 / k \), and \( Z_0 = \omega / k \), and, for the present case in particular, \( Z_0 = \mu / \epsilon \). (24)

For small amplitudes, the nonlinear effect is negligible. As the power increases, the nonlinear correction terms will start to play a role and can be computed from the change in the resonance frequency.

IV. CIRCULAR CYLINDRICAL STRUCTURES

It will be shown that this class of canonical problems is not merely a complicated mathematical extension of the rectangular case, in which Bessel functions replace the trigonometric expressions. In fact, if we adopt the general theory given above, then we find that curved metallic structures, as opposed to rectangular geometries, will usually suppress phase matching, and consequently also suppress coherent nonlinear interaction. The general treatment of linear vector waves is given by Stratton [22], who cites original work by Hansen. For completeness, and since the subject is mathematically more complicated than the rectangular case, the general theory is summarized. The general expressions for nonsingular fields in cylindrical coordinates [22, see p. 361] is

\[
\psi_n = e^{in\phi} J_n(\lambda r) e^{ihz-i\omega t}.
\]

where \( \psi_n \) in denoting the nonsingular Bessel functions and \( \lambda^2 = k^2 - h^2 \), the summation extends on \( -\infty < n < \infty \) and \( a_n, b_n \) are coefficients. The structure (26) and (27) can be split into even and odd parts by defining

\[
\psi_n = \begin{cases} 
\cos(n\phi) J_n(\lambda r) e^{ihz-i\omega t} \\
\sin(n\phi) J_n(\lambda r) e^{ihz-i\omega t}
\end{cases}
\]

In order to express (26) in terms of (28), note that \( \psi_n \) in (26) corresponds to \( \partial / \partial \phi \), and apply this operator to (28) (see Stratton [22, p. 395]). The orthogonal vector wave functions \( M \) and \( N \) are defined in Stratton [22, p. 392 ff.]

\[
M_n = (\nabla \psi_n) \times Z \quad \nabla \times N_n = k N_n
\]

The TE \( E \) field in (26) is recognized as \( i \omega \mu \Sigma b_n M_n \), and the mate \( H \) field corresponds to \( k \Sigma \partial / \partial \phi \); the TM \( H \) field is recognized as \( -i k^2 / \mu \Sigma a_n N_n \) and the mate \( E \) field is

\[
F\left( \frac{n \pi}{a}, \frac{m \pi}{b}, \frac{\pi}{d}, \omega, A \right) = 0 \tag{24}
\]

which is amplitude dependent. Consequently, the resonance frequency is also amplitude dependent. This is the key to measuring

\[
\epsilon_{\text{eff}}(E) = k^2 / (\mu \omega^2) \tag{25}
\]
given by \((k\omega/c)\sum a_n N_n\). Relevant to our subject is the representation of \(M\) and \(N\) in terms of sums (integrals) of plane waves
\[
M_n = \frac{1}{2\pi} \int_0^{2\pi} (ik \times \xi) e^{i\lambda r \cos(\beta - \phi) + i n \beta + i h z - i\omega t} d\beta
\]
\[
N_n = \frac{1}{2\pi} \int_0^{2\pi} (k \zeta - h k) e^{i\lambda r \cos(\beta - \phi) + i n \beta + i h z - i\omega t} d\beta.
\]
(30)

(See Stratton [22, pp. 396, 397]).

Thus far the linear problem has been summarized. Strictly speaking, the nonlinear problem does not admit superposition of plane waves. However, making the same assumption as before, that it is legitimate to superpose waves which are not phase matched, structures like (30) are admissible. But exactly this argumentation also leads to the prediction that the significance of nonlinear interaction will be very small, since for each amplitude \(ik \times \xi e^{i n \beta}\) or \((k \zeta - h h) e^{i n \beta}\), the nonlinear dispersion equation (18) yields a different value for \(k\), and in the absence of some concerted effort of all the plane waves in the integrand (30), the effect of nonlinearity will be negligible. One way of dealing with this assumption is to assume the presence of many modes, and let the amplitudes affect the dispersion equation such that a combined coherent effect will emerge. Thus, we define the \(E\) field corresponding to (30) as
\[
\frac{1}{2\pi} \int_0^{2\pi} d\beta e^{i\lambda r \cos(\beta - \phi) + i h z - i\omega t} E(\beta)
\]
where \(E(\beta) = \sum E_n(\beta) e^{i n \beta}\) stands for a sum of amplitudes of various modes. Assuming again the isotropic medium as in (19) and (21) prescribes
\[
\frac{1}{2\pi} \int_0^{2\pi} d\beta e^{i\lambda r \cos(\beta - \phi) + i h z - i\omega t} \{k^2 E(\beta) - \omega^2 \mu (\beta) E(\beta)\}
\]
\[
- \omega^2 \mu (\beta) E(\beta) \hat{E}(\beta) - \omega^2 \mu (\beta) E(\beta) \hat{E}(\beta) - \ldots \} = 0.
\]
(32)

What we are trying to do now is to find conditions for the vanishing of the expression in braces in (32) for all \(\beta\). For simplicity, let us assume first that only \(\epsilon^{(1)}\) and \(\epsilon^{(2)}\) are nonzero. The condition for the vanishing of the braces in (32) for all \(\beta\) prescribes
\[
(\lambda^2 - \omega^2 \mu (\beta)) \sum b_n e^{i n \beta} - \omega^2 \mu (\beta) \sum b_n b_n' e^{i(n + n') \beta} = 0
\]
(33)

for the TE field, for example. Hence, using the orthogonality of \(e^{i n \beta}\), (33) yields
\[
(\lambda^2 - \omega^2 \mu (\beta)) b_n - \omega^2 \mu (\beta) \sum b_n b_n' = 0.\]
(34)

For each \(n\), an equation of the type (34) is obtained, and since all factors except \(k\) are given beforehand, each equation yields a value \(h_n\). Since \(\lambda\) is determined by boundary conditions, (34) in general means that different \(\lambda_n\) are associated with different \(h_n\); hence, our assumption (31) (with one \(h\) applying to all modes \(n\)) is invalid. Still, our argument may be applied as an approximation. If, for \(\epsilon^{(2)} = 0\), a value \(h_0\) is computed, then the nonlinear effect is
\[
\Delta h_n = h_n - h_0.
\]
(35)

If \(\Delta h_n\) is small compared to \(h_0\), then the phase mismatch \(e^{i n \lambda_h}\) will be negligible for some range of \(z\), for which \(h\) in (31) can be replaced by \(h_0\). Naturally, this means that for high-order modes, where \(h\) is getting smaller compared to \(\lambda\), the approximation is increasingly improved. This is to be expected, because for higher modes, corresponding to larger arguments in the Bessel functions, the asymptotic representation of the Bessel functions constitutes a good approximation. This is tantamount to saying that as \(\lambda\) increases, the waves become more and more similar to plane waves. The above argument shows the interaction of modes, which also means that if one mode \(n\) is injected into the system, then powers of \(e^{i n \beta}\) will produce higher modes, whose amplitudes contain the nonlinear \(e^{i n}\) as factors. Subject to the restrictions mentioned above, the field in circular cylindrical waveguides can be represented by (27) and (28) with the proper modifications, i.e., \(h, k\), are replaced by \(h_n, k_n\), obtained from (34) and the boundary condition on \(\lambda\), according to \(k_n^2 = \lambda^2 + h_n^2\). Similarly to the previous case of rectangular cavities, here too the resonance frequency of a cylindrical cavity will depend on the amplitude of the injected signal. For this case, \(h\) is determined by the length \(z\) of the cavity, and the only degree of freedom is offered by \(\omega\). If only one mode is present, which we can represent by \(\pm n\), \(|b_n| = |b_{-n}|\), then the same argument that led to (34) now yields, for \(\epsilon^{(3)}\), a relation of the form
\[
(\lambda^2 - \omega^2 \mu (\beta)) b_n - \omega^2 \mu (\beta) b_n^3 = 0
\]
(36)

which is somewhat oversimplified, but shows the dependence of \(\omega_n\) on the amplitude, represented here by \(b_n\).

Thus far only the fundamental frequency has been considered. Inasmuch as we were able to recast the fields in plane-wave integrals, it is clear that for frequency \(\omega\) we will not have \(\phi k\) (i.e., \(\phi k\) and \(\phi h\)) to maintain the phase-matching requirement. However, if the boundary conditions prescribe \(J_n(\lambda a) = 0\) or \((d/d\alpha) J_n(\lambda a) = 0\), for TM and TE modes, respectively, for a guide of radius \(r = a\), then in general \(J_n(\lambda a), (d/d\alpha) J_n(\lambda a)\) will not vanish due to the fact that the zeros of the functions are not evenly spaced. This means that harmonic production is suppressed in cylindrical waveguides and cavities. For high-order modes, the zeros become increasingly evenly spaced, because the waves resemble more and more plane waves. Hence, for large \(\lambda a\), the harmonic waves will be present.

V. SPHERICAL STRUCTURES

At this stage, where a lot of experimentation is needed to check the fundamentals of the theory, there is no point in bringing in all the heavy machinery for the scalar and vector spherical wave functions. These are comprehensively covered by Stratton [22, see ch. 7]. The ideas are identical, and therefore we expect the same conclusions. The repre-
sentation of spherical vector waves in terms of plane-wave integrals [22, pp. 416, 417], and the application of dispersion equations similar to (33) and (34) will lead to the results of \( k_{nm} \) for mode \( n, m \) which constitutes an approximate solution, increasingly improving as \( n \) increases (on account of the spherical Bessel functions behavior for large arguments). The conclusions for the behavior of harmonics follows the same lines.

V. SUMMARY AND CONCLUSIONS

The problem of nonlinear wave propagation is extremely complicated, physically and mathematically. The present study concentrates on weak nonlinear effects which provide the correction terms for the leading linear results. This is described mathematically by a model based on the Volterra series, and plane-wave dispersion relations are obtained by assuming periodic solutions. This theory is briefly recapitulated. Applications to rectangular waveguides and cavities are given. This problem is easy because the linear fields are given as combinations of a few (at most eight, for the fully developed case of a rectangular cavity) plane waves. Once the stipulation is made that nonphase-matched waves do not interact, the extension to the nonlinear case is straightforward. Results are given, and practical aspects of analyzing nonlinear devices, or measuring the properties of nonlinear media, are discussed.

The presence of curved metallic boundaries is shown to suppress nonlinear interaction and harmonic production. This effect is increasingly pronounced as the waves depart more and more from plane waves, i.e., when the curvature of wavefronts increases. Cylindrical waves are considered in some detail, the treatment for spherical structures is only delineated, but the above conclusions seem to be valid in general.

There are many heuristic assumptions in the basic theory, and experimental data is necessary to check its validity.

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Dan Censor was born in Tel-Aviv, Israel, in 1936. He studied for the B.Sc. E.E., M.Sc. E.E., and D.Sc. Eng. at the Technion—Israel Institute of Technology from 1957–1967. From 1967–1969, he was with the Department of Information Engineering, University of Illinois, Chicago, 1969–1975 in the Department of Environmental Sciences (now Geophysical and Planetary Sciences), Tel-Aviv University, 1975–1976 in the Institute of Theoretical Physics, University of Düsseldorf, West Germany. Since 1975, he has been Professor of Electrical Engineering in the Department of Electrical and Computer Engineering, of the Ben Gurion University of the Negev, Beer Sheva, Israel. In 1979–1980, he held a visiting appointment with the Department of Electrical Engineering and Computer Science, University of California San Diego, La Jolla, CA, and in 1980–1981 was a National Academy of Sciences NRC Senior Research Associate of the NASA Goddard Space Flight Center, Greenbelt, MD.

His research activities and publications in wave and ray (especially electromagnetic) theory are in the areas of propagation and scattering involving moving media and moving obstacles, theory of ray and solitary wave propagation in dispersive, absorptive, and nonlinear media.

Dr. Censor is a member of the Israel National Committee for Radio Science and presently the Official Member of Israel in URSI Commission B.