Short Papers

Topological Rules for Linear Networks and Their Application

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***Abstract*—Nathan's rules are stated and their applications are illustrated. The rules provide the expansion of admittance functions of linear networks which may include reciprocal and nonreciprocal branches, infinite-gain operational amplifiers, dependent current or voltage sources, vacuum tubes, transistors, and ideal transformers. Kirchhoff's rules for reciprocal networks as well as signal-flow-graph rules are included as special cases.**

The rules are purely topological and, in contradistinction to some prior work, do not include sign rules dependent upon an arbitrary assisgnment of labels to nodes, or directions to branches.

The applications include transistor, vacuum tube, analog computation, and transformer networks.

I. INTRODUCTION

A detailed bibliography pertinent to the subject of topological rules for linear networks is given by Nathan [1] and by Seshu and Reed.[2] Kirchhoff[3] has provided topological rules for impedance functions of linear reciprocal networks. Maxwell[4] has stated the dual admittance rules. Percival[5] attempted to extend these rules for active networks. His ideas were further developed by Coates[6] and Mayeda[7] who provide a set of rules for nonreciprocal networks. Their work suffers, however, from two major drawbacks. In the first place, it becomes necessary to replace the given network by two networks; second, the signs of terms in the expansions of network functions can only be found by the application of a sign rule which depends upon the numbering of nodes. Other approaches are Mason[8] and Mason and Zimmermann,[9] using bilateral nonreciprocal elements, and Mason,[10] Robichaud and Boisvert,[11] and Boisvert,[12] who use signal-flow graphs.

Proper topological rules should not use nontopological elements that are not inherent in a problem; for example, there must be no arbitrary numbering of nodes or branches, and no direction should be assigned to bilateral branches. Furthermore, the rules should operate as far as possible upon the graph of a given network, rather than upon an equivalent (more complicated) graph. Finally, they must show how to handle constraints. It follows that such rules must include Kirchhoff's as well as signal-flow-graph rules as special cases, and must be applicable to electrical networks which include reciprocal branches, nonreciprocal branches, and constraints in combination.

The new rules were first derived by Nathan in 1960 (unpublished notes) for constrained networks, extended to networks including finite transadmittances by Censor,[13] under Nathan's guidance, and finally formulated and proved by Nathan.[1]

The rules are here formulated in a different manner from that given in Nathan[1] where they are proved. Though clearly equivalent to those, the present form appears to be somewhat easier to apply in manual computations.

II. STATEMENT OF THE RULES

We deal with a linear, lumped, finite and time-invariant network N whose graph may include four kinds of weighted elements, represented by directed or undirected solid or dashed branches. An undirected solid branch represents an admittance and is weighted with the value of the admittance; and an undirected dashed line represents a conditional weighted short circuit. A directed solid line represents a voltage constraint and is weighted by the "gain" of the constraint; a directed dashed line represents a transadmittance and is weighted by its value. Table I lists the representations, in terms of these branches, of common network elements. One of the nodes of N is the reference node r. Zero voltage is assigned to r and voltage constraints are defined with respect to it.

The table includes graphs and instructions which, in conjunction with the following definitions and rules, permit the determination of network functions.

The following definitions either add to or modify those commonly used in linear graph theory.

1) *Loops:* The term loop (or "directed loop") shall designate a simple loop that does not pass through the reference node, includes at least one directed branch, and is assigned the same sense by all its directed branches.

2) *Paths:* An st path ($s \neq t$) is a simple (i.e., loopless) path leading from node s to node t without passing through r. If an st path contains any directed branches, they must all point towards t.

3) *Trees:* Trees consist only of undirected branches and must include reference node r. The isolated reference node is regarded as a (degenerate) tree of weight one.

4) *Weight:* The weight of a subgraph of N is the product of the weights of its branches.

5) *Removal of a Branch:* As far as network functions are concerned, the removal of a branch is equivalent to letting its weight become zero. Thus, a removed undirected branch and a removed directed and dashed branch are to be disregarded, and a directed solid branch is removed by the grounding of its output node (cf., Def. 6). A branch which is not removed will be called *present* or *active*.

6) *Grounding:* A node is grounded when it is short-circuited to reference node r. A loop or a path is grounded by merging all its nodes with r and subsequently removing its branches.

7) *Configurations and Terms:* The rules give network functions as sums of terms. To each term there corresponds a subnetwork of N, which will also be called a configuration. All configurations must be complete, i.e., they must touch all nodes of N. The value of a term is the weight of the associated configuration.

In general, we shall not distinguish between a configuration and the associated weight or term.

Next, we state the rules for the determination of the determinant and the cofactors of the admittance matrix Y of N, with respect to reference node r.

Rule 1

$$|Y| = T_0 - \sum_k L_k^{(1)} T_{1,k} + \sum_k L_k^{(2)} T_{2,k} - \sum_k L_k^{(3)} T_{3,k} + \cdots, \quad (1)$$

where $L_k^{(l)}$ is the weight of the kth set of l nontouching loops in N, and $T_{l,k}$ is the sum of weights of all trees in the residual network that is produced from N by grounding this kth set of loops; the summation is over all possibilities $k = 1, 2, 3, \cdots$.

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Note that $T_{l,k}$ is the determinant of the network produced from N by grounding the kth set of l nontouching loops and subsequently removing all directed branches. Equation (1) also gives the principal minor Y^{ss} if applied to the network produced from N by grounding node s.

Rule 2

Cofactor Y^{st} is given by

$$Y^{st} = \sum_k P_k^{st} Y_k^{st}, \qquad (2)$$

where P_k^{st} is the weight of the kth st path, and Y_k^{st} is the determinant of the residual network produced from N by grounding the kth st path; the summation is over all possibilities.

These rules, in conjunction with the graphs of network elements, yield expressions for network functions. Thus, the input impedance looking into node s (with respect to the reference node) is given by

$$Z_{(\text{in})s} = Y^{ss}/|Y| \qquad (3)$$

and the transmittance from node s to node t is

$$H_{st} = e_t/e_s = Y^{st}/Y^{ss}. \qquad (4)$$

III. The Graphs of Network Elements

Table I provides the weighted graphs of network elements and instructions for their use in connection with the rules.

Items 5 (ii) and 8 (ii) are instructions which are not essential, but which serve to eliminate cancelling terms.

Item 13 gives simplified rules for some of the terms of item 11.

Item 14 is a special case of 13.

IV. Applications

It is desirable to use a systematic procedure when applying the rules.

In general, we follow the prescription of Rule 1 and Rule 2 and successively consider all terms having 0, 1, 2, $\cdots$, loops. A method of "condensation" around some branch b of N is often helpful: we consider separately i) all terms containing b and ii) all those that do not. i) yields all terms having the weight b as a factor, and corresponds to all combinations in the network produced from N by short-circuiting b (i.e., merging its nodes). ii) are the terms in N when b is removed.

The following abbreviation will be found convenient. We denote by

$$\sum{}^{(m)} (a_1a_2 \cdots a_n) = a_1a_2 \cdots a_{m-1}a_m + a_1a_2 \cdots a_{m-1}a_{m+1} + a_{n-m+1}a_{n-m+2} \cdots a_n; \qquad m \le n \qquad (5)$$

the sum of the $\binom{n}{m}$ products formed from all possible combinations of m factors out of n. For example,

$$\sum{}^{(2)} (abcd) = ab + ac + ad + bc + bd + cd.$$

An Analog Integrator

The transmittance $H_{12} = Y^{12}/Y^{11}$ in Fig. 1 is required. First, consider $A = \infty$. From item 3 in Table I, $c = 1$ and $Y^{12} = ac = a$. The residual network consists of the isolated reference node; i.e., it is a tree of unity weight. $Y^{11} = (-1)^1 cb = -b$ is found by grounding node 1. Therefore,

$$H_{12} = -a/b = G_i/(G_0 + sC).$$

Next, consider finite A, item 2, Table I. The only allowed 12 path is ac, since node 2 must be grounded when c is not in the path, which eliminates ab from consideration. Thus $Y^{12} = ac$. To find Y^{11} ground node 1. From item 2, either c is active or node 2 is grounded. With c inactive we have the trees $a+b$, with c active we have the loop $(-1)^1 bc$; therefore $Y^{11} = a+b-bc$ and

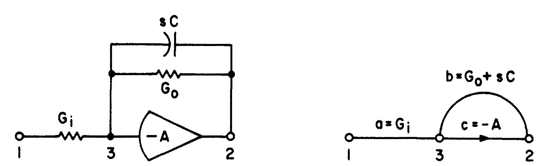
$$H_{12} = Y^{12}/Y^{11} = \frac{ac}{a+b-bc} = -\frac{G_i}{G_0 + sC + (1/A)(G_i + G_0 + sC)}.$$

The Double Integrator

For brevity, the weights of branches are denoted by the subscript letters. We determine $H_{14} = H_{14}(A)$.

i) $A = -\infty$. From item 3, Table I, set $A = 1$, and, since A must appear in each term, $Y^{14} = acA = ac$. To find Y^{11} ground node 1 and consider $a+b$ as a single branch. $Y^{11} = (-1)^1[Ae(a+b+c+d) + Adc]$. Therefore,

$$H_{14}(\infty) = Y_\infty^{14}/Y_\infty^{11} = -\frac{ac}{e(a+b+c+d)+dc} = -\frac{G_aG_c}{sC_e(G_a + sC_b + G_c + G_d) + G_cG_d}.$$



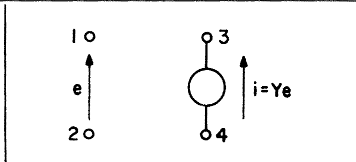
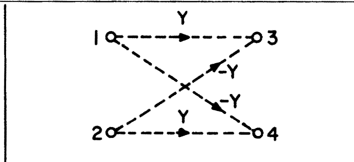
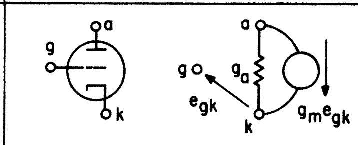
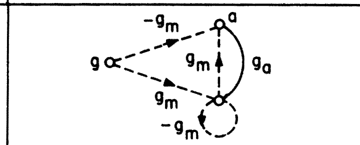
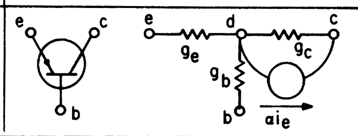
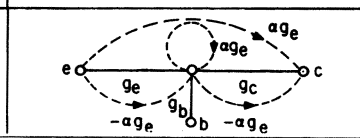
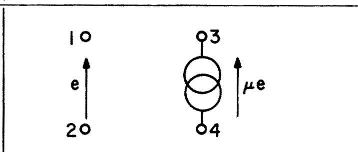
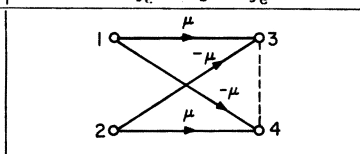
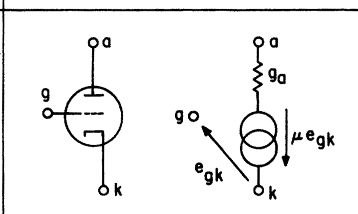
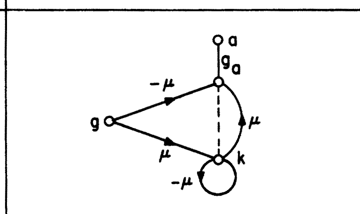
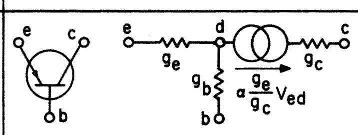
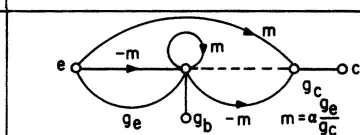
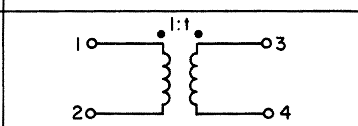
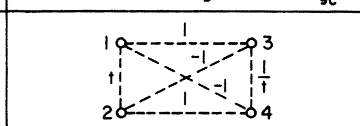
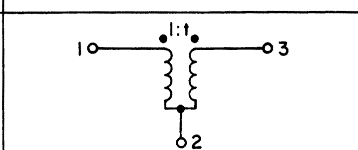
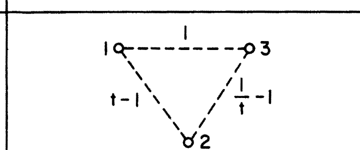
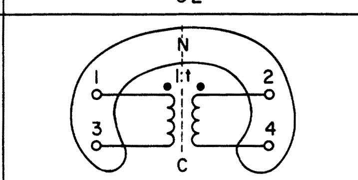
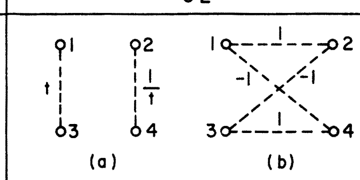
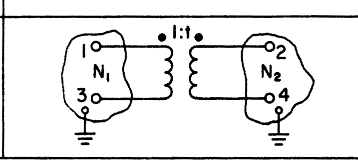
(a) (b)

Fig. 1. Integrator circuit: (a) schematic diagram and (b) its graph.

TABLE I

No.	Description	Symbol	Graph	Instructions
1	Admittance Y	Y = 1/sL; Y = sC; Y = G	Y	May be active in a term.
2	Voltage Constraint, (Ideal operational amplifier having infinite input impedance).	1, A, 2; e_1, $e_2 = Ae_1$; r, r	1 A 2; r	Either the directed branch or the short-circuit (dashed branch) is active, i.e. either A is active or node 2 is grounded. For brevity the dashed branch is often omitted
3	Constraint of infinite gain. (Special case of 2)	1, ∞, 2; or	1 ∞ 2	This branch is active in each term. Weight is arbitrary and is set equal to 1 for simplicity.
4	Transadmittance	1, 2; e_1, $i_2 = Ye_1$; r, r	1 Y 2	May be active in a term. (Superfluous if touching r.)

TABLE I (*Con'd*)

5	Floating Transadmittance			(i) Not more than one branch may be active in each term. (ii) Omit configurations that contain an undirected path bridging the arrowheads or shafts, simultaneously with one of these branches.
6	Triode (current equivalent)			cf. #5
7	Transistor (current equivalent)			cf. #5
8	Floating voltage constraint			(i) Not more than one branch may be active in each term. (ii) Omit configurations that contain an undirected path bridging the arrowheads or shafts, simultaneously with one of the directed branches.
9	Triode (voltage equivalent)			cf. #8
10	Transistor (voltage equivalent)			c.f. #8
11	Ideal transformer			In each term one and only one weighted short circuit (dashed line) is active
12	Ideal auto-transformer			cf. #11
13	Ideal transformer			cf. #11, but for terms such that the inactive elements include a cut-set C of N separating the windings of the transformer: in $\|y\|$, y^{ss} and in y^{st} if C does not separate nodes s and t-use graph (a); otherwise use graph (b).
14	Network separated by ideal transformer			For $\|y\|$, y^{ss} and y^{st} if both s and t are in N_1 or N_2, use graph #13 (a); otherwise use graph #13 (b).

ii) *If A is finite* we obtain Y^{11} by condensation around A, which is evidently tantamount to adding to AY_∞^{11} the terms corresponding to inactive A: viz. $(a+b+d)(c+e)+ce$. Moreover, by inspection of Fig. 2, $Y^{14}=AY_\infty^{14}$, and thus,

$$H_{14}(A) = -\frac{G_aG_c}{sC_e(G_a+sC_b+G_c+G_d)+G_cG_d - (1/A)\left[(G_a+sC_b+G_d)(G_c+sC_e)+G_csC_e\right]}.$$

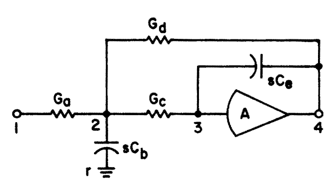


Fig. 2. Double integrator.

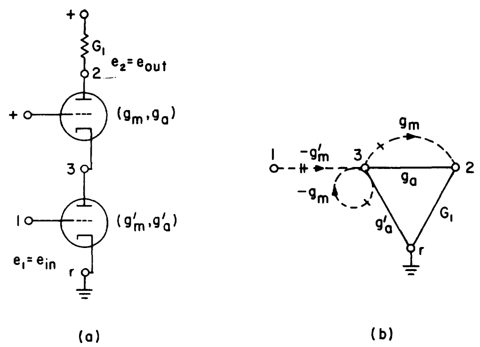


Fig. 3. (a) Cascode amplifier and (b) its graph.

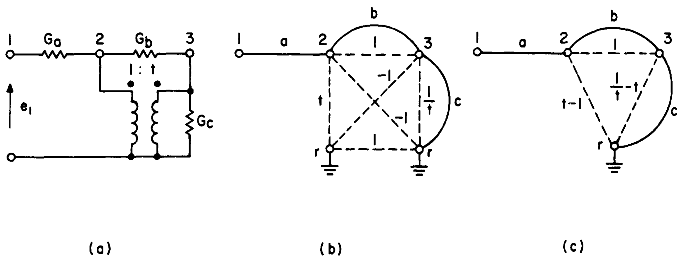


Fig. 4. Transformer circuit: (a) schematic diagram, (b) its graph, and (c) reduced graph.

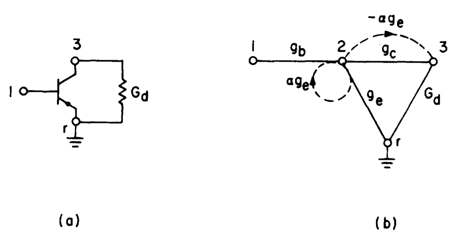


Fig. 5. (a) Common emitter amplifier and (b) its graph.

Cascode Amplifier

Referring to Fig. 3, the triodes are represented by their current graph item 6, Table I. Directed branches touching the reference node are omitted because they are never active. There are two *12* paths, and $Y^{12}=-g_m'(g_a+g_m)$. The $l=0$ (no loop) terms in Y^{11} are $\sum^{(2)}G_1g_ag_a'$. According to item 6, g_a cannot be active with $+g_m$ or $-g_m$ since it bridges their arrowheads. There remains the single $l=1$ term $(-1)^1(-g_m)G_1$. Therefore,

$$H_{12} = Y^{12}/Y^{11} = [-g_m'(g_a+g_m)]/[G_1g_a + G_1g_a' + g_ag_a' + g_mG_1].$$

A Transformer Circuit

The circuit of Fig. 4(a) has a graph (b) which is equivalent to the reduced graph (c). From item 11 in Table I, $Y^{12}=a\cdot 1=a$. To find Y^{11}, ground node 1, yielding

$$Y^{11} = 1\cdot(a+c) + (t-1)(b+c) + \left(\frac{1}{t}-1\right)(a+b) = \frac{a}{t} + ct + b\left(t-2+\frac{1}{t}\right).$$

Therefore,

$$H_{13} = Y^{13}/Y^{11} = tG_a/[G_a + t^2G_c + (t^2-2t+1)G_b].$$

Common Emitter Transistor Amplifier

For the amplifier of Fig. 5(a), the current graph, item 7 in Table I, provides its graph [Fig. 5(b)]. Thus, $Y^{13}=g_b(g_c-\alpha g_e)$ and, after grounding node 1,

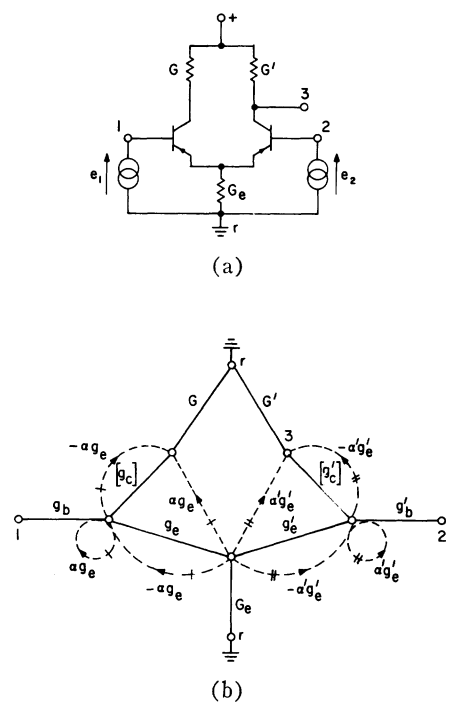


Fig. 6. Differential transistor amplifier: (a) schematic diagram and (b) its graph.

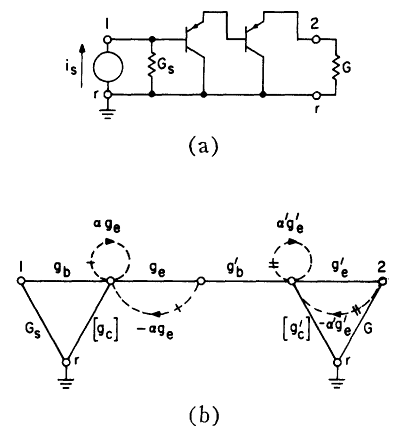


Fig. 7. (a) Composite transistor and (b) its graph.

$$Y^{11} = [(g_b+g_e)(g_c+G_d) + G_dg_c] + (-1)^1\alpha g_eG_d$$

since, by ii) in item 5 of Table I, g_c must not appear simultaneously with $\pm\alpha g_e$. Therefore,

$$H_{13} = g_b(g_c-\alpha g_e)/[(g_b+g_e)(g_c+G_d) + G_dg_c - \alpha g_eG_d].$$

Differential Transistor Amplifier

In the differential amplifier of Fig. 6(a): a) prove that $A_{13}=-A_{23}$, neglecting g_c, g_c' and G_e, where A_{13}, A_{23} are the gains from input nodes 1, 2, to output node 3; and b) calculate the common mode effect, neglecting g_c and g_c', but not G_e.

a) Since $Y^{11,22}=Y^{22,11}$ it is sufficient to prove $Y^{13,22}=Y^{23,11}$. From the graph of Fig. 6(b), neglecting G_e, with node 2 grounded, there is but one *13* path, and the residual tree is G, so that

$$Y^{13,22} = g_b g_e \alpha' g_e' g_b' G.$$

Similarly, having resort to ii) in item 5 of Table I,

$$Y^{23,11} = g_b'(-\alpha' g_e') G g_b g_e.$$

It follows that $A_{13} = -A_{23}$.

b) If G_e is not neglected, $Y^{13,22}$ is not affected, but $Y^{23,11}$ becomes

$$g_b'(-\alpha' g_e') G \left[\sum^{(2)} g_b g_e G_e + (-1)^1 \alpha g_e G_e \right].$$

It follows that the ratio of common mode to differential gain is, approximately,

$$\tfrac{1}{2} G_e [g_b + (1-\alpha) g_e]/(g_b g_e) \cong \tfrac{1}{2} G_e / g_e.$$

Composite Transistor

The composite transistor of Fig. 7(a) is loaded by G and fed by current source i_s, having input admittance G_s. The input impedance $Z_{in} = Z_{(in)1}$ with G connected, and the output impedance $Z_{out} = Z_{(in)2}$, with G_s connected, are required; g_c and g_c' will be neglected.

By inspection of the graph [Fig. 7 (b)], and disconnecting G_s,

$$|Y|_{(in)} = g_b g_e g_b' g_e' G - [(\alpha g_e) g_b g_b' g_e' G + (\alpha' g_e')(g_b g_e g_b') G] + (\alpha g_e)(\alpha' g_e') g_b g_b' G$$

and

$$Y_{(in)}^{11} = \sum^{(4)} (g_b g_e g_b' g_e' G) - \left[(\alpha g_e) g_b' g_e' G + (\alpha' g_e') \sum^{(2)} (g_b g_e g_b') \cdot G\right] + (\alpha g_e)(\alpha' g_e') g_b' G.$$

If G is sufficiently small, $Y_{(in)}{}^{11}$ tends to $g_b g_e g_b' g_e'$, which could have been written down by inspection of the graph, and

$$Z_{in} = Y_{(in)}^{11} / |Y|_{(in)} \cong [G(1 - \alpha - \alpha' + \alpha\alpha')]^{-1}$$
$$= [G(1-\alpha)(1-\alpha')]^{-1} = (1+\beta)(1+\beta')/G,$$

where

$$\beta = \alpha/(1-\alpha); \qquad \beta' = \alpha'/(1-\alpha').$$

Reconnecting G_s, but disconnecting G,

$$|Y|_{out} = G_s g_b g_e g_b' g_e'$$

and, for sufficiently small G_s,

$$Y_{out}^{22} \cong g_b g_e g_b' g_e' - [(\alpha g_e) g_b g_b' g_e' + (\alpha' g_e') g_b g_e g_b'] + (\alpha g_e)(\alpha' g_e') g_b g_b',$$

which results in

$$Z_{out} \cong 1/[G_s(1+\beta)(1+\beta')].$$

References

[1] A. Nathan, "Topological rules for linear networks," *IEEE Trans. Circuit Theory*, vol. CT-12, pp. 344–358, September 1965.

[2] S. Seshu and M. B. Reed, *Linear Graphs and Electrical Networks*. Reading, Mass.: Addison Wesley, 1961.

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The Laplace Transformation of the Impulse Function for Engineering Problems

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Abstract—**The concept and properties of the impulse or delta function have been presented in many books and periodicals. However, when the impulse function arises in undergraduate courses dealing with transform methods, great confusion can exist in the general application. The purpose here is to discuss the Laplace transform of the impulse function in such a manner that will not cause confusion when applied to system problems. Examples indicating difficulty with the normal presentation of this problem are given. A formulation of the Laplace transformation of the impulse function suited for undergraduate and first-year graduate students is presented.**

Introduction

The knowledge of the Laplace transformation is now considered essential for electrical and mechanical undergraduate engineering students, and in many schools all engineering students take a course using transform methods. There is a decided trend[5] to teach this subject immediately after first-year calculus, and preferably in the sophomore year. However, the classroom time spent at this level is not long. In future years, most junior- and senior-year material will be based on the student already having mastered the Laplace transformation. Even in fundamental courses the student comes face to face with the impulse function, and realizes that it is possible to change the currents in inductors and the voltages across capacitors in zero time, under certain conditions, by application of impulse functions. While it is possible in many networks not to use the impulse function, a short discussion of convolution and its application brings this function back to the student's attention. Therefore, we will define this function and give its normal Laplace transformation and proceed to show where trouble can develop. This has been discussed in the excellent text of J. Aseltine.[2] However, to the authors' knowledge, there is no other text or reference material that properly discusses this topic.

Basic Discussion

The definition of the impulse function is as follows:

$$\delta(t-a) = \infty \qquad t = \alpha$$
$$= 0 \qquad t \neq \alpha$$
$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1.$$

An important property is

$$\delta(t) \triangleq \frac{du(t)}{dt}.$$

If we now use

$$\delta(t-a) = \frac{d}{dt} u(t-a) \tag{1}$$

and take the Laplace transform as follows:[2]

$$\delta\left[\frac{d}{dt} u(t-a)\right] = s\mathfrak{L}[u(t-a)] - u(0^+).$$

Now $u(0^+) = 0$ and

$$\mathfrak{L}u(t-a) = \int_0^\infty u(t-a) e^{-st} dt = \int_a^\infty e^{-st} dt = \frac{e^{-as}}{s}$$

$$\mathfrak{L} \frac{d}{dt} u(t-a) = e^{-as} \tag{2}$$

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