THE QUASI LORENTZ TRANSFORMATION FOR ROTATING OBJECTS

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Abstract—The quasi Lorentz transformation introduced recently, is a differential first order \( \nu/c \) approximation to the Lorentz transformation, providing a tool for analyzing electromagnetic scattering problems involving non-uniform motion. Presently the QLT is applied to rotating systems of reference, using the slip-shells model. It is shown that an observer rotating relative to a monochromatic plane wave will measure a Bessel series type frequency spectrum. The problem of scattering by rotating circular cylinders is analyzed. As expected, the scattered wave in the initial frame shows no Doppler frequency shifts. However, for material cylinders, the scattering amplitude (radiation pattern) and the associated scattering coefficients show velocity effects. Moreover, these effects depend on the sense of rotation. Due to the frequency spectrum in the cylinder’s rest frame, dispersion effects will be displayed. It is noted that unlike the instantaneous velocity approximation model, based on the Minkowski constitutive relation, the present model has the potential of dealing with non axially symmetric rotating scatterers.

1. Introduction and Rationale

Einstein’s Special Relativity (SR) theory [1] postulates a propagation speed \( c \) of electromagnetic waves as an invariant constant for all observers in inertial reference systems. From \( c=\text{const.} \) follows the Lorentz Transformation (LT). The other postulate of SR is the “principle of relativity” (PR), stipulating the validity of form-invariance of the ME, subject to the LT. Under such circumstances, the Maxwell’s Equations (ME) render global wave solutions, facilitating the investigation of boundary value problems involving uniformly moving scatterers.

The exact SR model is incompatible with varying velocity, in particular with rotating systems, where constant angular velocity leads to the infinity catastrophe (IC), i.e., with increasing distance from the axis of rotation \( r \to \infty \), the linear speed increases \( v \to \infty \), eventually exceeding \( c \), which is unacceptable in the context of SR. Approximate modeling was suggested e.g., by van Bladel [2], (see p. 206 ff.) to obviate the IC. Presently a velocity field is chosen, commensurate with the differential QLT model, which also avoids the IC.

Recently [3] the QLT model has been introduced and applied to the problem of harmonically moving scatterers. In spite of the complexity of such problems, the feasibility of deriving explicit closed solutions was demonstrated. In order to recapture the essentials of the model, consider first the LT taken to the first order in the velocity, yielding

\[
\mathbf{r}' = \mathbf{r} - \mathbf{v}t, t' = t - \mathbf{v} \cdot \mathbf{r} / c^2
\]

whereas the exact expression for the LT involved factors \( \gamma = (1 - v^2 / c^2)^{-1/2} \), approximated here by \( \gamma \approx 1 \). The transformations in (1) relate the two coordinate
quadruplets \((r, t)\), and \((r', t')\). Upon rewriting (1) in terms of the dimensionless 
\(\beta = v / c, \tilde{v} = v / |v|\) and normalized time \(\tau = ct\), we obtain

\[r' = r - \beta \tilde{v} \tau, \quad \tau' = \tau - \beta \tilde{v} \cdot r\]  
(2)

demonstrating the first order dependence on \(\beta\) in (1). Sometimes one encounters phrases like: “the Galilei Transformation (GT) is the limit of the LT for low velocities”. Obviously (1), (2), refute this assertion. The correct transition from the LT to the GT is by taking the limit \(c \to \infty\), which also explains the incompatibility of the GT with the ME, the latter, contrary to \(c \to \infty\), predicts wave phenomena with finite propagation speeds. This remark is important because some authors consider Galilean electrodynamics in analyzing rotating systems, e.g., [5], which is valid only locally, in a very restrictive sense.

Consider the differential representation

\[dr' = dr - v dt, \quad dt' = dt - \mathbf{v} \cdot dr / c^2\]  
(3)

The integration of (3) leads to (1), differing only within a trivial constant of integration that can be taken as zero, hence (1), (3) are equivalent. However if in (1) constant \(v\) is replaced by a space-time varying function \(v(r, t)\), then taking differentials leads to

\[dv' = dv - v dt - d\mathbf{v}, \quad dt' = dt - \mathbf{v} \cdot dr / c^2 - \mathbf{r} \cdot d\mathbf{v} / c^2\]  
(4)

involving \(d\mathbf{v}\), i.e., acceleration terms expressed as derivatives of \(v(r, t)\).

The QLT is intended to deal with systems where acceleration terms per se are negligible, hence they are excluded from (4), and the ME are considered to hold in moderately accelerated systems as well. This provides a framework for a consistent approximation of the exact SR and ME, facilitating the analysis of scattering problems involving non-uniformly moving objects. This paradigm of allowing the kinematics of varying velocity but neglecting acceleration effects in the ME is widely accepted for moderate acceleration, e.g., see [4], where the Minkowski Constitutive Relations (MCR), originally applicable to constant velocities, are exploited as an approximation for varying velocities in rotating systems.

Accordingly, we stipulate the QLT in the form (3), i.e.,

\[dr' = dr - v(r, t) dt, \quad dt' = dt - c^2 v(r, t) \cdot dr\]  
(5)

where for completeness \(v(r, t)\) depends on all space-time coordinates. The integration of (5) leads to

\[r' = r - \int v(r, t) dt, \quad t' = t - c^{-2} \int v(r, t) \cdot dr\]  
(6)

where \(r, t\), denote integration variables. Taking differentials in (6) yields back (5), thus (6) is verified. However, true to the assumptions leading to (5), terms involving \(d\mathbf{v}\) are neglected. To facilitate the differentiation with respect to \(r\) of the line integral in (6), we need to use the Leibniz rule for differentiation of integrals, applicable only when the path integral depends on the endpoints only. Accordingly the model must stipulate \(v\) as an irrotational (conservative) vector field

\[\partial_t \times v(r, t) = 0\]  
(7)

It follows that \(v\) can be represented as the gradient of a scalar field

\[v(r, t) = \partial_t \Phi(r, t)\]  
(8)

Hence (5) can be recast as

\[dr' = dr - \partial_t \Phi(r, t) dt, \quad dt' = dt - c^{-2} d\Phi(r, t)\]  
(9)
where \( d\Phi(r, t) = v(r, t) \cdot dr \), and consistently with (5), the time derivative \( \partial_t \Phi(r, t) \) is neglected hence \( d\Phi \) is an exact differential in \( r \)-space. Consequently

\[
\Phi(r, t) = \int^\Phi d\Phi = \int^r \partial_t \Phi(r, t) \cdot dr = \int^t v(r, t) \cdot dr
\]  
(10)

In terms of (10), we can rewrite (6) as

\[
r' = r - \int^t \partial_r \Phi(r, t) dt, t' = t - c^2 \Phi(r, t)
\]  
(11)

For the special case when \( v = v(r) \) is independent of time, (11) reduces to

\[
r' = r - \partial_r \Phi(r) t, t' = t - c^2 \Phi(r)
\]  
(12)

It is noted that space and time derivatives of (1), leading to (3), and of (11), leading to (5), yield the same terms, hence the same metric is involved.

The velocity transformation subject to (5) is given by

\[
\begin{align*}
\frac{d}{dt} &= \frac{v}{u} \\
\frac{d}{dr} &= \frac{d}{dt} + \frac{v}{u} \frac{d}{dr}
\end{align*}
\]  
(13)

To the first order in \( \beta \), (13) can be written as

\[
\frac{d}{dt} = \frac{u(1 + c^2 v \cdot u)}{1 - v \cdot u} - v
\]  
(14)

We must retain the term in parentheses in (14) because \( u \) is not limited to low velocities. For example, \( u \) may denote the phase velocity of a wave, which can be close to \( c \).

### 2. The Slip-Shells Rotational Model

The rotational mode involves a velocity \( v \) pointing in the azimuthal (angular) direction \( \hat{\phi} \), with angle \( \phi \) around the cylindrical axis \( z \). In order to obviate the IC and at the same time satisfy (7), we choose a slip-shells rotational model, where the angular velocity decreases with the distance from the rotation axis \( r = 0 \). In circular-cylindrical coordinates

\[
v(r, t) = \hat{\phi} \Omega(t) / r
\]  
(15)

where for completeness the time dependence is retained in the rotation parameter \( \Omega(t) \). The model (15) obviates the IC, because the linear speed \( \Omega / r \) diminishes with the distance, eventually \( v \to 0 \) for \( r \to \infty \). Obviously, we have to avoid the vicinity of the singularity \( r = 0 \), otherwise the first order \( v / c \) stipulation of the QLT is violated. The incident and scattered waves are usually exterior to this region, usually occupied by the scatterer.

The potential function (10) presently becomes

\[
\Phi = \Omega(t) \phi
\]  
(16)

dependent on \( \phi \). Obviously \( \phi \) must be restricted to the range \( 2\pi \) in order for \( \Phi \) to be unique, but this does not affect its derivative, i.e., the velocity.

The line element \( dr \) in circular-cylindrical coordinates, is given by

\[
dr = \hat{r} dr + \hat{\phi} \Omega dt + \hat{z} dz
\]  
(17)

Substituting from (15), (17), into (5) yields the present QLT

\[
dr' = dr - \hat{\phi} \Omega(t) dt / r, dt' = dt - c^2 \Omega(t) d\phi
\]  
(18)

According to the original QLT (5), (12), in (18) space coordinates perpendicular to the velocity are invariant, i.e.,

\[
r' = r, z' = z
\]  
(19)

It follows that the azimuthal component of (18) reduces to
\[ d\varphi' = d\varphi - \Omega(t)dt / r^2 \]  

(20)

In accordance with (11), (16), the integral form of (18) becomes

\[ t' = t - c^{-2}\Omega(t)\varphi, \quad r' = r - (\dot{\varphi}/r)\int_0^t \Omega(t)dt \]  

(21)

which for time independent constant \( \Omega \) reduces to

\[ t' = t - c^{-2}\Omega\varphi, \quad r' = r - \dot{\varphi}t/r \]  

(22)

Note that for \( c \to \infty \) the first equation (22) reduces to a Galilean transformation with \( t' = t \). In view of (22), on a constant radius \( r \) the azimuthal component becomes

\[ \varphi' = \varphi - \Omega t / r^2 = \varphi - \bar{\Omega}t \]  

(23)

where \( \bar{\Omega} = \Omega / r^2 \) is the angular frequency of rotation, inversely proportional to \( r^2 \).

The present results will be applied below to the problem of scattering by axially rotating circular cylinders.

3. Form-Invariance of the ME

As mentioned above, SR theory stipulates the form-invariance of the ME, subject to the LT. This means that given the ME in one reference-frame (e.g., in source free domains)

\[ \partial_r \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \partial_r \times \mathbf{H} = \partial_t \mathbf{D}, \quad \partial_r \cdot \mathbf{D} = 0, \quad \partial_r \cdot \mathbf{B} = 0 \]  

(24)

where \( \mathbf{E} = \mathbf{E}(r, t) \) etc. implies in another reference frame

\[ \partial_r \times \mathbf{E}' = -\partial_t \mathbf{B}', \quad \partial_r \times \mathbf{H}' = \partial_t \mathbf{D}', \quad \partial_r \cdot \mathbf{D}' = 0, \quad \partial_r \cdot \mathbf{B}' = 0 \]  

(25)

with \( \mathbf{E}' = \mathbf{E}'(r', t') \) etc. “Subject to the LT” means that application of the differential-operator form of the LT (e.g., see [13])

\[ \partial_r = \bar{U} \cdot (\partial_{r'} - \mathbf{v} \partial_{r'} / c^2), \quad \partial_i = \gamma (\partial_i - \mathbf{v} \cdot \partial_i) \]  

(26)

\[ \bar{U} = \bar{I} + (\gamma - 1)\mathbf{\hat{v}} \mathbf{\hat{v}}, \quad \gamma = (1 - \beta^2)^{-1/2} \]

to (24), and regrouping of terms

\[ \mathbf{E}' = \bar{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{B}' = \bar{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2) \]  

\[ \mathbf{D}' = \bar{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H} / c^2), \quad \mathbf{H}' = \bar{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \]  

(27)

\[ \bar{\mathbf{V}} = \gamma \bar{I} + (1 - \gamma)\mathbf{\hat{v}} \mathbf{\hat{v}} \]

yields (25). It has been shown [3] that to the first order in \( \beta \) the QLT prescribes the field transformations (FT) corresponding to (27) in the form

\[ \mathbf{E}' = \mathbf{E} + \mathbf{v}(r, t) \times \mathbf{B}, \quad \mathbf{B}' = \mathbf{B} - \mathbf{v}(r, t) \times \mathbf{E} / c^2 \]  

\[ \mathbf{D}' = \mathbf{D} + \mathbf{v}(r, t) \times \mathbf{H} / c^2, \quad \mathbf{H}' = \mathbf{H} - \mathbf{v}(r, t) \times \mathbf{D} \]  

(28)

In the present case of rotation the FT (28) are specialized to the velocity (15).

The various transformations discussed above are given for the primed reference system values in terms of the unprimed ones. The inverse relations, in the context of the approximations stipulated, prescribe interchanging primed and unprimed symbols, and inverting the sign of the velocity. For example, consider the last equation (28)

\[ \mathbf{H}' = \mathbf{H} - \mathbf{v}(r, t) \times \mathbf{D} \Rightarrow \mathbf{H} = \mathbf{H}' + \mathbf{v}(r', t') \times \mathbf{D}' \]  

(29)

In (29) we have exploited the fact that the present formalism is a first order approximation in \( \beta \), hence the arguments of \( \mathbf{v}(r', t') \) can be take to zero order in \( \beta \).
Similarly in (29) \( \mathbf{D}' = \mathbf{D} \) to the zero order in the velocity. This applies to all expressions discussed above.

4. Transformation of Plane and Cylindrical Waves

Intuitively speaking, we ask the following question: An observer sits on the rotating merry-go-round and observes an incoming monochromatic plane wave. While moving around, he sometimes moves towards the incoming wave, then recedes from it, and between these extremes are instances where the normal component of the relative velocity vanishes. Therefore the periodic rotational motion leads in some manner to what is known as frequency modulation (usually referred to as FM). Like the situation encountered in harmonic motion [3], we expect the transformed wave to display some spectrum defined in terms of Bessel functions, as in FM.

Plane waves are characterized by constant vector amplitudes obeying the pertinent FT, in our case (28). Space and time variations are delegated to the appropriate phase exponentials

\[
\theta(r, t) = \theta'(r', t')
\]

although in general the phase is not form-invariant.

Consider an incident plane wave

\[
\mathbf{E} = \hat{z}E_0 e^{i\vartheta}, \quad \mathbf{H} = -\hat{\mathbf{y}}H_0 e^{i\alpha}, \quad E_0 / H_0 = (\mu_0 / \varepsilon_0)^{1/2} = \xi_0
\]

\[
\theta = kr - \omega t = kr \cos \varphi - \omega t, \quad \omega / k = (\mu_0 \varepsilon_0)^{1/2}
\]

propagating in free space (vacuum) with the \( \mathbf{E} \)-field polarized along the cylindrical axis. The constitutive relations in free space are

\[
\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D}' = \varepsilon_0 \mathbf{E}', \quad \mathbf{B}' = \mu_0 \mathbf{H}'
\]

By substitution of (32) in (28), the FT reduce to two equations

\[
\mathbf{E}' = \mathbf{E} + \mu_0 \mathbf{v} \times \mathbf{H}, \quad \mathbf{H}' = \mathbf{H} - \varepsilon_0 \mathbf{v} \times \mathbf{E}
\]

Subject to the FT (33), the plane wave (31) transforms into

\[
\mathbf{E}' = \hat{z}E_0 e^{i\vartheta}(1 + \beta \sin \varphi), \quad \mathbf{H}' = -H_0 e^{i\alpha}(\hat{\mathbf{y}} + \beta \hat{\mathbf{r}}), \quad \beta = \Omega / (cr)
\]

For arbitrary direction of propagation \( \alpha \) in the plane normal to \( \hat{z} \), (34) becomes

\[
\mathbf{E} = \hat{z}E_0 e^{i\vartheta}, \quad \theta = k \cdot r - \omega t = kr \cos(\varphi - \alpha) - \omega t, \quad \mathbf{H} = -\hat{z} \times \hat{k}H_0 e^{i\alpha}
\]

Instead of (34) we now have

\[
\mathbf{E}' = \hat{z}E_0 e^{i\vartheta} - \beta \hat{\mathbf{r}} \times \hat{z} \times \hat{k}E_0 e^{i\alpha} = \hat{z}E_0 e^{i\vartheta}(1 + \beta \sin(\varphi - \alpha))
\]

Hence the only formal difference between (31), (34), and (35), (36), is that \( \varphi \) is replaced by \( \varphi - \alpha \).

In (34) the factor \( e^{i\vartheta}(1 + \beta \sin \varphi) \) is recast as

\[
e^{i(kr \cos \varphi - \omega t)}(1 + \beta \sin \varphi) = e^{-i\alpha} \sum_n J_n(kr)(1 - i \frac{\beta}{2} (e^{i\varphi} - e^{-i\varphi})) e^{i\varphi}
\]

\[
e^{-i\alpha} \sum_n (i^n J_n e^{i\varphi} - i^{n+1} \frac{\beta}{2} J_n e^{i(n+1)\varphi} + i^{n+1} \frac{\beta}{2} J_n e^{i(n-1)\varphi}) = e^{-i\alpha} \sum_n \tilde{J}_n e^{i\varphi}
\]

\[
\tilde{J}_n(kr) = J_n - \frac{\beta}{2} J_{n-1} - \frac{\beta}{2} J_{n+1} = (1 - n\beta / (kr))J_n(kr) = Q_{n\alpha} J_n(kr)
\]

\[
Q_{n\alpha} = 1 - n\Omega k / (c(kr)^2), \quad \Sigma_n = \sum_{n=-\infty}^\infty
\]

where \( J_n(kr) \) denotes the regular Bessel functions and the series is rearranged by raising and lowering the running indices \( n \). The last expression uses a known recurrence relation for the cylindrical functions (e.g., see[7]).

In view of (22), (23), (37), in (34) we now have
where the last expression appears similar to (37) with $\beta = 0$, i.e., the conventional representation for systems at rest. From (38) it follows that we get a discrete frequency spectrum with the central frequency $\omega$ for $n = 0$ and discrete sidebands involving integral values $n$, in agreement with the anticipated discrete Bessel spectrum, as conjectured above. For an arbitrary direction of propagation (36), in (38) $\phi$ is replaced by $\phi - \alpha$.

Arbitrary wave solutions can be constructed as sums or integrals of plane waves with their appropriate amplitudes and propagation directions. In general the directions can be complex, which leads to the Sommerfeld representations of the cylindrical functions \cite{7, 11}, also allowing for the functions that are singular at $r' = 0$.

Consider a superposition integral given as

$$E_s = \frac{\hat{E}_0}{\pi} \int_{\phi_0}^{\phi} e^{i\beta} g(\tau) d\tau$$

where $g(\phi)$ in terms of the real azimuthal angles $\phi$ is now a Fourier series describing the far field scattering amplitude, also referred to as the complex radiation pattern.

According to (36), the transformation of (39) is effected on the plane waves in the integrand, yielding

$$E'_s = \frac{\hat{E}_0}{\pi} \int_{\phi_0}^{\phi} e^{i\beta} g(\tau)(1 + \beta \sin(\phi - \tau)) d\tau, \quad g(\tau) = \Sigma_n a_n e^{i\tau}$$

which can be recast in the form

$$E'_s = \frac{\hat{E}_0}{\pi} \int_{\phi_0}^{\phi} e^{i\beta} \overline{g(\tau)} d\tau, \quad \overline{g(\tau)} = \Sigma_n \overline{a_n} e^{i\tau}$$

achieved by judiciously raising and lowering indices in (42).

The Sommerfeld integral (39) can also be recast as a Bessel-Fourier series \cite{7}, in terms of the Hankel functions of the first kind. Together with the time factor $e^{-i\omega t}$ this defines outgoing waves as in (40)

$$E_s = \frac{\hat{E}_0}{\pi} \int_{\phi_0}^{\phi} e^{i\beta} a_n H_n(kr) e^{i(\omega - \alpha)t}$$

Similarly, (42) prescribes

$$E'_s = \frac{\hat{E}_0}{\pi} \int_{\phi_0}^{\phi} \overline{a_n} H_n(kr') e^{i(\omega - \alpha)t}$$

However as shown in (42), the coefficients $\overline{a_n}$ are still dependent on the azimuthal angle $\phi$, therefore (44) is modified by once again raising and lowering indices.
\[ \Sigma_n \beta \alpha_n H_n e^{i\omega t} = \Sigma_n (i^n a_n H_n e^{i\omega t} - i^{n+1} \frac{\beta}{2} a_{n+1} H_n e^{i(\omega + \phi + \beta) t} + i^{n+1} \frac{\beta}{2} a_{n-1} H_n e^{i(\omega - \phi + \beta) t}) \\
= \Sigma_n (i^n a_n H_n e^{i\omega t} - i^n \frac{\beta}{2} a_{n+1} H_n e^{i\omega t} - i^n \frac{\beta}{2} a_{n-1} H_n e^{i\omega t}) = \Sigma_n i^n a_n \tilde{H}_n e^{i\omega t} \]

(45)

\[ \tilde{H}_n = H_n - \frac{\beta}{2} H_{n+1} - \frac{\beta}{2} H_{n-1} = Q_n H_n(kr) \]

\[ E'_t = \tilde{E}_t \sum_n i^n a_n \tilde{H}_n e^{i\omega t} \]

It is noted that the formulas for \( \tilde{J}_n \) in (37) and \( \tilde{H}_n \) in (45) are identical in structure. This is not surprising in view of the fact that all the cylindrical functions are related, i.e., \( J_n \) is the real part of \( \tilde{H}_n \), and the Neumann function \( N_n \) is defined by an expression (becoming indeterminate for integer \( n \)) in terms of \( J_n, J_{-n} \) (see for example [7] p. 357).

Finally, by inspection of (38), we recast (45) as

\[ E'_t = \tilde{E}_t \sum_n i^n a_n \tilde{H}_n(kr') e^{i\omega t} = \tilde{E}_t \sum_n i^n a_n \tilde{H}_n(kr') e^{i\omega t} e^{-i\phi} = \tilde{E}_t \sum_n i^n a_n \tilde{H}_n e^{i\omega t} e^{-i\phi} \]

(46)

revealing identical spectral and azimuthal structure.

The present analysis demonstrated that by subjecting the ME (24) solutions (35), (43), to the FT (33), we derive the corresponding solutions (38), (46) of the ME (25). The transformations consist of two parts: (a) application of the appropriate FT, and, (b) expressing the results in terms of the native coordinates of the reference system into which the fields are transformed.

5. Scattering by Circular Cylinders

Quite a few studies of scattering by rotating cylinders and spheres have been published, see [2, 4], and references therein. Configurations involve geometries of revolution rotating about a symmetry axis, e.g., a circular cylinder rotating about its axis, a sphere rotating about its polar axis. Such problems have been solved by exploiting the MCR, originally derived for uniformly moving media, see Sommerfeld’s recapitulation [8] of Minkowski’s original work [9] (see in particular [9], p. 75).

The MCR are based on the SR theory [1] which assumes constant velocity between reference systems. The adaptation to scattering by rotating objects is therefore an approximation. It is heuristically based on the presumed validity of the MCR for local instantaneous velocities. Thus in the relevant analyses the scattering problem is solved as if we are dealing with an object at rest, with the velocity effects delegated to the constitutive relations and their effects on the boundary conditions at the surface between free space and the rotating object. Inasmuch as the motion of the scatterer’s surface is parallel to itself, no Doppler frequency shifts appear in the solution. The frequency is determined by the incident monochromatic plane wave and therefore no dispersion effects due to a frequency spectrum are apparent. Thus far this model, based on the MCR, has not been tested, or compared to other models. The present model, based on the QLT for rotation, provides an alternative that must be accepted or rejected, based on future empirical data.

In the present analysis the transformation of the incident excitation wave into the scattering object’s rest frame results in a discrete frequency spectrum (38). Consequently, the scattered and internal waves preserve this spectral structure, and reveal dispersion effects associated with the scatterer’s geometry and material composition.
It is interesting to note (38) that according to \( \omega_n' = \omega - n\Omega / r'^2 = \omega - n\beta / r' \), in the reference frame of the scatterer at rest, the spectral factors \( e^{-in\phi'} \) in the scattered wave (46) reduce to a single frequency \( e^{-in\phi'} \) with increased distance \( r' \) from the scatterer. Thus in the far field Doppler frequency shifts are not present. In this respect the present results are similar with those based on the MCR. However, in the near field at the scatterer’s surface the spectral components exist and must be taken into account. Interestingly, other cases where local Doppler shifts diminish in a global setting, blending into a single frequency, have been noted previously [10].

The simplest problem is suggested by a perfectly conducting circular cylinder of radius \( r' = \rho \) rotating about the axis \( r' = 0 \). The only dispersion effect here is due to the geometry, because material parameters do not feature in solutions involving external parameters \( \varepsilon, \mu \) (approaching 0 or \( \infty \)). An incident plane wave (31) is transformed into the cylinder’s rest frame yielding (37), (38). The scattered wave is chosen as (43), transforming into (45), (46). At the surface \( r' = \rho \) the boundary conditions prescribe a vanishing total tangential field \( E' \), and (38), (46), both refer to the same spectral components \( \omega_n' = \omega - n\Omega / \rho^2 \). Consequently, we can compare the series term by term on a temporal basis. The orthogonality of the conventional Bessel-Fourier series [7], guaranteed by the azimuthal dependence \( e^{in\phi'} \), is not invoked here. The azimuthal variations \( \phi_n' = n\phi' - \omega e^{-2\Omega}\phi' \) are also identical in the present problem, effecting the cancellation of the factor \( e^{in\phi'} \). The coefficients are therefore

\[
a_n = -\frac{J_n(k\rho)}{P_n(k\rho)} = -\frac{J(k\rho)}{P_n(k\rho)} = -\frac{J(k\rho)}{H_n(k\rho)} \quad (47)
\]

where in (47) the factor \( 1 - n\beta / (k\rho) \) established in (37), (45), cancels out in the numerator and denominator. Consequently the coefficients \( a_n \) reduce to the classical solution for a non-rotating cylinder. Inasmuch as (46) has been derived by transforming (43), for the present configuration rotation effects cannot be detected in the initial (unprimed) reference frame. This is expected from the fact that a perfectly conducting surface moving parallel to itself facilitates arbitrary surface currents, i.e., motion of charges, as prescribed by the fields at the surface.

Consider now the interior domain \( r' \leq \rho \), characterized by constitutive parameters \( \varepsilon, \mu \). For an arbitrary single frequency \( \omega_n' \) a pertinent solution of (25) is chosen as

\[
E'_{i,n} = \hat{z}E_0\sum_m i^m b_m J_m(K_n r')e^{im\phi'} , \quad K_n' = \omega_n'\sqrt{\varepsilon_i\mu_i} , \quad \omega_n' = \omega - n\Omega / \rho^2 \quad (48)
\]

with yet undetermined constant coefficients \( b_m \). In a dispersive medium \( \varepsilon_i = \varepsilon_i(\omega_n') , \mu_i = \mu_i(\omega_n') \), i.e., the constitutive parameters depend on the frequency. Note that in (48) we are in the scatterer’s medium rest frame, hence the circular cylindrical solutions of (25) prescribe an azimuthal dependence \( e^{im\phi'} \). For an ensemble of discrete frequencies \( \omega_n' \) the solution is augmented, becoming

\[
E'_{i} = \hat{z}E_0\sum_m i^m b_m J_m(K_n r')e^{im\phi'} \quad (49)
\]

On account of the linearity of the ME, each member of the double series (49) is a proper solution of (25) as well. We now collapse (49) by retaining only terms for which \( m = n \), obtaining

\[
E'_{i} = \hat{z}E_0\sum_i i^i b_i J_{nk} e^{ik\phi'} , \quad K_n' = \omega_n'\sqrt{\varepsilon_i\mu_i} , \quad \omega_n' = \omega - n\Omega / \rho^2 , \quad J_{nk} = J_n (K_n r') \quad (50)
\]
Similarly to the classical problem of scattering by a cylinder (see a similar analysis [12]), the scattering coefficients are obtained by equating the tangential electric and magnetic fields at the surface \( r^\prime = \rho \). For the present case of electric field polarization we get

\[
\bar{J}_n + a_n \bar{H}_n = \bar{b}_n J_{nk}, \quad \bar{J}^\prime_n + a_n \bar{H}^\prime_n = Z \bar{b}_n J_{nk}^\prime
\]

with the prime denoting differentiation of the pertinent functions with respect to the arguments involving \( \rho \). Solving (51) for the coefficients yields

\[
a_n = [J_{nk} \bar{b}_n - Z J_{nk}^\prime \bar{b}_n]/\Delta_n, \quad \bar{b}_n = [\bar{J}_n \bar{H}_n - \bar{J}^\prime_n \bar{H}^\prime_n]/\Delta_n
\]

(52)

In contradiction to (47), in the present case the factor \( Q_{nk} \) introduced in (37), (45), and its derivative with respect to \( k \rho \), where applicable, are not canceled out and velocity effects in \( a_n \) are present. Thusly as the distance \( r^\prime \) from the origin is increased, \( \bar{H}_n (kr) \) approaches \( H_n (kr) \), and \( \bar{E}_n \), (46), approaches \( E_n \), (43). Indeed there are no Doppler frequency shifts that can be detected, but the velocity effects in \( a_n \), (52), are present. By experimentally measuring \( E_n \) or the associated scattering amplitude \( g(\phi) = \Sigma a_n e^{i\omega \phi} \) the Fourier coefficients \( a_n \) can be calculated. Comparing the coefficients \( a_n \) for non-rotating cylinders \( g(\phi, \Omega = 0) \), to those obtained for the rotating cylinder \( g(\phi, \Omega) \) provides information on specific velocity effects.

Consider (52) for cylinders at rest \( \Omega = 0 \). All various cylindrical functions and their derivatives appear in products of two satisfy [7]

\[
Z_n = (-1)^n Z_n, \quad Z_n^\prime = (-1)^n Z_n^\prime, \quad Z_n = J_n, \quad N_n, \quad H_n
\]

(53)

Hence when replacing \( n \) by \( -n \) in (52), sign changes will cancel out, \( a_n = a_{-n} \), and therefore the fields are symmetrical with respect to \( \phi = 0 \)

\[
g(\phi, \Omega = 0) = g(-\phi, \Omega = 0)
\]

(54)

This symmetry breaks down for the rotating cylinders due to \( Q_{nk} \neq Q_{nk} \), hence rotation can be detected.

Furthermore, it is noted (37), (38), (45), (46), (48)-(50), that the factor \( n\Omega \) retains its sign upon simultaneously replacing \( n \) by \( -n \) and \( \Omega \) by \( -\Omega \), therefore

\[
a_n (\Omega) = a_{-n} (-\Omega), \quad g(\phi, \Omega) = g(-\phi, -\Omega)
\]

(55)

Thus upon inverting the direction of rotation, the radiation pattern is flipped through the line \( \phi = 0 \), or in other words, becomes a mirror image. This means that an inversion of the direction of rotation can be detected by measurement.

6. Concluding Remarks

Scattering by rotating objects is of interest for engineering as well as theoretical applications. The present study exploits the QLT for investigating problems involving objects of rotation, in particular circular cylinders, axially rotating about the axis of symmetry.

As expected, there are no Doppler shifts and new frequencies in the scattered wave in the reference system in which the monochromatic incident plane wave is launched. However, in the cylinder’s rest frame the excitation consists of a discrete
multi frequency spectrum, therefore the scattered wave coefficients and the scattering amplitude show velocity effects: for scatterers at rest the scattering amplitude is symmetrical with respect to $\phi = 0$, the direction of the incident wave. Rotation introduces asymmetry with respect to $\phi = 0$, with coefficients and scattering amplitude displaying opposite characteristics, as shown in (55).

The present method is not based on the MCR, hence it is adequate for investigating scattering by arbitrarily shaped rotating objects. Such cases are expected to be characterized by non-vanishing Doppler effects, essentially similar to the vibrating plane reflector [3].

Future work is suggested by eccentrically rotating circular cylinders, and rotating cylinders of elliptical cross section. Rotating eccentric scatterer will provide analytical results simulating such phenomena as scattering by helicopter wings, rotating machinery, etc. It would be interesting to compare results of existing ad-hoc engineering approximations with the mathematically consistent approach.

7. References