RELATIVISTIC INVARIANCE OF DISPERSION-RELATIONS AND THEIR ASSOCIATED WAVE-OPERATORS AND GREEN-FUNCTIONS

Dan Censor

Department of Electrical and Computer Engineering,
Ben-Gurion University of the Negev
84105 Beer-Sheva, Israel

Abstract—Identifying invariance properties helps in simplifying calculations and consolidating concepts. Presently the Special Relativistic invariance of dispersion relations and their associated scalar wave operators is investigated for general dispersive homogeneous linear media. Invariance properties of the four-dimensional Fourier-transform integrals is demonstrated, from which the invariance of the scalar Green-function is inferred.

Dispersion relations and the associated group velocities feature in Hamiltonian ray tracing theory. The derivation of group velocities for moving media from the dispersion relation for these media at rest is discussed. It is verified that the group velocity concept satisfies the relativistic velocity-addition formula. In this respect it is considered to be ‘real’, i.e., substantial, physically measurable, and not merely a mathematical artifact. Conversely, if we assume the group velocity to be substantial, it follows that the dispersion relation must be a relativistic invariant.

Keywords: Relativistic Electrodynamics, Electromagnetic Theory, Electromagnetic Wave Propagation.

1. INTRODUCTION AND OUTLINE

Relativistic Electrodynamics, based on Einstein’s monumental 1905 article [1], facilitates the discussion of wave propagation and scattering in the presence of moving objects and moving media. In this context, invariance properties (in the sense explained below) are of great interest, providing for simple calculations to be carried out on dispersion relations pertinent to reference-frames where media are at rest, and applicable to other reference-frames.

Such invariance property for the dispersion relation has been discussed previously [2]. Recently [3] the invariance property for the dispersion relation in simple media has been discussed and explicitly verified. Presently the invariance properties for a general class of media is studied. This includes the invariance of dispersion relations, the associated wave operators, Green functions in unbounded regions, and properties of relevant group-velocities.

A short outline of Relativistic Electrodynamics, also serving to introduce notation used throughout, is followed by a discussion of the four-fold Fourier transform integrals and their relativistic properties. This allows us to relate results in the spatiotemporal and spectral domains. A discussion of the dispersion relations and wave operators in media
at-rest is followed by the corresponding forms in moving media. The invariance properties are then derived as the confluence of all these subjects. Finally the relativistic invariance properties are implemented in the study of the group velocity concept for arbitrary inertial reference-frames.

2. RELATIVISTIC ELECTRODYNAMICS

Consider the Maxwell equations in source-free domains, given in a reference-frame $\Gamma'$

\[
\partial_t \times \mathbf{E}'(\mathbf{R}') = -\partial_t \cdot \mathbf{B}'(\mathbf{R}'), \quad \partial_t \times \mathbf{H}'(\mathbf{R}') = \partial_t \cdot \mathbf{D}'(\mathbf{R}')
\]

\[
\partial_t \cdot \mathbf{D}'(\mathbf{R}') = 0, \quad \partial_t \cdot \mathbf{B}'(\mathbf{R}') = 0
\]

(1)

where the fields are functions of the native spatiotemporal coordinates $\mathbf{R}'$, grouped into a Minkowski four-space quadruplet $[4, 5]$ $\mathbf{R}' = (t',ict')$, and $\partial_t$ stands for the Nabla symbol, see also [6].

Einstein’s Special Relativity theory [1] postulates the “principle of relativity”, prescribing that in all inertial reference-frames the Maxwell equations are form-invariant, i.e., have the same functional form in terms of the fields and coordinates native to the specific reference-system. Thus in another reference-system $\Gamma$ we have

\[
\partial_t \times \mathbf{E}(\mathbf{R}) = -\partial_t \cdot \mathbf{B}(\mathbf{R}), \quad \partial_t \times \mathbf{H}(\mathbf{R}) = \partial_t \cdot \mathbf{D}(\mathbf{R})
\]

\[
\partial_t \cdot \mathbf{D}(\mathbf{R}) = 0, \quad \partial_t \cdot \mathbf{B}(\mathbf{R}) = 0
\]

(2)

where in (2) the fields are functions of the appropriate spatiotemporal coordinates $\mathbf{R} = (r,ict)$.

The Lorentz transformation, relating coordinates in $\Gamma$ and $\Gamma'$, follows from Einstein’s postulate [1] that $c$, the speed of light in empty space (vacuum), is a constant in all inertial reference-frames. Accordingly

\[
r' = \tilde{U} \cdot (r - v t), \quad t' = \gamma (t - v \cdot r / c^2)
\]

\[
\gamma = (1 - \beta^2)^{-1/2}, \beta = v / c, \quad \tilde{U} = \mathbf{I} + (\gamma - 1) \hat{v} \hat{v}, \quad \hat{v} = \mathbf{v} / v, v = |\mathbf{v}|
\]

(3)

where in (3) $\mathbf{v}$ is the velocity of $\Gamma'$ (the velocity of its origin $r' = 0$, say) as observed in $\Gamma$. The dyadic $\tilde{U}$ multiplies components in the direction of $\mathbf{v}$ by $\gamma$, and $\mathbf{I}$ is the idemfactor dyadic. For sake of brevity (3) can be symbolized as $\mathbf{R}' = \mathbf{R} '[\mathbf{R}]$. Solving (3) for the primed quantities yields the inverse Lorentz transformation $\mathbf{R} = \mathbf{R}'[\mathbf{R}']$, which can be written in form-invariant form with respect to (3) as

\[
r = \tilde{U}' \cdot (r' - v' t'), \quad t = \gamma' (t' - v' \cdot r' / c^2)
\]

\[
v' = -v, \quad \tilde{U}' = \tilde{U}, \quad \gamma' = \gamma
\]

(4)
The application of the chain rule of calculus to (3), (4), yields the Lorenz transformations for space and time differential operators

\[
\frac{\partial_e}{\partial_e} = \hat{U} \cdot (\hat{\partial}_r + \mathbf{v} \cdot \hat{e}_r/c^2), \quad \frac{\partial_e}{\partial_e} = \gamma (\hat{\partial}_r + \mathbf{v} \cdot \hat{e}_r/c^2) \\
\frac{\partial_e}{\partial_e} = \hat{U}' \cdot (\hat{\partial}_r' + \mathbf{v}' \cdot \hat{e}_r'/c^2), \quad \frac{\partial_e}{\partial_e} = \gamma (\hat{\partial}_r' + \mathbf{v}' \cdot \hat{e}_r'/c^2) \tag{5}
\]

where the second line (5) is the inverse with primes and unprimed quantities interchanged, as in (4). Similarly to the Minkowski four-vector \( \mathbf{R} = (r, \, ict) \), the differential operators in (5) can be grouped into a four-gradient Minkowski vector (e.g., see [6])

\[
\hat{e}_R = (\hat{\partial}_r, -i \hat{\partial}_t / c), \quad \hat{e}_R' = (\hat{\partial}_r', -i \hat{\partial}_t' / c) \tag{6}
\]

hence the two lines (5) can be symbolized by \( \hat{e}_R' = \hat{e}_R[\hat{e}_R], \quad \hat{e}_R = \hat{e}_R[\hat{e}_R'] \), respectively.

Combining (1)-(6) yields the transformation equations for the fields

\[
\mathbf{E}' = \tilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{B}' = \tilde{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2) \\
\mathbf{D}' = \tilde{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H} / c^2), \quad \mathbf{H}' = \tilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \tag{7}
\]

\[
\tilde{\mathbf{V}} = \gamma \mathbf{i} + (-1 - \gamma) \mathbf{v} \mathbf{v}
\]

where in (7) the dyadic \( \tilde{\mathbf{V}} \) multiplies all components perpendicular to \( \mathbf{v} \) by \( \gamma \). The formulas given by [1] for free-space (vacuum) have since been extended to material media as well, e.g., see [7]. The inverse of (7) is once again effected by exchanging primed and unprimed quantities, e.g., \( \mathbf{E} = \tilde{\mathbf{V}}' \cdot (\mathbf{E}' + \mathbf{v}' \times \mathbf{B}') \), \( \tilde{\mathbf{V}}' = \tilde{\mathbf{V}}, \mathbf{v}' = -\mathbf{v} \), etc. It is important to note the arguments in (7): all the primed fields, observed (i.e., measured) in \( \mathbf{\Gamma}' \) are originally expressed in terms of the native spatiotemporal coordinates \( \mathbf{R}' \), while the unprimed fields, observed in \( \mathbf{\Gamma} \), are expressed in terms of \( \mathbf{R} \). The Lorentz transformation \( \mathbf{R}' = \mathbf{R} \mathbf{[R]}, \) (3), or \( \mathbf{R} = \mathbf{R} \mathbf{[R]}, \) (4), mediate between spatiotemporal events in the two reference-frames.

3. RELATIVISTIC PROPERTIES OF FOURIER TRANSFORM INTEGRALS

So far we have been working in the spatiotemporal domain. By effecting four-dimensional Fourier transforms we are able to include various constitutive properties, especially dispersion, which are naturally expressed in the spectral domain, and investigate their spatiotemporal domain counterparts.

A four-dimensional Fourier transform is introduced in the form of a four-fold integral [6]

\[
f(x, y, z, i\omega) = q \int dk_x e^{i k_x x} \int dk_y e^{i k_y y} \int dk_z e^{i k_z z} \int d(i\omega / c) e^{i (\omega / c)(i\omega / c)} \\
\mathcal{F}(k_x, k_y, k_z, i\omega / c), \quad q = (2\pi)^{-4} \tag{8}
\]
where in (8) \( f(x, y, z, ict) \) is an arbitrary scalar spatiotemporally-dependent function, and \( \tilde{f}(k_x, k_y, k_z, i\omega / c) \) indicates the corresponding transformed function; the integration limits extend from \(-\infty\) to \(\infty\). Using four-vector notation, (8) can be compacted in the form

\[
f(R) = q \int (d^4K) \tilde{f}(iK)e^{ik \cdot R}, \quad d^4K = dk_xdk_ydk_zd(\text{i}\omega / c) 
\]

\[
R = (r, ict), \quad K = (k, \text{i}\omega / c), \quad K \cdot R = k \cdot r - \omega t 
\]

where in (9) \( K \) is the spectral domain Minkowski four-vector. The notation \( \tilde{f}(iK) \) with an argument \( iK \) will prove to be convenient for subsequent analysis. Henceforth, the dot will indicate an inner product both in the three-, and four-dimensional spaces. No ambiguity can result because the factor vectors refer to the space involved in an obvious manner.

The corresponding inverse Fourier transformation is therefore written as

\[
\tilde{f}(iK) = \int (d^4R)f(R)e^{-ik \cdot R}, \quad d^4R = dx dy dz d(ict) 
\]

where the limits of integration are formally taken from \(-\infty\) to \(\infty\). If \( f(R) \) is causal, existing only for \( t \geq 0 \) within the light cone \( R \cdot R \leq 0 \), the regions where the function vanishes will not contribute to the integral.

Exploiting the properties of the exponential, (9) can be formally recast [6, 8, 9] as

\[
f(R) = q \int (d^4K) \tilde{f}(iK)e^{ik \cdot R} = q \int (d^4K) \tilde{f}(i\partial) e^{i\partial \cdot R} 
\]

\[
= \tilde{f}(i\partial)q \int (d^4K)e^{i\partial \cdot R} = \tilde{f}(i\partial)\delta(\text{i}) 
\]

(11)

describing the four-dimensional delta-function expansion of \( f(R) \). The function \( \tilde{f}(iK) \) and the associated operator function \( \tilde{f}(i\partial) \) are simply related by a formal exchange of symbols \( iK \leftrightarrow i\partial \). The mathematical representation (11) is useful for approximations if the function \( \tilde{f}(i\partial) \) can be truncated to a series of a few operators \( i\partial \). This prescribes slowly varying functions \( \tilde{f}(i\partial) \), represented by a low order polynomial in \( i\partial \).

Rapidly changing \( \tilde{f}(iK) \) may be better approximated by rational functions with numerator and denominator polynomials [9, 10] \( \tilde{f}(iK) = h_n(iK) / h_d(iK) \), leading to

\[
h_d(i\partial) f(R) = h_n(i\partial) \delta(R) 
\]

(12)

Thus far all mention of Minkowski four-vectors involved notation only. At this point the relativistic import of Minkowski’s four-space is included. Accordingly the inner product of two Minkowski four-vectors is an invariant, hence for two reference-frames \( \Gamma \) and \( \Gamma' \) we have in (9)-(11)
In studies discussing Special Relativity where the Minkowski four-space is not used in this context, (13) is still needed. It must then be elevated to a status of a separate postulate, usually referred to as the “phase invariance principle”, e.g., see [11]. It is noted that Einstein [1] assumed (13) tacitly, omitting its postulational status. This was blurred by the soon appearing Minkowski work which introduced four-vectors. On the other hand, Einstein’s postulate regarding the invariance of \( c \), the speed of light in empty space (vacuum), in all inertial reference-frames was blurred by the fact that it is inherent in the Minkowski theory, by virtue of the invariance of the inner product, as in (13), when applied to \( \mathbf{R} \cdot \mathbf{R} = \mathbf{R}' \cdot \mathbf{R}' \).

Substituting \( \mathbf{R}' = \mathbf{R}[\mathbf{R}] \) from (3) into (13) yields the Lorentz transformation for the spectral parameters \( \mathbf{K}' = \mathbf{K}[\mathbf{K}] \) and its inverse \( \mathbf{K} = \mathbf{K}[\mathbf{K}'] \)

\[
\mathbf{k}' = \tilde{\mathbf{U}} \cdot (\mathbf{k} - \mathbf{v} \mathbf{\omega} / c^2), \quad \mathbf{\omega}' = \gamma (\mathbf{\omega} - \mathbf{v} \cdot \mathbf{k})
\]

\[
\mathbf{k} = \tilde{\mathbf{U}} \cdot (\mathbf{k}' - \mathbf{v}' \mathbf{\omega}' / c^2), \quad \mathbf{\omega} = \gamma (\mathbf{\omega}' - \mathbf{v}' \cdot \mathbf{k}')
\]

respectively. In (14) the formulas for \( \mathbf{k}', \mathbf{k} \), are referred to as the Relativistic Fresnel Drag Effect, the corresponding formulas for \( \mathbf{\omega}', \mathbf{\omega} \), are referred to as the Relativistic Doppler Effect. Corresponding to (6) we can derive from (14) four-vectors for derivatives in the spectral domain [6], and similarly to (5), we exploit (14) to construct Minkowski space four-vectors and derive appropriate transformations

\[
\partial_{\mathbf{k}} = \left( \partial_{\mathbf{k}}, -i c \partial_{\mathbf{\omega}} \right), \quad \partial_{\mathbf{k}'} = \left( \partial_{\mathbf{k}'}, -i c \partial_{\mathbf{\omega}'} \right)
\]

\[
\partial_{\mathbf{k}'} = \tilde{\mathbf{U}} \cdot (\partial_{\mathbf{k}} + \mathbf{v} \partial_{\mathbf{\omega}}), \quad \partial_{\mathbf{\omega}} = \gamma (\partial_{\mathbf{\omega}} + \mathbf{v} \cdot \partial_{\mathbf{k}} / c^2)
\]

\[
\partial_{\mathbf{k}} = \tilde{\mathbf{U}} \cdot (\partial_{\mathbf{k}'} - \mathbf{v} \partial_{\mathbf{\omega}'}, \partial_{\mathbf{\omega}} = \gamma (\partial_{\mathbf{\omega}'} - \mathbf{v} \cdot \partial_{\mathbf{k}'} / c^2)
\]

Exploiting (6), (15), the dyadics (matrices) \( \partial_{\mathbf{k}} \mathbf{K}', \partial_{\mathbf{k}} \mathbf{R}' \), respectively, are constructed. The associated Jacobian determinants involved in changing variables in multiple integrals. The value of the Jacobians in the present case is unity, hence we have

\[
d^4 \mathbf{K}' = \det \left[ \partial_{\mathbf{k}} \mathbf{K}' \right] d^4 \mathbf{K} = d^4 \mathbf{K}
\]

\[
d^4 \mathbf{R}' = \det \left[ \partial_{\mathbf{k}} \mathbf{R}' \right] d^4 \mathbf{R} = d^4 \mathbf{R}
\]

From (13), (16), it follows that (9), (10), can be rewritten in the form
\[ f(R) = f(R[R']) = q \int (d^4 \mathbf{K}') \bar{f}(iK[K']) e^{iK[R']} \]
\[ = q \int (d^4 \mathbf{K}') \bar{p}(iK) e^{iK[R']} = p(R') \]
\[ \bar{f}(iK) = \bar{f}(iK[K']) = \int (d^4 \mathbf{R}') f(R[R']) e^{-iK[R']} \]
\[ = \int (d^4 \mathbf{R}') p(R') e^{-iK[R']} = \bar{p}(iK') \] (17)

In (17) we started in \( \Gamma \), expressing \( f(R), \bar{f}(iK) \), in terms of the native coordinates \( R, i\mathbf{K} \). Making the relevant substitutions, everything was then recast in terms of \( \Gamma' \) native coordinates \( R', i\mathbf{K}' \). Note carefully that such a formal conversion of coordinates does not mean that \( p(R'), \bar{p}(iK') \) are the physical fields observed (measured) in \( \Gamma' \): spatiotemporally dependent functions \( f = p \) and their spectral domain transforms \( \bar{f} = \bar{p} \), no matter what their arguments are, are still measured in \( \Gamma \). Starting in \( \Gamma' \) reference-frame yields the analog of (17)

\[ f'(R') = q \int (d^4 \mathbf{K}') \bar{f}'(iK') e^{iK[R']} = f'(R[R]) \]
\[ = p'(R) = q \int (d^4 \mathbf{K}) \bar{p}'(iK)e^{iK[R]} \]
\[ \bar{f}'(iK') = \int (d^4 \mathbf{R}') f'(R') e^{-iK[R']} = \bar{f}'(iK[K]) \]
\[ = \bar{p}'(iK) = \int (d^4 \mathbf{R}) p'(R)e^{-iK[R]} \] (18)

The results (17), (18), are kinematical in nature. The relation of \( f, \bar{f} \), to \( f', \bar{f}' \), respectively requires a dynamic principle, like Einstein’s principle of relativity [1] that led to the field transformations (7), valid in the spatiotemporal as well as the spectral domains [2, 6].

Recapturing the idea of (11), whereby the integrals are replaced by differential operators, (17), (18), can be written as

\[ p(R') = \bar{p}(\partial_R) \delta(R'), p'(R) = \bar{p}'(\partial_R) \delta(R) \] (19)

respectively.

4. DISPERSION RELATIONS AND WAVE OPERATORS IN MEDIA AT-REST

Thus far no medium has been assumed, hence all reference-frames are equivalent, in particular in the present notation, \( \Gamma \) takes no priority over \( \Gamma' \). This changes in the presence of material media. Henceforth \( \Gamma' \) will be the reference-frame for media at-rest.

A quite general class of constitutive relations is characterized in the spectral domain by
with constitutive dyadics $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, which in general are dispersive, i.e., depend on the spectral coordinates $K'$.

Strictly speaking, expressions such as (20), involving a product of terms, are expressed in the spatiotemporal domain in terms of four-fold convolution integrals [6, 9]. Such integrals extend over the entire spatiotemporal domain. To provide for causality, the integrals must be limited to the positive time interval part of the light cone as discussed after (10).

Assuming that a local technique as in (11) is applicable, (20) is substituted in the spectral integral (11) and $iK \Rightarrow \epsilon_r$ is effected in the constitutive dyadics $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, which are then taken outside the integral sign. What remains are the four-dimensional Fourier transforms for the fields (11), finally yielding in the spatiotemporal domain

\begin{align*}
D'(K') &= \tilde{\alpha}(iK') \cdot E'(K') + \tilde{\beta}(iK') \cdot H'(K') \\
B'(K') &= \tilde{\gamma}(iK') \cdot H'(K') + \tilde{\delta}(iK') \cdot E'(K')
\end{align*}  \tag{20}

In the spectral domain (1) takes the form

\begin{align*}
\vec{k} \times E'(K') &= B'(K'), \vec{k} \times H'(K') = -D'(K'), \vec{k}' = k'/\omega'
\end{align*}  \tag{22}

where the vector equations (1) are retained in (22), and the superfluous scalar equations are omitted. Combining (20) with (22) provides a determinate system of twelve scalar homogeneous equations for the Cartesian components of the fields. Eliminating $\vec{B}'$ and $\vec{D}'$ reduces the number of scalar homogeneous equations to six, involving the Cartesian components of $\vec{E}', \vec{H}'$, taking the form

\begin{align*}
\tilde{\phi}' \cdot E'(K') - \tilde{\gamma} \cdot H'(K') &= 0, \tilde{\psi}' \cdot H'(K') + \tilde{\alpha} \cdot E'(K') = 0 \\
\tilde{\phi}' &= \tilde{\phi}'(iK') = \vec{k}' \times \vec{I} - \tilde{\delta}, \tilde{\psi}' = \tilde{\psi}'(iK') = \vec{k}' \times \vec{I} + \tilde{\beta}
\end{align*}  \tag{23}

Inasmuch as the constitutive dyadics $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, are not explicitly specified here, all we can do now is to proceed along general guidelines. Combining $\vec{E}', \vec{H}'$, into a single six-vector $\vec{M}'$, (23) is recast as

\begin{align*}
\begin{bmatrix}
\tilde{\phi}'(iK') & -\tilde{\gamma}(iK') \\
\tilde{\alpha}(iK') & \tilde{\psi}'(iK')
\end{bmatrix}
\begin{bmatrix}
E'(K') \\
H'(K')
\end{bmatrix} = \tilde{G}'(iK') \cdot \vec{M}'(K') = 0
\end{align*}  \tag{24}

The condition of non-triviality, i.e., the assumption that at least one of the six field components comprising $\vec{M}'$ is non-vanishing, prescribes that the determinant of $\tilde{G}'$ vanishes.
The scalar function \( G' \) in (25) is referred to as the dispersion relation. An equivalent derivation of \( G' = 0 \) starts with the six scalar equations (24), which are reduced by successively eliminating unknowns. The last irreducible equation includes any one of the six Cartesian field components denoted by \( M' \), comprising \( M' \). Therefore

\[
G'(iK')M'(K') = 0
\]  

Clearly \( M' \neq 0 \) implies \( G' = 0 \). Obviously we can also write (26) as

\[
G'(iK')M'(K') = 0
\]  

Applying to (26), (27), the four-dimensional integral and implementing the technique (11) yields the spatiotemporal domain scalar wave equation

\[
q \int (d^4K')M'(K')G'(iK')e^{ik'R'} = G'(\partial_{R'})M'(R') = 0
\]

where in (28) \( M'(R') \) is the spatiotemporal domain Fourier-transformed function associated with the spectral domain \( M'(K') \). The same symbol \( M' \) has been used for functions in the spatiotemporal and spectral domains for brevity. It follows that \( G'(\partial_{R'}) \) is the wave operator, applicable to all six Cartesian field components.

5. DISPERSION RELATIONS AND WAVE OPERATORS IN MOVING MEDIA

In \( \Gamma \) the medium is observed in motion. The pertinent Maxwell equations are now given by (2). When it comes to constitutive relations, our aim is not to try and provide transformations for the constitutive dyadics \( \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \), (20) per se, as attempted elsewhere [12-15], but rather to provide relations between fields in \( \Gamma \), which will render the system of equations determinate, as given for \( \Gamma' \) in (23). To this end we use Minkowski’s theory [4], also discussed, with some historical notes, by Sommerfeld [16].

Minkowski’s approach is to assume the constitutive relations as given, characterizing the medium in question in its rest frame \( \Gamma' \), i.e., (20) in our case. Substituting (7) in (20) yields the Minkowski constitutive relations [4, 16] for the present case, in the spectral domain in \( \Gamma \).
\[ D(K) + v \times H(K) / c^2 \]
\[ = \tilde{\alpha}_x(iK) \cdot (E(K) + v \times B(K)) + \tilde{\beta}_y(iK) \cdot (H(K) - v \times D(K)) \]
\[ B(K) - v \times E(K) / c^2 \]
\[ = \tilde{\gamma}_y(iK) \cdot (H(K) - v \times D(K)) + \tilde{\delta}_y(iK) \cdot (E(K) + v \times B(K)) \]
\[ \tilde{\eta}_x(iK) = \tilde{V}^{-1} \cdot \tilde{\eta}(iK[K]) \cdot \tilde{V}, \quad \tilde{V}^{-1} = \tilde{U} / \gamma, \quad \tilde{\eta} = \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \]

In (29) the modified constitutive dyadics are created by pre-, post-, multiplication of the original dyadics by \( \tilde{V}^{-1}, \tilde{V} \), respectively. The modified constitutive parameters are initially functions of \( iK' \). By substitution of \( K' = K[K] \) from (14), the modified constitutive dyadics are obtained as functions of the argument \( iK \). We have already discussed the fact that the fields, and not their arguments expressed in terms of either \( \Gamma \) or \( \Gamma' \) coordinates, determine whether observations, i.e., measurements, are performed in \( \Gamma \) or \( \Gamma' \). In (29) we are dealing with unprimed fields, therefore we are residing in reference frame \( \Gamma \).

The analog of (21) is now
\[ D(R) + v \times H(R) / c^2 \]
\[ = \tilde{\alpha}_x(\partial_r) \cdot (E(R) + v \times B(R)) + \tilde{\beta}_y(\partial_r) \cdot (H(R) - v \times D(R)) \]
\[ B(R) - v \times E(R) / c^2 \]
\[ = \tilde{\gamma}_y(\partial_r) \cdot (H(R) - v \times D(R)) + \tilde{\delta}_y(\partial_r) \cdot (E(R) + v \times B(R)) \]
\[ D(R) + v \times H(R) / c^2 \]

Instead of (22) we have its \( \Gamma \) analog by simply removing apostrophes
\[ \bar{k} \times E(K) = B(K), \quad \bar{k} \times H(K) = -D(K), \quad \bar{k} = k / \omega \]

Combining (31) with (29) yields the analog of (23)
\[ \phi \cdot E(K) - \tilde{\gamma}_y \cdot \tilde{\mu} \cdot H(K) = 0, \quad \psi \cdot H(K) + \tilde{\alpha}_x \cdot \tilde{\mu} \cdot E(K) = 0 \]
\[ \phi = \tilde{\phi}(iK) = \bar{k}' \times \bar{I} - \tilde{\delta}_y \cdot \tilde{\mu}, \quad \psi = \tilde{\psi}(iK) = \bar{k}' \times \bar{I} + \tilde{\beta}_y \cdot \tilde{\mu} \]
\[ \bar{k} \times E(K) = B(K), \quad \bar{k} \times H(K) = -D(K), \quad \bar{k} = k / \omega \]

Note the formal similarity of the last expression in (32) and the Fresnel Drag Effect (14).

In the limit \( v = 0 \) (32) reduces to (23). Once again (32) constitutes a system of six scalar homogeneous equations in terms of the six Cartesian components of the field vectors \( E, H \).

The analog of (24) is now
leading to the analogs of (25)-(28)

\[
\det[\hat{G}'(iK')] = G'(iK') = 0
\]

\[
G(iK)M(K) = 0, G(\hat{\nu})M(R) = 0
\] (34)

With this we finish the derivation of the dispersion relations, and move on to discuss the problem of relativistic invariants.

6. RELATIVISTIC INVARIANCE PROPERTIES

Recognizing invariance properties is an important part of any mathematical physics discipline, usually leading to simplified manipulation and notation, much like solving a complicated differential equation by implementation of a clever change of variable.

Incorporating (31) and (7) in (27) yields

\[
G'(iK')M'(K') = G'(iK')\hat{V} \cdot \hat{\mu} \cdot M(K) = \hat{V} \cdot \hat{\mu} \cdot [G'(iK')M(K)] = 0
\] (35)

where in (35) \( G' \) and \( \hat{V} \cdot \hat{\mu} \) have been interchanged. This is feasible because of the scalar nature of \( G' \). Inasmuch as the determinants of \( \det[\hat{V}] \neq 0, \det[\hat{\mu}] \neq 0 \) are non-vanishing, therefore similarly to (24) it follows that

\[
G'(iK')M(K) = G'(iK'[K])M(K) = \widetilde{G}(iK)M(K) = 0
\] (36)

where by virtue of the unprimed fields \( M \), the dispersion-relation \( \widetilde{G}(iK) \) applies in the \( \Gamma' \) reference-frame. By comparing (34), (36), it becomes clear that

\[
G'(iK') = G'(iK'[K]) = \widetilde{G}(iK) = G(iK) = 0
\] (37)

i.e., the \( \Gamma' \) dispersion-relation \( G' \) uniquely determines the \( \Gamma \) counterpart \( G = G' = 0 \).

Expressing either in terms of \( K \) or \( K = K[K'] \) is simply a matter of convenience. An explicit calculation of the dispersion-relation invariance in simple media has been carried out in [3].

For the wave operator it follows in a consistent manner
\[ G'(\partial_R) = G'(\partial_R[\partial_R]) = G(\partial_R) \]
\[ G'(\partial_R)M'(R') = G'(\partial_R[\partial_R])M'(R'[R]) \]
\[ = G(\partial_R)M'(R'[R]) = \vec{V} \cdot \vec{\mu} \cdot [G(\partial_R)M(R)] = 0 \]  \hspace{1cm} (38)
\[ G(\partial_R)M(R) = G(\partial_R[\partial_R])M(R[R']) \]
\[ = G'(\partial_R)M(R[R']) = \vec{V} \cdot \vec{\mu}' \cdot [G'(\partial_R)M'(R')] = 0, \quad \vec{\mu}' = \vec{I} - v \times \vec{R} \times \vec{I} \]

hence the scalar wave operators are invariants as well, and the consequent scalar wave equations are shown in (38).

7. INVARIANCE OF THE FREE SPACE SCALAR GREEN FUNCTION

For an bounded less region in the rest-frame \( \Gamma \), the appropriate scalar Green function is defined by the inhomogeneous wave equation

\[ G'(\partial_R)g'(R') - \delta(R') = 0 \]  \hspace{1cm} (39)

together with some causality condition or radiation condition [17], in order to distinguish waves emanating from the singularity \( R' = 0 \), from the case of (mathematically possible) waves converging onto the singularity from infinity. The definition (39) used here is slightly different from the widely used \( G'(\partial_R)g'(R') = -\delta(R') \). Also note that usually instead of \( \partial_R \) we have \( \partial_{R'} - \rho' \) with some fixed \( \rho' \), such that \( \partial_{R' - \rho'} = \partial_R \). All this does not affect the generality of our argument.

Applying to (39) the Fourier transform (11) in \( \Gamma' \), we obtain

\[ G'(\partial_R)g'(R') - \delta(R') = q \int (d^4K') [G'(iK')g'(K') - 1] e^{K \cdot R'} = 0 \]  \hspace{1cm} (40)

implying in the spectral domain

\[ g'(K') = 1 / G'(iK') \]  \hspace{1cm} (41)

Applying the integration operator in (40) to (41) yields in the spatiotemporal domain

\[ q \int (d^4K') g'(K') e^{K \cdot R'} = g'(R') = q \int (d^4K') G^{-1}(iK') e^{K \cdot R'} \]  \hspace{1cm} (42)

where we do not address some finer points regarding the convergence of the integral [17]. Suffice it to say that the zeroes of \( G' \) become poles in (42), determining \( g' \) in the spatiotemporal domain.

The invariance of the dispersion relation (37), and the properties of the four-fold integrals discussed above, e.g., (17), prescribe for (42)
\[ g'(\mathbf{R}') = q \int (d^4\mathbf{K}) G^{-1}(i\mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{R}} = g'(\mathbf{R}[\mathbf{R}]) = q \int (d^4\mathbf{K}) G^{-1}(i\mathbf{K}[\mathbf{K}]) e^{i\mathbf{K} \cdot \mathbf{R}} = q \int (d^4\mathbf{K}) G^{-1}(i\mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{R}} = g(\mathbf{R}) \] (43)

i.e., having been given \( g'(\mathbf{R}') \), by substituting \( \mathbf{R}'[\mathbf{R}] \) we obtain \( g'(\mathbf{R}') = g'(\mathbf{R}[\mathbf{R}]) = g(\mathbf{R}) \), and thus the invariance of the unbounded domain scalar Green function is established. In electromagnetics, \( g', g \), can be associated with Cartesian components of the fields.

8. THE GROUP VELOCITY IN \( \Gamma' \) AND \( \Gamma \)

One reason for deriving dispersion relations is to trace rays in various media, exploiting the Hamiltonian ray-tracing theory, e.g., see [6, 17-20]. By definition, rays are paths whose tangent at any point is parallel to the group velocity direction at that point.

It is tempting to think of rays as geometrical entities, and assume that in a moving medium \( \Gamma \) we observe the same rays of the medium at rest in \( \Gamma' \), except that the ray paths are now moving, subject to the Lorentz transformation (3). This is a misconception. The moving medium in \( \Gamma' \) behaves as a different medium, and rays in this medium will be governed by the appropriate dispersion relation (34). However, the associated group velocity does behave as a substantial physical quantity, i.e., \( \mathbf{u} = \mathbf{dr}/dt, \mathbf{u}' = \mathbf{dr'}/dt' \), in \( \Gamma, \Gamma' \), respectively, and is thus subject to the relativistic velocity-addition formula, as demonstrated below.

Group velocities are computed from the relevant dispersion relations according to

\[ \mathbf{v}_g = \partial_{\mathbf{k}} \Omega(\mathbf{k}) = -\partial_{\mathbf{\omega}} G(i\mathbf{K}) / \partial_{\mathbf{\omega}} G(i\mathbf{K}) \]
\[ \mathbf{v}'_g = \partial_{\mathbf{k}} \Omega'(\mathbf{k}') = -\partial_{\mathbf{\omega}} G'(i\mathbf{K}') / \partial_{\mathbf{\omega}} G'(i\mathbf{K}') \] (44)

where in (44) \( \Omega, \Omega' \), are relevant if the dispersion-relations can be recast as \( G(i\mathbf{K}) = \omega - \Omega(\mathbf{k}) = 0, G'(i\mathbf{K}') = \omega' - \Omega'(\mathbf{k}') = 0 \) in \( \Gamma, \Gamma' \), respectively.

Usually it is simpler to evaluate \( G' \) (25) in the medium rest-frame \( \Gamma' \) rather than directly solve for \( G \) (34) in \( \Gamma \). Hence the invariance \( G = G' \) in (37) and the transformations (15) are exploited to calculate \( \mathbf{v}_g \) in (44)

\[ \mathbf{v}_g = -\partial_{\mathbf{k}} G(i\mathbf{K}) / \partial_{\mathbf{\omega}} G(i\mathbf{K}) \]
\[ = -[\mathbf{\bar{U}} \cdot (\partial_{\mathbf{k}} - \mathbf{v} \partial_{\mathbf{\omega}}) G'(i\mathbf{K}')] / [\gamma (\partial_{\mathbf{\omega}} - \mathbf{v} \cdot \partial_{\mathbf{k}} / c^2) G'(i\mathbf{K}')] \] (45)

The relativistic velocity-addition formula relating velocities in \( \Gamma, \Gamma' \), respectively, follows from (3), (4). Upon taking differentials and dividing, we obtain

\[ \mathbf{u}' = \mathbf{\bar{U}} \cdot (\mathbf{u} - \mathbf{v}) / (\gamma (1 - \mathbf{v} / c^2)) \]
\[ \mathbf{u} = \mathbf{\bar{U}} \cdot (\mathbf{u}' + \mathbf{v}) / (\gamma (1 + \mathbf{v} \cdot \mathbf{u}' / c^2)) \] (46)
Identifying \( \mathbf{u} = \mathbf{v}_g \), \( \mathbf{u}' = \mathbf{v}'_g \) and substituting from (44) into (46), then exploiting (15) and the invariance (37), verifies that group velocities are ‘real’, i.e., substantial physical velocities, satisfying the relativistic velocity-addition formula (46)

\[
\mathbf{v}'_g = \mathbf{\bar{U}} \cdot (\mathbf{v}_g - \mathbf{v}) / (\gamma - \mathbf{v} \cdot \mathbf{v}_g / c^2)
\]

\[
= -[\mathbf{\bar{U}} \cdot (\partial_k + \mathbf{v} \partial_{\omega}) G(i\mathbf{K})] / [\gamma (\partial_{\omega} + \mathbf{v} \cdot \partial_k / c^2) G(i\mathbf{K})]
\]

\[
= -\partial_k G'(i\mathbf{K}') / \partial_{\omega} G'(i\mathbf{K}')
\]

Conversely, if we assume that the group velocity is a substantial measurable quantity, obeying the relativistic velocity-addition formula, then it follows that the dispersion-relation is a relativistic invariant.

9. SUMMARY AND CONCLUDING REMARKS

In the present study the relativistic invariance properties of dispersion-relations are investigated. The media chosen are very general: dispersive, homogeneous, bi-anisotropic (in a very broad sense), and linear.

The invariance of the dispersion-relations in arbitrary inertial reference systems has been demonstrated. The invariance properties of the four-dimensional Fourier-transform integrals facilitated the discussion of the invariance of the associated wave operators. Furthermore, the properties of the integrals enabled us to show that the scalar Green functions associated with the dispersion-relations also constitute relativistic invariants.

Dispersion-relations are needed for the implementation of the Hamiltonian ray tracing theory. At the heart of this theory is the fact that rays are paths whose tangent at any point is parallel to the group-velocity at that point. Exploiting the invariance \( G = G' \), it has been shown that the group-velocity in the moving medium, as observed in \( \Gamma' \), can be derived from the medium’s rest-frame dispersion-relation in \( \Gamma' \). This offers a simpler method for computing the group-velocity. While ray paths are not geometrical entities subject to the spatiotemporal Lorentz transformation, the group-velocities computed in \( \Gamma', \Gamma' \), do obey the relativistic velocity-addition formula, subject to the invariance \( G = G' \). Conversely, if the transformation formula for velocities is assumed, the invariance \( G = G' \) follows.

ACKNOWLEDGMENT

I am grateful to Prof. Martin W. McCall, Department of Physics, The Blackett Laboratory, Imperial College London, for reading the article and making many helpful comments. Any errors are of course my responsibility.

REFERENCES