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The Need for a First-order Quasi Lorentz Transformation

D. Censor

Department of Electrical and Computer Engineering, Ben–Gurion University of the Negev
84105 Beer–Sheva, Israel

Abstract. Solving electromagnetic scattering problems involving non-uniformly moving objects or media requires an approximate but consistent extension of Einstein’s Special Relativity theory, originally valid for constant velocities only. For moderately varying velocities a quasi Lorentz transformation is presented. The conditions for form-invariance of the Maxwell equations, the so-called “principle of relativity”, are shown to hold for a broad class of motional modes and time scales. A simple example of scattering by a harmonically oscillating mirror is analyzed in detail. Application to generally orbiting objects is mentioned.

Keywords: Maxwell equations, special relativity, wave scattering, moving objects.

PACS: 03.50.De, 03.30.+p, 41.20.Jb

INTRODUCTION AND RATIONALE

Einstein’s Special Relativity (SR) theory [1] is based on the invariance of \( c \), the speed of light in free space (vacuum), for all inertial observers. The other postulate so called “principle of relativity” (PR) for the Maxwell Equations (ME), is discussed below. From the invariance of \( c \) follow [1] the spatial and temporal Lorentz transformations (LT) for inertial reference systems \( \Gamma \) and \( \Gamma' \) and their related space-time coordinates \( r, t \), and \( r', t' \), respectively

\[
\begin{align*}
    r' &= \hat{U} \cdot (r - vt), \\
    t' &= \gamma(t - v \cdot r / c^2) \\
    \hat{U} &= \hat{I} + (\gamma - 1)\hat{v} \hat{v}, \\
    \hat{v} &= v / v, \quad \beta = v / c, \quad c = (\mu_0 \varepsilon_0)^{1/2}, \quad v = |v|
\end{align*}
\]

(1)

written in coordinate-independent form. In (1) \( \hat{I} \) is the idemfactor dyadic, and \( \hat{v} \hat{v} \) is a nine component dyadic denoting the normalized metric outer product \([v][v]^{T} / |v|^2\) of the row vector \([v]\). The dyadic \( \hat{U} \) serves to multiply all terms parallel to \( v \) by \( \gamma \).

The corresponding inverse transformations \((1*)\) are derived by algebraically solving the initial formulas (1) for the sought variables \( r, t \). Formally, inverse transformations follow by simply interchanging primed and unprimed symbols and setting \( v' = -v \). Thus

\[
\begin{align*}
    r &= \hat{U} \cdot (r' - v't'), \\
    t &= \gamma(t' - v' \cdot r' / c^2), \\
    v' &= -v
\end{align*}
\]

(1*)

Henceforth inverse formulas will be referred to by an asterisk, whether they are written out explicitly or only mentioned.
So far the LT (1) is restricted to constant velocities \( \mathbf{v} \). Upon trying to formulate a systematic model for varying velocities, at least one reference frame ceases to be inertial. We cannot claim that such an approximate model is applicable to arbitrarily large velocities, therefore the model validity will be restricted to a first order approximation in \( \beta \). Furthermore, for the model to be simple and tractable, the effects of the varying velocities are viewed as kinematical only, i.e., no attempt will be made to incorporate the acceleration effects into the media properties and/or the ME.

Such a model based on the d’Alembert solution of the one-dimensional wave equation has been proposed in the literature [2]. It was exploited for analyzing scattering by harmonically moving mirrors [3]. Other attempts of incorporating varying velocities, based on different modeling, have been made too, e.g., see [4] and references mentioned therein.

A very suggestive methodology is to approximate the motion by a sequence of discrete systems, each possessing its local/instantaneous velocity \( \mathbf{v}_i = \mathbf{v}_i(\mathbf{r}_i, t_i) \). Thus the instantaneous velocity \( \mathbf{v}_i \) is assumed to be valid for a limited region of space-time in the vicinity of the world event \( \mathbf{r}_i, t_i \). When the discrepancy becomes too large the velocity is updated. Figuratively this is like having a sequence of discrete frames in a movie. Accordingly (1) would be replaced by

\[
\mathbf{r}' = \mathbf{\bar{U}}(\mathbf{v}_i) \cdot (\mathbf{r} - \mathbf{v}_i t),
\]

\[
t' = \gamma(\mathbf{v}_i)(t - \mathbf{v}_i \cdot \mathbf{r} / c^2).
\]

As a multi-scale method (2) might be plausible for systems where the velocity changes slowly and monotonically over space-time regions large in comparison to other system parameters. However, this assumption holds only for a restricted class of configurations. It does not hold, for example, in the case of a wave emitted by an oscillating antenna. Intuitively one expects the antenna motion to cause a Doppler effect which during the mechanical cycle creates higher and lower frequencies. The results would then be akin to a frequency modulated carrier wave of frequency \( \omega \), creating sidebands at frequencies \( \omega \pm n\Omega \), with \( \Omega \) corresponding to the mechanical frequency. This is a result that cannot be displayed when the local instantaneous velocity concept is employed, because for each incremental change of \( \mathbf{v}_i \) a different Doppler frequency shift is created, and the resulting continuous spectrum does not merge into the expected discrete sideband frequencies. Our goal is to find a first order quasi LT (QLT) that will satisfactorily deal with scattering by objects moving at varying velocities.

**FIRST ORDER SPECIAL RELATIVITY AND THE QUASI LORENTZ TRANSFORMATION**

First order SR is derived by keeping in the LT and the ME only first order terms in \( \beta \). Due care is called for, to guarantee that the ensuing approximations are mathematically consistent. Application to (1) prescribes \( \gamma = 1 \), yielding

\[
\mathbf{r}' = \mathbf{r} - \mathbf{v} t, t' = t - \mathbf{v} \cdot \mathbf{r} / c^2
\]
It is inconsistent to simply neglect terms involving $c^2$, arguing that these are negligible for low velocities. This becomes clear upon rewriting (3) in terms of normalized and dimensionless $\tau = ct$, $\beta$, $\mathbf{v}$

$$
\mathbf{r}' = \mathbf{r} - \beta \mathbf{v} \tau, \tau' = \tau - \beta \mathbf{v} \cdot \mathbf{r}
$$

(4)

showing that a consistent approximation leads to (3). Consequently the Galilean transformation (GT)

$$
\mathbf{r}' = \mathbf{r} - \mathbf{v} t, \ t' = t
$$

(5)

cannot be considered as the limiting case of the LT (1) for small $\beta$, rather it constitutes a limiting case for (3) for $c \to \infty$ [5]. The ME predict the existence of electromagnetic waves propagating in free space at a finite speed $c$, encompassing lightwave phenomena which are supported by numerous empirical observations. From this point of view, the GT is incompatible with the ME and inadequate for describing global space-time phenomena.

Consider the differential representation

$$
\frac{d\mathbf{r}'}{d\tau'} = \mathbf{U} \cdot (d\mathbf{r} - \mathbf{v} dt), \ dt' = \gamma (dt - \mathbf{v} \cdot d\mathbf{r} / c^2)
$$

(6)

differing from (1) only within a trivial constant of integration that can be taken as zero. Hence (1), (6) are equivalent. Similarly the first order representation (3) is equivalent to

$$
\frac{d\mathbf{r}'}{d\tau'} = d\mathbf{r} - \mathbf{v} dt, \ dt' = dt - \mathbf{v} \cdot d\mathbf{r} / c^2
$$

(7)

within a trivial integration constant, and can replace (3) for all purposes.

Any digression from constant $\mathbf{v}$ constitutes an approximation. The choice of an approximate model is not unique, and its range of validity is subject to arguments based on available physical experimental data. In the absence of such data, the model is tentative and only its mathematical validity can be scrutinized.

Consider replacing in (2) $\mathbf{v}$ or in (3) $\mathbf{v}$ by a varying velocity $\mathbf{v}$. For completeness let $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ depend on space and time. The differential form of (3) would then become

$$
\frac{d\mathbf{r}'}{d\tau'} = d\mathbf{r} - \mathbf{v} dt - td\mathbf{v}, \ dt' = dt - \mathbf{v} \cdot d\mathbf{r} / c^2 - r \cdot d\mathbf{v} / c^2
$$

(8)

involving higher order differentials $d\mathbf{v} = d(d\mathbf{r} / dt)$, which in the present case, based on physical arguments, is considered as negligible for the problems at hand. It boils down to assuming negligible acceleration effects and treating varying velocities as a kinematical phenomenon, both in in the LT and the ME.

Thus the differential first order QLT is postulated as

$$
\frac{d\mathbf{r}'}{d\tau'} = d\mathbf{r} - \mathbf{v}(\mathbf{r}, t) dt, \ dt' = dt - \mathbf{v}(\mathbf{r}, t) \cdot d\mathbf{r} / c^2.
$$

(9)

The choice was inspired by the fact that for constant $\mathbf{v}$ (9), reduce as a limiting case to (7), but does not involve $d\mathbf{v}$ as in (8).

Furthermore, in the present model $\mathbf{v}(\mathbf{r}, t)$ is stipulated as an irrotational (conservative) vector field

$$
\mathbf{D} \times \mathbf{v}(\mathbf{r}, t) = 0
$$

(10)

where $\mathbf{D}$ denotes the “Nabla” operator $\nabla$. From (10) it follows that $\mathbf{v}$ is representable as the gradient of a scalar field

$$
\mathbf{v}(\mathbf{r}, t) = \mathbf{D} \Phi(\mathbf{r}, t)
$$

(11)
hence in (9)
\[ \mathbf{v}(\mathbf{r}, t) \cdot d\mathbf{r} = \partial_t \Phi(\mathbf{r}, t) \cdot d\mathbf{r} = d\Phi(\mathbf{r}, t) \]  
(12)
is an exact differential. Consequently (9), can be recast in terms of \( \Phi(\mathbf{r}, t) \) as
\[ d\mathbf{r}' = d\mathbf{r} - \partial_t \Phi(\mathbf{r}, t) dt, \quad dt' = dt - d\Phi(\mathbf{r}, t) / c^2. \]  
(13)

Inasmuch as \( \Phi(\mathbf{r}, t) \), defined according to (11), (12), is already of first order in \( \beta \), associated factors and arguments can be taked to order \( \beta^0 \), i.e.,
\[ \mathbf{v}'(\mathbf{r}', t') = -\mathbf{v}(\mathbf{r}, t), \quad \Phi'(\mathbf{r}', t') = -\Phi(\mathbf{r}, t). \]  
(14)
Hence the inverse transformations are given by
\[ d\mathbf{r} = d\mathbf{r}' - \partial_r \Phi(\mathbf{r}', t') dt', \quad dt = dt' - d\Phi'(\mathbf{r}', t') / c^2. \]  
(13*)

The integration of the differentials in the second equation (13) yields immediately
\[ t' = t - \Phi(\mathbf{r}, t) / c^2 \]  
(15)
which in view of \( \Phi = \int d\Phi \) is rewritten as
\[ t' = t - \int d\Phi / c^2 = t - \int' \mathbf{v}(\mathbf{r}, t) \cdot d\mathbf{r} / c^2. \]  
(16)
In (16) \( \mathbf{r} \) denotes the integration variable, vanishing after integration, and in view of \( d\Phi \) being an exact differential, the last integration in (16) depends on the end points only. It should be noted that the integration in (16) is performed with respect to \( \mathbf{r} \) while \( t \) is treated as unvarying, i.e., \( \mathbf{v}(\mathbf{r}, t) |_{\mathbf{r}} \). Consequently taking differentials in (16), using the Leibnitz rule differentiating integrals, yields back the differential form in the second equation (13), hence also (9).

The first equation (13) can be rewritten as
\[ d(\mathbf{r}' - \mathbf{r}) / dt = -\partial_r \Phi(\mathbf{r}, t). \]  
(17)
Integrating both sides of (17) with respect to \( t \) the yields
\[ \mathbf{r}' = \mathbf{r} - \int \partial_r \Phi(\mathbf{r}, t) dt = \mathbf{r} - \partial_r \int \Phi(\mathbf{r}, t) dt = \mathbf{r} - \int' \mathbf{v}(\mathbf{r}, t) dt \]  
(18)
where upon integration the integration variable \( t \) vanishes. Once again it is noted that in (18) the integration is with respect to \( t \) while \( \mathbf{r} \) is held unvarying during integration, i.e., \( \mathbf{v}(\mathbf{r}, t) |_{\mathbf{r}} \). Accordingly, taking differentials in (18) yields back the differential form in the first equation (13), hence also (9).

One might wish to consider \( \mathbf{v}(\mathbf{r}, t) \) as a function in four-space. Accordingly, (18) prescribes an integration along the \( t \)-axis with \( \mathbf{r} \) held constant, while (16) prescribes an integration in the subspace \( \mathbf{r} \) with \( t \) held constant. Like in (1), (3), etc. for constant \( \mathbf{v} \), in equations (16), (18), we deal with two quadruplets \( \mathbf{r}, t \), and \( \mathbf{r}', t' \), related by the two equations. To refer to these quadruplets as points in four-space is optional.

A less restrictive case is afforded by spatially-independent, time-dependent velocities \( \mathbf{v}(t) \). For this case the temporal transformation (16) trivially integrates into the corresponding first equation (3), with constant \( \mathbf{v} \) replaced by \( \mathbf{v}(t) \). Note that \( \mathbf{v}(t) \) trivially satisfies (10), hence \( \mathbf{v}(t) \) is not restricted to be an irrotational (conservative) vector field. This point is important for special cases where the velocity of scatterers depends on time only.
THE PRINCIPLE OF RELATIVITY

Einstein [1] postulated the PR, namely the form-invariance of the ME. For simplicity we confine the discussion to source-free regions. Thus in $\Gamma$ the ME are given by
\[
\partial_r \times E + \partial_t B = 0, \quad \partial_r \times H - \partial_t D = 0, \quad \partial_r \cdot D = 0, \quad \partial_r \cdot B = 0.
\] (19)
In general all the fields are space and time dependent, e.g., $E = E(r, t)$.

According to the PR, in another inertial system $\Gamma'$ the form-invariance of the ME (19) is preserved
\[
\partial_{r'} \times E' + \partial_{t'} B' = 0, \quad \partial_{r'} \times H' - \partial_{t'} D' = 0, \quad \partial_{r'} \cdot D' = 0, \quad \partial_{r'} \cdot B' = 0
\] (20)
formally obtained from (19) by replacing $\partial_r$, $\partial_t$, by $\partial_{r'}$, $\partial_{t'}$, respectively, and the fields, e.g., $E = E(r, t)$, by $E' = E'(r', t')$, etc.

The mere postulation that the PR must exist is incomplete. The implementation of the PR to the ME prescribes field transformation (FT) formulas, relating fields observed in $\Gamma$ and $\Gamma'$. The derivation of the FT facilitates the discussion of problems involving reference systems in relative motion. A full SR derivation is given for example by Kong [5]. Presently we need the FT only to the first order in $\beta$, which is simpler to derive. The chain rule of calculus in space and time is written as
\[
\partial_{r'} = (\partial_r \cdot r') \partial_r + (\partial_t \cdot t') \partial_t, \quad \partial_{t'} = (\partial_t \cdot t') \partial_t + (\partial_r \cdot r') \partial_r.
\] (21)
Like $\hat{\mathbf{v}}$ in (1), $\partial_{r'} \mathbf{r}$ in (21) denotes the outer product of the vectors $\partial_r$, $\mathbf{r}$. Computing the partial derivatives in parenthesis (21) from (1*), (3*), yields
\[
\partial_r = \hat{U} \cdot (\partial_r + \mathbf{v} \partial_r / c^2), \quad \partial_{t'} = \gamma (\partial_{t'} + \mathbf{v} \cdot \partial_{t'})
\] (22)
\[
\partial_{r'} = \partial_r + \mathbf{v} \partial_r / c^2, \quad \partial_{t'} = \partial_t + \mathbf{v} \cdot \partial_r
\] (23)
respectively, specifying the LT for the space-time differential operators. The corresponding differential operators for the GT are obtained by taking in (22), (23), the limit $c \to \infty$.

Substituting (22*) in (19) and collecting terms, (20) is obtained subject to the FT
\[
\mathbf{E}' = \hat{\mathbf{V}} \cdot (E + \mathbf{v} \times \mathbf{B}), \quad \mathbf{B}' = \hat{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times E / c^2),
\] (24)
\[
\mathbf{D}' = \hat{\mathbf{V}} \cdot (D + \mathbf{v} \times H / c^2), \quad \mathbf{H}' = \hat{\mathbf{V}} \cdot (H - \mathbf{v} \times D),
\]
where $\hat{\mathbf{V}} = \gamma \hat{\mathbf{I}} + (1-\gamma) \mathbf{v} \mathbf{v}$ in (24) multiplies field components perpendicular to $\mathbf{v}$ by $\gamma$. Conversely, substituting (22) in (20) and collecting terms, (19) is obtained subject to the FT (24*). Keeping in (24) only first order $\beta$ terms yields
\[
\mathbf{E}' = E + \mathbf{v} \times \mathbf{B}, \quad \mathbf{B}' = B - \mathbf{v} \times E / c^2,
\] (25)
\[
\mathbf{D}' = D + \mathbf{v} \times H / c^2, \quad \mathbf{H}' = H - \mathbf{v} \times D.
\]
In (24), (25), because both coordinate systems $\mathbf{r}', t'$, and $\mathbf{r}, t$, appear simultaneously in the same formula, the LT is required to mediate between the coordinates. The GT analog of (24), (25) is obtained by effecting the limit $c \to \infty$
\[
\mathbf{E}' = E + \mathbf{v} \times \mathbf{B}, \quad \mathbf{B}' = B, \quad \mathbf{D}' = D, \quad \mathbf{H}' = H - \mathbf{v} \times D.
\] (26)

In free space (vacuum), in $\Gamma$, $\Gamma'$, the constitutive relations are
\[
\mathbf{D} = \varepsilon_0 E, \quad \mathbf{B} = \mu_0 H, \quad \mathbf{D}' = \varepsilon_0 \mathbf{E}', \quad \mathbf{B}' = \mu_0 \mathbf{H}',
\] (27)
respectively. Substitution of (24) or (25) in (27) demonstrates that in free space (vacuum) the same constitutive relations with parameters $\mu_0, \epsilon_0$, apply to all inertial observers.

However, substitution of (26), into the second equation (27) implies in $\Gamma$
\[D = \epsilon_0(E + v \times B), \quad B = \mu_0(H - v \times D)\] (28)
amounting to a statement that while in $\Gamma'$, in free space the constitutive relations are velocity independent, as in (27), in all other inertial, moving with relative velocity $v$, the constitutive relations (28) do depend on $v$. This sets aside $\Gamma'$ as a preferred reference system, and allows observers in all other reference systems $\Gamma$ to compute their absolute velocity relative to $\Gamma'$ solely by measuring fields in the system $\Gamma$ where they reside. Obviously (28) takes us back to Newtonian absolutism and the luminiferous aether theory, nowadays superseded by SR. See related work [6, 7].

**MAXWELL EQUATIONS FORM-INVARiance FOR VARYING VELOCITIES**

Thus far in (22)-(28), and the corresponding inverse expressions, constant $v$ was assumed. Consider now the application of the chain rule (21) to (18*). With the proviso that (10) is satisfied, the Leibniz rule for differentiation of integrals yields, similarly to (23)
\[\partial_{\tau'} = \partial_{\tau} + v(r, t)\partial_{\tau} / c^2, \quad \partial_{\tau'} = \partial_{\tau} + v(r, t) \cdot \partial_{\tau}\] (29)
where (14) is exploited in (29).

Application of (29) to the last equation (20) yields
\[2(\partial_{\tau'} + v(r, t) \partial_{\tau} / c^2) \cdot B' = \partial_{\tau} \cdot B' + v(r, t) \cdot \partial_{\tau} B' / c^2 = 0.\] (30)

Note the the last term (30) contains $v$ as a factor and is therefore already of first order in $\beta$. This allows us to represent $\partial_{\tau} B' = \partial_{\tau} B$, approximated to order $\beta^2$. Anticipating a result similar to (24), we add and subtract $\partial_{\tau} \cdot B$, obtaining
\[\partial_{\tau} \cdot (B' - B) + v(r, t) \cdot \partial_{\tau} B / c^2 = 0.\] (31)
Substituting from the ME (19) yields
\[\partial_{\tau} \cdot (B' - B) - v(r, t) \cdot \partial_{\tau} \times E / c^2 = 0.\] (32)

Furthermore, exploiting the vector formula for $\partial_{\tau} \cdot (v \times E)$, we obtain
\[\partial_{\tau} \cdot (B' - B + v(r, t) \times E / c^2) - E \cdot \partial_{\tau} \times v(r, t) / c^2 = 0.\] (33)

Subject to (10) the last term (33) vanishes, hence (33) reduces to
\[B' = B - v(r, t) \times E / c^2\] (34)
corresponding to the second equation in (25) with constant $v$ replaced by $v(r, t)$. The analog of (33) is
\[\partial_{\tau} \cdot (D' - D - v(r, t) \times H / c^2) + H \cdot \partial_{\tau} \times v(r, t) / c^2 = 0.\] (35)

Subject to (10) the analog of (34) follows as
\[D' = D + v(r, t) \times H / c^2.\] (36)

Hence subject to (10) the PR applies to the scalar ME in (19), (20).
We turn now to the vector ME in (19), (20). Application of (23) to the first equation (20) yields

$$\partial_r \times \mathbf{E}' + \mathbf{v}(r, t) \times \partial_r \mathbf{E}' / c^2 + \partial_r \mathbf{B}' + \mathbf{v}(r, t) \cdot \partial_r \mathbf{B}' = 0. \quad (37)$$

Once again, we may approximate $\partial_r \mathbf{E}' = \partial_r \mathbf{E}$ and $\partial_r \mathbf{B}' = \partial_r \mathbf{B}$ because of the factor $\mathbf{v}$. Tentatively assuming the first equation of (19) and subtracting it from (37) yields

$$\partial_r \times (\mathbf{E}' - \mathbf{E}) + \mathbf{v}(r, t) \times \partial_r \mathbf{E} / c^2 + \partial_r (\mathbf{B}' - \mathbf{B}) + \mathbf{v}(r, t) \cdot \partial_r \mathbf{B} = 0. \quad (38)$$

Substituting for $\mathbf{B}' - \mathbf{B}$ from (34) and using the rule for the derivative of a product leads to

$$\partial_r \times (\mathbf{E}' - \mathbf{E}) + \mathbf{E} \times \partial_r \mathbf{v}(r, t) / c^2 + \mathbf{v}(r, t) \cdot \partial_r \mathbf{B} = 0. \quad (39)$$

Exploiting $\partial_r \cdot \mathbf{B} = 0$ in (19) and the formula for $\partial_r \times (\mathbf{v} \times \mathbf{B})$, (39) is recast in the form

$$\partial_r \times (\mathbf{E}' - \mathbf{E} - \mathbf{v}(r, t) \times \mathbf{B}) + \mathbf{E} \times \partial_r \mathbf{v}(r, t) / c^2 + \mathbf{B} \cdot \partial_r \mathbf{v}(r, t) - \mathbf{B} \cdot \partial_r \cdot \mathbf{v}(r, t) = 0. \quad (40)$$

For constant $\mathbf{v}$ (40) reduces to the first equation in (25) as expected. In general, for space-time dependent velocities $\mathbf{v}(r, t)$ the last three terms in (33) become negligible only if the variations in space and time are slow compared to field variations. Thus if the time variation of the field can be described by some representative frequency $\omega$, the condition $\partial_r \mathbf{v} \ll \omega \mathbf{v}$ must be satisfied, or if the time variations $\partial_r \mathbf{v}$ can be represented by some representative frequency $\Omega$, then $\Omega \ll \omega$ is required. Similarly the spatial variations of the fields over a representative wavelength $\lambda$ must be much smaller compared to the representative length scale $\Lambda$ of the velocity field, i.e., $\Lambda \gg \lambda$. However, according to the initial stipulation of the model, these terms should be ignored because like (8) they involve differentials of $\mathbf{v}$. Thus (40) becomes

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}(r, t) \times \mathbf{B}, \quad (41)$$

i.e., the first equation (25) for varying $\mathbf{v}(r, t)$.

The counterpart of (40) for the other vector equation is given by

$$\partial_r \times (\mathbf{H}' - \mathbf{H} + \mathbf{v}(r, t) \times \mathbf{D}) + \mathbf{H} \times \partial_r \mathbf{v}(r, t) / c^2 - \mathbf{D} \cdot \partial_r \mathbf{v}(r, t) + \mathbf{D} \partial_r \cdot \mathbf{v} = 0. \quad (42)$$

The same observations made on (40) apply here too, hence we have

$$\mathbf{H}' = \mathbf{H} - \mathbf{v}(r, t) \times \mathbf{D}, \quad (43)$$

i.e., the last equation (25) for varying $\mathbf{v}(r, t)$.

### PLANE WAVE TRANSFORMATIONS

Plane waves solutions of the ME (19), (20) feature in many theoretical problems and engineering applications. Moreover the linearity of the ME facilitates the representation of arbitrary fields as a superposition or integral of such base solutions. Plane waves are characterized by constant vector amplitudes obeying the pertinent FT (24), (25). Space and time variation are delegated to the appropriate relativistic invariants phase exponentials $e^{\imath \theta} = e^{\imath \omega t}$. It follows that

$$\theta(r, t) = \theta(r', t') \quad (44)$$

and except for a trivial constant of integration, (44) also implies the invariance of the differentials.
\[ d\theta = \partial_r \theta \cdot dr + \partial_t \theta dt = k(r, t) \cdot dr - \omega(r, t) dt \]
\[ = d\theta' = \partial_r \theta' \cdot dr' + \partial_t \theta' dt' = k'(r', t') \cdot dr' - \omega'(r', t') dt'. \]

For constant \( v \) the integrals of (45) are trivial, and the phase retains its form-invariance

\[ \theta = k \cdot r - \omega t = \theta' = k' \cdot r' - \omega' t'. \]  

For varying velocities this conclusion does not hold because the wave parameters \( k, \omega \), cease to be constants. Instead, a spectrum of frequencies is created, as shown below. On a more sophisticated level (46) represents the inner product of two four-vectors in Minkowski space, which is an invariant [8].

Substituting the LT (1) in (45) and regrouping terms leads to

\[ k' = \tilde{U} \cdot (k - v \omega / c^2), \omega' = \gamma (\omega - v \cdot k) \]  

the relativistic Fresnel drag and Doppler shift formulas, respectively. The corresponding first order velocity expressions are then

\[ k' = k - v \omega / c^2, \omega' = \omega - v \cdot k \]  

which can also be obtained by substituting the LT (3) in (46).

Wave parameters are subject to Snell’s law of refraction, which in general can be written [9] as

\[ \partial_r \times k(r, t) = 0, \partial_t \omega(r, t) + \partial_r k(r, t) = 0. \]  

Like (10), this entails a conservative field for \( k \), hence \( d\theta \) (45) is an exact differential. Obviously if we deal with \( k(t) \) depending on time only, then the first equation (49) is trivially satisfied.

In (45) we have \( d\theta = \partial_r \theta \mid_r \cdot dr + \partial_t \theta \mid_t \, dt \), where we specifically indicate the variables that are kept constants. The integration of (45) is therefore given by

\[ \theta = \int d\theta = \int' k(r, t) \mid_r \cdot dr - \int' \omega(r, t) \mid_t \, dt \]  

where \( r, t \), indicate the integration variables in the respective integrals. Subject to the first equation (49) taking the differential \( d\theta \) in (50), using the Leibnitz rule for differentiation of integrals, yields back (45).

Substituting the QLT (8) into (45) and regrouping terms leads to the analog of (48)

\[ k'(r', t') = k(r, t) - v(r, t) \omega(r, t) / c^2, \]
\[ \omega'(r', t') = \omega(r, t) - v(r, t) \cdot k(r, t). \]  

**AN EXAMPLE: SCATTERING BY A PLANE OSCILLATING MIRROR**

In order to demonstrate the method, a simple example of a harmonic plane wave incident on a plane oscillating mirror is analyzed. Free space (vacuum), normal incidence, and a perfect mirror are assumed.

The mirror is at rest in \( \Gamma' \) at \( z' = 0 \), moving harmonically with respect to the origin of \( \Gamma \) according to

\[ z = \zeta_0 \sin \Omega t \]  

with a maximum span \( \zeta_0 \). The corresponding time dependent velocity is
\[ \mathbf{v}(t) = \hat{\mathbf{v}}_0 \cos \Omega t, \quad v_0 = \zeta c = \beta_0 c \]  
(52)

and in view of (14)
\[ \mathbf{v}'(t') = -\hat{\mathbf{v}}_0 \cos \Omega t' . \]  
(53)

The incident wave is given in \( \Gamma \) as
\[ \mathbf{E} = \hat{\mathbf{E}} e^{i\theta}, \quad \mathbf{H} = \hat{\mathbf{H}} e^{i\phi} \]
(54)

where \( \theta \) for constant \( k, \omega \), is obtained by a trivial integration in (50).

According to (52), (53), the reference system \( \Gamma' \) as a whole is moving with a time dependent velocity relative to \( \Gamma \). Therefore any changes effected in the wave parameters will be time dependent as well subject to (51). Similarly to (50), in \( \Gamma' \) the phase is prescribed by
\[ \theta' = \int k'(t')dt' - \int \omega'(t')dt' \]
(55)

\[ k'(t') / k = \omega'(t') / \omega = P, \quad P = 1 - \beta_0 \cos \Omega t = 1 - \beta_0 \cos \Omega t' \]
where in terms of order \( \beta_0 \) we may use either \( t \) or \( t' \). The integration (55) yields
\[ \theta' = k'z' - \omega' + (\omega \beta_0 / \Omega) \sin \Omega t', \quad k' = k - (\omega v_0 / c^2) \cos \Omega t = kP. \]
(56)

The amplitudes are transformed according to (25), with the velocity given by (52)
\[ \mathbf{E}' = \hat{\mathbf{E}} e^{i\phi} = E_0 \hat{\mathbf{E}} e^{i\phi}, \quad \mathbf{H}' = \hat{\mathbf{H}} e^{i\phi} = H_0 \hat{\mathbf{H}} e^{i\phi}, \]
\[ E'_0 / E_0 = H'_0 / H_0 = P, \quad E'_0 / H'_0 = Z. \]
(57)

At the mirror location \( z' = 0 \) the fields are time-varying in accordance with (56)
\[ M = Pe^{-i\xi e^{-\omega z' + \omega \beta_0 / \Omega}}, \]
(58)

vindicating the assertion made above that at the oscillating mirror we have a new spectrum that was not present in the incident wave.

In order to calculate the reflected wave and transform it back from \( \Gamma' \) to \( \Gamma \), we first need to represent the spectrum as a superposition of time harmonic signals. Expanding the exponential in (58) in a Bessel functions series [10] we get
\[ M = Pe^{-i\xi e^{-\omega z' + \omega \beta_0 / \Omega}}, \]
(59)

Further manipulation of (59) results in
\[ M = e^{-i\xi} (1 - \beta_0 (e^{i\Omega t} + e^{-i\Omega t}) / 2) \sum_n J_n e^{i\omega_n t} \]
\[ = e^{-i\xi} \sum_n J_n (e^{i\omega_n t} \beta_0 (e^{i\Omega t} + e^{-i\Omega t} / 2) = \sum_n e^{-i\xi} I_n \]
\[ I_n = (J_n - \beta_0 (J_{n-1} + J_{n+1}) / 2) (1 - \beta_n n / \xi) J_n = (1 - n\Omega / \omega) J_n \]
(60)

where \( I_n \) is compacted by using a well-known formula for \( J_n \) [10]. Thus we end up with a discrete spectrum of frequencies \( \omega_n' \) associated with amplitudes \( I_n \). Finally (57) can be recast as
\[ \mathbf{E}' = \hat{\mathbf{E}} e^{i\xi} \sum_n I_n e^{i\phi'}, \quad \mathbf{H}' = \hat{\mathbf{H}} e^{i\phi'} \]
(61)

constituting a superposition of space-time harmonic plane waves.

In order to satisfy the boundary conditions at a perfectly conducting mirror, the total \( \mathbf{E} \) field must vanish, therefore the reflected wave is given by a superposition of space-time harmonic plane waves.
Inverse-transforming the fields (62) from $\Gamma'$ to $\Gamma$ follows the same procedure applied to the incident wave (54). Applying the corresponding inverse transformations of (25*) to the field vectors (62), and exploiting the phase invariance (44) yields (cf. (57))

$$
E_R = -\hat{x}E_0\Sigma_n I_n e^{ik_{R,n}z},\quad H_R = \hat{y}H_0\Sigma_n I_n e^{ik_{R,n}},
$$

$$
\theta_{R,n} = -k'_{R,n}z' - \omega_{R,n}t',\quad \omega_{R,n} = \omega_n,\quad k'_{R,n} = k_n.
$$

(62)

with the same factor $P$ defined in (55).

Noting that the reflected waves propagate in the $-\hat{z}$ direction, the inverse (48) now becomes

$$
k_{R,n} / k'_{R,n} = \omega_{R,n} / \omega'_{R,n} = P.
$$

(64)

The phase $\theta_{R,n}$ in (63) is derived by integrating (50), using the inverses (51*), (53*), (64*), (cf. (55), (56))

$$
\theta_{R,n} = \int d\theta_{R,n} = -\int_{z_0}^{z} k_{R,n}(t)dz - \int_{t_0}^{t} \omega_{R,n}(t)dt'
$$

$$
= -k_{R,n}z - \omega'_{R,n}t + (\omega'_{R,n}\beta_0 / \Omega)\sin\Omega t.
$$

(65)

By inspection of (58), (62), the time signal for index $n$ at some location in $\Gamma$, say $z = 0$, is

$$
M_n = Pe^{-i\xi_{0,n}t + i\omega_{R,n}\xi_{0} / c} = \alpha_{R,n} / \Omega
$$

yielding the analog of (59)

$$
M_n = Pe^{-i\xi_{0,n}t + i\omega_{R,n}\xi_{0} / c} = \alpha_{R,n} / \Omega
$$

(66)

which similarly to (60) is recast as

$$
M_n = \sum_m e^{-i\xi_{0,n}t} I_{n,m},\quad I_{n,m} = (J_{n,m} - \beta_0(J_{n,m-1} + J_{n,m+1}) / 2),
$$

$$
= (1 - \beta_0 m / \xi_{0,n})J_{n,m} = (1 - m\Omega / \omega_{R,n})J_{n,m}.
$$

(67)

Similarly to (61), the space dependence is appended to (68), and substituted in (63), leading to a double summation

$$
E_R = -\hat{x}E_0\Sigma_{n,m} L_{n,m} e^{ik_{R,n}z},\quad H_R = \hat{y}H_0\Sigma_{n,m} L_{n,m} e^{ik_{R,n}}
$$

$$
\theta_{R,n} = -k_{R,n}z - \omega_{R,n}t,\quad \omega_{R,n} / k_{R,n} = c
$$

$$
\Sigma_{n,m} = \sum_m L_{n,m} = I_n I_{n,m}
$$

(69)

and with that the problem is solved.

**DISCUSSION AND SUMMARY**

The QLT discussed above is based on Einstein’s SR theory [1] as a limiting case, to the first order in $\beta$, and an on the heuristic assumption that the ME are valid for moderate acceleration. The integration (18) has been tentatively suggested previously [11]. The validity of the PR subject to the QLT has been discussed here in some detail.
The investigation of the QLT is expected to facilitate future discussion of problems involving varying velocity, heretofore not tackled. Presently the results have been applied to a very simple representative problem of scattering by a plane mirror in time-harmonic lineal motion. It has been shown that the scattered wave constitutes a discrete spectrum of sidebands separated by the mechanical oscillation frequency $\Omega$. 

REFERENCES