FREE SPACE MULTIPLE SCATTERING BY MOVING OBJECTS

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Abstract—Presently we study multiple-scattering problems involving moving objects. This also covers the class of problems of single-scattering from moving objects excited by arbitrary sources, e.g., spherical or cylindrical elementary antennas, as opposed to plane-wave excitation.

Uniform motion, i.e., constant velocities, are assumed, and the wave propagation medium is taken as free space (vacuum), allowing for relatively simple transformations from one inertial reference-frame to another.

A consistent use of plane-wave integral representations is conducive to a systematic and trackable relativistic formalism. The far-field forms, which are the leading terms of the inverse-distance differential-operator representations, facilitate a simple check, comparing them by inspection with the exact plane-wave integrals.

To derive numerical results the plane wave integrals can be recast in terms of the differential-operator representations which are easier to evaluate. This is especially convenient when the moving objects recede to- or arrive from- large distances.

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1. INTRODUCTION AND GENERAL REMARKS

Since Einstein’s monumental article [1] (see also [2], and especially [3] for present notation), electromagnetic scattering of waves in the presence of moving objects continues to be a subject of interest. For a gateway to the relevant literature see [4]. Usually the technique used involves transforming the excitation wave into the co-moving reference-frame of the object at rest, solving the scattering problem, then transforming the scattered wave back to the original reference-frame. This has been dubbed as the “frame hopping” method (e.g., see [4] p. 149), and is applicable to scattering problems in free space.

The class of problems investigated here involves an ensemble of moving objects, scattering the initial excitation wave, which can be an arbitrary wave, and waves arriving from other scatterers. In order to gain more insight into the subject, we start with a simple one-dimensional problem of scattering by two slabs.

Subsequently this is followed by two- and three-dimensional successive-scattering examples. The pattern emerging from such examples will be used to define a complete algorithm for multiple-scattering.

In a series of articles Twersky [5–8] developed the mathematical tools necessary for discussing single- and multiple-scattering. Recently, the mathematical tools involved in this class of problems have been reviewed, and application to relativistic scattering stated [9]. Unlike problems involving an ensemble of objects at rest, where the initial frequencies do not change, in velocity-dependent scattering the Doppler effect is involved, creating new spectral components. These must be traced throughout the analysis from one successive-scattering process to another. This explains why scattering by moving objects is amenable to successive-scattering techniques, but the whole ensemble cannot be considered as a single, global object.

From the relativistic point of view, it is shown that plane waves are the easiest to handle, hence subsequently they are treated preferentially. We also mention inverse-distance differential-operator representations, for which the far-field forms are the leading terms, and which provide means for numerical calculations.

One of the problems arising in this study is the need for careful book-keeping of indices, in order to keep track of the various scattering processes. We have three classes of indices: Each scatterer is identified by an index $1, 2, 3, \cdots$. Next, we define spatiotemporal reference-frames $\Gamma_\chi, \chi = a, b, c, \cdots$. Lastly, we identify waves, whose parameters need indexing regarding the sequence of their origin, their directions of polarization and propagation, and the reference-frame with which they
2. A SIMPLE EXAMPLE — SCATTERING BY TWO SLABS

The problem of successive-scattering by an ensemble of objects, at rest or in motion, is analyzed. Consider a static configuration first. Here one can start with a global surface-integral representation for the whole ensemble, selecting a surface which surrounds all the objects. One can also contract this surface to a number of disconnected surfaces, as long as all the objects are surrounded. Departing from this approach, Twersky [5–8] continued to develop a multiple-scattering algorithm. By iterative substitution of the resulting self-consistent equations, the sequence of successive-scattering emerges, whereby the scattered wave by one object provides the excitation for another.

In order to gain more insight, we start with a simple static case, where the global closed form and the successive-scattering sequences can easily be recognized. Consider two slabs, thin relative to their separation, so that the transit time through them is negligible, situated at \( x = 0, \ x = d \). The excitation is provided by a plane-wave propagating along the \( x \)-axis, normal with respect to the slabs’ interfaces. Actually, the problem we are considering is very similar to the classical problem of scattering by a single slab [10], except that now the slab interfaces are replaced by the two thin slabs. The regions \( x < 0, \ 0 < x < d, \ x > d \), are denoted by \( i, \ ii, \ iii \), respectively. The excitation and reflected waves, observed in region \( i \), are given, respectively, by

\[
\begin{align*}
E_0 &= \hat{z}E_0 e^{i\theta_0}, \quad \theta_0 = kx - \omega t, \quad E_r = \hat{z}E_r e^{i\theta_r}, \\
\theta_r &= -kx - \omega t, \quad E_r/E_0 = R \tag{1} \\
R &= (r_{12} + \Delta)/(1 - \Delta r_{12}) = (r_{12} + \Delta)\Sigma_n, \\
\Sigma_n &= \Sigma_{n=0}(\Delta r_{21})^n, \quad \Delta = r_{23}e^{2kd}
\end{align*}
\]

where \( R \) in (1) is the global reflection coefficient, obtained by solving the boundary-value problem as done in [10] for the single thick slab. Thus far (1) is a formal solution of a mathematical problem. Upon interpreting the factor \( e^{2kd} \) in (1) as due to the extra transit time of the wave reflected between the interfaces in region \( ii \), we are starting to fission the result (1) into sequences of successive-scattering processes. In (1) \( r_{12} = -r_{21}, \ r_{23} = -r_{32} \) denote the reflection coefficients of the individual slabs for single-scattering reflections, i.e., in the absence of the other slab.
As long as we go from the global solution (1) to the successive-scattering interpretation, assuming in (1) that the expansion into a geometrical progression infinite series is convergent, everything is done rigorously. It will be shown that for velocity-dependent systems the global solution of the boundary-value problem cannot be derived, and we have to be contented with constructing the solution by describing the successive-scattering scheme.

There are two equivalent ways of interpreting (1). We can assume that each object is situated in free space, or we can assume that in order to reach the region $ii$ the wave must first be transmitted from region $i$ through the left slab. Thus for $n = 0$ in (1) the first term involving $r_{12}$ in the numerator corresponds to reflection into region $i$ by the left slab itself. The second term in the numerator and the $n = 0$ term in the series yield $\Delta$, signifying reflection from the right slab into region $ii$. Combining factors of $e^{i2kd}$ from the numerator and the terms $n = 0, 1$ of the series yields

$$e^{i2kd}(r_{23} + r_{23}r_{12}r_{21}) = \Delta(1 - r_{12}^2) = \Delta(1 + r_{12})(1 - r_{12})$$

$$= \Delta(1 + r_{12})(1 + r_{21}) = \Delta \tau_{12} \tau_{21}, \quad (2)$$

where in (2) $\tau_{12}, \tau_{21}$ denote the single-transmission coefficients for the left slab from region $i$ to $ii$, and $ii$ to $i$, respectively.

Similarly to (1), the global transmitted wave, or the wave scattered into region $iii$ is given by

$$E_t = \hat{z}E_{te}e^{i\theta_e}, \quad \theta_e = kx - \omega t, \quad \frac{E_t}{E_0} = T$$

$$T = \tau_{12}\tau_{23}/(1 - \Delta r_{21})$$

$$= \tau_{12}\tau_{23}\Sigma_n = (1 + r_{12} + r_{23} + r_{12}r_{23})\Sigma_n \quad (3)$$

In (3), the sum shows once again the scattering between the slabs in region $ii$. The two formats, i.e., involving $r$ only, or both $r$ and $\tau$ factors, are quite mnemonic, and either one can be chosen.

In contradistinction to (1), (3), in a configuration involving moving objects new frequencies are created, hence a global solution of the boundary-value problem is not readily available. Let us now investigate the simple case of successive-scattering processes created by two slabs, when the left slab is at rest, but the right slab moves with velocity $v = v_x$ along the $x$-axis.

In order to emphasize the parameters associated with the initial reference-frame $\Gamma_a$, the excitation wave (1) is rewritten as

$$E_{0a} = \hat{z}E_{0ae}e^{i\theta_{0a}}, \quad \theta_{0a} = k_0a x_a - \omega_0a t_a \quad (4)$$
Accordingly the reflection from the left slab at rest is given in (1) with the proper indexing. To compute the first reflection mode from the right slab into region $\mathcal{R}_{ii}$, we first transform (4) to $\Gamma_b$, obtaining
\[E_0^b = \hat{z}E_0^b e^{i\theta_{k0b}}, \quad \theta_{k0b} = k_{0b} x_b - \omega_{0b} t_b\]
\[\frac{E_{0b}}{E_{0a}} = \frac{k_{0b}}{k_{0a}} = \frac{\omega_{0b}}{\omega_{0a}} = \delta = \left(\frac{1 - \beta}{1 + \beta}\right)^{1/2}\] (5)

where in (5) the amplitude factor $\delta$ is prescribed by the relativistic field transformations, $x_b, t_b$ are related to $x_a, t_a$ through the Lorentz transformation, and $k_{0b}, \omega_{0b}$ are obtained when the relativistic Doppler effect formulas are applied to $k_{0a}, \omega_{0a}$. The general formulas are given below. One can also adopt the phase conservation principle $\theta_{k0b} = \theta_{k0a}$.


Translating the wave (5) to the local coordinates of the right slab, situated in $\Gamma_b$ at $d_b$ takes into account the phase shift due to the new location, yielding
\[E_0^b = \hat{z}E_0^b e^{i k_{0b} d_b} e^{i \theta_{k0b2}}, \quad x_b = x_{b2} + d_b, \quad \theta_{k0b2} = k_{0b} x_{b2} - \omega_{0b} t_b\] (6)

The wave reflected by the right slab into region $\mathcal{R}_{ii}$, translated back to the original coordinates of $\Gamma_b$, is
\[\hat{z}E_0^b r_{23} e^{i k_{0a} d_b} e^{i k_{0a} x_a - i \omega_{0a} t_a}\] (7)

and the relativistic transformation of the wave in (7) back to $\Gamma_a$ yields
\[\hat{z}E_0^a \Delta_v e^{i k_{0a} x_a - i \omega_{0a} t_a}, \quad \Delta_v = \delta^2 r_{23} e^{i k_{0a} d_b} e^{i \left(1 - \delta^2\right) (k_{0a} x_a - \omega_{0a} t_a)}\] (8)

Obviously, the factor $\Delta_v$ in (8) will appear every time there is a reflection from the right slab towards the left one, and reduces to $\Delta$, (1)–(3), for $v = 0$. Unlike $\Delta$, depending only on the geometry, material properties, and parameters of the excitation wave, now we have $\Delta_v$ also dependent on the spatiotemporal parameters $x_a, t_a$, i.e., depending on when and where the scattered waves are observed. This implies that we cannot define global reflection and transmission coefficients intrinsic to a global object, as we did in the velocity-independent case. This is not surprising, because the velocity effect and changing position of the moving slab affect the scattering processes. Consequently, even though for this simple case $\Delta$ in (1)–(3) can be replace by $\Delta_v$, (8), the closed form expressions cannot qualify as global reflection and transmission coefficients.

The conclusion of this discussion is that in a configuration involving moving objects, we can only follow successive-scattering processes, but global scattering amplitudes will not be available.
3. SINGLE-SCATTERING BY OBJECTS AT REST

In order to establish the needed notation, consider single-scattering by two- and three-dimensional objects. For simplicity we start by assuming the far field approximations.

The excitation wave, say the \( E \)-field, is given in the initial reference-frame \( \Gamma_a \) by (4), with arbitrary orientation

\[
E_{0a} = \hat{E}_{0a} E_{0a} e^{i\theta_{k0a}}, \quad \theta_{k0a} = k_{0a} \cdot r_a - \omega_{0a} t_a
\]

In the two-dimensional case the incident field is assumed to be polarized along the cylindrical axis \( \hat{E}_{0a} = \hat{z} \). The field (9) can be considered as the zero-order successive-scattering mode.

The next mode is single-scattering. We consider scattering by object 1, possessing a local coordinates system denoted by \( r_{a1} \). The wave (9) is translated to the local system, yielding

\[
E_{0a} = \hat{E}_{0a} E_{0a} e^{i\theta_{k0a}}, \quad r_{a1} = r_a - \rho_{a1}, \quad \theta_{k0a1} = k_{0a1} \cdot r_{a1} - \omega_{0a1} t_a
\]

At large distances from object 1, the scattered wave is given by

\[
E_{0;1a} = \hat{E}_{0a} E_{0a} e^{i\theta_{k0a1}} \rho_{a1} e^{i\theta_{k0a1}}, \quad \eta_{0;1a} = \eta_{0a} (\hat{E}_{0a}, k_{0a1}, \hat{r}_{a1})
\]

constituting the leading terms in the inverse-distance representations discussed below. In (11) indices 0;1a identify the source, or the previous scatterer, the scattering object, and the inertial reference-frame, and 0;1a1 denote the local coordinate system of object 1. The far-field wave \( \eta_{0;1a} \) with the corresponding \( f_\zeta(K_{0;1a}) \), \( \zeta = 2, 3 \) denoting, two-, three-dimensional cases, respectively. The scattering amplitude is \( g_{0;1a} \), with arguments \( \hat{E}_{0a} \), \( k_{0a} \), \( k_{0;1a} \), indicating the directions of polarization, the propagation vector (direction and associated frequency) of the excitation wave, and the direction of propagation of the scattered wave, respectively. The wave thus created is transversal, in that \( g_{0;1a} \), \( k_{0;1a} \) and the associated \( H \) field all are mutually perpendicular in space. This seemingly complex notation is necessary in order to identify the specific process. The notation \( \omega_{0;1a} = k_{0;1a} c = \omega_{0a} = k_{0a} c \) is trivial, indicating that there is no change in frequency and wavelength when there is no motion.

In the cylindrical-spherical-case, we have respectively

\[
H_0^{(1)}(K) = f_2(K)e^{iK}, \quad h_0^{(1)}(K) = f_3(K)e^{iK}
\]
where in (12) $H_0^{(1)}$, $h_0^{(1)}$ denote the zero-order Hankel functions of the first kind, for the cylindrical-spherical-case, respectively, the tilde indicating the asymptotic nature of the far-field approximation.

In (11), the amplitude varies slowly in the far field, hence the exponential can be treated as a plane wave, as in (9), (10)

$$K_{0;1a} - \omega_{0;1a} t_a \theta_{0;1a} = k_{0;1a} \cdot r_{1a} - \omega_{0;1a} t_a, \quad \hat{k}_{0;1a} = \hat{r}_{1a}$$

a property which is important for the subsequent relativistic transformations. The method of indexing is clarified in (13): $0;1a$ denotes that the scattering mode involves excitation by source 0 and scattering by object 1, in reference-system $\Gamma_a$, and the local coordinate system of object 1.

For arbitrary distances, external with respect to the surface (circle or sphere) circumscribing the object, Twersky [5, 7, 8], shows that the scattered wave can be represented by a plane-wave integral, based on the corresponding Sommerfeld representations for the cylindrical or spherical Hankel functions

$$E_{0;1a} = E_{0;1a} e^{ik_{0;1a} \cdot r_{1a}}$$

$$g_{0;1a} = g_{0;1a} (\hat{E}_{0a}, \hat{k}_{0a}, \hat{p}_{1a})$$

$$\int d\Omega_{p_{1a}} e^{i k_{0;1a} \hat{p}_{1a} \cdot r_{1a}} = \frac{1}{2\pi} \int_{\tau=\theta_{1a}+\pi/2-i\infty}^{\tau=\theta_{1a}-\pi/2+i\infty} d\tau e^{iK_{0;1a} C\theta_{1a} - \tau}$$

$$C_\lambda = \cos \lambda, \quad K_{0;1a} = k_{0;1a} r_{1a}$$

$$\int d\Omega_{p_{1a}} e^{i k_{0;1a} \hat{p}_{1a} \cdot r_{1a}} = \frac{1}{2\pi} \int_{\mu_{1a}=-\pi}^{\mu_{1a}=\pi} d\mu_{1a} \int_{\nu_{1a}=0}^{\nu_{1a}=\pi/2-i\infty} d\nu_{1a} e^{iK_{0;1a} C_{1a} S_{1a}}, \quad S_\lambda = \sin \lambda$$

where in (14) the integration is over azimuthal angles $\tau$ in the two-dimensional case, and azimuthal, polar angles, $\mu_{1a}$, $\nu_{1a}$, respectively, for the three-dimensional case. Corresponding to the plane-wave integrals (14) one can also recast the scattered waves in terms of the inverse-distance differential operators, and the special functions series [5, 7, 8], see also a short review in [9].

4. SINGLE-SCATTERING BY MOVING OBJECTS

After the advent of Einstein’s Special-Relativity theory, single-scattering by moving objects has been amply studied. For a gateway to the literature see for example [4]. Usually such problems are
characterized by objects whose local coordinate system coincides with the origin of the inertial reference-system, which is not the general case. Similarly to (10), here objects are arbitrarily located.

Consider the single-scattering mode provided by an object at rest in inertial reference-frame $\Gamma_b$, whose origin, when observed from $\Gamma_a$, moves with relative velocity $\mathbf{v}_b$. At time $t_a = t_b = 0$ the origins of $\Gamma_a$, $\Gamma_b$, coincide. To analyze this problem, the excitation wave (9) must first be transformed into $\Gamma_b$. Coordinates are transformed by exploiting the Lorentz transformation

$$
\mathbf{r}_b = \hat{\mathbf{U}}_a^b \cdot (\mathbf{r}_a - \mathbf{v}_a^b t_a), \quad t_b = \gamma_a^b(t_a - \mathbf{v}_a^b \cdot \mathbf{r}_a/c^2)
$$

$$
\gamma_a^b = (1 - (\beta_a^b)^2)^{-1/2}, \quad \beta_a^b = \mathbf{v}_a^b/c, \quad \mathbf{v}_a^b = |\mathbf{v}_a^b|
$$

$$
\hat{\mathbf{U}}_a^b = \hat{\mathbf{I}} + (\gamma_a^b - 1)\hat{\mathbf{v}}_a^b \hat{\mathbf{v}}_a^b, \quad \hat{\mathbf{v}}_a^b = \mathbf{v}_a^b/|\mathbf{v}_a^b|
$$

where in (15) a tilde denotes dyadics, $\hat{\mathbf{I}}$ is the unit dyadic.

Assuming (15), and the phase invariance principle (e.g., see [11, 12]) leads to the relativistic Doppler effect formulas

$$
\theta_{k0a} = \mathbf{k}_{0a} \cdot \mathbf{r}_a - \omega_{0a} t_a = \theta_{k0b} = \mathbf{k}_{0b} \cdot \mathbf{r}_b - \omega_{0b} t_b
$$

$$
\mathbf{k}_{0b} = \hat{\mathbf{U}}_a^b \cdot (\mathbf{k}_{0a} - \mathbf{v}_a^b \omega_{0a}/c^2),
$$

$$
\omega_{0b} = \gamma_a^b(\omega_{0a} - \mathbf{v}_a^b \cdot \mathbf{k}_{0a})
$$

(16)

For the plane waves in free space, the relativistic transformation formulas for the field prescribe (e.g., see [1, 3, 9])

$$
\mathbf{E}_{0b} = \hat{\mathbf{E}}_{0a} \mathbf{E}_{0a} e^{i\theta_{k0a}} = \hat{\mathbf{W}}_{k0a} \cdot \mathbf{E}_{0a} e^{i\theta_{k0a}}
$$

$$
\hat{\mathbf{W}}_{k0a} = \hat{\mathbf{V}}_a^b \cdot (\hat{\mathbf{I}} + \hat{\mathbf{v}}_a^b \times \mathbf{k}_{0a} \times \hat{\mathbf{I}})
$$

(17)

Similarly to (10), we consider scattering by object 2, whose local coordinate system is defined by $\mathbf{r}_{b2}$, hence a coordinate translation is effected

$$
\mathbf{E}_{0b} = \hat{\mathbf{E}}_{0b} \mathbf{E}_{0b} e^{i\theta_{k0b} + \rho\epsilon} e^{i\theta_{k0b2}}, \quad \mathbf{r}_{b2} = \mathbf{r}_b - \mathbf{r}_{b2}
$$

$$
\theta_{k0b2} = \mathbf{k}_{0b} \cdot \mathbf{r}_{b2} - \omega_{0b} t_b
$$

(18)

Similarly to (11), (13), the excitation wave (18) gives rise to a far-field scattered wave

$$
\mathbf{E}_{02b} = \mathbf{E}_{0b} e^{i\theta_{k0b} \rho\epsilon} e^{i\theta_{k0b2}}, \quad \mathbf{g}_{02b} = \mathbf{g}_{0b} \cdot (\mathbf{E}_{0b}, \mathbf{k}_{0b}, \mathbf{r}_{b2})
$$

$$
\eta_{02b} = (\mathbf{K}_{02b}) e^{i\theta_{k02b} + \mathbf{r}_{b2} \cdot \mathbf{v}_{02b} \cdot \mathbf{k}_{02b}}, \quad \mathbf{K}_{02b} = \mathbf{k}_{02b} \mathbf{r}_{b2}, \quad \mathbf{w}_{02b} = \mathbf{k}_{02b} \mathbf{c} = \omega_{0b} = \mathbf{k}_{0b} \mathbf{c}
$$

(19)
The plane-wave integral representation follows from (14), (19)

\[
E_{0;2b} = E_{0b} e^{i(k_0 - k_{0;2b}) \rho_{2b} \int d\Omega_{p_{0;2}} e^{i\theta_{p_{0;2}} g_{0;2b}},}
\]

\[
g_{0;2b} = g_{0;2b}(E_{0b}, k_{0b}, \hat{p}_{02})
\]

\[
\theta_{p_{0;2b}} = k_{0;2b} \rho_{2b} \cdot r_b - \omega_{0;2b} t_b, \quad \omega_{0;2b} = k_{0;2b} c = \omega_{0b} = k_{0b} c
\] (20)

The fields (19), (20) must now be transformed back to \( \Gamma_c \) in order to complete the discussion for this mode, or to a different reference-frame if a higher order successive-scattering mode is sought. To facilitate the transformation, the waves (19), (20) are expressed in terms of \( r_b \) coordinates

\[
E_{0;2b} = E_{0b} e^{i(k_0 - k_{0;2b}) \rho_{2b} \int d\Omega_{p_{0;2}} e^{i\theta_{p_{0;2}} g_{0;2b}},}
\]

\[
\eta_{0;2b} = f_2(K_{0;2b}) e^{i\theta_{k_{0;2b}}, \quad f_3(K_{0;2b}) e^{i\theta_{k_{0;2b}}}
\]

\[
\theta_{k_{0;2b}} = k_{0;2b} \rho_{2b} \cdot r_b - \omega_{0;2b} t_a, \quad \theta_{p_{0;2b}} = k_{0;2b} \rho_{2b} \cdot r_b - \omega_{0;2b} t_a
\] (21)

\[
E_{0;2c} = E_{0b} \int d\Omega_{p_{0;2}} e^{i(k_0 - k_{0;2b}) \rho_{2b} \cdot \hat{p}_{02} e^{i\theta_{p_{0;2}} g_{0;2b}}}
\]

We will consider now a more general transformation to a reference-frame \( \Gamma_c \), where \( \Gamma_c = \Gamma_a \), with its associated \( \nu_b^c = \nu_a^b = -\nu_b^a \) should be considered as a special case. In (21) we have plane waves with phase conservation treated according to (16), yielding in the present case

\[
\theta_{k_{0;2b}} = k_{0;2b} \rho_{2b} \cdot r_b - \omega_{0;2b} t_a = \theta_{k_{0;2c}} = k_{0;2c} \rho_{2c} \cdot r_c - \omega_{0;2c} t_c \quad \theta_{p_{0;2b}} = k_{0;2b} \rho_{2b} \cdot r_b - \omega_{0;2b} t_a = \theta_{p_{0;2c}} = k_{0;2c} \rho_{2c} \cdot r_c - \omega_{0;2c} t_c
\] (22)

For the amplitudes we apply to the plane waves a dyadic as in (17), yielding in \( \Gamma_c \)

\[
E_{0;2c} = E_{0b} e^{i(k_0 - k_{0;2b}) \rho_{2b} \cdot \hat{p}_{02} e^{i\theta_{p_{0;2}} g_{0;2b}},}
\]

\[
g_{0;2c} = g_{0;2b}(\hat{E}_{0b}, k_{0b}, \hat{p}_{02})
\]

\[
\hat{W}_{k_{0;2b}} = \hat{V}_b^c \cdot (\hat{I} + \beta_b^c \hat{v}_b^c \times \hat{p}_{02} \times \hat{I})
\]

\[
E_{0;2c} = E_{0b} \int d\Omega_{p_{0;2}} e^{i(k_0 - k_{0;2b}) \rho_{2b} \cdot \hat{p}_{02} e^{i\theta_{p_{0;2}} g_{0;2b}}}
\] (23)

\[
g_{0;2c} = g_{0;2b}(\hat{E}_{0b}, k_{0b}, \hat{p}_{02})
\]

\[
\hat{W}_{p_{0;2b}} = \hat{V}_b^c \cdot (\hat{I} + \beta_b^c \hat{v}_b^c \times \hat{p}_{02} \times \hat{I})
\]

In (23) \( E_{0;2c} \) describes the field corresponding to \( E_{0;2b} \), measured in \( \Gamma_c \). The question of expressing all terms, or some of them, using \( \Gamma_b \) or \( \Gamma_c \) native coordinates is a matter of convenience remaining mute here, and will come up in connection with the evaluation of the integral.
5. SUCCESSIVE-SCATTERING BY MOVING OBJECTS

In order to demonstrate successive-scattering, we start with the singly-scattered wave (23) and assume an object 3 in $\Gamma_c$. The scattering of the waves in (23) by this object is by now a straightforward question requiring book-keeping of the relevant indices. The results are increasingly cumbersome therefore only one such mode will be demonstrated.

The phase of the waves in (23) is first translated to the local coordinate system $r_{c3}$ of object 3

$$\begin{align*}
\theta_{k_{0;2c}} &= k_{0;2c} \cdot r_c - \omega_{0;2c} t_c = k_{0;2c} \cdot r_{c3} + k_{0;2c} \cdot r_{c3} - \omega_{0;2c} t_c \\
\theta_{p_{0;2c}} &= k_{0;2c} \cdot p_c - \omega_{0;2c} t_c \\
\theta_{p_{0;2c}} &= k_{0;2c} \cdot p_{c3} \cdot r_c - \omega_{0;2c} t_c \\
\theta_{p_{0;2c}} &= k_{0;2c} \cdot p_{c3} - \omega_{0;2c} t_c
\end{align*} \tag{24}$$

The waves thus defined in (23), (24) give rise to the far-field scattered waves

$$\begin{align*}
E_{0;2;3c} &= E_{0b} e^{i(k_{0b} - k_{0;2b}) \cdot r_{c}} f_{c}(K_{0;2b} e^{ik_{0;2b} \cdot r_{c3}}) \\
G_{0;2;3c} &= G_{0;2;3c} (\hat{g}_{0;2b}^c, k_{0;2c}, \hat{r}_{c3}) \\
\eta_{0;2;3c} &= f_{c}(K_{0;2;3c} e^{iK_{0;2;3c} \cdot r_{c3}}) e^{iK_{0;2;3c} \cdot r_{c3}} \\
K_{0;2;3c} - \omega_{0;2;3c} t &= k_{0;2;3c} \cdot r_{c3} - \omega_{0;2;3c} t_c \\
\omega_{0;2;3c} &= k_{0;2;3c} \cdot r_{c3} - \omega_{0;2;3c} t_c
\end{align*} \tag{25}$$

Corresponding to (25), the plane-wave integral representations are

$$\begin{align*}
E_{0;2;3c} &= \int d\Omega_{p_{0;2}} e^{i(k_{0b} - k_{0;2b}) \cdot r_{c}} \rho_{p_{0;2}} e^{i(k_{0;2c} \cdot p_{0;2} \cdot r_{c3})} \\
G_{0;2;3c} &= \int d\Omega_{p_{0;3}} e^{i\theta_{p_{0;2;3c}}} g_{0;2;3c} \\
\omega_{0;2;3c} &= \omega_{0;2c} = k_{0;2c} \\
\theta_{p_{0;2;3c}} &= k_{0;2;3c} \cdot p_{c3} \cdot r_{c3} - \omega_{0;2;3c} t_c
\end{align*} \tag{26}$$

and the process (26) can be continued. It is now obvious that to find the next order of successive-scattering, we first have to translate the phases $\theta_{k_{0;2;3c}}$ in (25) and $\theta_{p_{0;2;3c}}$ in (26) to the origin of $\Gamma_c$, this will introduce appropriate phase factors. Using again the phase invariance,
we have $\theta_{k0;2;3c} = \theta_{k0;2;3d}$, and $\theta_{p0;2;3c} = \theta_{p0;2;3d}$, i.e., the phases in the next reference-frame $\Gamma_d$. The field transformations will have to be applied to $g_{0;2;3c}$ in (25), (26) respectively, and thus $E_{0;2;3d}$ is available, providing the excitation for the next successive-scattering mode, etc., etc. Eventually all fields will have to be transformed into the reference-frame in which the observer is situated.

6. TWERSKY’S INVERSE-DISTANCE DIFFERENTIAL OPERATORS

The plane-wave representations used above have been chosen because of the relative ease offered in following various velocity-dependent successive-scattering modes. It does not seem likely that the chains of integrals over complex paths thus found will submit to straightforward analytic evaluation or numerical computation.

For numerical evaluation we need more amenable representations. Some initial results were reported previously for a simple case of multiple-scattering by two cylindrical scatterers, one moving and one at rest [13].

It has been mentioned above that the far-field expressions as in (11) are the leading term of asymptotic or exact representations in terms of series of inverse powers of the distance and differential operators acting on the integral kernel. Such representations have been investigated by Twersky [5–8], see also [9] for summary and notation, and adaptation to relativistic problems.

For example, in terms of these operators, (14) for the three-dimensional case will become

$$E_{0;1a} = E_{0a} e^{i\theta_{0a}} \int d\Omega_{p_{1a}} e^{i\theta_{p_{0;1a}}} g_{0;1a}(\hat{E}_{0a}, k_{0a}, \hat{p}_{1a})$$

$$O(K_{0;1a}, \vec{D}) = \sum_{\nu=0}^{\infty} \rho_\nu \vec{D} \cdot (\vec{D} - 1 \cdot 2\hat{I}) \cdot (\vec{D} - 2 \cdot 3\hat{I})$$

$$\eta_{0;1a} = f_3(K_{0;1a}) e^{iK_{0;1a} - i\omega_{0;1a} t_a}$$

$$K_{0;1a} = k_{0;1a} r_{a1}, \quad \rho_\nu = (i/(2K_{0;1a}))^\nu / \nu!$$

where in (27) $\vec{D} = \vec{D}(\partial_{r_{a1}}, \hat{r}_{a1})$ is a complicated dyadic expression (e.g., see [9]), involving derivatives symbolized by the gradient operator $\partial_{r_{a1}}$, and functions and unit vectors associated with the directions in the coordinate system $\hat{r}_{a1}$. 

In a consistent manner, (25) contains a chain of far-field expressions, and applying the differential operator representations to (26) will similarly produce a chain of operators. Let us first consider the inner integral

\[
E_{0;2;3c} = E_{0b} e^{ik_{0b} \rho_{b2} - i\omega_{0;2;3}t} \int d\Omega_{p_{c2}} e^{i(k_{0;2c}\rho_{c3} - k_{0;2b}\rho_{b2})} \rho_{p_{c2}} \\
\times \int d\Omega_{p_{c3}} e^{ik_{0;2c}\rho_{c3} - \kappa_{0;2c}\rho_{c3}} g_{0;2;3c}(\hat{g}_{p0;2b}^c, k_{0;2c}, \hat{p}_{c3}) \\
= E_{0b} e^{ik_{0b} \rho_{b2} - i\omega_{0;2;3}t} \int d\Omega_{p_{c2}} e^{i(k_{0;2c}\rho_{c3} - k_{0;2b}\rho_{b2})} \rho_{p_{c2}} \\
f_3(K_{0;2c}) \tilde{O}_{c3} \cdot g_{0;2;3c}(\hat{g}_{p0;2b}^c, k_{0;2c}, \hat{r}_{c3}) \\
\tilde{O}_{c3} = \hat{O}_{c3}(K_{0;2c}, \tilde{D}_{c3}(\partial_{r_{c3}}, \hat{r}_{c3})), \quad \hat{g}_{p0;2b}^c = \hat{g}_{p0;2b}^c(\hat{p}_{b2})
\]

We are left now with one integral which we rewrite as

\[
E_{0;2;3c} = E_{0b} e^{ik_{0b} \rho_{b2} - i\omega_{0;2;3}t} \int d\Omega_{p_{b2}} e^{i\tilde{\kappa} \rho_{b2}} G(\hat{p}_{b2}) \\
\tilde{\kappa} = k_{0;2c}\rho_{c3} - k_{0;2b}\rho_{b2} = \kappa \hat{\kappa} \\
G(\hat{p}_{b2}) = f_3(K_{0;2c}) \tilde{O}_{c3} \cdot g_{0;2;3c}(\hat{g}_{p0;2b}^c, k_{0;2c}, \hat{r}_{c3}), \quad \hat{g}_{p0;2b}^c = \hat{g}_{p0;2b}^c(\hat{p}_{b2})
\]

Thus when it comes to actual evaluation of the fields, we do not need to compute the complex contour integrals, instead we can exploit the differential operator series. This is particularly convenient when the object is moving away from the observer, or approaching from intermediate and large distances, facilitating good approximations with just a few leading terms in the differential-operator series.

7. SUMMARY AND CONCLUDING REMARKS

The question of scattering by a collection of objects moving in free space is considered. Single-scattering problems have been considered in the literature, with the excitation usually provided by a plane-wave. The “frame-hopping” method is usually used, whereby relativistic transformation are applied to the amplitude, frequency, and propagation vector, in order to have a plane wave in the reference-frame of the object at rest. There the problem is solved, and the fields are transformed into the initial reference-frame.

Using a simple example, it was shown above that, in general, a collection of moving objects will not submit to a description in terms of a single global object. It follows that a-priori only successive-scattering approaches will enable us to solve such scattering problems.
Moreover, the scattered waves providing the excitation for the next scattering process, are in general not planar, and therefore the problem becomes even more complicated. To obviate this difficulty, Sommerfeld-type integrals, constituting plane-wave integral representations, are used. Typically, this requires indexing of various parameters, like the reference-frame, the parameters characterizing a specific excitation and scattered waves, and their frequency, direction of propagation, and polarization.

The formalism for successive-scattering is thus obtained in terms of complex path integrations. Such complex integrals are usually very inconvenient for numerical evaluation. Therefore the integrals are recast in terms of Twersky’s inverse-distance differential-operator representations. Such representations are particularly convenient when the objects in question are receding into the far field, or coming in from large distances, allowing for good approximations in terms of a small number of terms.

REFERENCES


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