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THE THEORY OF LOW-FREQUENCY WAVE PHYSICS REVISITED

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Abstract—The Stevenson approach to low-frequency time-harmonic wave scattering, that expanded the electric and magnetic fields in power series of k, essentially the inverse wavelength, is scrutinized. Stevenson's power series approach perforce implies a variable frequency ω, i.e., a variable wave-number k, an assumption challenged here.

Presently the three major linear wave physics models: acoustics, electromagnetics, and elastodynamics, are put on an equal footing by introducing the self-consistent system concept. Accordingly any low-frequency series expansion starts with the pertinent Helmholtz equation. Far-field surface-integrals are derived for each case.

To verify our approach, an example of low-frequency electromagnetic scattering by a long cylinder is elaborated, the results are compared to, and agree with the exact Hankel-Fourier series solution.

1. INTRODUCTION AND OVERVIEW OF THE OLD THEORY

Low-Frequency wave theory is important in physics, used for example in the Rayleigh scattering theory that explains the blue color of the sky in daytime, and the red color at sunset. It was therefore recognized that a concise mathematical theory is very desirable.

One mathematical advantage of the formalisms discussed below is that they replace solutions of Helmholtz wave equations with solutions of corresponding (scalar and vector) Laplace equations, studied in potential theory. Moreover, the latter are separable in more curvilinear coordinate systems, thus leading to canonical solutions which are not available for the Helmholtz equation [1].

As a prototype of low-frequency theories consider the analysis of Morse and Feshbach (see [1], p. 1085), who represent the solution of the Helmholtz equation in terms of a series in ascending powers of the *constant* $ik = i\omega/c$. This constant may assume different values for different problems, i.e., be a *parameter* in the context of a family of solutions. Unfortunately they use the term "power series", but evidently this is not a series in ascending powers of a *variable*. We contend that the need to appreciate this distinction is crucial to the development of a consistent low-frequency theory. Accordingly we refer to the two kinds of series as *constant-coefficient*, and *variable-coefficient* series, respectively.

In 1953 Stevenson [2] introduced a basic approach to low-frequency scattering that expanded the electric and magnetic fields in variable-coefficient series of the inverse wavelength $k/2\pi = \omega/(2\pi c)$, and by comparing equal powers of k, used these series to derive recurrence relations and distant field surface-integral representations. Many authors have subsequently used this method [3–5]. A similar approach has been adopted by [6]. The implicit assumption in Stevenson's source paper is that these series are of the *variable-coefficient* type. We show here that this assumption is incorrect.

We also show that another ansatz inherent in the traditional theory, of simultaneously expanding the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as independent series, leads to inconsistencies. The correct procedure is to expand just one field ($\boldsymbol{E}$ say), and then to obtain the expansion of the other field ($\boldsymbol{H}$) via the appropriate Maxwell *curl* relation. Only if this procedure is followed is a solution of Maxwell's equations, or an equivalent self-consistent system of equations, guaranteed. A similar procedure applies to low frequency scattering in acoustics and elastodynamics.

Finally, the new theory is illustrated and verified by analyzing the problem of scattering from a perfectly conducting cylinder. We compare the new results obtained, using our low-frequency algorithm, with the well studied Hankel-Fourier series solution.

In order to contrast the present results with the Stevenson approach [2–5], and to introduce some useful notation and formulas, we present the derivation for the electromagnetic field, based on the legitimacy (which we contest) of using variable-coefficient series expansion in ik.

We start with the source-free Maxwell equations in MKS units, (see e.g., Stratton [7]), for time-harmonic fields with the time factor $e^{-i\omega t}$, which is henceforth suppressed, in homogeneous, linear, isotropic media (e.g., free space):

$$\begin{gathered}\partial_{\boldsymbol{r}} \times \boldsymbol{E} = i\omega\mu\boldsymbol{H},\ \partial_{\boldsymbol{r}} \times \boldsymbol{H} = -i\omega\varepsilon\boldsymbol{E},\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E} = \partial_{\boldsymbol{r}} \cdot \boldsymbol{H} = 0 \\ \boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r}), \quad \boldsymbol{H} = \boldsymbol{H}(\boldsymbol{r}), \quad \boldsymbol{D} = \varepsilon\boldsymbol{E}, \quad \boldsymbol{B} = \mu\boldsymbol{H}\end{gathered} \tag{1}$$

where in (1) $\partial_{\boldsymbol{r}}$ denotes the Nabla operator and ε, μ are constant scalars.

Stevenson's theory starts with the assumption that the fields can be *simultaneously* expanded in variable-coefficient series in ik

$$\boldsymbol{E}(\boldsymbol{r}) = e_0\Sigma_{n=0}^{\infty}(ik)^n\boldsymbol{E}_n(\boldsymbol{r})/n! \tag{2}$$
$$\boldsymbol{H}(\boldsymbol{r}) = h_0\Sigma_{n=0}^{\infty}(ik)^n\boldsymbol{H}_n(\boldsymbol{r})/n! \tag{3}$$

with constant amplitudes e_0, h_0 bearing the pertinent physical units.

Substituting (2) and (3), into (1) and equating powers of ik yields the recurrence relations

$$\begin{aligned}&e_0\partial_{\boldsymbol{r}} \times \boldsymbol{E}_n = h_0cn\mu\boldsymbol{H}_{n-1},\ h_0\partial_{\boldsymbol{r}} \times \boldsymbol{H}_n = -\varepsilon e_0cn\boldsymbol{E}_{n-1} \\ &\partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n(\boldsymbol{r}) = \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_n(\boldsymbol{r}) = 0 \\ &c = (\mu\varepsilon)^{-1/2}, \quad \boldsymbol{E}_n = \boldsymbol{E}_n(\boldsymbol{r}), \quad \boldsymbol{H}_n = \boldsymbol{H}_n(\boldsymbol{r})\end{aligned} \tag{4}$$

Appropriate re-indexing has been done in the above equations to ensure that the summation range can be maintained as starting with $n = 0$. For cases where choosing $e_0/h_0 = Z = (\mu/\varepsilon)^{1/2}$ is allowed, (4) simplifies to

$$\begin{aligned}\partial_{\boldsymbol{r}} \times \boldsymbol{E}_n &= n\boldsymbol{H}_{n-1},\ \partial_{\boldsymbol{r}} \times \boldsymbol{H}_n = -n\boldsymbol{E}_{n-1} \\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n &= \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_n = 0\end{aligned} \tag{5}$$

The mathematical operations leading to (4) and (5) are only valid for variable-coefficient series where ik is a variable.

The results are far reaching, culminating in surface-integrals which can be found elsewhere [5].

Here is one way of seeing why (5) is inapplicable: Apply the *curl* operation to one of the equations in (5), note the *div* relations, and substitute the remaining *curl* relation. Repeat this for the other equation. We thus derive *simultaneously*

$$\begin{aligned} \partial_{\boldsymbol{r}}^2 \boldsymbol{E}_n(\boldsymbol{r}) &= n(n-1)\boldsymbol{E}_{n-2}(\boldsymbol{r}),\ \partial_{\boldsymbol{r}}^2 \boldsymbol{H}_n(\boldsymbol{r}) = n(n-1)\boldsymbol{H}_{n-2}(\boldsymbol{r}) \\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n(\boldsymbol{r}) &= \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_n(\boldsymbol{r}) = 0, \quad \partial_{\boldsymbol{r}}^2 = \partial_{\boldsymbol{r}} \cdot \partial_{\boldsymbol{r}} \end{aligned} \tag{6}$$

Inasmuch that (5) provides the same number of equations as (6), we expect the two sets to be equivalent. However, it is well-known that reduced sets, obtained by substituting from the original equations, lose information and allow for solutions which do not satisfy the initial set. Solving the recurrence relations for $\boldsymbol{E}_n$ and $\boldsymbol{H}_n$ in (6) does not guarantee that (5) will be satisfied. The correct procedure is to solve for one of the fields, say $\boldsymbol{E}_n$, and derive the associated $\boldsymbol{H}_n$ from the appropriate Maxwell *curl* relation. Alternatively first solve for $\boldsymbol{H}_n$ and then derive the associated $\boldsymbol{E}_n$. It follows that (2) and (3) cannot be stated *simultaneously*.

Notably for the treatment of acoustics, Dassios [5] followed a different route. The fundamental equations for time-harmonic fields are

$$\partial_{\boldsymbol{r}} \cdot \boldsymbol{v}(\boldsymbol{r}) - i\omega\gamma p(\boldsymbol{r}) = 0, \quad \partial_{\boldsymbol{r}} p(\boldsymbol{r}) - i\omega\rho \boldsymbol{v}(\boldsymbol{r}) = 0 \tag{7}$$

with $p, \boldsymbol{v}, \gamma, \rho$, denoting acoustical pressure, velocity, compressibility, and mass-density, respectively.

Consistently with (2) and (3) one has to state

$$p(\boldsymbol{r}) = q_0 \Sigma_{n=0}^{\infty} (ik)^n p_n(\boldsymbol{r})/n! \tag{8}$$

$$\boldsymbol{v}(\boldsymbol{r}) = v_0 \Sigma_{n=0}^{\infty} (ik)^n \boldsymbol{v}_n(\boldsymbol{r})/n! \tag{9}$$

with $k = \omega/c = \omega(\gamma\rho)^{1/2}$, and obtain in a similar manner to (4)

$$v_0 \partial_{\boldsymbol{r}} \cdot \boldsymbol{v}_n(\boldsymbol{r}) = nc\gamma q_0 p_{n-1}(\boldsymbol{r}), \quad q_0 \partial_{\boldsymbol{r}} p_n(\boldsymbol{r}) = nc\rho v_0 \boldsymbol{v}_{n-1}(\boldsymbol{r}) \tag{10}$$

Similarly to (6), we derive from (10)

$$\partial_{\boldsymbol{r}}^2 p_n(\boldsymbol{r}) = n(n-1)p_{n-2}(\boldsymbol{r}), \quad \partial_{\boldsymbol{r}}^2 \boldsymbol{v}_n(\boldsymbol{r}) = n(n-1)\boldsymbol{v}_{n-2}(\boldsymbol{r}) \tag{11}$$

and all the reservations that applied to the set (6) follow for the set (11).

Interestingly, for acoustics the low-frequency analysis in [5] starts from the Helmholtz equation on $(\partial_{\boldsymbol{r}}^2 + k^2)p(\boldsymbol{r}) = 0$ for the acoustic pressure p, leading to the first recurrence relation (11), with the tacit assumption that for the field $\boldsymbol{v}(\boldsymbol{r})$ one has to return to (7). This is the correct procedure as we understand it, following the prototype [1].

Unfortunately the inconsistency of the two methods was not realized so far. Similar confusing arguments exist for elastodynamics too, with the unwarranted equating of powers of ik in different series, leading to results that do not tally with the consistent systems discussed bellow.

2. CONSISTENT MAXWELL SYSTEMS

Starting again from (1), we derive by substitution the Helmholtz equations

$$\left(\partial_{\boldsymbol{r}}^2 + k^2\right)\boldsymbol{E} = 0, \quad \left(\partial_{\boldsymbol{r}}^2 + k^2\right)\boldsymbol{H} = 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{E} = \partial_{\boldsymbol{r}} \cdot \boldsymbol{H} = 0 \qquad (12)$$

Clearly the system (1) cannot be retrieved from (12). However, the set

$$\left(\partial_{\boldsymbol{r}}^2 + k^2\right)\boldsymbol{E} = 0, \quad \partial_{\boldsymbol{r}} \times \boldsymbol{E} = i\omega\mu\boldsymbol{H}, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{E} = \partial_{\boldsymbol{r}} \cdot \boldsymbol{H} = 0 \qquad (13)$$

is necessary and sufficient for re-deriving the missing equation $\partial_{\boldsymbol{r}} \times \boldsymbol{H} = -i\omega\varepsilon\boldsymbol{E}$. The sets (1) and (13) are therefore equivalent and we refer to (13) as the *first consistent Maxwell system*. The *second consistent Maxwell system* is given by

$$\left(\partial_{\boldsymbol{r}}^2 + k^2\right)\boldsymbol{H} = 0, \quad \partial_{\boldsymbol{r}} \times \boldsymbol{H} = -i\omega\mu\boldsymbol{E}, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{E} = \partial_{\boldsymbol{r}} \cdot \boldsymbol{H} = 0 \qquad (14)$$

from which $\partial_{\boldsymbol{r}} \times \boldsymbol{E} = i\omega\mu\boldsymbol{H}$ is readily derived. The choice of one consistent system over another is a matter of convenience, when applying boundary conditions in a scattering problem, for example. It is easily verified that mixing consistent systems can result in over-determined systems in which a solution found by solving some of the equations might not be a solution to the remaining equations.

A similar situation exists for acoustics and elastodynamics, to which we refer later.

3. PLANE WAVES AND LOW-FREQUENCY EXPANSIONS

Since the wave equation is linear, any solution can be constructed by combining basis solutions. The simplest example is a superposition (integral) of plane waves. However, in general such summations need to include wave solutions that are singular at the origin (e.g., spherical waves), that require complex $\boldsymbol{k}$, even when studying lossless media characterized by real $k^2 = \boldsymbol{k} \cdot \boldsymbol{k}$. This is explained below in connection with the Sommerfeld-type integral representations of the special functions involved .

To illustrate these ideas in their simplest context, consider a plane electromagnetic wave at a single frequency, i.e., $\boldsymbol{k}$ a constant vector, and its associated Taylor expansion. For simplicity choose $\boldsymbol{k} = k\hat{\boldsymbol{x}}$. A plane wave can serve as the incident wave for scattering problems

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= \hat{\boldsymbol{e}}e_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \hat{\boldsymbol{e}}e_0 \Sigma_{n=0}^{\infty}(ik)^n(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})^n/n! \\ \boldsymbol{E}(x) &= \hat{\boldsymbol{e}}e_0 e^{ikx} = \hat{\boldsymbol{e}}e_0 \Sigma_{n=0}^{\infty}(ik)^n(x)^n/n!,\ \boldsymbol{k} = k\hat{\boldsymbol{x}} \\ \boldsymbol{H}(\boldsymbol{r}) &= \hat{\boldsymbol{h}}h_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \hat{\boldsymbol{h}}h_0 \Sigma_{n=0}^{\infty}(ik)^n(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})^n/n! \\ \boldsymbol{H}(x) &= \hat{\boldsymbol{h}}h_0 e^{ikx} = \hat{\boldsymbol{h}}h_0 \Sigma_{n=0}^{\infty}(ik)^n(x)^n/n!,\ \boldsymbol{k} = k\hat{\boldsymbol{x}} \end{aligned} \tag{15}$$

polarized along the constant unit vectors $\hat{\boldsymbol{e}}, \hat{\boldsymbol{h}}$.

In the language of complex variables the functions in (15), both in the exponential and the series representations, are regular for small k, i.e., no poles are present (see [1], p. 1085). This is an essential test for the validity of representations similar to (15), discussed below.

It must be emphasized that a plane-wave (15), even for real wavenumber k, can have complex wavevector $\boldsymbol{k}$ if it propagates in complex directions (e.g., complex $\hat{\boldsymbol{x}}$). Such waves are called inhomogeneous plane waves (see [7], p. 360), and appear for example in the plane wave integrals below. Consider a complex propagation vector

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{k}_R + i\boldsymbol{k}_I \\ k^2 &= \boldsymbol{k}\cdot\boldsymbol{k} = \boldsymbol{k}_R\cdot\boldsymbol{k}_R - \boldsymbol{k}_I\cdot\boldsymbol{k}_I + 2i\boldsymbol{k}_R\cdot\boldsymbol{k}_I \end{aligned} \tag{16}$$

with subscripts R and I denoting real and imaginary components. Prescribing for complex $\boldsymbol{k}$ that k^2, and therefore also k, be real, implies that the real and imaginary components must be perpendicular

$$\boldsymbol{k}_R\cdot\boldsymbol{k}_I = 0 \tag{17}$$

Therefore for real k we have in general

$$\boldsymbol{k} = \boldsymbol{k}_R + i\boldsymbol{k}_I = k\hat{\boldsymbol{k}} = k(\boldsymbol{k}_R/k + i\boldsymbol{k}_I/k) \tag{18}$$

with a complex directional unit vector $\hat{\boldsymbol{k}} = (\boldsymbol{k}_R/k + i\boldsymbol{k}_I/k)$.

Inasmuch as (15) is a solution of the Maxwell system (1), $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the corresponding Helmholtz equations (12), e.g.,

$$\begin{aligned} \left(\partial_{\boldsymbol{r}}^2 + k^2\right)\boldsymbol{E}(\boldsymbol{r}) &= \hat{\boldsymbol{e}}e_0\Sigma_{n=0}^{\infty}(ik)^n\left(\partial_{\boldsymbol{r}}^2 + k^2\right)(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}r)^n/n! = 0 \\ \left(d_x^2 + k^2\right)\boldsymbol{E}(x) &= \hat{\boldsymbol{e}}e_0\Sigma_{n=0}^{\infty}(ik)^n\left(d_x^2 + k^2\right)x^n/n! = 0,\ \boldsymbol{k} = k\hat{\boldsymbol{x}} \end{aligned} \tag{19}$$

In order that (19) be satisfied as an identity, we have to equate powers of the variable x (not ik) and re-index terms accordingly. Thus

$$\begin{aligned} \Sigma_{n=0}^{\infty}(ik)^n d_x^2 x^n/n! &= \Sigma_{n=0}^{\infty}(ik)^n n(n-1)x^{n-2}/n! \\ &= \Sigma_{n=0}^{\infty}(ik)^{n+2}x^n/n! \end{aligned} \tag{20}$$

where it is noted that in (20) $n(n-1)$ vanishes for $n = 0$, $n = 1$ allowing the second summation to start at $n = 0$. This treatment also agrees with the prototype scheme [1] mentioned above.

In a trivial manner we rewrite (19) and (20), and obtain a recurrence relation

$$\begin{gathered}\boldsymbol{E}(x) = e_0 \Sigma_{n=0}^{\infty} (ik)^n \boldsymbol{E}_n(x)/n!,\ \boldsymbol{E}_n(x) = \hat{\boldsymbol{e}} x^n \\ d_x^2 \boldsymbol{E}_n(x) = n(n-1)\boldsymbol{E}_{n-2}(x) \\ \boldsymbol{E}(\boldsymbol{r}) = e_0 \Sigma_{n=0}^{\infty} (ik)^n \boldsymbol{E}_n(\boldsymbol{r})/n!,\ \boldsymbol{E}_n(\boldsymbol{r}) = \hat{\boldsymbol{e}}(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}} r)^n \\ \partial_{\boldsymbol{r}}^2 \boldsymbol{E}_n(\boldsymbol{r}) = n(n-1)\boldsymbol{E}_{n-2}(\boldsymbol{r})\end{gathered} \tag{21}$$

with partial waves in (21) denoted by $\boldsymbol{E}_n$.

This elementary example provides the key for generalizing to arbitrary wave fields. The elementary plane wave can be used as a basis for defining arbitrary wave fields in terms of Sommerfeld-type integrals [7–12]. Whether the integral denotes outgoing, incoming, or standing waves, depends on the specific integration contour C. In general this contour is complex, prescribing complex directions of propagation for inhomogeneous waves, as explained in (16)–(18).

Accordingly, an arbitrary wave-function is represented as a plane-wave integral of the form

$$\boldsymbol{E}(\boldsymbol{r}) = e_0 \int_C e^{ik\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \boldsymbol{g}(\hat{\boldsymbol{k}}) d\Omega_{\hat{\boldsymbol{k}}} \tag{22}$$

with $d\Omega_{\hat{\boldsymbol{k}}}$ in (22) indicating the integration over all directions prescribed by the contour C, and $\boldsymbol{g}(\hat{\boldsymbol{k}})$ being the amplitude associated with the wave propagating in direction $\hat{\boldsymbol{k}}$. For example, for the two- and three-dimensional cases we have

$$\int_C d\Omega_{\hat{\boldsymbol{k}}} = \frac{1}{\pi} \int_{\beta=-\pi/2+i\infty}^{\beta=\pi/2-i\infty} d\beta,\ \int_C d\Omega_{\hat{\boldsymbol{k}}} = \frac{1}{2\pi} \int_{\beta=-\pi}^{\beta=\pi} d\beta \int_{\alpha=0}^{\alpha=\pi/2-i\infty} \sin\alpha d\alpha \tag{23}$$

respectively, with β and α, denoting complex azimuthal and polar angles, respectively [9–12].

In view of the Taylor expansion (15), which can also be continued into the complex domain, (23) can be recast in a series of partial waves

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= e_0 \int_C e^{ik\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \boldsymbol{g}(\hat{\boldsymbol{k}}) d\Omega_{\hat{\boldsymbol{k}}} = e_0 \Sigma_{n=0}^{\infty} (ik)^n \boldsymbol{E}_n(\boldsymbol{r})/n! \\ \boldsymbol{E}_n(\boldsymbol{r}) &= \int_C \boldsymbol{g}(\hat{\boldsymbol{k}})(\hat{\boldsymbol{k}} \cdot \boldsymbol{r})^n d\Omega_{\hat{\boldsymbol{k}}}\end{aligned} \tag{24}$$

where in (24) the series representation, derived from a superposition of plane waves, satisfies the condition of analyticity discussed after (15), i.e., that $\boldsymbol{E(r)}$, considered as a complex function of k, is regular in the vicinity of $k = 0$.

Applying the same argument that led from (19) to (21), we arrive at the identical recurrence relation

$$\partial_{\boldsymbol{r}}^2 \boldsymbol{E}_n(\boldsymbol{r}) = n(n-1)\boldsymbol{E}_{n-2}(\boldsymbol{r}) \tag{25}$$

4. LOW-FREQUENCY SERIES FOR CONSISTENT MAXWELL SYSTEMS

Considering (24) and the first consistent Maxwell system (13) gives

$$\partial_{\boldsymbol{r}} \cdot \boldsymbol{E}(\boldsymbol{r}) = e_0 \Sigma_{n=0}^{\infty} (ik)^n \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n(\boldsymbol{r})/n! = 0 \tag{26}$$

On account of our maxim that (26) is a constant-coefficient series in ik, we *cannot* claim that *a necessary condition* for (26) to be valid is that each term in the series individually vanishes. However, since $\partial_{\boldsymbol{r}} \cdot \boldsymbol{E}(\boldsymbol{r}) = 0$ vanishes for *arbitrary* $\boldsymbol{r}$, *a sufficient condition* is that the series vanishes term by term. Thus (26) is implied by

$$\partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n(\boldsymbol{r}) = 0 \tag{27}$$

The converse argument does not follow: (26) does not imply (27) in general. This key argument, henceforth referred to as the *sufficiency condition*, will be met in similar circumstances below.

Let us now attempt to define $\boldsymbol{H}(\boldsymbol{r})$ according to (3) and seek for the relation between the partial fields $\boldsymbol{E}_n$ and $\boldsymbol{H}_n$. Subject to the first consistent Maxwell system (13) we have

$$\begin{aligned} \partial_{\boldsymbol{r}} \times \boldsymbol{E} &= e_0 \Sigma_{n=0}^{\infty} (ik)^n \partial_{\boldsymbol{r}} \times \boldsymbol{E}_n(\boldsymbol{r})/n! \\ &= i\omega\mu \boldsymbol{H} = i\omega\mu h_0 \Sigma_{n=0}^{\infty} (ik)^n \boldsymbol{H}_n(\boldsymbol{r})/n!, \; e_0/h_0 = Z \end{aligned} \tag{28}$$

implying according to the sufficiency condition that

$$\partial_{\boldsymbol{r}} \times \boldsymbol{E}_n(\boldsymbol{r}) = ik\boldsymbol{H}_n(\boldsymbol{r}) \tag{29}$$

Summarizing, the partial-wave representation of the first consistent Maxwell system (13) is given by

$$\begin{aligned} \partial_{\boldsymbol{r}}^2 \boldsymbol{E}_n(\boldsymbol{r}) &= n(n-1)\boldsymbol{E}_{n-2}(\boldsymbol{r}), \; \partial_{\boldsymbol{r}} \times \boldsymbol{E}_n(\boldsymbol{r}) = ik\boldsymbol{H}_n(\boldsymbol{r}) \\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n(\boldsymbol{r}) &= 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_n(\boldsymbol{r}) = 0 \end{aligned} \tag{30}$$

Using the same line of arguments, the second consistent Maxwell system (14) results in the partial-wave representation

$$\begin{aligned} \partial_{\boldsymbol{r}}^2 \boldsymbol{H}_n(\boldsymbol{r}) &= n(n-1)\boldsymbol{H}_{n-2}(\boldsymbol{r}), \; \partial_{\boldsymbol{r}} \times \boldsymbol{H}_n(\boldsymbol{r}) = -ik\boldsymbol{E}_n(\boldsymbol{r}) \\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_n &= 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_n = 0 \end{aligned} \tag{31}$$

5. SURFACE-INTEGRAL REPRESENTATIONS

Stevenson [2] already observed that we need to extend the results to the far-field, which can be achieved by Kirchhoff-type surface-integrals [1, 7, 9–12]. These constitute a rigorous mathematical expression of the primitive Huygens' principle used in optics. Essentially, sources known on a surface serve to compute the field in other locations in space. In the context of low-frequency partial waves, representations developed subject to the assumption of variable-coefficient series in ik have been devised, e.g., see [5], but need revision subject to the present theory.

Stevenson [2] was also aware of the limitations of this method, which becomes increasingly inaccurate with distance. Indeed, this is an ill-defined method because small changes of fields on the surface cause large diffraction effects at distant points. Numerical computations therefore need to include artificial "regularization" schemes.

We start with the scalar case which is the simplest example and will also serve for the acoustical pressure, and with adequate modifications, for the two-dimensional electromagnetic case. We need the free-space scalar Greens-function G, and a scalar field p satisfying the following Helmholtz equations

$$\begin{gathered}(\partial_{\boldsymbol{r}}^2 + k^2)\, G(k|\boldsymbol{r}-\boldsymbol{\rho}|) = -\delta(\boldsymbol{r}-\boldsymbol{\rho}),\ (\partial_{\boldsymbol{r}}^2 + k^2)\, p(\boldsymbol{r}) = 0 \\ G(k|\boldsymbol{r}-\boldsymbol{\rho}|) = e^{ik|\boldsymbol{r}-\boldsymbol{\rho}|}/4\pi|\boldsymbol{r}-\boldsymbol{\rho}|\end{gathered} \tag{32}$$

It can be shown, e.g., [1], that inside a closed surface

$$\begin{gathered}p(\boldsymbol{r}) = \oint_S [p(\boldsymbol{\rho})\partial_{\boldsymbol{\rho}} G(k|\boldsymbol{r}-\boldsymbol{\rho}|) - G(k|\boldsymbol{r}-\boldsymbol{\rho}|)\partial_{\boldsymbol{\rho}} p(\boldsymbol{\rho})] \cdot d\boldsymbol{S} \\ d\boldsymbol{S}(\boldsymbol{\rho}) = \hat{\boldsymbol{n}}(\boldsymbol{\rho}) dS(\boldsymbol{\rho})\end{gathered} \tag{33}$$

with the change of sign with respect to [1], who use a normal unit vector pointing away from the interior of the volume, while we adhere to [10–12] and others, employing $\hat{\boldsymbol{n}}$ pointing away from the scatterer, i.e., into the volume.

For the vector analog of (32), (33), we need the dyadic Green-function $\tilde{\boldsymbol{G}}$ and a vector field $\boldsymbol{F}$. The analog of (22) becomes [1]

$$\begin{gathered}(\partial_{\boldsymbol{r}}^2 + k^2)\, \tilde{\boldsymbol{G}}(k|\boldsymbol{r}-\boldsymbol{\rho}|) = -\tilde{\boldsymbol{I}}\delta(\boldsymbol{r}-\boldsymbol{\rho}),\ (\partial_{\boldsymbol{r}}^2 + k^2)\, \boldsymbol{F}(\boldsymbol{r}) = 0 \\ \tilde{\boldsymbol{G}}(k|\boldsymbol{r}-\boldsymbol{\rho}|) = \tilde{\boldsymbol{I}}G = \tilde{\boldsymbol{I}}e^{ik|\boldsymbol{r}-\boldsymbol{\rho}|}/4\pi|\boldsymbol{r}-\boldsymbol{\rho}|\end{gathered} \tag{34}$$

For the transversal electromagnetic field characterized by $\partial_{\boldsymbol{r}} \cdot \boldsymbol{F}(\boldsymbol{r}) = 0$, we need the transversal part $\tilde{\boldsymbol{G}}_T$ of the dyadic Green-function. Thus (34) becomes

$$\begin{gathered}\left(\partial_{\boldsymbol{r}}^2+k^2\right)\tilde{\boldsymbol{G}}_T(k|\boldsymbol{r}-\boldsymbol{\rho}|)=-\tilde{\boldsymbol{I}}\delta(\boldsymbol{r}-\boldsymbol{\rho}),\ \left(\partial_{\boldsymbol{r}}^2+k^2\right)\boldsymbol{F}(\boldsymbol{r})=0\\ \partial_{\boldsymbol{r}}\cdot\tilde{\boldsymbol{G}}_T(k|\boldsymbol{r}-\boldsymbol{\rho}|)=0,\quad \partial_{\boldsymbol{r}}\cdot\boldsymbol{F}(\boldsymbol{r})=0,\quad \boldsymbol{F}=\boldsymbol{E},\boldsymbol{H}\\ \tilde{\boldsymbol{G}}_T(k|\boldsymbol{r}-\boldsymbol{\rho}|)=\left(\tilde{\boldsymbol{I}}+\partial_{\boldsymbol{r}}\partial_{\boldsymbol{r}}/k^2\right)G(k|\boldsymbol{r}-\boldsymbol{\rho}|)=\left(\tilde{\boldsymbol{I}}+\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}}/k^2\right)G(k|\boldsymbol{r}-\boldsymbol{\rho}|)\end{gathered} \tag{35}$$

By inspection of equation (13.1.10), p. 1770 of [1], noting that one of the terms there can be recast as

$$\tilde{\boldsymbol{G}}\cdot[\hat{\boldsymbol{n}}\times(\partial_{\boldsymbol{\rho}}\times\boldsymbol{F})]=(\partial_{\boldsymbol{\rho}}\times\boldsymbol{F})\cdot(\tilde{\boldsymbol{G}}\times\hat{\boldsymbol{n}})=-(\partial_{\boldsymbol{\rho}}\times\boldsymbol{F})\cdot(\hat{\boldsymbol{n}}\times\tilde{\boldsymbol{G}}) \tag{36}$$

the analog of (33) for transversal vectors, with the same sign agreement, is found in the form

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r})&=\oint_S\left[(d\boldsymbol{S}\times\boldsymbol{F}(\boldsymbol{\rho}))\cdot(\partial_{\boldsymbol{\rho}}\times\tilde{\boldsymbol{G}}_T)-(\partial_{\boldsymbol{\rho}}\times\boldsymbol{F}(\boldsymbol{\rho}))\cdot(d\boldsymbol{S}\times\tilde{\boldsymbol{G}}_T)\right]\\ d\boldsymbol{S}&=d\boldsymbol{S}(\boldsymbol{\rho})=\hat{\boldsymbol{n}}dS,\ \tilde{\boldsymbol{G}}_T=\tilde{\boldsymbol{G}}_T(k|\boldsymbol{r}-\boldsymbol{\rho}|)=(\tilde{\boldsymbol{I}}+\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}}/k^2)G(k|\boldsymbol{r}-\boldsymbol{\rho}|)\end{aligned} \tag{37}$$

For $\boldsymbol{F}=\boldsymbol{E}$ and $\boldsymbol{F}=\boldsymbol{H}$, expressed in series (2) and (3) respectively, and substituted into (37) we obtain

$$\Sigma_{n=0}^{\infty}(ik)^n/n!\left\{\boldsymbol{F}_n(\boldsymbol{r})+\oint_S[\partial_{\boldsymbol{\rho}}\times\boldsymbol{F}_n(\boldsymbol{\rho}))\cdot(d\boldsymbol{S}\times\tilde{\boldsymbol{G}}_T)-(d\boldsymbol{S}\times\boldsymbol{F}_n(\boldsymbol{\rho}))\cdot(\partial_{\boldsymbol{\rho}}\times\tilde{\boldsymbol{G}}_T)]\right\}=0 \tag{38}$$

Once more, we appeal to the sufficiency condition. Accordingly, since $\boldsymbol{r}$ can assume arbitrary values, each term in the series (38) vanishes individually, yielding

$$\boldsymbol{F}_n(\boldsymbol{r})=\oint_S\left[(d\boldsymbol{S}\times\boldsymbol{F}_n(\boldsymbol{\rho}))\cdot(\partial_{\boldsymbol{\rho}}\times\tilde{\boldsymbol{G}}_T)-(\partial_{\boldsymbol{\rho}}\times\boldsymbol{F}_n(\boldsymbol{\rho}))\cdot(d\boldsymbol{S}\times\tilde{\boldsymbol{G}}_T)\right] \tag{39}$$

This ends the discussion of the low-frequency formalism for the Maxwell electromagnetic theory. The acoustics and elastodynamics analogs are subsequently discussed. Inasmuch as the electromagnetic part above was carried out in great detail, these physical models follow with very few modifications.

6. ACOUSTICAL LOW-FREQUENCY THEORY

The fundamental equations for time-harmonic acoustical fields have been given above (7). By substitution and elimination of $\boldsymbol{v}$, the first consistent acoustical system is derived as

$$\left(\partial_{\boldsymbol{r}}^2 + k^2\right) p(\boldsymbol{r}) = 0,\ \partial_{\boldsymbol{r}} p(\boldsymbol{r}) - i\omega p \boldsymbol{v}(\boldsymbol{r}) = 0,\ k^2 = \omega^2 \gamma \rho,\ \partial_{\boldsymbol{r}} \times \boldsymbol{v}(\boldsymbol{r}) = 0 \tag{40}$$

The second consistent acoustical system is similarly obtained by eliminating p in (7)

$$\left(\partial_{\boldsymbol{r}}^2 + k^2\right) \boldsymbol{v} = 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{v} - i\omega\gamma p = 0 \tag{41}$$

In view of the above discussion concerning plane-wave integrals, especially (22)–(25), for the first consistent acoustical system (8) is assumed. Furthermore, invoking the sufficiency condition, (40) yields for partial waves

$$\begin{gathered} \partial_{\boldsymbol{r}}^2 p_n(\boldsymbol{r}) = n(n-1) p_{n-2}(\boldsymbol{r}) \\ \partial_{\boldsymbol{r}} p_n(\boldsymbol{r}) - i\omega\rho \boldsymbol{v}_n(\boldsymbol{r}) = 0,\ \partial_{\boldsymbol{r}} \times \boldsymbol{v}_n(\boldsymbol{r}) = 0 \end{gathered} \tag{42}$$

which is the analog of (30).

The corresponding surface-integral representation for partial waves is (c.f. (33))

$$p_n(\boldsymbol{r}) = \oint_S \left[p_n(\boldsymbol{\rho}) \partial_{\boldsymbol{\rho}} G(k|\boldsymbol{r} - \boldsymbol{\rho}|) - G(k|\boldsymbol{r} - \boldsymbol{\rho}|) \partial_{\boldsymbol{\rho}} p_n(\boldsymbol{\rho})\right] \cdot d\boldsymbol{S} \tag{43}$$

In a similar manner the second consistent acoustic system (41) becomes, in terms of partial waves (c.f. (42)),

$$\partial_{\boldsymbol{r}}^2 \boldsymbol{v}_n(\boldsymbol{r}) = n(n-1) \boldsymbol{v}_{n-2}(\boldsymbol{r}), \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{v}_n - i\omega\gamma p_n = 0 \tag{44}$$

Similarly to (35) we need now Helmholtz equations for longitudinal fields

$$\begin{gathered} \left(\partial_{\boldsymbol{r}}^2 + k^2\right) \tilde{\boldsymbol{G}}_L(k|\boldsymbol{r} - \boldsymbol{\rho}|) = -\tilde{\boldsymbol{I}} \delta(\boldsymbol{r} - \boldsymbol{\rho}),\ \left(\partial_{\boldsymbol{r}}^2 + k^2\right) \boldsymbol{v}(\boldsymbol{r}) = 0 \\ \tilde{\boldsymbol{G}}_L = \tilde{\boldsymbol{I}} G(k|\boldsymbol{r} - \boldsymbol{\rho}|) - \tilde{\boldsymbol{G}}_T = -\partial_{\boldsymbol{r}} \partial_{\boldsymbol{r}} G(k|\boldsymbol{r} - \boldsymbol{\rho}|)/k^2,\ \partial_{\boldsymbol{r}} \times \tilde{\boldsymbol{G}}_L(k|\boldsymbol{r} - \boldsymbol{\rho}|) \end{gathered} \tag{45}$$

The same equation (13.1.10), p. 1770 of [1], mentioned above, prescribes the surface integral representation for longitudinal vector waves in the form

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{G}}_L)(\boldsymbol{v}(\boldsymbol{\rho}) \cdot d\boldsymbol{S}) - (\partial_{\boldsymbol{\rho}} \cdot \boldsymbol{v}(\boldsymbol{\rho}))(\tilde{\boldsymbol{G}}_L \cdot d\boldsymbol{S})\right] \\ \tilde{\boldsymbol{G}}_L &= -\partial_{\boldsymbol{r}} \partial_{\boldsymbol{r}} G(k|\boldsymbol{r} - \boldsymbol{\rho}|)/k^2 = -\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\rho}} G(k|\boldsymbol{r} - \boldsymbol{\rho}|)/k^2 \end{aligned} \tag{46}$$

Finally the surface-integral representation for the longitudinal partial-waves is obtained by invoking the sufficiency condition

$$\boldsymbol{v}_n(\boldsymbol{r}) = \oint_S \Big[(\partial_{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{G}}_L)(\boldsymbol{v}_n(\boldsymbol{\rho}) \cdot d\boldsymbol{S}) - (\partial_{\boldsymbol{\rho}} \cdot \boldsymbol{v}_n(\boldsymbol{\rho}))(\tilde{\boldsymbol{G}}_L \cdot d\boldsymbol{S})\Big] \tag{47}$$

and this ends the discussion of the acoustical low-frequency theory.

7. ELASTODYNAMIC LOW-FREQUENCY THEORY

Following [1, 5], the basic equation of elastodynamics for time-harmonic fields are given by

$$\begin{aligned} \partial_{\boldsymbol{r}} \cdot \tilde{\boldsymbol{\tau}} &= -i\omega\rho\boldsymbol{v}, \ -i\omega\tilde{\boldsymbol{\tau}} = \lambda(\partial_{\boldsymbol{r}} \cdot \boldsymbol{v})\tilde{\boldsymbol{I}} + \mu\tilde{\boldsymbol{V}} \\ \tilde{\boldsymbol{V}} &= \partial_{\boldsymbol{r}}\boldsymbol{v} + (\partial_{\boldsymbol{r}}\boldsymbol{v})^T, \quad \boldsymbol{v} = -i\omega\boldsymbol{w} \end{aligned} \tag{48}$$

where in (48) λ, μ are the Lamé constants, $\tilde{\boldsymbol{\tau}} = \tilde{\boldsymbol{\tau}}^T$ is the symmetric stress tensor. Using $\boldsymbol{v} = \partial_t \boldsymbol{w} = -i\omega\boldsymbol{w}$, the velocity field associated with the displacement field $\boldsymbol{w}$, facilitates comparison with acoustics, the limiting case of elasticity for vanishing shear.

The velocity $\boldsymbol{v}$ (or the associated displacement $\boldsymbol{w}$) is now split into two components

$$\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_s, \quad \partial_{\boldsymbol{r}} \times \boldsymbol{v}_p = 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{v}_s = 0 \tag{49}$$

where in (49) the indices p and s denote the pressure (vortex free), and the shear (source free), fields, respectively.

Substituting from (49) into (48) yields

$$\begin{aligned} \partial_{\boldsymbol{r}} \cdot \tilde{\boldsymbol{\tau}}_p &= -i\omega\rho\boldsymbol{v}_p, \quad -i\omega\tilde{\boldsymbol{\tau}}_p = \lambda(\partial_{\boldsymbol{r}} \cdot \boldsymbol{v}_p)\tilde{\boldsymbol{I}} + \mu\tilde{\boldsymbol{V}}_p \qquad (50) \\ \partial_{\boldsymbol{r}} \cdot \tilde{\boldsymbol{\tau}}_s &= -i\omega\rho\boldsymbol{v}_s, \ -i\omega\tilde{\boldsymbol{\tau}}_s = \mu\tilde{\boldsymbol{V}}_s, \ \tilde{\boldsymbol{V}}_\alpha = \partial_{\boldsymbol{r}}\boldsymbol{v}_\alpha + (\partial_{\boldsymbol{r}}\boldsymbol{v}_\alpha)^T, \ \alpha = p, s \end{aligned}$$

Elimination of $\tilde{\boldsymbol{\tau}}_p$, $\tilde{\boldsymbol{\tau}}_s$ in the pairs of equations in (50), and noting that

$$\partial_{\boldsymbol{r}} \cdot \tilde{\boldsymbol{V}} = 2\partial_{\boldsymbol{r}}^2\boldsymbol{v} + \partial_{\boldsymbol{r}} \times (\partial_{\boldsymbol{r}} \times \boldsymbol{v}) \tag{51}$$

yields, after some additional manipulation, the first consistent elastodynamic system

$$\begin{aligned} \left(\partial_{\boldsymbol{r}}^2 + k_p^2\right)\boldsymbol{v}_p = 0, \ \lambda(\partial_{\boldsymbol{r}} \cdot \boldsymbol{v}_p)\tilde{\boldsymbol{I}} + \mu\tilde{\boldsymbol{V}}_p + i\omega\tilde{\boldsymbol{\tau}}_p = 0 \\ k_p^2 = \omega^2/c_p^2, \quad c_p^2 = (\lambda + 2\mu)/\rho \\ (\partial_{\boldsymbol{r}}^2 + k_s^2)\boldsymbol{v}_s = 0, \quad \mu\tilde{\boldsymbol{V}}_s + i\omega\tilde{\boldsymbol{\tau}}_s = 0 \\ k_s^2 = \omega^2/c_s^2, \quad c_s^2 = \mu/\rho \end{aligned} \tag{52}$$

Note that in (52), constituting the first consistent system, we have two Helmholz equations describing modes for $\boldsymbol{v}_p$ and $\boldsymbol{v}_s$. Similarly to the electromagnetic systems (2) and (3), and the acoustic systems (8) and (9), series for $\boldsymbol{v}_p$, $\boldsymbol{v}_s$ are introduced

$$\begin{aligned} \boldsymbol{v}_\alpha(\boldsymbol{r}) &= v_{0,\alpha}\Sigma_{n=0}^{\infty}(ik_\alpha)^n \boldsymbol{v}_{\alpha,n}(\boldsymbol{r})/n! \\ \tilde{\boldsymbol{\tau}}_\alpha(\boldsymbol{r}) &= \tau_{0,\alpha}\Sigma_{n=0}^{\infty}(ik_\alpha)^n \tilde{\boldsymbol{\tau}}_{\alpha,n}(\boldsymbol{r})/n!,\ \alpha = p, s \end{aligned} \tag{53}$$

Inspection of (50) and (53) suggests introducing the compact notation

$$\tilde{\boldsymbol{V}}_\alpha(\boldsymbol{r}) = v_{0,\alpha}\Sigma_{n=0}^{\infty}(ik_\alpha)^n \tilde{\boldsymbol{V}}_{\alpha,n}(\boldsymbol{r})/n!,\ \alpha = p, s \tag{54}$$

Similarly to (30), (31) and (42), and by invoking the sufficiency condition, we have for the partial waves

$$\begin{gathered} \partial_{\boldsymbol{r}}^2 \boldsymbol{v}_{p,n} = n(n-1)\boldsymbol{v}_{p,n-2} \\ v_{0,p}\left[\lambda(\partial_{\boldsymbol{r}}\cdot\boldsymbol{v}_{p,n})\tilde{\boldsymbol{I}} + \mu\tilde{\boldsymbol{V}}_{p,n}\right] + i\omega\tau_{0,p}\tilde{\boldsymbol{\tau}}_{p,n} = 0 \\ \partial_{\boldsymbol{r}}^2 \boldsymbol{v}_{s,n} = n(n-1)\boldsymbol{v}_{s,n-2},\ v_{0,s}\mu\tilde{\boldsymbol{V}}_s + i\omega\tau_{0,s}\tilde{\boldsymbol{\tau}}_s = 0 \end{gathered} \tag{55}$$

Corresponding to the Helmholtz equations (52), surface-integral representations are given by (46) and (37) for the longitudinal and transversal, fields, respectively

$$\begin{aligned} \boldsymbol{v}_p(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}}\cdot\tilde{\boldsymbol{G}}_L)(\boldsymbol{v}(\boldsymbol{\rho})\cdot d\boldsymbol{S}) - (\partial_{\boldsymbol{\rho}}\cdot\boldsymbol{v}_p(\boldsymbol{\rho}))(\tilde{\boldsymbol{G}}_L\cdot d\boldsymbol{S})\right] \\ \tilde{\boldsymbol{G}}_L &= -\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}} G(k_p|\boldsymbol{r}-\boldsymbol{\rho}|)/k_p^2 \\ \boldsymbol{v}_s(\boldsymbol{r}) &= \oint_S \left[(d\boldsymbol{S}\times\boldsymbol{v}_s(\boldsymbol{\rho}))\cdot(\partial_{\boldsymbol{\rho}}\times\tilde{\boldsymbol{G}}_T) - (\partial_{\boldsymbol{\rho}}\times\boldsymbol{v}_s(\boldsymbol{\rho}))\cdot(d\boldsymbol{S}\times\tilde{\boldsymbol{G}}_T)\right] \\ \tilde{\boldsymbol{G}}_T &= \left(\tilde{\boldsymbol{I}} + \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}}/k_s^2\right) G(k_s|\boldsymbol{r}-\boldsymbol{\rho}|) \end{aligned} \tag{56}$$

According to the sufficiency condition, for the partial fields we have

$$\begin{aligned} \boldsymbol{v}_{p,n}(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}}\cdot\tilde{\boldsymbol{G}}_L)(\boldsymbol{v}_{p,n}(\boldsymbol{\rho})\cdot d\boldsymbol{S}) - (\partial_{\boldsymbol{\rho}}\cdot\boldsymbol{v}_{p,n}(\boldsymbol{\rho}))(\tilde{\boldsymbol{G}}_L\cdot d\boldsymbol{S})\right] \\ \boldsymbol{v}_{s,n}(\boldsymbol{r}) &= \oint_S \left[(d\boldsymbol{S}\times\boldsymbol{v}_{s,n}(\boldsymbol{\rho}))\cdot(\partial_{\boldsymbol{\rho}}\cdot\tilde{\boldsymbol{G}}_T) - (\partial_{\boldsymbol{\rho}}\times\boldsymbol{v}_{s,n}(\boldsymbol{\rho}))\cdot(d\boldsymbol{S}\times\tilde{\boldsymbol{G}}_T)\right] \end{aligned} \tag{57}$$

with the appropriate dyadic Green-functions given in (56).

We need now the second consistent elastic system and its expression in terms of partial waves. The Helmholtz operators $(\partial_{\boldsymbol{r}}^2 + k_p^2)$

and $(\partial_{\boldsymbol{r}}^2 + k_s^2)$ in (52) are scalars and therefore commute with vectors operators. Applying these operators to the second and fourth equation of (40) therefore yields

$$(\partial_{\boldsymbol{r}}^2 + k_p^2)\tilde{\boldsymbol{\tau}}_p = 0, \quad (\partial_{\boldsymbol{r}}^2 + k_s^2)\tilde{\boldsymbol{\tau}}_s = 0 \tag{58}$$

Consequently the second elastodynamic consistent system for the two modes is

$$(\partial_{\boldsymbol{r}}^2 + k_\alpha^2)\tilde{\boldsymbol{\tau}}_\alpha = 0, \quad \partial_{\boldsymbol{r}} \cdot \tilde{\boldsymbol{\tau}}_\alpha = -i\omega\rho \boldsymbol{v}_\alpha, \quad \alpha = p, s \tag{59}$$

When the partial waves (53) are substituted into (59), the Helmholtz equation prescribes the recurrence relation as in (30), (31), (42), (55), and by the sufficiency condition we obtain

$$\begin{aligned} \partial_{\boldsymbol{r}}^2 \tilde{\boldsymbol{\tau}}_{\alpha,n} &= n(n-1)\tilde{\boldsymbol{\tau}}_{\alpha,n-2} \\ \tau_{0,\alpha}\partial_{\boldsymbol{r}} \cdot \tilde{\boldsymbol{\tau}}_{\alpha,n} &= -i\omega\rho v_{0,\alpha}\boldsymbol{v}_{\alpha,n}, \ \alpha = p, s \end{aligned} \tag{60}$$

Surface integrals for transversal dyadics have already been given previously [11], and prescribe for our present case (cf. (56))

$$\begin{aligned} \tilde{\boldsymbol{\tau}}_s(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}} \times \tilde{\boldsymbol{G}}_L)^T \cdot (d\boldsymbol{S} \times \tilde{\boldsymbol{\tau}}_s(\boldsymbol{\rho})) - (d\boldsymbol{S} \times \tilde{\boldsymbol{G}}_T)^T \cdot (\partial_{\boldsymbol{\rho}} \times \tilde{\boldsymbol{\tau}}_s(\boldsymbol{\rho}))\right] \\ \tilde{\boldsymbol{G}}_T &= \left(\tilde{\boldsymbol{I}} + \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}}/k_s^2\right) G(k_s|\boldsymbol{r} - \boldsymbol{\rho}|) \end{aligned} \tag{61}$$

Inasmuch as $\tilde{\boldsymbol{\tau}}$ is symmetric, and no cross-products appear in the longitudinal surface-integral (46), (56), we now have

$$\begin{aligned} \tilde{\boldsymbol{\tau}}_p(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{G}}_L)(\tilde{\boldsymbol{\tau}}_p(\boldsymbol{\rho}) \cdot d\boldsymbol{S}) - (\partial_{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{\tau}}_p(\boldsymbol{\rho}))(\tilde{\boldsymbol{G}}_L \cdot d\boldsymbol{S})\right] \\ \tilde{\boldsymbol{G}}_L &= \tilde{\boldsymbol{G}}_L(k_p|\boldsymbol{r} - \boldsymbol{\rho}|) = -\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}} G(k_p|\boldsymbol{r} - \boldsymbol{\rho}|)/k_p^2 \end{aligned} \tag{62}$$

Using the series (53) and once more invoking the sufficiency condition, the expression of (61), (62) in terms of partial waves is given by

$$\begin{aligned} \tilde{\boldsymbol{\tau}}_{s,n}(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}} \times \tilde{\boldsymbol{G}}_T)^T \cdot (d\boldsymbol{S} \times \tilde{\boldsymbol{\tau}}_{s,n}(\boldsymbol{\rho})) - (d\boldsymbol{S} \times \tilde{\boldsymbol{G}}_T)^T \cdot (\partial_{\boldsymbol{\rho}} \times \tilde{\boldsymbol{\tau}}_{s,n}(\boldsymbol{\rho}))\right] \\ \tilde{\boldsymbol{\tau}}_{s,n}(\boldsymbol{r}) &= \oint_S \left[(\partial_{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{G}}_T)(\tilde{\boldsymbol{\tau}}_{p,n}(\boldsymbol{\rho}) \cdot d\boldsymbol{S}) - (\partial_{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{\tau}}_{p,s}(\boldsymbol{\rho}))(\tilde{\boldsymbol{G}}_L \cdot d\boldsymbol{S})\right] \end{aligned} \tag{63}$$

respectively, with the appropriate dyadic Green-function understood. With that the discussion of the theory is completed.

8. ELECTROMAGNETIC SCATTERING FROM A CYLINDER

This simple scattering problem provided for us a prime motivator for scrutinizing the low-frequency theory. As will be shown below, the fields prescribed by (4) for $n = 0$, namely

$$\partial_{\boldsymbol{r}} \times \boldsymbol{E}_0(\boldsymbol{r}) = 0,\ \partial_{\boldsymbol{r}} \times \boldsymbol{H}_0(\boldsymbol{r}) = 0,\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_0(\boldsymbol{r}) = \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_0(\boldsymbol{r}) = 0 \quad (64)$$

cannot simultaneously exist as solutions of one and the same scattering problem.

Consider a perfectly conducting infinitely long circular-cylinder of radius $r = a$, with the cylindrical axis $r = 0$ being along the z-coordinate. The incident wave is given by (15) and (21), with $\hat{\boldsymbol{e}} = \hat{\boldsymbol{z}},\ \hat{\boldsymbol{h}} = -\hat{\boldsymbol{y}},\ \hat{\boldsymbol{k}} = \hat{\boldsymbol{x}}$. This means that we consider here the TM (transverse magnetic) scalar problem. Accordingly we choose (2) and the first consistent Maxwell system (30) to describe the scattered field. Thus the incident and scattered fields are denoted by

$$\begin{aligned} \boldsymbol{E}_i(\boldsymbol{r}) &= \hat{\boldsymbol{z}} e_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \hat{\boldsymbol{z}} e_0 e^{ikr\cos\psi} = \hat{\boldsymbol{z}} e_0 \Sigma_{n=0}^{\infty} (ik)^n (r\cos\psi)^n / n! \\ \boldsymbol{E}_s(\boldsymbol{r}) &= \hat{\boldsymbol{z}} e_0 \Sigma_{n=0}^{\infty} (ik)^n E_{s,n}(\boldsymbol{r})/n!,\ \cos\psi = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}} \end{aligned} \quad (65)$$

respectively, with ψ in (65) denoting the azimuthal angle, and $\hat{\boldsymbol{r}}$ pointing away from the cylindrical axis.

The boundary-condition prescribes the vanishing of the tangential electric field at the boundary. For the present case, with fields polarized in the $\hat{\boldsymbol{z}}$ direction

$$\hat{\boldsymbol{r}} \times (\boldsymbol{E}_i(\boldsymbol{r}) + \boldsymbol{E}_s(\boldsymbol{r})) = 0\Big|_{r=a} \quad (66)$$

According to the sufficiency condition, invoked for arbitrary ψ, the series satisfy (66) term by term, or in other words, (66) must be satisfied for any ψ, hence powers of $\cos\psi$ provide the orthogonal basis for the series.

In addition to these constraints, $\boldsymbol{E}_s(\boldsymbol{r})$ in conjunction with the harmonic time variation factor $e^{-i\omega t}$ must describe outgoing waves, i.e., the fields must satisfy a radiation condition. This will be discussed in the context of the exact solution presented below.

According to (30), for $n = 0$ (65) prescribes $E_{s,0}(\boldsymbol{r})$ as a solution of the Laplace equation, independent of ψ. The electrostatic potential of a uniformly charged infinite line suggests a monopole logarithmic solution, which is the only solution satisfying the above provisos

$$\begin{aligned} \boldsymbol{E}_{s,0}(\boldsymbol{r}) = \hat{\boldsymbol{z}} A_0 \ln(\kappa r),\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_{s,0} &= 0 \\ \hat{\boldsymbol{z}} \partial_{\boldsymbol{r}}^2 E(\boldsymbol{r}) = \hat{\boldsymbol{z}} A_0 r^{-1} d_r (r d_r \ln(\kappa r)) &= 0,\ \kappa = 1 \end{aligned} \quad (67)$$

where in (67) A_0 is a constant coefficient, and *kappa* is a unit inverse-length constant carrying the physical units, in order for the argument of the logarithm function to be dimensionless.

According to (66)

$$\boldsymbol{E}_{i,0} + \boldsymbol{E}_{s,0} = 0\Big|_{r=a}, \; A_0 = -1/\ln(\kappa a), \; \boldsymbol{E}_{s,0}(\boldsymbol{r}) = -\hat{\boldsymbol{z}} \ln(\kappa r)/\ln(\kappa a) \tag{68}$$

In accordance with (30), the field $\boldsymbol{H}_{s,0}$ associated with (68) is given by

$$\begin{aligned} \boldsymbol{H}_{s,0}(\boldsymbol{r}) &= \partial_{\boldsymbol{r}} \times \boldsymbol{E}_{s,0}(\boldsymbol{r}) = \hat{\boldsymbol{\psi}}/[ikr \ln(\kappa a)] \\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{H}_{s,0} &= 0, \quad \partial_{\boldsymbol{r}} \times \boldsymbol{H}_{s,0} = 0 \end{aligned} \tag{69}$$

It is obvious from (69) that $\partial_{\boldsymbol{r}} \times \boldsymbol{H}_{s,0} = 0$ but $\partial_{\boldsymbol{r}} \times \boldsymbol{E}_{s,0}(\boldsymbol{r}) \neq 0$, otherwise we cannot have associated electric and magnetic fields satisfying (30), see also (64) and following remarks.

Comparing (68) and (69), we see that for increasing r the ratio of field amplitudes behaves like

$$|\boldsymbol{E}_{s,0}|/|\boldsymbol{H}_{s,0}| \propto |\ln(\kappa r)/r| \sim r \tag{70}$$

i.e., the electric field becomes dominant over the magnetic field, as expected.

For the next term $n = 1$, (65) prescribes a factor $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}} = \cos\psi$. Hence in order to satisfy the boundary-conditions on the cylinder at $r = a$ for all ψ, the factor $\cos\psi$ must feature in the solution. We therefore choose from the electrostatic potential theory repertoire a dipole term

$$\boldsymbol{E}_{s,1}(\boldsymbol{r}) = \hat{\boldsymbol{z}} ikA_1 \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}}/r = \hat{\boldsymbol{z}} ikA_1 \cos\psi/r, \; \partial_{\boldsymbol{r}}^2 \boldsymbol{E}_{s,1} = 0, \; \partial_{\boldsymbol{r}} \cdot \boldsymbol{E}_{s,1} = 0 \tag{71}$$

The constant A_1 is determined from the boundary-condition, hence we have

$$\boldsymbol{E}_{i,1} + \boldsymbol{E}_{s,1} = 0\Big|_{r=a}, \quad A_1 = -a^2, \quad \boldsymbol{E}_{s,1}(\boldsymbol{r}) = -\hat{\boldsymbol{z}} ika^2 \cos\psi/r \tag{72}$$

According to (30), the corresponding magnetic field is given by

$$\begin{aligned} \boldsymbol{H}_{s,1}(\boldsymbol{r}) &= -\partial_{\boldsymbol{r}} \times \hat{\boldsymbol{z}} ika^2 \cos\psi/ikr \\ &= a^2 \hat{\boldsymbol{z}} \times \partial_{\boldsymbol{r}}(\cos\psi/r) = (a^2/r^2)(\hat{\boldsymbol{r}} \sin\psi - \hat{\boldsymbol{\psi}} \cos\psi) \end{aligned} \tag{73}$$

Similarly to (70) we have now

$$|\boldsymbol{E}_{s,1}|/|\boldsymbol{H}_{s,1}| \sim r \tag{74}$$

So once again the electric field becomes dominant over the magnetic for increasing r.

The exact solution for scattering from a perfectly conducting circular cylinder is a well known problem, e.g., see [13]. The incident wave (65) and scattered wave solution of (12) are stated in terms of cylindrical wave functions

$$\begin{aligned} \boldsymbol{E}_i(\boldsymbol{r}) &= \hat{\boldsymbol{z}} e_0 \Sigma_{m=-\infty}^{m=\infty} i^m J_m(kr) e^{im\psi} \\ \boldsymbol{E}_s(\boldsymbol{r}) &= \hat{\boldsymbol{z}} e_0 \Sigma_{m=-\infty}^{m=\infty} i^m a_m H_m(kr) e^{im\psi} \\ H_m &= H_m^{(1)} = J_m + iN_m \end{aligned} \tag{75}$$

The Hankel functions of the first kind $H_m = H_m^{(1)}$ are a combination of the nonsingular Bessel functions J_m, and the Neumann functions N_m which are singular at $kr = 0$ (e.g., see [7]). Combined with the time factor $e^{-i\omega t}$, outgoing scattered waves are stated in (75).

For cylindrical functions of real integer order we have [17, 18]

$$Z_{-m}(kr) = (-1)^m Z_m(kr), \quad Z = J, N, H^{(1)}, H^{(2)} \tag{76}$$

for Bessel, Neumann, and the first and second kind Hankel functions, respectively. The coefficients a_m (75) are computed by solving the pertinent boundary value problem prescribed by (66) (e.g., see [13]). Together with (76) we thus obtain

$$a_m = a_{-m} = -J_m(ka)/H_m(ka) \tag{77}$$

For small arguments, formulas for the cylindrical functions near the origin [7] are exploited, e.g., [9], yielding for $a_0(ka)$, $H_0(kr)$ in (75) for small a and r

$$\begin{aligned} a_0(ka) &\simeq -i\pi/[2\ln(2/(\delta ka))] \simeq i\pi[2\ln a] \\ H_0(kr) &= J_0(kr) + iN_0(kr) \simeq iN_0(kr) \\ &\simeq -i(2/\pi)\ln(2/(\delta kr)) \simeq i(2/\pi)\ln r \\ a_0(ka)H_0(kr) &\simeq -\ln r/\ln a \end{aligned} \tag{78}$$

where in (78) δ is a constant [7]. This is exactly the expression given in (68). Summing the terms for $m = \pm 1$ in (75) , (77) , we obtain

$$\begin{aligned} &a_1 = a_{-1} = -J_1(ka)/H_1(ka) \simeq -i\pi(ka)^2/4 \\ &H_1(kr) \simeq iN_1(kr) \simeq i2/(\pi kr) \\ &i2a_1(ka)H_1(kr)\cos\psi \simeq -ika^2\cos\psi/r \end{aligned} \tag{79}$$

in full agreement with (72).

For TE polarization, i.e., having the $\boldsymbol{H}$ field polarized along the z-coordinate, we follow a similar procedure. We use the second consistent Maxwell system (32). The analog of (75) is given by

$$\begin{aligned}
\boldsymbol{H}_i(\boldsymbol{r}) &= \hat{\boldsymbol{z}} h_0 \Sigma_{m=-\infty}^{m=\infty} i_m J_m(kr) e^{im\psi} \\
\boldsymbol{E}_i(\boldsymbol{r}) &= -e_0 \Sigma_{m=-\infty}^{m=\infty} i^m e^{im\psi} (\hat{\boldsymbol{r}} m/kr + i\hat{\boldsymbol{\psi}} \partial_{kr}) J_m(kr) \\
\boldsymbol{H}_s(\boldsymbol{r}) &= \hat{\boldsymbol{z}} h_0 \Sigma_{m=-\infty}^{m=\infty} i^m b_m H_m(kr) e^{im\psi} \\
\boldsymbol{E}_s(\boldsymbol{r}) &= -e_0 \Sigma_{m=-\infty}^{m=\infty} i^m b_m e^{im\psi} (\hat{\boldsymbol{r}} m/kr + i\hat{\boldsymbol{\psi}} \partial_{kr}) H_m(kr)
\end{aligned} \tag{80}$$

with the same boundary-conditions (66). The analog of (77) is now

$$b_m = -b_{-m} = -\partial_{ka} J_m(ka) / \partial_{ka} H_m(ka) \tag{81}$$

The details of the expansion of the leading terms of (80), (81), are somewhat more complicated and are omitted.

9. LOW-FREQUENCY FIELDS AND RADIATION CONDITIONS

Mathematically, outgoing scattered waves are defined in terms of radiation conditions. Sommerfeld [8] (see also [7] for citation of original reference and an extensive discussion) made the important observation that at large distances the scattered wave must be a bounded outgoing wave, in order to describe how energy supplied by the incident wave is carried away from the boundary by the scattered wave. Accordingly, two-, or three-dimensional scattered scalar waves, e.g., pressure in acoustics (39), have to satisfy the radiation condition

$$\lim_{r\to\infty} r^{(D-1)/2} (\partial_r p_s - ikp_s) = 0, \quad p_s = p_s(\boldsymbol{r}) \tag{82}$$

for dimensionality $D = 2$, $D = 3$, respectively. A weaker integral condition has been given by Wilcox [14]. In a sense, it presents the radiation condition as a statement of an averaging process. For further comments and references to original papers see the comprehensive article by Hayashy [15]. The electromagnetic counterpart of (82), often referred to as the Silver-Müller radiation condition, is even simpler in form [13–16]: Exploiting the fact that electromagnetic waves are transverse, and noting that at large distances the direction of propagation coincides with the radius-vector, $\hat{\boldsymbol{k}} = \hat{\boldsymbol{r}}$, the radiation condition becomes one of the following

$$\lim_{r\to\infty} r^{(D-1)/2} (\boldsymbol{H}_s - \hat{\boldsymbol{r}} \times \boldsymbol{E}_s / Z) = 0, \quad \lim_{r\to\infty} r^{(D-1)/2} (\boldsymbol{E}_s + Z\hat{\boldsymbol{r}} \times \boldsymbol{H}_s) = 0 \tag{83}$$

and the appropriate Wilcox integral radiation conditions [14] follow. By substitution from (1), conditions (77) can be recast in the form

$$\begin{aligned} \lim_{r\to\infty} r^{(D-1)/2}(\hat{\boldsymbol{r}} \times \partial_{\boldsymbol{r}} \times \tilde{\boldsymbol{I}} + ik) \cdot \boldsymbol{H}_s &= 0, \\ \lim_{r\to\infty} r^{(D-1)/2}(\hat{\boldsymbol{r}} \times \partial_{\boldsymbol{r}} \times \tilde{\boldsymbol{I}} + ik) \cdot \boldsymbol{E}_s &= 0 \end{aligned} \tag{84}$$

The radiation condition is satisfied by the exact solutions (75) and (80) of the Helmholtz equation, and will be satisfied in the distant field by the surface-integrals representations given above. On he other hand, it must be kept in mind that the low-frequency partial-field series apply to the near field, and it should not come as a surprise that some forms fail to satisfy the radiation conditions. An example is the logarithmic monopole solution (67), (68) and (78), where the logarithm function diverges at infinity.

10. CONCLUDING REMARKS

Low-frequency theory is important for various branches of wave physics. Accordingly [1] treats k in the Helmholtz equation as a perturbation parameter. Other researchers, notably [2–6] extended the validity of the series on the assumption that fields can be expanded in *variable-coefficient series* in terms of the inverse wavelength. The wavenumber, k, was treated as a variable. However, this assumption is inconsistent with the frequency ω, hence also k, being constants characterizing the harmonic time variation and the associated Helmholtz wave equations.

The present study does not invoke variable k and variable-coefficient series properties. In fact, it has been argued and verified by a simple example of scattering by a perfectly-conducting circular cylinder that the old theory does not lead to correct results.

Instead, our study is based on the concept of consistent systems, which are equivalent to the basic physical models in electromagnetics, acoustics, and elasticity. The present consistent systems contain Helmholtz equations for which series manipulation and re-labeling of indices are shown to hold. In expressions relating more than one field, invoking the idea of "equating powers of ik" implies variable ik, therefore variable-coefficient series, as done previously [2–6]. The present arguments rely on Sommerfeld-type plane-wave integrals, which allow for a consistent derivation of the pertinent series. In general the contour of integration is complex, involving inhomogeneous plane-waves possessing complex propagation vectors $\boldsymbol{k}$ but real values of $k^2 = \boldsymbol{k} \cdot \boldsymbol{k}$, a situation that can be achieved even in non-absorbing media, such as free-space (vacuum), when the real and imaginary components are perpendicular, i.e., $\boldsymbol{k}_R \cdot \boldsymbol{k}_I = 0$.

Appropriate surface-integral representations are provided for computing the far field, although it is known that this method is ill-behaved, the results being very sensitive to small changes on the surface involved.

Further work is planned in comparing the low-frequency results with exact solutions, e.g., the Mie solution (e.g., see [7]) for scattering by a sphere. This will involve more complicated special functions associated with vector spherical-waves.

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