TEACHING SPECIAL-RELATIVITY: KINEMATICAL DERIVATION OF THE LORENTZ TRANSFORMATION

Dan Censor
Department of Electrical and Computer Engineering,
Ben-Gurion University of the Negev, Beer-Sheva, Israel

Martin McCall
Department of Physics, Imperial College London, United Kingdom SW7 2AZ

Abstract—Special Relativity is traditionally based on the two postulates introduced in Einstein's 1905 paper. This often proves to be pedagogically problematic, especially in relation to the Lorentz Transformation of time. We derive here the low-velocity approximation of the Lorentz Transformation via simple kinematical ideas based on the propagation of tagged light pulses. This approach provides continuity with what students are familiar from elementary mechanics. The similarity and dissimilarity of the Lorentz and Galilean Transformations are discussed. Finally, the exact Lorentz Transformation and the prevalent axiomatic approach are discussed.

1. Introduction and Statement of the Problem

The prevalent textbook approach (e.g., see [1]) towards teaching the Lorentz Transformation (LT) of Einstein's Special Relativity (SR) theory follows the methodology of his celebrated 1905 paper [2]. Einstein introduced SR *via* the two Postulates of Relativity stating that for all inertial observers: (i) the laws of physics (Einstein [2] specifically focused on Electromagnetism) take the same form; and (ii) the speed of light c is invariant. This axiomatic approach, sometimes with a few variations (see for example [3, 4]), is universally employed in teaching SR.

As well as its aesthetic appeal, the axiomatic approach has the advantage that it quickly confronts students with ideas such as time dilation and length contraction. However, to enable students to assimilate ideas incrementally, an alternative approach built around Newtonian kinematics may be beneficial.

Actually, Einstein's postulates emerged only after his predecessors had grappled with relativistic ideas *via* kinematical arguments that nowadays seem to us as somewhat naïve [5]. Those arguments, notably Poincaré's paper [6], were based on exchange of light signals. Arguments based on light propagation must a priori assume the kinematics of light propagation in free space (vacuum). Below we consider relatively moving observers, but assume the light waves to move in one frame (the S frame described below) only. Consequently Postulate (ii) does not feature at this stage.

Our goal is to explain in simple terms the elements involved in the LT without immediately invoking the invariance of the speed of light. Such a program can take various forms, e.g., see [7]. Like the forerunners of SR, we start with the low velocity approximation whereby the well-known relativistic factor $\gamma = (1 - v^2/c^2)^{-1/2}$ is approximated by $\gamma = 1$. Crucially, we do not assume the Galilean approximation $t = t'$ but rather *derive* the correct low-velocity time transformation $t' = t - vx/c^2$.

Once students have assimilated the low-velocity transformations of space and time, these can be refined to their relativistically exact form by accounting for the symmetry of inertial systems and Postulate (ii) above. While the space transformation is straightforward, the time transformation is counter-intuitive and requires detailed expounding.

As with the early historical discussions [5], the arguments presented will be purely kinematical. But in contradistinction, our discussion is based on measurements of space and time made within a single inertial reference system S, referred to as the Lab System. We avoid velocity addition forms like $c \pm v$, obviously contradicting Postulate (ii), often appearing in early discussions [5].

The use of light signals facilitates the synchronization of clocks in arbitrary inertial systems. Accordingly, in each such system a latticework of rods is posited for establishing distances and locations. By emitting a pulse from a Master Clock (MC) placed, say, at the origin of the Lab system, all clocks in that system can be synchronized. This standard construction (e.g., see [1]) is discussed at the beginning of Einstein's paper [2]. The process of clock synchronization within a single inertial system is performed *without* the assumption that the speed of light c is the same for all inertial observers in relative motion, i.e., Postulate (ii) is not required at this stage.

Length separations and time durations measured in a system S', moving with constant velocity relative to S will be deduced through their relationship to space-time coordinates in S.

2. Lorentz and Galilei Transformations

Consider frame S' moving with velocity v with respect to S along their co-aligned x-axes. When $v << c$, we assume $\gamma = \left(1 - v^2 / c^2\right)^{-1/2} \approx 1$. In retrospect, already knowing Einstein's SR, we note that the LT takes the form (cf. [5])

$$x' = x - vt \tag{1}$$
$$t' = t - vx / c^2 \tag{2}$$
$$y' = y,\ z' = z \tag{3}$$

where c is the vacuum speed of light observed in S. Since we will obtain (1) and (2) without the second postulate we make no assumption about the speed of light in S'.

The Galilei transformation is obtained from (1)-(3) by taking in (2) the limiting case $c \to \infty$, leading to $t' = t$, i.e., becoming a statement that time is identical for all observers in relatively moving inertial systems. The Galilean approximation $t = t'$ is often ascribed to low velocities and/or small values of x. Mathematically this means that in (2) the condition $vx / c^2 << t$ must be satisfied, implying that at some arbitrary x the condition is satisfied only later than some time value t. For small time values $t \to 0$ the condition only holds near the origin $x \to 0$. Obviously this is too restrictive if we are seeking a description in which the spatial and temporal separation of events is arbitrarily. For such a description the limit $c \to \infty$ must be taken to arrive at $t = t'$. However, infinite light speed, with its attendant connotation of instantaneous communication (or instantaneous transmission of information, akin to action at a distance) is inconsistent with experiment and with theory in the context of Maxwell's equations. In other words, the Galilean Transformation $t = t'$ and Maxwell's equations are incompatible in the sense that we cannot at the same time insist that $c \to \infty$ and discuss a theory that predicts light propagation at some finite speed*.

3. The Spatial Transformation

A kinematical explanation of (1), dubbed as the spatial transformation, is straightforward. Written in the form

$$x = x' + vt \tag{4}$$

(1) describes the path of motion (aka equation of motion) along the x-axis of a point A whose initial position at time $t = 0$ is $x = x'$. See Fig. 1. In this sense the parameter x' in S is a constant. Differentiating (4) yields

$$dx' = 0 = dx - vdt \tag{5}$$

Therefore, by definition, v is the velocity of the point A when observed from S.

For a special choice $x' = 0$ (4) becomes

$$x = vt \tag{6}$$

the path for a point moving at velocity v, that at $t = 0$ coincided with the origin $x = 0$, depicted by the solid line in Fig. 1. Later on this point will be identified with the location of the Slave Clock (SC), introduced below.

So far (4) and (6) merely describe the paths of points moving according to generic (1). The key to defining another system of reference S' is the fact that the distance between arbitrary points moving at velocity v remains a constant. Thus if an observer is attached to one of these points, all the other points will appear at rest relative to his position.

Incorporating (3), the above arguments can be extended to three-dimensional space. Thus instead of the single path (1) one can assume the three-dimensional counterpart

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t \tag{7}$$

Designating some arbitrary point to be the origin, and considering $\mathbf{r}'$ as an arbitrary location defines the frame of reference S'.

At this stage one cannot talk about (1) and (3), or (7), as a complete space-time transformation of coordinates, because we are not yet in possession of the associated temporal transformation (2). For that reason the question of simultaneity in its SR context is not yet applicable.

The analysis of the temporal transformation (2), is more complicated and needs a more detailed narrative.

4. The Temporal Transformation

---

* Of course we cannot rule out the possibility of infinite speeds and "instantaneous action at a distance" in general.

To establish the temporal transformation (2), assume a master clock (MC) located at the origin $x=0$, transmitting a discrete sequence of tagged electromagnetic pulses propagating at the velocity c in S. Thus each pulse actually consists of a spiked burst serving as marker and an associated signal occupying part of the dead time between pulses, coding the MC time at which the burst was emitted. Hence "the pulse emitted by the MC at $t=0$" is understood to mean a pulse associated with the coding tag $t'=0$.

The main idea here is that the SC situated at $x'=0$ is actuated by the tagged pulses received from the MC. The tag t' detected by the SC is then used to establish the 'official' time

$$t'' = t'|_{x'=0} \tag{8}$$

at the SC located at $x'=0$. This information is then used to synchronize the time to arbitrary locations in S', as explained below.

As depicted by the dashed lines in Fig. 1, The n-th pulse in the sequence is described by the world line

$$x = c(t - t'_n) \tag{9}$$

i.e., $t = t'_n$ at $x=0$. In general we indicate the tag time as t' and (9) is rewritten as

$$x = c(t - t') \tag{10}$$

Solving (6) and (10), yields the intersection of the lines (Fig. 1) at

$$t' = t(1 - v/c) = t''|_{x'=0} \tag{11}$$

where t' is the time tag detected by the SC and ascribed as the corresponding time t'' in S' for $x'=0$ (later generalized for arbitrary x' as explained subsequently). For arbitrary points x' the intersection of the lines (1) and (10) yields

$$t' = t(1 - v/c) - x'/c \tag{12}$$

showing that for paths like (1), having at $t=0$ an offset position $x = x'$, there is an additional delay of x'/c for the pulse tagged by t', namely the time needed for the pulse to cover the extra distance x'. But instead of putting slave clocks in various locations x', detecting different tags according to (12), only the SC at $x'=0$ is considered for defining the time t'' at arbitrary locations x'. Such a statement begs the question: "how is this synchronization performed?". Obviously we have to compensate for the extra time delay, i.e., knowing x', and t' at arbitrary points according to (12), the time t'' is assigned throughout S' by computing

$$t'' = t' + x'/c \tag{13}$$

Note that the for the synchronization, only S data is exploited, hence questions of the velocity of propagation in S', or Postulate (ii), are irrelevant.

Physically (11) is a manifestation of the Doppler Effect in its simplest form. It tells us that the motion of the SC relative to the pulses causes a delay in the reception time. At time t the SC already moved out a distance vt, therefore a pulse with an earlier tag emitted vt/c seconds earlier, is needed for the pulse to reach the SC at time t.

Pick a specific event occurring in S at space-time coordinates $\{x_e, t_e\}$ (for brevity the coordinates themselves are referred to as the 'event') such that

$$t_e = x_e / c \tag{14}$$

i.e., this event is chosen on the dashed line identified by the tag $t' = t'_0 = 0$ in Fig. 1, as given by (10) for $t' = 0$. Substituting (14) in (11) yields

$$t'_e = t_e - vx_e / c^2 = t''_e |_{x'=0} \tag{15}$$

Equation (15) relates the time t''_e at the location of the SC, which is also the time ascribed to all arbitrary points x' at rest with respect to $x' = 0$, i.e., all points defined as belonging to S', to the space-time coordinates of the event $\{x_e, t_e\}$. Consequently (15) provides the temporal transformation (2) for the present specific case.

Arbitrary events $\{x, t\}$ are located on different dashed lines in Fig. 1, satisfying (10) instead of (14). For the same x_e we now have $\bar{t}_e$, shifted according to

$$\bar{t}_e = t_e + \bar{t}' \tag{16}$$

i.e., it is located on the world line of the pulse tagged by

$$x_e = c(\bar{t}_e - \bar{t}') \tag{17}$$

where (17) should be compared to (10) and (14). The later pulse, with its delayed tag $\bar{t}'$ will also reach the SC at a later time, therefore in (15) the delay will be added to the two sides of the equation. Incorporating (16) we now have

$$t_e + \bar{t}' - vx_e / c^2 = \bar{t}_e - vx_e / c^2 = (t''_e + \bar{t}') |_{x'=0} \tag{18}$$

Defining

$$\bar{t}''_e = t''_e + \bar{t}' \tag{19}$$

We finally have

$$\bar{t}''_e |_{x'=0} = \bar{t}_e - vx_e / c^2 \tag{20}$$

once again recognized as (2), but now applying to arbitrary events $\{x, t\}$. The analog of (7) is the three-dimensional low velocity time transformation

$$t' = t - \mathbf{v} \cdot \mathbf{r} / c^2 \tag{21}$$

5. The Need for Symmetry and the Principle of Relativity

So far, our narrative has been based on the existence of a preferred Lab System S. This is *par excellence* a pre-relativistic notion. It served to establish (1)-(3) without Postulate (ii) and with a minimal appeal to Postulate (i), invoking the kinematics of light pulses. Once the non-Galilean time transformation (2) is established, introducing the rest of the SR fundamentals is straightforward. To order v/c inverting (1)-(3) yields

$$x = x' - v't' \tag{22}$$

$$t = t' - v'x' / c^2 \tag{23}$$

$$y = y',\ z = z' \tag{24}$$

showing that the privileged status of S used in the derivation of (1)-(3) was just temporary, since the transformation of spacetime coordinates from S' to S is the same as from S to S' with $v' = -v$ as required by the symmetry dictated by postulate (i).

The introduction of the γ-factor into the transformations now does require the second postulate and is easily assimilated by the discussion of light clocks in relative motions (see e.g. [1] p. 138). We then arrive at the usual Lorentz transformations

$$x' = \gamma(x - vt) \tag{25}$$

$$t' = \gamma(t - vx / c^2) \tag{26}$$

$$y' = y\ ,\ z' = z \tag{27}$$

$$\gamma = (1 - v^2 / c^2)^{-1/2} \tag{28}$$

We can then, as above, appeal to the symmetry between S and S' to establish the inverse LT, involving $v' = -v$ and the same γ factor containing $v'^2 = v^2$. The three-dimensional analog of (25)-(28) is recast similarly to (7) and (21)

$$\mathbf{r}' = \tilde{\mathbf{U}} \cdot (\mathbf{r} - \mathbf{v}t),\ t' = \gamma(t - \mathbf{v} \cdot \mathbf{r} / c^2),\ \tilde{\mathbf{U}} = \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}} \tag{29}$$

where $\tilde{\mathbf{U}}$ is a dyadic (matrix) multiplying the coordinates perpendicular to $\mathbf{v}$ by γ.

The complete LT leads to a discussion of concepts usually arising in this context, such as the light cone, sub-luminal and super-luminal velocities, length contraction and time dilation, which will not be revisited here.

In order to check consistency with Einstein's Postulate (ii), consider now the LT (25)-(27) in differential form

$$dx' = \gamma(dx - vdt) \tag{30}$$

$$dt' = \gamma(dt - vdx / c^2) \tag{31}$$

$$dy' = dy,\ dz' = dz \tag{32}$$

Define arbitrary sub-luminal speeds according to

$$u^2 = u_x^2 + u_y^2 + u_z^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 \tag{33}$$
$$u'^2 = u_x'^2 + u_y'^2 + u_z'^2 = (dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2 \tag{34}$$

Substitution from (30)-(32) yields

$$u'^2 = [(u_x - v)^2/(1 - vu_x/c^2)^2 + (u_y^2 + u_z^2)/\gamma^2]/(1 - vu_x/c^2)^2 \tag{35}$$

Upon assuming $u^2 = c^2$, i.e., that in S the speed of a point is c, or equivalently

$$u_y^2 + u_z^2 = c^2 - u_x^2 \tag{36}$$

we obtain from (34)

$$u'^2 = [(u_x - v)^2 + (c^2 - u_x^2)/\gamma^2]/(1 - vu_x/c^2)^2 = c^2 \tag{37}$$

So the complete LT (25)-(28) is compatible with Postulate (ii), namely if the speed is c in one inertial system, it is also c in another, showing that c is an invariant. Einstein [2] started with Postulate (ii) and derived the LT, which is of course aesthetically more elegant, but sometimes more difficult for students to assimilate on their first encounter with SR.

Finally, it is noted that if both velocity components perpendicular to v vanish, i.e.

$$u_y = 0, u_z = 0 \tag{38}$$

then (28)-(30), for low velocities, leads to

$$\begin{aligned} u' &= dx'/dt' = (dx - vdt)/(dt - vdx/c^2) \\ &= (u - v)/(1 - vu/c^2) \end{aligned} \tag{39}$$

and for $u = c$ we obtain $u' = c$. Therefore caution must be exercised when dealing with such a specialized case.

6. Simultaneity And Moving Observers—An Example

According to the GT, time is identical in all reference systems: I am riding my horse and watching the time on the town’s clock tower on the hill. Surely it is “logical” that the person sitting at the roadside will see the same time? We are, after all, watching the same clock. In hindsight, being already familiar with SR, we of course know the answer. *Watching the time* on the clock tower entails propagation of light waves, and unless we take into account the time retardation due to the finite speed of light propagation, we cannot be sure we are talking about the same time for all observers. The important distinction between the low velocity LT time transformation (2) and the Galilean $t' = t$ can be appreciated by considering the following problem taken from [1]

> Two individuals S' and S'' are walking towards each other along a road each at a speed of 3ms^{-1} relative to the road. They cross at a site occupied by a stationary third observer, S. All agree to set their time origin at the crossover point, i.e. $t = t' = t'' = 0$. Near a star that lies on the line of the road, four light years away, a space ship at rest in the frame of S at location x, launches a missile at $t = 0$ destined to destroy the Earth some time in the future. Calculate the time when the missile launch occurs in the frames S' and S'', stating carefully in each case whether it is earlier or later than in S. Ignore the effects of gravity and ignore the rotation of the Earth. Comment on which, if any, of the earthbound observers can actually discuss the Earth's fate when they meet.

Assume that S' (respectively S'') moves with velocity $v = +3\text{ms}^{-1}$ ($v = -3\text{ms}^{-1}$) relative to S. With $t = 0$, $x = 4\times365\times24\times60\times60\times3\times10^{8} = 3.8\times10^{16}$ m and $v = \pm3\text{ms}^{-1}$, we obtain From (2) $t' = -1.3$ s and $t'' = +1.3$ s. Thus in S' (respectively S'') the missile is launched about one second before (after) S' (S'') meets S. The result illustrates the relativity of simultaneity occurring between frames moving at non-relativistic speeds. In the frame associated with S', the missile is launched *before* the individual at the origin of S' meets his counterparts in S and S''. However, since it would take at least four years for the information that the missile has been launched to reach S', he cannot inform the other individuals about the fate of the earth when they meet.

7. Summary and Concluding Remarks

The teaching of special relativity poses special challenges. In many undergraduate physics courses, SR is taught very near the beginning (in one author's institution it is taught in the first semester). Whilst many students enjoy the provocative challenges that are immediately encountered with the traditional 'two postulates' approach (time dilation, length contraction, twins paradox etc.), others may benefit from a more seamless construction building on what they know from Newtonian mechanics. It is to these latter students that the approach developed in this paper is directed. A skeletal form of the First Postulate is used in assuming only that signals propagate according to simple kinematics, and that the time information carried by such signals can be freely exchanged between frames. The presentation is necessarily one-sided initially, giving the Master Clock in the Lab Frame preferred status. However, as we have shown, this asymmetry is easily removed once the transformations (1)-(3) have been obtained. The time transformation of (2) is the first departure from many student's intuition, and we have therefore presented an alternative narrative to arrive at this. Only once the low-velocity transformations are developed is the second postulate invoking the invariance of the speed of light introduced, and the full LT derived in the standard way. The symmetry between frames can then be used again to show that the full LTs are consistent with the first postulate.

8. References

1. M. W. McCall, *Classical Mechanics: A Modern Introduction* (John Wiley & Sons, Chichester, 2000).
2. A. Einstein, “Zur Elektrodynamik bewegter Körper,” Ann. Phys. (Lpz.), **17**, 891–921, (1905); English translation: “On the Electrodynamics of moving bodies,” in “The Principle of Relativity,” (Dover, 1952).
   Available on the net, e.g., at
   http://www.ee.bgu.ac.il/~censor/relativity-directory/einstein.pdf
3. D. Censor, “Electrodynamics, topsy-turvy special relativity, and generalized Minkowski constitutive relations for linear and nonlinear systems”, *PIER-Progress In Electromagnetics Research*, vol. 18, pp. 261-284, Elsevier, 1998. Available at http://www.ee.bgu.ac.il/~censor/topsy-paper.pdf.
4. D. Censor, "Relativistic electrodynamics: various postulate and ratiocination frameworks", *PIER—Progress In Electromagnetic Research*, Vol. 52, pp.301-320, 2005.
   Available at http://www.ee.bgu.ac.il/~censor/games-paper.pdf
5. M.N. Macrossan, “A Note on Relativity before Einstein” *The British Journal for the Philosophy of Science*, Vol. 37, pp. 232-234, 1986.
   Available at http://eprint.uq.edu.au/archive/00002307/01/larmor.pdf
6. H. Poincaré, 1900, "La theorie de Lorentz et la Principe de Reaction", *Archives Neerlandaies*, V, 253-78, 1900
7. H. A. Atwater, “Non Simultaneity in the Aberration of Starlight,” *American Journal of Physics*, vol. 42, 1022–1024, 1974.

Figure Captions:

Figure 1: Spacetime diagram illustrating the motion of the SC (6) and the tagged light pulses.

Figure 2: Spacetime diagram illustrating the derivation of the time transformation.

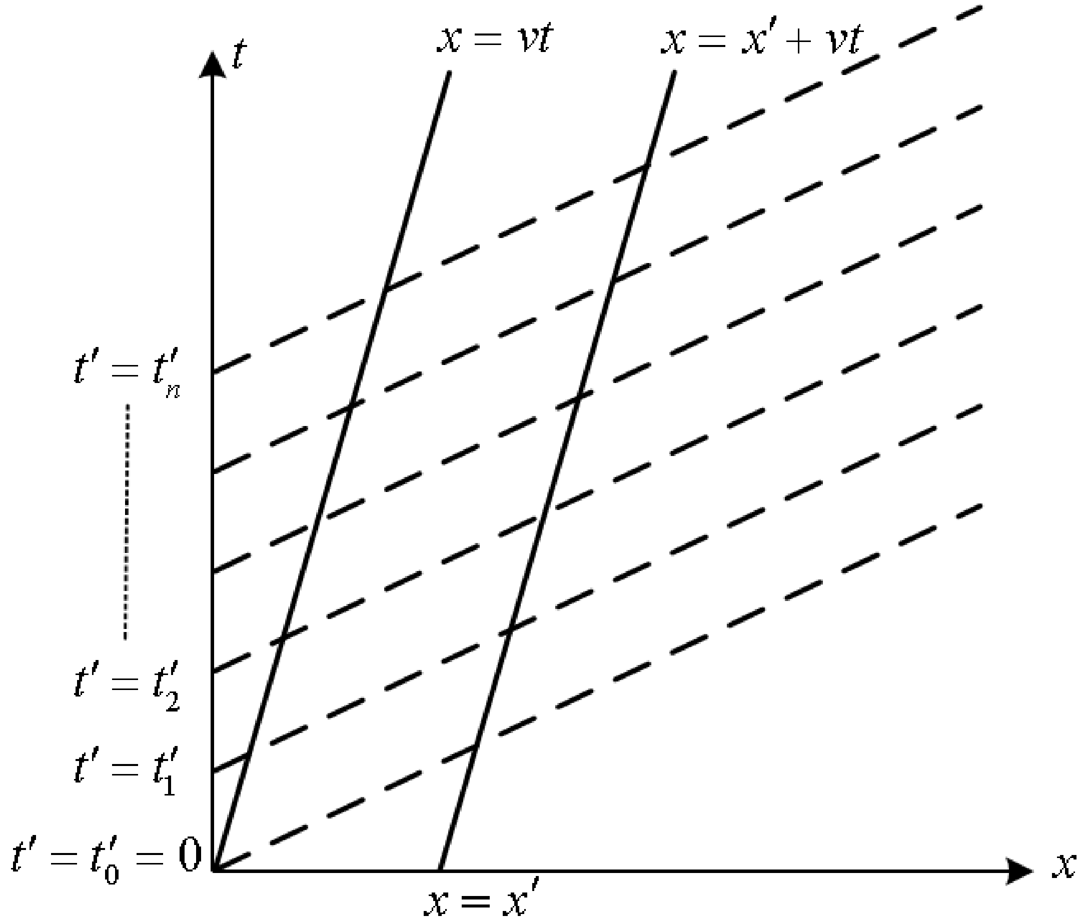

Figure 1

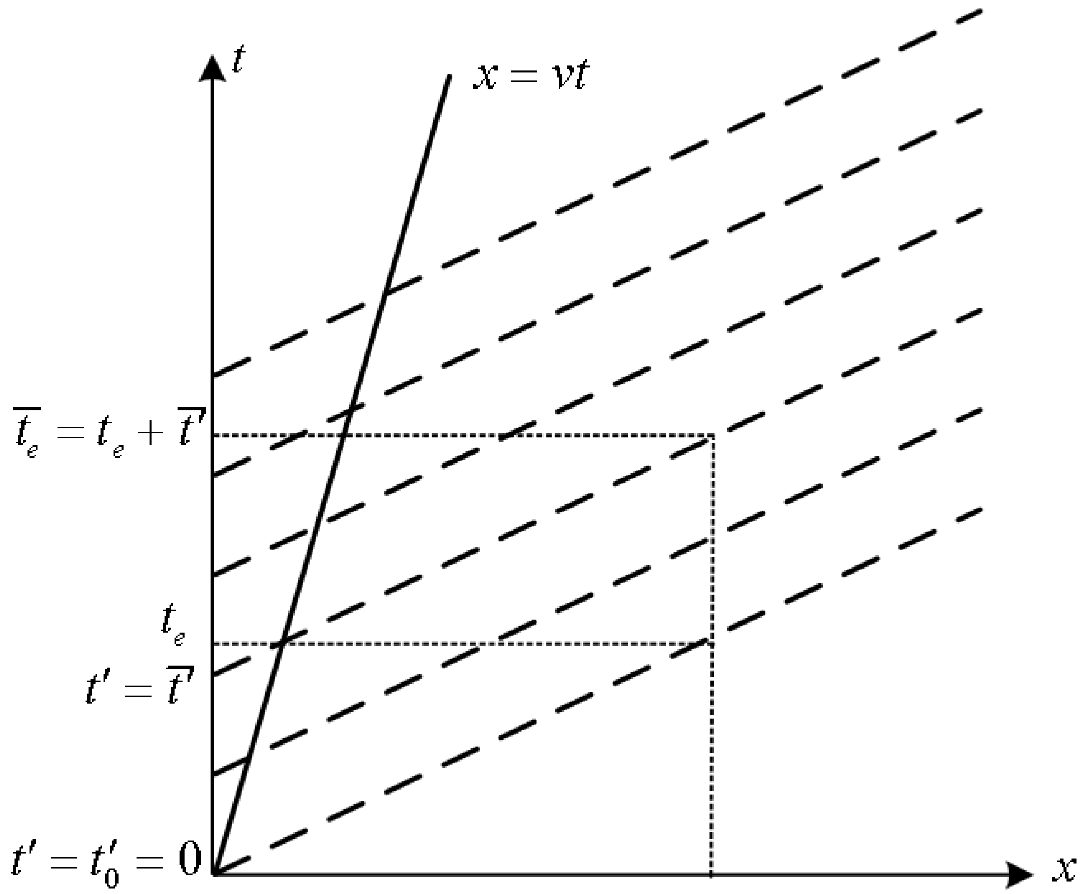

Figure 2