width is plotted as a function of deviation from the output frequency of the oscillator. The noise characteristics of a klystron (WECo type 459) which is now being used as a beat oscillator in a 6-GHz radio system are also shown in Fig. 1.

It appears that the use of avalanche diodes with high-Q temperature-stabilized cavities affords the opportunity to manufacture microwave signal sources with stability approaching that of quartz crystal oscillators, and with noise output lower than klystrons or varactor multiplier strings.

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First-Order Propagation in Moving Media

In a recent paper [1], vector relations were established, leading to the wave equation in moving media. Essentially, we derived Tai's wave equation [2] and three spatially independent vector solutions. Collier and Tai [3] and Du and Compton [4] discuss propagation in waveguides filled with axially moving media. Here we consider the first-order theory to get a relatively simple formalism, and discuss a few cases of interest.

Substituting Minkowski's constitutive relations [5] to the first order in v/c in Maxwell's equations yields, for harmonic time variation $e^{-i\omega t}$,

$$
\begin{gathered}
\nabla^* \times \boldsymbol{H} = -i\omega\epsilon \boldsymbol{E}, \\
\nabla^* \times \boldsymbol{E} = i\omega\mu \boldsymbol{H}, \\
\nabla^* = \nabla + i\omega\boldsymbol{\Lambda}, \quad \boldsymbol{\Lambda} = \left(\frac{1}{C^2} - \frac{1}{c^2}\right) v, \\
c = (\epsilon_0\mu_0)^{-1/2}, \quad C = (\epsilon\mu)^{-1/2}.
\end{gathered} \tag{1}
$$

In view of b) in Table I,

$$
\begin{gathered}
\nabla^* \cdot \boldsymbol{E} = 0, \\
\nabla^* \cdot \boldsymbol{H} = 0.
\end{gathered} \tag{2}
$$

Manipulating (1) and (2) and with c), Table I, we obtain

$$
\begin{gathered}
\nabla^{*2}\boldsymbol{\Psi} + k^2\boldsymbol{\Psi} = 0, \nabla^* \cdot \boldsymbol{\Psi} = 0, \\
\boldsymbol{\Psi} = E, H, \quad \nabla^{*2} = \nabla^* \cdot \nabla^*.
\end{gathered} \tag{3}
$$

Note that $\nabla^* \cdot \boldsymbol{\Psi} = 0$ generally implies $\nabla \cdot \boldsymbol{\Psi} \neq 0$. In the following, (1) and (2) will be considered as the physical model underlying our formalism.

In view of the analogy between ∇^* and ∇ exhibited in Table I and in (1)–(3), we follow the argument given by Stratton [6] (who cites original papers by Hansen) for the conventional wave equation. We seek three spatially perpendicular vector solutions of (3), generated by a scalar equation whose solution is presumably known. Define

$$
\boldsymbol{L}^* = \nabla^*\phi, \quad \nabla^* \times \boldsymbol{L}^* = 0, \tag{4}
$$

where ϕ is a scalar field. Inserting $\boldsymbol{L}^*$ into (3) and exploiting d), Table I, we get

$$
\nabla^{*2}\nabla^*\phi + k^2\nabla^*\phi = \nabla^*(\nabla^{*2}\phi + k^2\phi), \tag{5}
$$

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TABLE I

a) $\nabla^* \times \nabla^* \Phi = 0$,	Φ = arbitrary scalar.
b) $\nabla^* \cdot \nabla^* \times A = 0$,	A = arbitrary vector.
c) $\nabla^* \times \nabla^* \times A = \nabla^*\nabla^* \cdot A - \nabla^{*2}A$.	
d) $\nabla^{*2}\nabla^* = \nabla^*\nabla^{*2}$.	
e) $\nabla^* \times \nabla^{*2} = \nabla^{*2}\nabla^* \times$.	

See Nathan and Censor [1].

which vanishes subject to

$$
\nabla^{*2}\phi + k^2\phi = 0. \tag{6}
$$

Hence $\boldsymbol{L}^*$ is the analog of $\boldsymbol{L}$ [6]. Since $\nabla^* \cdot \boldsymbol{L} = \nabla^* \cdot \nabla^*\phi \neq 0$, but $\nabla^* \cdot \boldsymbol{\Psi} = 0$, $\boldsymbol{L}^*$ functions will not appear in our solutions and serve only to generate $\boldsymbol{M}^*$, $\boldsymbol{N}^*$, the analogs of $\boldsymbol{M}$, $\boldsymbol{N}$ [6]. Consider

$$
\begin{aligned}
\boldsymbol{M}^* &= \nabla^* \times \hat{a}\phi = \boldsymbol{L}^* \times \hat{a}, \\
\boldsymbol{N}^* &= \nabla^* \times \boldsymbol{M}^*/k, \\
\nabla^* \cdot \boldsymbol{M}^* &= 0, \quad \nabla^* \cdot \boldsymbol{N}^* = 0, \\
\boldsymbol{L}^* \cdot \boldsymbol{M}^* &= \boldsymbol{L}^* \cdot \boldsymbol{N}^* = \boldsymbol{M}^* \cdot \boldsymbol{N}^* = 0.
\end{aligned} \tag{7}
$$

Inserting $\boldsymbol{M}^*$ into (3) and incorporating e), Table I, yields

$$
\begin{aligned}
\nabla^{*2}(\nabla^* \times \hat{a}\phi) &+ k^2\nabla^* \times \hat{a}\phi \\
&= \nabla^* \times (\hat{a}\nabla^{*2}\phi + k^2\hat{a}\phi)
\end{aligned} \tag{8}
$$

which vanishes subject to (6). A similar argument applies to $\boldsymbol{N}^*$.

For spherical coordinates, in particular, we generate two solutions $\boldsymbol{M}^*$, $\boldsymbol{N}^*$ which are tangential over the entire surface of the sphere:

$$
\begin{aligned}
&\boldsymbol{M}^* = \nabla^* \times r\phi, \qquad \boldsymbol{N}^* = \nabla^* \times \boldsymbol{M}^*/k, \\
&\boldsymbol{M}^* \cdot \boldsymbol{N}^* = 0, \qquad \nabla^* \cdot \boldsymbol{M}^* = \nabla^* \cdot \boldsymbol{N}^* = 0.
\end{aligned} \tag{9}
$$

Substituting (9) into (3) and exploiting e), Table I, we obtain

$$
\begin{aligned}
\nabla^{*2}\nabla^* \times \boldsymbol{r}\phi &+ k^2\nabla^* \times \boldsymbol{r}\phi \\
&= \nabla^* \times (\nabla^{*2}\boldsymbol{r}\phi + k^2\boldsymbol{r}\phi) = 0.
\end{aligned} \tag{10}
$$

The expression in parentheses in (10) is equal to

$$
\boldsymbol{r}\nabla^{*2}\phi + k^2\boldsymbol{r}\phi + 2\phi i\omega\boldsymbol{\Lambda} = 0, \tag{11}
$$

which differs from (6), except for limiting cases where

$$
(1 - C^2/c^2) \ll \omega r. \tag{12}
$$

Thus far only constant velocities have been considered. If general-relativistic effects may be neglected, because the accelerations are low, we might accept (1) and (2) as our physical model, even for nonuniform velocities. The above formalism remains valid for v, satisfying

$$
\nabla^* \cdot v = 0, \quad \nabla^* \times v = 0, \tag{13}
$$

and securing the relations of Table I.

The scalar wave equation (6) is separable in Cartesian coordinates. Writing

$$
i\omega\boldsymbol{\Lambda} = i\lambda_x\hat{x} + i\lambda_y\hat{y} + i\lambda_z\hat{z}, \tag{14}
$$

and inserting into (3), we obtain

$$
\begin{aligned}
\phi = (A_1e^{i\alpha_x x} &+ A_2e^{i\beta_x x})(A_3e^{i\alpha_y y} + A_4e^{i\beta_y y}) \\
&\cdot (A_5e^{i\alpha_z z} + A_6e^{i\beta_z z}).
\end{aligned} \tag{15}
$$

Here $A_1 \cdots A_6$ are constants. Neglecting second-order terms in v/c, we get

$$
\begin{aligned}
\alpha_i &= -\lambda_i + a_i, \\
\beta_i &= -\lambda_i - a_i, \qquad i = x, y, z,
\end{aligned} \tag{16}
$$

where $-a_i^2$, $i = x$, y, z are the separation constants, such that $a_x^2 + a_y^2 + a_z^2 = k^2$. For a medium moving along a waveguide we take $\lambda_z = \lambda$. In the x, y directions the conventional solution is obtained; in the z direction we get

$$
c_1e^{i\gamma_1 z} + c_2e^{i\gamma_2 z}
$$

$$
\gamma_{\left\{ {1 \atop 2} \right\}} = -\lambda \pm \sqrt{k^2 - \left(\frac{\Pi n}{\xi}\right)^2 - \left(\frac{\Pi m}{\eta}\right)^2}; \tag{17}
$$

where (16) and (17) apply to $a_i \gg \lambda_i$; ξ, η are the dimensions of the waveguide in the x, y directions, respectively; and n, m specify the mode. The wavelength in (17) increases and decreases for waves propagating up and downstream, respectively. Since we have no first-order effect in the x, y directions

$$
\boldsymbol{E}_z = \hat{z}E_0 \sin\frac{\Pi n}{\xi} \times \sin\frac{\Pi m}{\eta} y e^{\gamma_{\left\{ {1 \atop 2} \right\}} z}, \tag{18}
$$

where E_0 is the amplitude of the field in question. In view of

$$
\nabla^* \times E_z\hat{z} = \nabla \times E_z\hat{z}, \tag{19}
$$

we obtain for the transverse fields H_x, H_y the same results as in the conventional case, with the velocity effects appearing in γ_1, γ_2.

Since (14) assumes arbitrary constant velocity, other cases suggest themselves. For example, consider a medium moving across the waveguide in the y direction; the walls where the medium is injected and withdrawn consists of a fine metallic mesh, such that the fine structure is small with respect to wavelength, and the conventional boundary conditions may be considered valid. In this case the dependence on y is of the form

$$
e^{-i\lambda_y} \sin\frac{\Pi m}{\eta} y \tag{20}
$$

for the TM case, and the propagation constant has a component in the y direction.

In cylindrical and spherical coordinates we consider nonuniform velocities, conforming with (13). A simple example is provided by an ideal incompressible fluid in rotational steady flow. For this case, Euler's equation of motion reduces to

$$
dv/v + dr/r = 0. \tag{21}
$$

The solution of (21) is $vr = \text{const}$; we therefore define

$$
i\omega\boldsymbol{\Lambda} = i\omega\left(\frac{1}{C^2} - \frac{1}{c^2}\right)\frac{V}{r}\hat{\theta} \equiv \frac{i\lambda}{r}\hat{\theta}, \tag{22}
$$

where (numerically) V is the velocity at unit radius. The solution of (6) involves

$$
2i\omega\boldsymbol{\Lambda} \cdot \nabla = \frac{2i\lambda}{r^2} \cdot \frac{\partial}{\partial\theta} \cdot \tag{23}
$$

The field must be single valued with respect to angle θ; hence we assume $e^{in\theta}$ dependence, getting

$$
\partial/\partial\theta = in. \tag{24}
$$

Separating off the z dependent part with γ^2 as the separation constant yields $e^{\pm i\gamma z}$, leaving for the r dependent part

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi_r\right) - \frac{n^2 + 2\lambda n}{r^2}\phi_r + (k^2 - \gamma^2)\phi_r = 0. \tag{25}$$

This is the conventional Bessel equation of order N:

$$N = \sqrt{n^2 + 2\lambda n}. \tag{26}$$

Consequently, the general solution for ϕ is

$$\phi = \sum_{n=-\infty}^{\infty} i^n J_N(\sqrt{k^2 - \gamma^2}\, r) e^{in\theta} e^{i\gamma z - i\omega t}. \tag{27}$$

For the $E_z\hat{z}$ along the guide (27) is the appropriate solution, taking $\phi = E_z/E_0$. Given $r = d$, the radius of the waveguide, the field vanishes on the walls of the waveguide when J_N vanishes. Denoting the zeros of J_N by $\rho_{n1} \cdots \rho_{nm}$, we have

$$\gamma = \sqrt{k^2 - \rho_{nm}^2/d^2}. \tag{28}$$

The associated transverse field, according to (1), is

$$\begin{aligned} H &= \frac{1}{i\omega\mu}\nabla^* \times E = \frac{1}{i\omega\mu}\nabla \times E_z\hat{z} \\ &+ \frac{\lambda}{\omega\mu r}\hat{r}E_z = \sum_{n=-\infty}^{\infty} E_0 i^n e^{in\theta} e^{i\gamma z - i\omega t} \\ &\cdot \left[\frac{n+\lambda}{\omega\mu r} J_N\left(\frac{\rho_{nm}}{d}r\right)\hat{r} \right. \\ &\left. + \frac{i\rho_{nm}}{\omega\mu d} J_N'\left(\frac{\rho_{nm}}{d}r\right)\hat{\theta}\right] \end{aligned} \tag{29}$$

where the prime denotes differentiation with respect to the argument.

Again we consider propagation in a rotating medium, defined in spherical coordinates by

$$i\omega\left(\frac{1}{C^2} - \frac{1}{c^2}\right)\frac{V}{r\sin\theta}\hat{\phi} \equiv i\lambda\hat{\phi}/r\sin\theta, \tag{30}$$

where (numerically) V is the velocity at a distance $r\sin\theta = 1$ from the polar axis, and $\hat{\phi}$ is the azimuthal direction. For a case where (12) is applicable we solve (6):

$$\left(\nabla^2 + \frac{2i\lambda}{r^2\sin\theta}\frac{\partial}{\partial\phi} + k^2\right)\Phi = 0, \tag{31}$$

and this equation is separable in spherical coordinates. Since the field must be single valued in terms of ϕ, we take the azimuthal function to be $e^{im\phi}$, i.e.,

$$\frac{\partial}{\partial\phi} = im, \tag{32}$$

where m is an integral number. This yields the conventional equation in θ, the polar angle, but with m^2 replaced by m_λ^2:

$$m_\lambda^2 = m^2 - 2\lambda m. \tag{33}$$

Therefore, the general solution for Φ is

$$\Phi = \sum_{m,n} i^n a_{mm} j_n(kr) P_n^{m_\lambda}(\cos\theta) e^{im\phi} e^{-i\omega t}, \tag{34}$$

where j_n is the spherical Bessel function of order n. The fact that in $P_n^{m_\lambda}$, m_λ is not an integer does not interfere with the periodicity in Π with respect to θ.

Neglecting terms of order λ^2, the vector solutions are given by (9):

$$\begin{aligned} M^* &= \nabla^* \times r\Phi = \left(\nabla + \frac{i\lambda\hat{\phi}}{r\sin\theta}\right) \times r\Phi = \left[\frac{\hat{\theta}}{\sin\theta}\left(\frac{\partial}{\partial\phi} + i\lambda\right) - \hat{\phi}\frac{\partial}{\partial\theta}\right]\Phi \\ &= \sum_{n,m} i^n a_{nm} j_n(kr) e^{im\phi - i\omega t}\left[\frac{\hat{\theta} i}{\sin\theta}(m+\lambda) - \hat{\phi}\frac{\partial}{\partial\theta}\right] P_n^{m_\lambda}(\cos\theta); \\ kN^* &= \nabla^* \times M^* = \left(\nabla + \frac{i\lambda}{r\sin\theta}\hat{\phi}\right) \times M^* \\ &= \sum_{n,m} i^n a_{nm} j_n(kr) e^{im\phi - i\omega t}\left[\frac{\hat{r}(m+\lambda)^2}{r\sin^2\theta} - \frac{\hat{r}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] P_n^{m_\lambda}(\cos\theta) \\ &+ \sum_{n,m} i^n a_{nm} e^{im\phi - i\omega t}\left[\frac{i(m+\lambda)\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\right)\right] r j_n(kr) P_n^{m_\lambda}(\cos\theta). \end{aligned} \tag{35}$$

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The Exponential Horn as a Uniform Waveguide with a Variable Index of Refraction

Abstract—**Plane-wave and transmission-line equations were applied to the interaction of a horn antenna with a stratified dissipative medium in its near field. The input impedances of an exponential horn terminated with a dissipative dielectric were calculated and measured at X-band frequencies.**

The calculation of the impedance of antennas covered by stratified dielectrics or plasmas by standard modal techniques [1]–[4] can get complicated and time consuming. It has been found that fairly good results can be obtained by a simple approximation method when the antenna is an exponential horn.

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It has been pointed out [5], [6] that, under some circumstances, a curved boundary and constant index of refraction can be very well approximated by a plane boundary and a variable index of refraction. This technique is applied here to the horn; i.e., the horn will be represented in terms of a uniform waveguide and an axially variable index of refraction. The system is shown in Fig. 1. Since in the horn the cutoff wavelength varies exponentially from the input waveguide to the mouth of the horn, the index of refraction in the z direction may be written as

$$n_z = [1 - (\lambda/2a)^2 e^{-2\alpha z}]^{1/2} \tag{1}$$

where λ is the free-space wavelength, a is the broad dimension of the waveguide, and α depends on the taper.

In the TE_{10} mode in a rectangular waveguide there is no variation with x, and the wave equation for the electric field component is

$$\frac{\partial^2 E_x}{dy^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0, \tag{2}$$

where

$$k^2 = \omega^2\mu_0\epsilon(z) = k_y^2 + k_z^2 = k_0^2(n_y^2 + n_z^2).$$

Separation of variables and application of the boundary conditions at $y = 0$ and $y = a$ (the walls of the guide are assumed to be perfectly conducting) yields, for the TE_{10} mode,

$$E_x = \psi(y)\Phi(z) \tag{3}$$

where

$$\psi(y) = A\sin k_y y = A\sin\frac{\pi y}{a},$$

and the equation for $\Phi(z)$ is

$$\frac{d^2\Phi}{dz^2} + \left[k_0^2 - \left(\frac{\pi}{a}\right)^2 e^{-2\alpha z}\right]\Phi = 0. \tag{4}$$

An analytic solution to (4) is

$$\Phi = A[H_p^{(2)}(x) + RH_p^{(1)}(x)] \tag{5}$$

where $H_p^{(1)}$, $H_p^{(2)}$ are the Hankel functions of the first and second kind,

$$x = \pm\frac{j\pi}{\alpha a}e^{-\alpha z} \quad \text{and} \quad p = \pm j\frac{k_0}{\alpha},$$

but since these functions of imaginary order are not tabulated, a direct numerical integration of (4) is the simplest way to obtain Φ. The use of Φ in (3), plus the relation for the magnetic field in the TE_{10} mode,

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z}, \tag{6}$$

allows the computation of the impedance a any position looking toward the load by