

DIFFERENTIAL-OPERATORS FOR CIRCULAR AND ELLIPTICAL WAVE-FUNCTIONS IN FREE-SPACE RELATIVISTIC SCATTERING

Dan Censor*, Iani Arnaoudov**, and George Venkov**

* Ben-Gurion University of the Negev
Department of Electrical and Computer Engineering
Beer Sheva, Israel 84105

** Department of Applied Mathematics and Informatics
Technical University of Sofia
1756 Sofia, Bulgaria

ABSTRACT—A novel method is presented for scattering by cylinders uniformly moving in free-space. The method is based on the application of differential-operators to the rest-state scattered waves. This method is applicable to any solution of the Helmholtz wave equation expressed in terms of a separable coordinate-system, in particular it is demonstrated here for circular and elliptical wave-functions.

The analysis applies to arbitrary cylindrical geometries and is based on plane-wave representations which facilitate simple and compact formulas.

- 1. Introduction**
 - 2. The Excitation Wave**
 - 3. Moving Cylinders, Circular Wave-Functions**
 - 4. Cylinders at Rest, Elliptical Wave-Functions**
 - 5. Moving Cylinders, Elliptical Wave-Functions**
 - 6. Concluding Remarks**
- References**

1. INTRODUCTION

Scattering of electromagnetic waves by circular and elliptical cylinders poses boundary-value problems involving Hankel, and Mathieu functions, respectively. For a good starting point for the theory of these special functions see Stratton [1]. For the functions of the elliptical cylinder we also draw on Stamnes [2], and Stamnes and Spjelkavik [3], who also cite [1].

In order to analyze scattering problems involving moving objects, Einstein's special-relativity theory [4, 5] must be incorporated. The mathematics associated with such problems, especially in free-space, has been reviewed recently [6], providing the necessary background and notation for the present investigation.

Scattering problems involving various geometries have been reviewed by Van Bladel, whose comprehensive book [7] provides a good starting point for this class of problems. See also Kong [8] for relevant subjects.

The method used here was dubbed as "frame-hopping" [7]: We start with the excitation wave in the "laboratory" inertial system, transform the wave into the "co-moving" reference-frame where the object is at rest. Here we have to solve the

scattering problem of an object at rest. The results are transformed back into the initial “laboratory” reference-frame, providing the velocity-dependent fields that are measured in this inertial system. Einstein [4] has already used this approach for analyzing the problem of scattering by a moving half-space, deriving the relativistic Doppler effect and aberration formula. However, his treatment is limited to free space. Special-relativity also accounts for material media, providing formulas for the Fresnel drag effect. Problems of this kind have been considered even before the advent of special-relativity, as mentioned by Pauli [5].

In order to introduce a new differential-operator representation for the scattered wave as observed in the reference-frame in which the objects are moving, and to provide some background, the problem of the moving cylinder is revisited [9, 10]. The idea is to go beyond the formal steps of applying the relativistic formulas to the fields in question and deriving the correct expressions: it is desired to be able to see how the velocity effects directly modify the rest-state results, and draw new conclusions. As in previous cases analyzed in terms of circular-cylindrical and spherical coordinates, also here the velocity-dependent mode-coupling is pointed out.

Basing the analysis on the behavior of plane-waves and plane-wave integrals facilitates a simple and compact presentation. Thus in subsequent sections \mathbf{E} -field polarization and the corresponding scattered waves are discussed almost exclusively. The associated \mathbf{H} -fields and similar problems involving \mathbf{H} -field polarization are left for the reader, and can easily be obtained from the Maxwell equations and the properties of the functions involved.

2. THE EXCITATION WAVE

The present section serves to introduce the basic ideas of relativistic electrodynamics, and the notation used throughout. The medium is free space, characterized by material parameters $\varepsilon = \varepsilon_0$, $\mu = \mu_0$. The excitation plane wave, possessing an electrical field \mathbf{E}_{ex} , and its associated magnetic field \mathbf{H}_{ex} , are defined in an inertial reference-frame Γ , in which the scattered wave is to be measured too. This frame is characterized by a coordinate-quadruplet x, y, z, t . The propagation vector \mathbf{k}_{ex} subtends an angle α measured from the x -coordinate towards the y -coordinate. Therefore

$$\begin{aligned} \mathbf{E}_{ex} &= \hat{\mathbf{z}} E_{ex} e^{i\theta_{ex}}, \quad \mathbf{H}_{ex} = \hat{\mathbf{k}}_{ex} \times \hat{\mathbf{z}} H_{ex} = \hat{\mathbf{k}}_{ex} \times \hat{\mathbf{z}} E_{ex} / \zeta \\ E_{ex} / H_{ex} &= (\mu / \varepsilon)^{1/2} = \zeta, \quad k_{ex} / \omega_{ex} = (\mu \varepsilon)^{1/2} = 1 / c \\ \theta_{ex} &= \mathbf{k}_{ex} \cdot \mathbf{r} - \omega_{ex} t = RC_{\varphi-\alpha} - \omega_{ex} t, \quad C_{\psi} = \cos \psi, \quad S_{\psi} = \sin \psi, \quad R = k_{ex} r \end{aligned} \quad (1)$$

where in (1) c denotes the light-speed in free-space, and throughout we compact the notation of trigonometric functions as shown.

Consider another inertial reference-frame Γ' , characterized by a coordinate-quadruplet x', y', z', t' , moving with constant velocity \mathbf{v} when observed from Γ . We start with a simpler example where $\mathbf{v} = v\hat{\mathbf{x}}$ and later generalize to arbitrary velocities. Einstein’s special-relativity theory [4, 5] prescribes the transformation of

(1) from Γ to Γ' . The mathematics associated with such problems has been reviewed recently [6]. The coordinates are related by the Lorentz transformation

$$\begin{aligned} \mathbf{r}' &= \tilde{\mathbf{U}} \cdot (\mathbf{r} - \mathbf{v}t), \quad t' = \gamma(t - \mathbf{v} \cdot \mathbf{r} / c^2) \\ \gamma &= (1 - \beta^2)^{-1/2}, \quad \beta = v/c, \quad v = |\mathbf{v}|, \\ \tilde{\mathbf{U}} &= \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}, \quad \hat{\mathbf{v}} = \mathbf{v}/v \end{aligned} \quad (2)$$

Assuming the phase-invariance principle, the relativistic Doppler-effect is derived

$$\mathbf{k}' = \tilde{\mathbf{U}} \cdot (\mathbf{k} - \mathbf{v}\omega/c^2), \quad \omega' = \gamma(\omega - \mathbf{v} \cdot \mathbf{k}) \quad (3)$$

The invariance of Maxwell's equations in inertial reference-frames prescribes the field-transformations

$$\begin{aligned} \mathbf{E}' &= \tilde{\mathbf{V}} \cdot (\mathbf{E} + \mu\mathbf{v} \times \mathbf{H}), \quad \mathbf{H}' = \tilde{\mathbf{V}} \cdot (\mathbf{H} - \varepsilon\mathbf{v} \times \mathbf{E}) \\ \tilde{\mathbf{V}} &= \gamma\tilde{\mathbf{I}} + (1 - \gamma)\hat{\mathbf{v}}\hat{\mathbf{v}} \end{aligned} \quad (4)$$

Accordingly, in Γ' we have

$$\begin{aligned} \mathbf{E}'_{ex} &= \hat{\mathbf{z}}E'_{ex}e^{i\theta'_{ex}}, \quad \mathbf{H}'_{ex} = \hat{\mathbf{k}}'_{ex} \times \hat{\mathbf{z}}H'_{ex} = \hat{\mathbf{k}}'_{ex} \times \hat{\mathbf{z}}E'_{ex} / \zeta \\ E'_{ex} / H'_{ex} &= (\mu / \varepsilon)^{1/2} = \zeta, \quad k'_{ex} / \omega'_{ex} = (\mu\varepsilon)^{1/2} = 1/c \\ \theta'_{ex} &= \mathbf{k}'_{ex} \cdot \mathbf{r}' - \omega'_{ex}t' = R'C_{\varphi'-\alpha'} - \omega'_{ex}t' \\ &= R'(C_{\varphi}C_{\alpha'} + S_{\varphi}S_{\alpha'}) - \omega'_{ex}t', \quad R' = k'_{ex}r' \end{aligned} \quad (5)$$

where the last expression (5) clearly demonstrates the symmetry between φ' , α' .

The primes in (5) and elsewhere, will be used throughout to avoid confusion, indicating that we are using Γ' native coordinates, irrespectively whether the actual measurements are performed in reference-frames Γ' or in Γ . It will be shown below that when the same Γ' native coordinates are retained for the scattered wave, we are dealing with less complex forms. The native coordinates pertaining to the "laboratory" reference-frame can always be substituted later.

To compute the spectral components in Γ' we used (3), which for the present case becomes

$$\mathbf{k}'_{ex} = \tilde{\mathbf{U}} \cdot (\mathbf{k}_{ex} - \hat{\mathbf{x}}\beta k_{ex}), \quad \omega'_{ex} = \gamma\omega_{ex}(1 - \beta\hat{\mathbf{x}} \cdot \hat{\mathbf{k}}_{ex}) = \gamma\omega_{ex}(1 - \beta C_{\alpha}) \quad (6)$$

In [6] it has been shown that for plane-waves in free-space (4) reduces to

$$\mathbf{E}' = \tilde{\mathbf{W}}_{\mathbf{k}} \cdot \mathbf{E}, \quad \mathbf{H}' = \tilde{\mathbf{W}}_{\mathbf{k}} \cdot \mathbf{H}, \quad \tilde{\mathbf{W}}_{\mathbf{k}} = \tilde{\mathbf{V}} \cdot (\tilde{\mathbf{I}} + \beta\hat{\mathbf{v}} \times \hat{\mathbf{k}} \times \tilde{\mathbf{I}}) \quad (7)$$

where in (7) it is understood that rightmost vectors act first, i.e., in the double-cross expression, $\hat{\mathbf{v}} \times$ acts last.

The fields for the present case are therefore

$$\begin{aligned}
\mathbf{E}'_{ex} &= \tilde{\mathbf{W}}_{\hat{\mathbf{k}}_{ex}} \cdot \mathbf{E}_{ex} = \gamma(\mathbf{E}_{ex} + \beta \hat{\mathbf{x}} \times \hat{\mathbf{k}}_{ex} \times \hat{\mathbf{z}} E_{ex} e^{i\theta_{ex}}) = \gamma(1 - \beta C_\alpha) \mathbf{E}_{ex} \\
&= \hat{\mathbf{z}} E'_{ex} e^{i\theta'_{ex}}, \quad E'_{ex} = \gamma(1 - \beta C_\alpha) E_{ex}, \quad \theta'_{ex} = \theta_{ex} \\
\mathbf{H}'_{ex} &= \tilde{\mathbf{W}}_{\hat{\mathbf{k}}_{ex}} \cdot \mathbf{H}_{ex} = \tilde{\mathbf{V}} \cdot (\mathbf{H}_{ex} + \beta \hat{\mathbf{y}} H_{ex} e^{i\theta_{ex}}) = \hat{\mathbf{H}}'_{ex} H'_{ex} e^{i\theta'_{ex}}
\end{aligned} \tag{8}$$

and thus the excitation in Γ' has been established.

3. MOVING CYLINDERS, CIRCULAR WAVE-FUNCTIONS

Scattering by moving circular cylinders has been discussed before [9, 10]. The scattered wave in Γ' is represented in the form

$$\begin{aligned}
\mathbf{E}'_{sc}(r', \varphi') &= \hat{\mathbf{z}} e'_{ex} \sum_m^{\pm\infty} i^m a_m(\alpha') H_m(R') e^{im\varphi'} \\
e'_{ex} &= E'_{ex} e^{-i\omega'_{ex} t'}, \quad H_m = H_m^{(1)}, \quad \sum_m^{\pm\infty} = \sum_{m=-\infty}^{m=\infty}
\end{aligned} \tag{9}$$

where in (9) $H_m = H_m^{(1)}$ denotes the Hankel function of the first kind, which in conjunction with the time factor in e'_{ex} secures outgoing waves; $a_m(\alpha')$ are the coefficients determined by solving the boundary-value problem, and their value depends on α' , the direction of propagation of the excitation wave.

Recasting H_m in terms of the relevant Sommerfeld integral, e.g., see [1], yields the plane-wave integral representation

$$\begin{aligned}
\mathbf{E}'_{sc}(r', \varphi') &= \hat{\mathbf{z}} e'_{ex} \frac{1}{\pi} \int_{\varphi' - (\pi/2) + i\infty}^{\varphi' + (\pi/2) - i\infty} e^{iR' C_{\varphi' - \tau'}} g(\tau') d\tau' \\
&\sim \hat{\mathbf{z}} e'_{ex} (2/i\pi R')^{1/2} e^{iR'} g(\varphi') \\
g(\varphi') &= \sum_m^{\pm\infty} a_m(\alpha') e^{im\varphi'} = \sum_m^{\pm\infty} \sum_\mu^{\pm\infty} a_{m\mu} e^{im\varphi'} e^{i\mu\alpha'}
\end{aligned} \tag{10}$$

where in (10) \sim denotes the far-field asymptotic approximation; $g(\varphi')$ is the scattering amplitude, and by expanding $a_m(\alpha')$ in a Fourier series, the symmetry between φ' , α' is demonstrated once more.

It is easy to verify that at the limits the integrand vanishes. Thus by formally substituting the upper limit the exponential in the integrand (10) we have [1]

$$\begin{aligned}
C_{-(\pi/2)+i\infty} &= C_{i\infty} C_{\pi/2} + S_{i\infty} S_{\pi/2} \\
C_{i\infty} &= (e^{ii\infty} + e^{-ii\infty})/2 \rightarrow \infty, \quad S_{i\infty} = -i(e^{ii\infty} - e^{-ii\infty})/2 \rightarrow i\infty \\
e^{ik'_{ex} r' i\infty} &\rightarrow 0
\end{aligned} \tag{11}$$

and clearly the exponential tends to zero. The same behavior occurs for the lower limit too.

The representation (10) is very convenient, facilitating a treatment of the appropriate relativistic transformations, according to (7), in terms of the plane-waves in (10). This in turn yields the scattered wave \mathbf{E}'_{sc} (note the absence of the

apostrophe, as opposed to \mathbf{E}'_{sc}), observed in Γ , but expressed in terms of the native coordinates r' , ϕ' of Γ'

$$\mathbf{E}_{sc}(r', \phi') = \hat{\mathbf{z}} \gamma e'_{ex} \frac{1}{\pi} \int_{\phi' - (\pi/2) + i\infty}^{\phi' + (\pi/2) - i\infty} (1 + \beta C_{\tau'}) e^{iR' C_{\phi' - \tau'}} g(\tau') d\tau' \quad (12)$$

Manipulation of (12) yields an expression of the form (9), but with different coefficients

$$\begin{aligned} \mathbf{E}_{sc}(r', \phi') &= \hat{\mathbf{z}} e'_{ex} \gamma \frac{1}{\pi} \int_{\phi' - (\pi/2) + i\infty}^{\phi' + (\pi/2) - i\infty} e^{iR' C_{\phi' - \tau'}} g'(\tau') d\tau', \quad H_m = H_m(R') \\ g'(\tau') &= (1 + \beta C_{\tau'}) g(\tau') = \sum_m^{\pm\infty} a_m [e^{im\tau'} + (e^{i(m+1)\tau'} + e^{i(m-1)\tau'}) \beta / 2] \\ \mathbf{E}_{sc}(r', \phi') &= \hat{\mathbf{z}} e'_{ex} \gamma \sum_m^{\pm\infty} i^m a_m e^{im\phi'} [H_m + i(H_{m+1} e^{i\phi'} - H_{m-1} e^{-i\phi'}) \beta / 2] \end{aligned} \quad (13)$$

By judiciously raising and lowering indices in (13) we achieve a more compact notation

$$\mathbf{E}_{sc} = \hat{\mathbf{z}} e'_{ex} \gamma \sum_m^{\pm\infty} i^m b_m H_m(R') e^{im\phi'}, \quad b_m = a_m + (a_{m-1} + a_{m+1}) \beta / 2 \quad (14)$$

The structure of (13), (14) clearly demonstrates the velocity-dependent mode-coupling between terms of various m .

For the purpose of subsequent applications, a different representation of (13), (14) is devised here. It is noted that in (10), (12)

$$\begin{aligned} R' C_{\phi' - \tau'} &= X' C_{\tau'} + Y' S_{\tau'}, \quad X' = k'_{ex} x', \quad Y' = k'_{ex} y' \\ C_{\tau'} e^{iR' C_{\phi' - \tau'}} &= -i \partial_{X'} e^{iR' C_{\phi' - \tau'}}, \quad S_{\tau'} e^{iR' C_{\phi' - \tau'}} = -i \partial_{Y'} e^{iR' C_{\phi' - \tau'}} \end{aligned} \quad (15)$$

where in (15) the factor $C_{\tau'}$, $S_{\tau'}$ are replaced by differential-operators $-i \partial_{X'}$, $-i \partial_{Y'}$, respectively, acting on the exponential $e^{iR' C_{\phi' - \tau'}}$.

Inasmuch as ϕ' appears in the integrand as well as in the limits, the order of integration and differentiation cannot be simply interchanged. However, the operator can indeed be taken outside the integral sign. To justify this step, take for example $-i \partial_{X'}$ outside the integral sign, and carefully apply the Leibnitz rule for differentiation of integrals. Because the integrand vanishes at the limits as manifested by (11), the operator acts on the exponential only. Another verification is provided by directly applying $-i \partial_{X'}$ to (9)

$$\begin{aligned} \mathbf{E}_{sc}(r', \phi') &= B \mathbf{E}'_{sc}(r', \phi') = \hat{\mathbf{z}} e'_{ex} B \sum_m^{\pm\infty} i^m a_m H_m(R') e^{im\phi'} \\ &= \hat{\mathbf{z}} e'_{ex} \gamma \sum_m^{\pm\infty} i^m a_m e^{im\phi'} [H_m + i(H_{m+1} e^{i\phi'} - H_{m-1} e^{-i\phi'}) \beta / 2] \\ B &= \gamma (1 - i \beta \partial_{X'}) = \gamma (1 + i \beta (S_{\phi'} \frac{1}{R'} \partial_{\phi'} - C_{\phi'} \partial_{R'})) \\ \frac{2m}{R'} H_m &= H_{m-1} + H_{m+1}, \quad 2 \partial_{R'} H_m = H_{m-1} - H_{m+1} \end{aligned} \quad (16)$$

where in (16) we transformed the operator to circular-polar coordinates and incorporated the indicated recurrence formulas, (e.g., see [1], p. 359 ff.), yielding (13) once again.

Inasmuch as in (16) the differential-operator B , involves $S_{\phi'}$, $C_{\phi'}$, affecting the angular behavior, the existence of mode-coupling is evident in this form too.

We now return to (1) in order to consider an arbitrary direction for the velocity, with the only restriction that it lies in the cross-sectional plane normal to the cylindrical axis $\hat{\mathbf{z}}$. Thus in Γ we have $\hat{\mathbf{v}} \cdot \hat{\mathbf{x}} = C_{\xi}$, i.e., the angle ξ is subtended by the two unit vectors $\hat{\mathbf{v}}$, $\hat{\mathbf{x}}$. This prescribes a few modifications to the above formulas. Equations (2), (3) are general formulas and remain the same, but the parameters appearing in (5) are modified. Accordingly we have to use the given direction $\hat{\mathbf{v}}$ in (6) in order to obtain the pertinent \mathbf{k}'_{ex} , ω'_{ex} . This yields

$$\mathbf{k}'_{ex} = \tilde{\mathbf{U}} \cdot (\mathbf{k}_{ex} - \hat{\mathbf{v}}\beta k_{ex}), \quad \omega'_{ex} = \gamma\omega_{ex}(1 - \beta\hat{\mathbf{v}} \cdot \hat{\mathbf{k}}_{ex}) = \gamma\omega_{ex}(1 - \beta C_{\alpha-\xi}) \quad (17)$$

Similarly (7) prescribes in (8)

$$\mathbf{E}'_{ex} = \tilde{\mathbf{W}}_{\hat{\mathbf{k}}_{ex}} \cdot \mathbf{E}_{ex} = \gamma(\mathbf{E}_{ex} + \beta\hat{\mathbf{v}} \times \hat{\mathbf{k}}_{ex} \times \hat{\mathbf{z}}E_{ex}e^{i\theta_{ex}}) = \gamma(1 - \beta C_{\alpha-\xi})\mathbf{E}_{ex} \quad (18)$$

Once (8) is established, there is no change in (9)-(11), however (12) must be modified. In (10) we have a superposition (integral) of plane waves. In order to establish the amplitude of each wave as measured in Γ , we need the inverse of (7), which is also the inverse of (18) in the form

$$\begin{aligned} \mathbf{E}_{sc}(r', \phi') &= \hat{\mathbf{z}}\gamma e'_{ex} \frac{1}{\pi} \int_{\phi'-(\pi/2)+i\infty}^{\phi'+(\pi/2)-i\infty} e^{iR'C_{\phi'-\tau}} g'(\tau) d\tau \\ g'(\tau) &= (1 + \beta C_{\tau'-\xi'}) g(\tau), \quad C_{\xi'} = \hat{\mathbf{v}} \cdot \hat{\mathbf{x}}' \end{aligned} \quad (19)$$

We now modify (13) according to (19)

$$\begin{aligned} \mathbf{E}_{sc} &= \hat{\mathbf{z}}e'_{ex} \gamma \frac{1}{\pi} \int_{\phi'-(\pi/2)+i\infty}^{\phi'+(\pi/2)-i\infty} e^{iR'C_{\phi'-\tau}} g'(\tau) d\tau \\ &= \hat{\mathbf{z}}e'_{ex} \gamma \sum_m^{\pm\infty} i^m a_m e^{im\phi'} [H_m + i(H_{m+1}e^{i(\phi'-\xi')} - H_{m-1}e^{-i(\phi'-\xi')})\beta/2] \\ g'(\tau) &= (1 + \beta C_{\tau'-\xi'}) g(\tau) \\ &= \sum_m^{\pm\infty} (a_m e^{im\tau'} + (a_m e^{i(m+1)\tau'-i\xi'} + a_m e^{i(m-1)\tau'+i\xi'})\beta/2) \end{aligned} \quad (20)$$

Consequently (14) becomes

$$\mathbf{E}_{sc} = \hat{\mathbf{z}}e'_{ex} \gamma \sum_m^{\pm\infty} i^m b_m H_m(R') e^{im\phi'}, \quad b_m = a_m + (e^{-i\xi'} a_{m-1} + e^{i\xi'} a_{m+1})\beta/2 \quad (21)$$

Of course one immediately wonders about the relation between the two angles ξ , ξ' . They are related through the Lorentz transformation (2), and the relation involves space and time coordinates. But this is immaterial at this point and comes up only when we seek to transform parameters and coordinates between Γ and Γ' .

In a consistent manner, we now exploit (15) to represent (20) in terms of the differential operators, thus obtaining

$$\begin{aligned}
 \mathbf{E}_{sc} &= \hat{\mathbf{z}} e'_{ex} \gamma \frac{1}{\pi} \int_{\phi'-(\pi/2)+i\infty}^{\phi'+(\pi/2)-i\infty} e^{iR'C_{\phi'-\tau'}} g'(\tau') d\tau' \\
 &= \hat{\mathbf{z}} e'_{ex} B_{\xi'} \frac{1}{\pi} \int_{\phi'-(\pi/2)+i\infty}^{\phi'+(\pi/2)-i\infty} e^{iR'C_{\phi'-\tau'}} g(\tau') d\tau' \\
 g'(\tau') &= (1 + \beta(C_{\xi'} C_{\tau'} + S_{\xi'} S_{\tau'})) g(\tau'), \quad B_{\xi'} = \gamma(1 - i\beta(C_{\xi'} \partial_{X'} + S_{\xi'} \partial_{Y'}))
 \end{aligned} \tag{22}$$

Like (16), we recast (22) in terms of a series of cylindrical waves

$$\begin{aligned}
 \mathbf{E}_{sc}(r', \phi') &= B_{\xi'} \mathbf{E}'_{sc} = \hat{\mathbf{z}} e'_{ex} B_{\xi'} \sum_m^{\pm\infty} i^m a_m H_m(R') e^{im\phi'} \\
 B_{\xi'} &= \gamma(1 - i\beta(C_{\xi'} \partial_{X'} + S_{\xi'} \partial_{Y'})) \\
 &= \gamma(1 - i\beta(C_{\phi'-\xi'} \partial_{R'} - S_{\phi'-\xi'} \frac{1}{R'} \partial_{\phi'})) \\
 \partial_{X'} &= C_{\phi'} \partial_{R'} - S_{\phi'} \frac{1}{R'} \partial_{\phi'}, \quad \partial_{Y'} = S_{\phi'} \partial_{R'} + \frac{1}{R'} C_{\phi'} \partial_{\phi'}
 \end{aligned} \tag{23}$$

Finally, if one wishes to express \mathbf{E}_{sc} explicitly in terms of the coordinates native to Γ then (2) can be substituted, however, this leads to very complicated forms that are not amenable to analytic work.

So far we have not addressed the geometry of the cylinder. This means that the cardinal results (14), (16), apply to arbitrary cylindrical cross-sections, provided the scattering amplitude $g(\phi')$ for the cylinder at rest in Γ' is known, i.e., $a_m(\alpha')$ or $a_{m\mu}$ are known, from measurement of the scattering amplitude in the far field, for example. The homogeneous circular cylinder in particular is obviously angularly-symmetric, i.e., the choice of the reference angle $\phi' = 0$ of the polar coordinate system is arbitrary, and the scattered field is symmetric with respect to the direction of incidence. $\alpha' = 0$. In other words, the scattering amplitude in (10) now becomes

$$\begin{aligned}
 g(\phi') &= \sum_m^{\pm\infty} a_m e^{im\phi'} = \sum_m^{\pm\infty} a_m C_{m\bar{\phi}'} = \sum_m^{0,\infty} \mathcal{E}_m a_m C_{m\bar{\phi}'} \\
 \mathcal{E}_m &= \begin{cases} 2, & m > 0 \\ 1, & m = 0 \end{cases}, \quad \sum_m^{0,\infty} = \sum_{m=0}^{m=\infty}, \quad \bar{\phi}' = \phi' - \alpha', \quad a_m = a_{-m}
 \end{aligned} \tag{24}$$

where in (24) the coefficients a_m are constants, independent of α' .

Our aim in the following sections will be to derive similar forms for \mathbf{E}_{sc} , for the moving cylinder, in terms of elliptical wave-functions.

4. CYLINDERS AT REST, ELLIPTICAL WAVE-FUNCTIONS

In the present section scattering by cylinders in terms of elliptical wave-functions in the co-moving reference-frame is recapitulated. Inasmuch as there are variations in notation amongst various authors, this summary also serves to define the special-functions used, the geometry, and the notation. See [1-3].

The cylindrical axis is taken in taken along the z -coordinate, as above. When actually solving a scattering problem the ellipse axes are usually chosen parallel to the x, y -coordinates. This is immaterial here.

The transformations relating Cartesian and elliptical coordinates are given by

$$\begin{aligned} x' &= C_u^h C_w^h d / 2 = r' C_{\varphi'}, y' = S_u^h S_w^h d / 2 = r' S_{\varphi'} \\ r' &= ((C_u^h)^2 - (S_w^h)^2) d / 2, \varphi' = \arctg(y'/x') \\ C_u^h &= \cosh u', S_u^h = \sinh u' \end{aligned} \quad (25)$$

where d is the inter-focal distance. The transformation of the Helmholtz wave equation leads to solutions in terms of Mathieu functions [1].

From the chain rule of calculus (25) yields the transformations from elliptical to Cartesian partial derivatives

$$\begin{aligned} \partial_{u'} &= (\partial_{x'} x') \partial_{x'} + (\partial_{y'} y') \partial_{y'}, \partial_{w'} = (\partial_{x'} x') \partial_{x'} + (\partial_{y'} y') \partial_{y'} \\ \partial_{x'} x' &= \partial_{y'} y' = S_u^h C_w^h d / 2, \partial_{w'} x' = -\partial_{u'} y' = -C_u^h S_w^h d / 2 \end{aligned} \quad (26)$$

The inverse transformations are given (e.g., see [2]) by

$$\begin{aligned} \partial_{x'} &= (\partial_{u'} u') \partial_{u'} + (\partial_{w'} w') \partial_{w'}, \partial_{y'} = (\partial_{u'} u') \partial_{u'} + (\partial_{w'} w') \partial_{w'} \\ \partial_{x'} u' &= \partial_{y'} w' = (2/d) S_u^h C_w^h / F, \partial_{y'} u' = -\partial_{x'} w' = (2/d) C_u^h S_w^h / F \\ F &= (C_u^h)^2 - (C_w^h)^2 = (S_u^h)^2 + (S_w^h)^2 \\ (S_w^h)^2 + (C_w^h)^2 &= 1, (C_u^h)^2 - (S_u^h)^2 = 1 \end{aligned} \quad (27)$$

Using the formulas at the bottom of (27) and substituting from (27) into (26) provides a check on the consistency of the two sets of transformations.

Following [1] (see equation (84), p.386) with slight changes in notation, the excitation wave (5) in terms of Mathieu functions is given by

$$\begin{aligned} \mathbf{E}'_{ex} &= \hat{\mathbf{z}} E'_{ex} e^{i\theta'_{ex}} = \hat{\mathbf{z}} e'_{ex} (8\pi)^{1/2} \sum_{n,p}^{0,\infty} (i^n / M_n^p) J_n^p S_{n,w}^p S_{n,\alpha'}^p \\ \theta'_{ex} &= R' C_{\varphi'-\alpha'} - \omega'_{ex} t' = s' (C_u^h C_w^h C_{\alpha'} + S_u^h S_w^h S_{\alpha'}) \\ J_n^p &= J_n^p(s', C_u^h), s' = k'_{ex} d / 2, S_{n,w}^p = S_{n,w}^p(s', C_w^h) \\ \sum_{n,p}^{0,\infty} &= \sum_{n=0}^{n=\infty} \sum_{p=e,o} \end{aligned} \quad (28)$$

In (28) (cf. [1], pp. 377-8) M_n^p are known normalization coefficients following from the orthogonality properties of $S_{n,w}^p$; $p = e, o$ denotes the even and odd parity terms, on which we sum, i.e., $\sum_{n,p}$ involves the sum of two terms p and all n as indicated; J_n^p denote the non-singular radial Mathieu functions, referred to as first kind by [1] and other literature sources, corresponding to the non-singular J_n Bessel functions for the zero-eccentricity circular-cylinder in the limiting case; $S_{n,w}^p$ denotes the angular Mathieu functions, involving the indicated

angle w' . Note in (28) the symmetry between w' , α' , corresponding to the symmetry between φ' , α' in (5), (10).

The general solution in terms of a series of elliptical wave-functions and arbitrary coefficients $L_n^p(\alpha')$ is given by

$$\begin{aligned} \mathbf{E}'_{sc}(u', w') &= \hat{\mathbf{z}}e'_{ex} (8\pi)^{1/2} \sum_{n,p}^{0,\infty} L_n^p(\alpha') (i^n / M_n^p) H_n^p S_{n,w'}^p \\ L_n^p(\alpha') &= \sum_{\bar{n}, \bar{p}}^{0,\infty} A_{n,\bar{n}}^{p,\bar{p}} S_{n,\alpha'}^{p,\bar{p}} = \sum_{\bar{n}}^{0,\infty} A_{n,\bar{n}}^{p,e} S_{n,\alpha'}^{p,e} + A_{n,\bar{n}}^{p,o} S_{n,\alpha'}^{p,o} \end{aligned} \quad (29)$$

where in (29) H_n^p denotes the radial Mathieu functions, referred to as third kind by [1] and other literature sources, corresponding to the singular H_n Hankel functions of the first kind for the zero-eccentricity circular-cylinder limiting case; $L_n^p(\alpha')$ are coefficients determined by the solution of the specific boundary-value problem, or measured in the far-field, for example. In any case, they are considered here as given. Obviously for a bounded cylindrical cross-section $L_n^e(\alpha')$, $L_n^o(\alpha')$ are periodic in α' , consequently each can be represented as a series of angular Mathieu functions of parity $\bar{p} = e, o$. The coefficients $A_{n,\bar{n}}^{p,\bar{p}}$ are constants.

Usually we find in the literature, e.g., [2, 3], solutions of scattering problems involving cylinders with the ellipse shaped cross-section axes aligned to the x, y -coordinates, as mentioned above. Then, corresponding to (28), the scattered wave is chosen with $p = \bar{p}$, i.e., $L_n^p(\alpha') = A_n^p S_{n,\alpha'}^p$ with presumably known constants coefficients A_n^p

$$\begin{aligned} \mathbf{E}'_{sc}(u', w') &= \hat{\mathbf{z}}e'_{ex} (8\pi)^{1/2} \sum_{n,p}^{0,\infty} A_n^p (i^n / M_n^p) H_n^p S_{n,w'}^p S_{n,\alpha'}^p \\ &\sim \hat{\mathbf{z}}e'_{ex} (2/i\pi s' C_{u'}^h)^{1/2} e^{is' C_{u'}^h} g(w') \sim \hat{\mathbf{z}}e'_{ex} (2/i\pi R')^{1/2} e^{iR'} g(\varphi') \\ g(w') &= \sum_{n,p}^{0,\infty} A_n^p (1/M_n^p) S_{n,w'}^p S_{n,\alpha'}^p \sim g(\varphi'), \quad s' C_{u'}^h \sim R' \end{aligned} \quad (30)$$

where $g(\varphi')$ was defined in (10). The specific choice of the angular Mathieu functions in (30), displaying symmetry between w' , α'

The above provides an adequate background for our subsequent discussion of velocity-dependent scattering in elliptical coordinates. More theoretical detail will be introduced as needed.

5. MOVING CYLINDERS, ELLIPTICAL WAVE-FUNCTIONS

The above theoretical background facilitates the representation of the scattered wave as measured in Γ , i.e., the wave scattered by the moving object in question, as measured in the "laboratory" reference-frame. The result is expressed in terms of elliptical coordinates and functions native to Γ' , as was done above for circular wave-functions.

To achieve this goal, all we have to do is to prove that the scattered waves as in (29), (30) can be expressed in terms of circular wave-functions as in (9) and *vice-*

versa. Then we can apply the technique displayed in (12)-(16) to the case of elliptical wave-functions.

Here we extensively used a report [11] which unfortunately cannot be found in an archival journal (there exists a shortened version which also incorporates another report [12], but does not include the theoretical review of [11]. It can be found under the same title [13]).

We can start by equating the even and odd parts of (28) to the even and odd parts of the of the circular-functions representation for the plane-wave (5). For this we need the relation between negative and positive order indices, in order to change the summation limits from 0 to ∞ ,

$$Z_{-\eta} = (-1)^\eta Z_\eta, \quad Z = J, N, H, H^{(2)} \quad (31)$$

Where in (31) η is a real integer and Z stands for the four solutions of the Bessel equation: the non-singular Bessel functions, the singular Neumann functions, and their combinations—first and second kind Hankel functions, respectively.

This leads to

$$\begin{aligned} \sum_m \varepsilon_m i^m J_m(R') \left\{ \begin{array}{l} C_{m\varphi} C_{m\alpha'} \\ S_{m\varphi} S_{m\alpha'} \end{array} \right\} &= (8\pi)^{1/2} \sum_n i^n \left\{ \begin{array}{l} J_n^e S_{n,w}^e S_{n,\alpha'}^e / M_n^e \\ J_n^o S_{n,w}^o S_{n,\alpha'}^o / M_n^o \end{array} \right\} \\ \varepsilon_m &= \begin{cases} 2, & m > 0 \\ 1, & m = 0 \end{cases}, \quad \Sigma_m = \Sigma_{m=0}^{m=\infty}, \quad \Sigma_n = \Sigma_{n=0}^{n=\infty} \end{aligned} \quad (32)$$

where in (32) upper and lower parts of the terms in braces correspond to even and odd functions.

The Fourier series expansions for $S_{n,\alpha'}^p$ are well known, e.g., see [1], p. 377

$$S_{n,\alpha'}^e = \bar{\Sigma}_m D_{m,e}^n C_{m\alpha'}, \quad S_{n,\alpha'}^o = \bar{\Sigma}_m D_{m,o}^n S_{m\alpha'} \quad (33)$$

where in (33) $D_{m,p}^n$ are known coefficients of the corresponding Fourier series, and $\bar{\Sigma}$ denotes that m, n have the same parity.

Multiplying the two sides of the upper, lower equation (31) by $C_{q\alpha'}, S_{q\alpha'}$, respectively, and integrating from 0 to 2π yields

$$\varepsilon_q i^q Z_q \left\{ \begin{array}{l} C_{q\varphi} \\ S_{q\varphi} \end{array} \right\} = (8\pi)^{1/2} \sum_n i^n \left\{ \begin{array}{l} D_{q,e}^n Z_n^e S_{n,w}^e / M_n^e \\ D_{q,o}^n Z_n^o S_{n,w}^o / M_n^o \end{array} \right\}, \quad Z_q = J_q, \quad Z_n^p = J_n^p \quad (34)$$

Alternatively, we can multiply the two sides of (32) by the angular Mathieu functions, namely $S_{q,\alpha'}^e, S_{q,\alpha'}^o$ for the upper, lower equations, respectively. Once again integrate from 0 to 2π , exploiting (33) and the orthogonality of the angular Mathieu functions [1]. This will lead to

$$(8\pi)^{1/2} i^q \left\{ \begin{array}{l} Z_q^e S_{q,w}^e \\ Z_q^o S_{q,w}^o \end{array} \right\} = \sum_m \varepsilon_m i^m Z_m(R') \left\{ \begin{array}{l} D_{m,e}^q C_{m\varphi} \\ D_{m,o}^q S_{m\varphi} \end{array} \right\}, \quad Z_q = J_q, \quad Z_n^p = J_n^p \quad (35)$$

and in a sense (34), (35) are inverses, describing how circular waves can be represented in terms of elliptical waves and *vice-versa*.

Consider the second radial solutions of the Mathieu equation, linearly independent of J_n^p , denoted by N_n^p . These functions are defined by using the same formula used to define the Neumann functions N_n in terms of J_n , e.g., see [1], p.357. Therefore this applies to their linear combinations as well, specifically to the Hankel functions $H_n^{(\pm)} = J_n \pm iN_n$. It follows that (34), (35) can be extended to the four kinds of functions

$$\begin{aligned} Z_\eta &= J_\eta, N_\eta, H_\eta = H_\eta^{(1)}, H_\eta^{(2)} \\ Z_\eta^p &= J_\eta^p, N_\eta^p, H_\eta^p = H_\eta^{(1)p}, H_\eta^{(2)p} \end{aligned} \quad (36)$$

where in (36) the terms in the first line correspond to those in the second line in an obvious manner.

Exploiting (35) we may now momentarily assume that the elliptical wave-functions $H_n^p S_{n,w}^p$ in (30) are expressed in terms of the circular wave-functions of $H_m(k'_{ex} r') \left\{ \begin{smallmatrix} C_{m\phi} \\ S_{m\phi} \end{smallmatrix} \right\}$ of (9), with the corresponding parity. This justifies the contour-integral representation (10), and in turn the use of the differential operators and the interchange of integration and differentiation, as discussed above.

It follows that the for the simpler case $\mathbf{v} = v\hat{\mathbf{x}}$, the operator $B = \gamma(1 - i\beta\partial_{x'})$ of (16) can be applied to (30) in order to derive $\mathbf{E}_{sc}(u', w')$ as measured in Γ , expressed in terms of Γ' native coordinates. Substituting from (27), elliptical coordinates and their associated partial derivatives can be employed

$$\begin{aligned} \mathbf{E}_{sc}(u', w') &= B\mathbf{E}'_{sc} = \hat{\mathbf{z}}e'_{ex}(8\pi)^{1/2} B\Sigma_{n,p}^{0,\infty} A_n^p(i^n / M_n^p) H_n^p S_{n,w}^p S_{n,\alpha}^p \\ B &= \gamma(1 - i\beta\partial_{x'}) = \gamma(1 - i\beta(u'_x \partial_{u'} + w'_x \partial_{w'})) \\ u'_{x'} &= \partial_{x'} u' = S_u^h C_{w'} / G, w'_{x'} = \partial_{x'} w' = -C_u^h S_{w'} / G, G = s'F \end{aligned} \quad (37)$$

where F, s' have been defined in (27), (28), respectively.

For arbitrary direction of motion normal to the cylindrical axis, we use $B_{\xi'}$, (23), yielding

$$\begin{aligned} \mathbf{E}_{sc}(u', w') &= B_{\xi'}\mathbf{E}'_{sc} = \hat{\mathbf{z}}e'_{ex}(8\pi)^{1/2} B_{\xi'}\Sigma_{n,p}^{0,\infty} A_n^p(i^n / M_n^p) H_n^p S_{n,w}^p S_{n,\alpha}^p \\ B_{\xi'} &= \gamma(1 - i\beta(C_{\xi'}\partial_{x'} + S_{\xi'}\partial_{y'})) \\ &= \gamma(1 - i\beta(C_{\xi'}u'_{x'}\partial_{u'} + C_{\xi'}w'_{x'}\partial_{w'} + S_{\xi'}u'_{y'}\partial_{u'} + S_{\xi'}w'_{y'}\partial_{w'})) \\ u'_{x'} &= \partial_{x'} u' = w'_{y'} = \partial_{y'} w' = S_u^h C_{w'} / G \\ u'_{y'} &= \partial_{y'} u' = -w'_{x'} = -\partial_{x'} w' = C_u^h S_{w'} / G \end{aligned} \quad (38)$$

where in (38) $C_{\xi'}, S_{\xi'}$ are considered as constants and there is no point in expressing them in terms of elliptical coordinates.

With (37), (38) at our disposal, we have achieved our goal of providing explicit formulas for the waves scattered by moving cylinders, expressed in terms of elliptical wave-functions. Obviously, the results apply also to the more general result (29), and therefore to cylinders with arbitrary, bounded, cross-section.

Once again, if one wishes to express \mathbf{E}_{sc} explicitly in terms of the coordinates native to Γ , (2) together with (17) can be substituted, which again leads to very complicated forms that are useful only for numerical simulations and limiting analytic cases.

6. CONCLUDING REMARKS

Velocity-dependent scattering problems involving moving objects are of interest mathematically and for engineering application. The problem of scattering by moving cylinders expressed in terms of circular-cylindrical wave-functions has been discussed before [9, 10], but was specific for this class of functions.

With these solutions as a starting-point, the present study suggests how to include other classes of wave-functions. The key element is to express first the wave-functions, i.e., radial Hankel times angular trigonometric functions, as a plane-wave superposition (integral). The relativistic transformations are then applied to the plane-waves in the integrand.

Presently this technique has been used to define differential-operators acting on the rest-state solution. All that needs to be done in order to apply the operator to other coordinate-systems wave-functions is to show that the latter wave-functions can be expressed in terms of the circular-cylindrical ones.

One can say in general that since the wave equation is linear, any solution can be expressed in terms of other solutions, e.g., plane-wave integrals should be the base for any representation of the wave equation. While this is true, the feasibility of adapting the new differential-operator representation should be demonstrated for each class of wave-functions.

This approach is verified above for wave-functions of the elliptical cylinder. Consequently it is expected that similar results, based on spherical wave-functions, will lead to an analogous differential operator for spheroidal wave-function—but this is yet to be shown.

REFERENCES

1. Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill, 1941.
2. Stamnes, J.J., “Exact two-dimensional scattering by perfectly reflecting elliptical cylinders, strips and slits”, *Pure and Applied Optics: Journal of the European Optical Society Part A*, 1995, vol. 4, no. 6, pp. 841-855.
3. Stamnes, J.J. and B. Spjelkavik, “New method for computing eigenfunctions (Mathieu functions) for scattering by elliptical cylinders”, *Pure and Applied Optics: Journal of the European Optical Society Part A*, vol. 4, no. 3, pp. 251-262, 1995.
4. Einstein, A., “Zur Elektrodynamik bewegter Körper”, *Ann. Phys. (Lpz.)*, Vol. 17, 891-921, 1905; English translation: “On the Electrodynamics of moving bodies”, *The Principle of Relativity*, Dover .
5. Pauli, W., *Theory of Relativity*, Pergamon, 1958, also Dover Publications.
6. Censor, D., "The mathematical elements of relativistic free-space scattering", *JEMWA—Journal of Electromagnetic Waves and Applications*, Vol. 19, pp. 907-923, 2005.
7. Van Bladel, J., *Relativity and Engineering*, Springer, 1984.

8. Kong, J.A., *Electromagnetic Wave Theory*, Wiley, 1986.
9. Censor, D., "Scattering in velocity dependent systems", *Radio Science*, Vol. 7, pp. 331 - 337, 1972.
10. Censor, D., "Application-oriented relativistic electrodynamics (2)", *PIER, Progress In Electromagnetics Research*, Vol. 29, 107–168, 2000.
11. Burke, J.E., and V. Twersky, "Scattering of waves by an elliptical cylinder and by a semi-elliptical protuberance on a ground plane" Report EDL-M-266, Sylvania Defense Laboratories, Mountain View, California, 1960.
12. Burke, J.E., E.J. Christensen, and S.B. Lyttle, "Numerical results for low frequency scattering by elliptic cylinders and by isolated semi-elliptic protuberances", Report EDL-M455, Sylvania Defense Laboratories, Mountain View, California, 1962.
13. Burke, J.E., and V. Twersky, "Scattering of waves by an elliptic cylinder and by a semielliptic protuberance on a ground plane", *Journal of the Optical Society of America*, vol. 54, pp. 732-744, 1964.