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Eikonal Equation for Moving Media and Its Relation to Dynamic Programming

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***Abstract*—The eikonal equation for moving media (first order in V/c) is derived for the electromagnetic case. Maxwell's equations with Minkowski's constitutive relations, and the dynamic programming approach, lead to essentially the same result.**

The acoustical eikonal equation for moving media is derived by means of the dynamic programming method.

A result obtained by Kritikos, believed to be wrong, is shown to be valid as an approximation.

INTRODUCTION

The derivation of the eikonal equation for anisotropic media via the method of dynamic programming is given by Brandstatter [1], [2]. This is an extension of an earlier work by Kalaba [3].

Presently we are interested in the eikonal equation for moving media. In the acoustical case, this equation has been known for quite a long time (see Heller [4] and Kornhauser [5]; see also Ugincius [6] for additional references). Subsequently, it is shown that the same eikonal equation can be obtained as a special case of the anisotropic medium treatment [2].

The electromagnetic case has been considered by Kritikos [7], who simply replaced the refractive index by the (first order in V/c) velocity-dependent analog for plane waves.

Unfortunately, Kritikos failed to state the frame of reference in which the properties of the medium, located at a point r, are time-invariant. This raised a comment by Unz [9]. However, subject to the present statement of the problem, it is shown that Kritikos' result constitutes a valid approximation to the eikonal equation in moving media. The eikonal equation for the electromagnetic case will be derived here: 1) directly from Maxwell's equations, with Minkowski's constitutive relations (to the first order in V/c) incorporated; and 2) by using the dynamic programming approach for anisotropic media.

ACOUSTICS

The eikonal equation for moving acoustical media is given by Heller [4] and by Kornhauser [5]. The same result is obtained here by using the dynamic programming method for anisotropic media [2]. This will provide a basis for comparison of the acoustical and the electromagnetic cases.

The properties of the medium (density, etc.) are given (Eulerian description) as a function of the location. Thus we define the speed of sound $C(\boldsymbol{r})$ as a property of the medium measured instantaneously in a comoving frame of reference. The velocity field is given by $\boldsymbol{V}(\boldsymbol{r})$. In the observer's (laboratory) frame of reference, $C(\boldsymbol{r})$, $\boldsymbol{V}(\boldsymbol{r})$ depend on $\boldsymbol{r}=(x_1, x_2, x_3)$ and are time-independent, which applies to a steady flow.

We consider the problem of determining a path of shortest time which connects the point $P(\boldsymbol{r})$ with a fixed point $P_0(\boldsymbol{r}_0)$. The motion of the medium introduces a preferred direction, hence the ray velocity is allowed to be a function of position and direction $\boldsymbol{u}$, $u_iu_i=1$, $i=1, 2, 3$ (summation convention understood throughout). We define $t(\boldsymbol{r})$ as the time taken by the ray to traverse an optimal path between P and P_0.

Using the principle of optimality, the initial decision is the choice of the initial direction $\boldsymbol{u}$. If the ray starts at $P(\boldsymbol{r})$ continues a short time in the direction $\boldsymbol{u}$, it will arrive at the point $Q(\boldsymbol{r}+v(\boldsymbol{r}, \boldsymbol{u})\Delta t\boldsymbol{u})$, where $v(\boldsymbol{r}, \boldsymbol{u})$ is the ray velocity. It must continue from Q to P_0 in such a way as to minimize the time for this part of the path. This yields the functional equation

$$t(\boldsymbol{r}) = \min_{\boldsymbol{u}} \left[\Delta t + t(\boldsymbol{r} + v\Delta t\boldsymbol{u})\right] + 0(\Delta t) \tag{1}$$

where the minimum is taken over all directions of $\boldsymbol{u}$, and $0(\Delta t)$ represents powers of Δt higher than one. Expanding (1) by Taylor's theorem, dividing by Δt, and letting $\Delta t\rightarrow 0$, we obtain

$$0 = \min_{\boldsymbol{u}} \left[1 + vu_it_i\right], \qquad t_i = \partial t/\partial x_i. \tag{2}$$

The direction $\boldsymbol{u}$ which minimized (2) is found in [2]

$$u_i = \pm\, (t_i - \beta_i)/\left[(t_j - \beta_j)^2\right]^{1/2}, \qquad \beta_i = v^{-2}\partial v/\partial u_i \tag{3}$$

where the sign is chosen such that (2) is minimized. From (3) it is clear that $\boldsymbol{u}$ does not coincide with $\boldsymbol{n}$, a unit vector normal to equal time surfaces, where

$$n_i = t_i/(t_jt_j)^{1/2}. \tag{4}$$

An exception is the case $\beta=0$, corresponding to the isotropic medium.

Combining (2), (3), and (4), we get

$$vu_in_i(t_jt_j)^{1/2} = 1. \tag{5}$$

Thus far the properties of the medium in question have not been considered, except for the general assumption that $v=v(\boldsymbol{r}, \boldsymbol{u})$ may depend on position and direction. For the nonrelativistic acoustical case, the physical assumption will be

$$vu_i = Cn_i + V_i. \tag{6}$$

That is, we assume that the ray propagates with the speed C in the direction $\boldsymbol{n}$, as seen by an observer following the medium, at the same time the ray is dragged with the moving medium at a velocity $\boldsymbol{V}$. Substitution of (6) into (5) yields

$$Cn_in_i(t_it_j)^{1/2} + V_it_i = 1 \tag{7}$$

which after some algebraic manipulation becomes

$$(t_jt_j)^{1/2} = C^{-1}(1 - V_it_i) \tag{8}$$

and this is immediately identified as the eikonal equation, for example, [4, eq. (16)], where Heller's eqs. (1) and (2) relate the notation to ours.

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ELECTRODYNAMICS

For the electromagnetic case we assume that special relativity is valid locally and instantaneously to the first order in V/c, where c is the speed of light in free space. This assumption is necessary since we assume a flow field $V(r)$, and, strictly speaking, special relativity applies to $V=\text{const}$. As in the acoustical case, we assume that $C=C(r)=[\mu(r)\epsilon(r)]^{-1/2}$ and that $C(r)$ and $V(r)$ are time-independent in the laboratory frame of reference.

Substitution of the Minkowski [9] constitutive relations (see also Sommerfeld [10]), to the first order in V/c, into Maxwell's equations for sourceless domains, with time variation $\exp(-i\omega t)$, yields

$$\begin{aligned}\nabla^* \times \boldsymbol{H} &= -i\omega\epsilon \boldsymbol{E}\\ \nabla^* \times \boldsymbol{E} &= i\omega\mu \boldsymbol{H}\\ \nabla^* &= \nabla + i\omega\boldsymbol{\Lambda}\\ \boldsymbol{\Lambda} &= \boldsymbol{V}(C^{-2} - c^{-2}).\end{aligned} \tag{9}$$

The operator ∇^* is a special case of the "extended ∇" discussed by Nathan and Censor [11]. It is easy to show that, provided $\nabla\times\boldsymbol{\Lambda}=0$, we can add to (9):

$$\nabla^*\cdot\boldsymbol{E} = \nabla^*\cdot\boldsymbol{H} = 0. \tag{10}$$

The transformation [12], [13]

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{E}_1 \exp(i\omega\Phi)\\ \boldsymbol{H} &= \boldsymbol{H}_1 \exp(i\omega\Phi)\\ \boldsymbol{\Lambda} &= -\nabla\Phi\end{aligned} \tag{11}$$

converts (9) and (10) into

$$\begin{aligned}\nabla \times \boldsymbol{H}_1 &= -i\omega\epsilon\boldsymbol{E}_1\\ \nabla \times \boldsymbol{E}_1 &= i\omega\mu\boldsymbol{H}_1\\ \nabla\cdot\boldsymbol{H}_1 &= \nabla\cdot\boldsymbol{E}_1 = 0.\end{aligned} \tag{12}$$

This is the usual form of Maxwell's equations in the medium at rest. Consequently, (12) yields the well-known (vector) Helmholtz wave equation. If a locally plane wave $\boldsymbol{E}_1=\boldsymbol{A}\exp[i\omega\psi_1(r)-i\omega t]$ is substituted, and with the usual provisions incorporated for media at rest, we obtain the eikonal equation

$$|\nabla\psi_1|^2 = C^{-2}. \tag{13}$$

Therefore, for $\boldsymbol{E}=\boldsymbol{E}_1$ ext $(i\omega\Phi)$, the eikonal equation is

$$|\nabla(\psi - \Phi)|^2 = C^{-2}, \qquad \psi = \psi_1 + \Phi \tag{14}$$

which to the first order in the velocity yields

$$|\nabla\psi| = C^{-1}[1 - \boldsymbol{V}\cdot\nabla\psi(1-\mu^{-2})], \qquad \mu = c/C. \tag{15}$$

For the limiting case $C \ll c$, this becomes the acoustical eikonal equation (8). The transition from (15) to Kritikos' equation from [7] is effected by comparing $|\nabla\psi|$ and $\boldsymbol{V}\cdot\nabla\psi(1-\mu^{-2})/C$. To the first order in V/C we may replace $\nabla\psi$ by $\boldsymbol{n}/C$.

The argument leading to (5) is valid here, too. However, instead of (6), we have to choose the first-order relativistic expression for the velocity as it appears to an observer in the laboratory. Thus taking into account the drag coefficient [14],

$$vu_i = Cn_i + V_i(1 - \mu^{-2}) \tag{16}$$

leading to

$$(t_jt_j)^{1/2} = C^{-1}[1 - V_it_i(1-\mu^{-2})] \tag{17}$$

which is similar to (8) and identical with (15).

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A Note on "Backscattering by Turbulent Irregularities: A New Analytical Description"

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Abstract—**An extremely useful and very complete analytical description of backscattering by turbulent irregularities has recently been given by Wheelon. Some confusing features in his treatment of the practical example of a rectangularly pulsed narrow-beam transmitted wave are being discussed.**

Recently, Wheelon [1] has shown with a general and very useful formalism how the traditional description of backscattering from turbulent irregularities may be extended to include inhomogeneous turbulence and arbitrary radar pulse shapes and antenna patterns. This letter points out a series of confusing features in what will probably be the most commonly used example in that paper [1], the narrow-beam rectangularly pulsed radar. Understanding and correction of these errors yields a natural interpretation of Wheelon's analysis.

The first problem is the use of a rectangular pulse in such a detailed analysis since the sharp corners of the rectangular pulse cause unphysical results. In Wheelon's equations [1, eq. (82) and (83)], essentially the Fourier transform of the pulse shape is calculated, and as is well known, the Fourier transform of the rectangular pulse is $\sin x/x$ which, when squared to calculate power in Wheelon's analysis, has large positive sidelobes. These frequency sidelobes represent the infinite bandwidth necessary to transmit a rectangular pulse; they appear automatically in [1, eq. (85)] and when the integral in that equation is correctly evaluated cause unphysical answers.

Second, there is an error in the derivation of eq. (84) from (83). Equation (84) should read

$$\Phi(K) = \frac{4}{R^2\beta^2}\left[\frac{\sin^2(2k+K)\Delta/2}{(2k+K)^2} + \frac{\sin^2(2k-K)\Delta/2}{(2k-K)^2}\right] \tag{84A}$$

plus terms proportional to

$$\frac{\sin[(2k+K)\Delta/2]\sin[(2k-K)\Delta/2]\cos[2K(R+\Delta)]}{(2k+K)(2k-K)}.$$

These last terms do oscillate to zero for large pulse length Δ but the term

$$\frac{\sin^2(2k+K)\Delta/2}{(2k+K)^2}$$

does not. The term in brackets in eq. (85) then reads

$$I = \frac{2}{\pi\Delta}\int_0^\infty dKS(K)\left[\frac{\sin^2(2k+K)\Delta/2}{(2k+K)^2} + \frac{\sin^2(2k-K)\Delta/2}{(2k-K)^2}\right]. \tag{85A}$$

A simple example which shows both $\sin^2$ terms are the same size is the Kolmogorov spectrum

$$\begin{aligned}S(K) &= \frac{15.5L_0^3}{(1+K^2L_0^2)^{11/6}}, &\quad K &< \kappa_{\max}\\ &= 0, & K &> \kappa_{\max}\end{aligned}$$

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