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Dispersion Equations in Moving Media

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***Abstract*—The derivation of dispersion equations for electromagnetic waves in moving media is considered. Absorption is included. It becomes clear that conduction current or polarization current models lead to identical dispersion equations in moving media. The invariance of dispersion equations is defined and discussed. The question of velocity induced wave modes in moving media is considered, and conditions for appearance of these new modes are stated.**

I. INTRODUCTION

Relativistic electrodynamics of moving media has been formulated by Minkowski, [1] see also Sommerfeld [2] for historical review and introduction. Subsequently other theories appeared, however, it is now widely accepted [3]–[5] that all relativistically correct theories can be traced back to Minkowski's [1] original formulation.

A well-known difficulty in establishing dispersion equations for moving media is the question whether conduction or polarization current models should be used (see Chawla and Unz [6] for discussion and further references). Essentially the difficulty stems from the fact that the two types of currents obey different relativistic transformation laws, and this becomes more complicated when we wish to include absorption in our model. Presently it is shown that one can work directly with the macroscopic parameters, without separating them into susceptance and conductance, and consequently all consistent approaches lead to identical dispersion equations.

II. RELATIVISTIC ELECTRODYNAMICS

Maxwell's equations (in MKS units) are given in an inertial system of reference (henceforth the "primed" system) by

$$\partial_{x'} \times H' = j' + \partial_{t'} D' \qquad \partial_{x'} \cdot B' = 0$$
$$\partial_{x'} \times E' = -\partial_{t'} B' \qquad \partial_{x'} \cdot D' = \rho' \tag{1}$$

where $\partial_{t'} = \partial/\partial t'$, and $\partial_{x'} \times, \partial_{x'} \cdot$ are the rot, div operators, respectively. The following transformation formulas are applied to the coordinates and fields in (1):

$$\begin{aligned} x' &= \tilde{U} \cdot (x - vt) & \partial_{x'} &= \tilde{U} \cdot (\partial_x + v\partial_t/c^2) \\ t' &= \gamma(t - x \cdot v/c^2) & \partial_{t'} &= \gamma(\partial_t + v \cdot \partial_x) \\ E' &= \tilde{V} \cdot (E + v \times B) & H' &= \tilde{V} \cdot (H - v \times D) \\ B' &= \tilde{V} \cdot (B - v \times E/c^2) & j' &= \tilde{U} \cdot (j - \rho v) \\ D' &= \tilde{V} \cdot (D + v \times H/c^2) & \rho' &= \gamma(\rho - j \cdot v/c^2) \end{aligned} \tag{2}$$

where v is the velocity of the primed system as seen from another (the "unprimed") system of reference, and

$$\begin{aligned} \gamma &= (1 - \beta^2)^{-1/2} & \tilde{U} &= \tilde{I} - (\gamma - 1)\,\hat{v}\hat{v} \\ \beta &= v/c & \hat{v} &= v/v \\ v &= (v \cdot v)^{1/2} & \tilde{V} &= \gamma\tilde{I} + (1 - \gamma)\,\hat{v}\hat{v} \\ c &= (\mu_0 \epsilon_0)^{-1/2} \end{aligned} \tag{3}$$

and $\tilde{I}$ is the idemfactor dyadic. This yields (1) without primes in the unprimed system.

In order to discuss dispersion equations in homogeneous media, we need the Fourier transform of the equations for E', B', D', H', j', and ρ' in (2). Consider for example the equation for $E'(X')$, $X' = (x', ict')$ is the space-time position four-vector. Similarly in the unprimed system we have $E(X)$, etc., where X', X are related by the Lorentz transformation, i.e., first two lines of (2). Thus we have

$$E'(X') = \tilde{V} \cdot (E(X) + v \times B(X)) \tag{4}$$

to which we wish to apply a four-fold Fourier transformation. On the left side of (4) we have

$$E'(X') = \int d^4K' E'(K')\, e^{iK' \cdot X'} \tag{5}$$

where $K' = (k', i\omega'/c)$. The right side of (4) is written as

$$\tilde{V} \cdot (E(X) + v \times B(X)) = \int d^4K \tilde{V} \cdot (E(K) + v \times B(K))\, e^{iK \cdot X}. \tag{6}$$

Now comes a crucial step in our argument. The relation between K and K' is yet undetermined, but in order that the integrals in (5), (6) be equal for all X, X', which are related by the Lorentz transformation, we must identify $K' \cdot X' = K \cdot X$. The latter implies that K, K' are four vectors related according to

$$k' = \tilde{U} \cdot (k - \omega v/c^2) \qquad \omega' = \gamma(\omega - v \cdot k) \tag{7}$$

and it follows that the volume element d^4K is invariant to rotation in Minkowski's four space, hence $d^4K = d^4K'$. Relations (7) are commonly referred to as the relativistic Doppler effect. Consequently we arrive at the transformation formula

$$E'(K') = \tilde{V} \cdot (E(K) + v \times B(K)) \tag{8}$$

where K, K' are related by (7). The same argument applies to the formulas for B', D', H', j', ρ', obtained by inspection of (2), where the transforms of the fields are now understood.

III. CONSTITUTIVE RELATIONS AND DISPERSION EQUATIONS

We identify now the primed system (1) as pertaining to the comoving system of reference in which the medium is at rest. The unprimed equations pertain to observations performed in the laboratory frame of reference, in which the medium is observed to move at a velocity v. The central problem here is the fact that constitutive relations are provided in the comoving system, while observations are performed in the laboratory (unprimed) system.

The constitutive relations are usually derived in the form

$$ik' \times H' = \tilde{\sigma}' \cdot E' \qquad k' \times E' = \omega' \tilde{\mu}' \cdot H' \tag{9}$$

given in K' transform space; the form (9) is sufficiently general for our

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subsequent argument. Note that $\tilde{\sigma}'$, $\tilde{\mu}'$ may be non-Hermitian (or complex, in the case of nongyrotropic media) thus taking into account absorption mechanisms.

To derive dispersion equations in the comoving system of reference we can eliminate in (9) E', or H' deriving the transformed wave equations

$$[k' \times \tilde{\mu}'^{-1} \cdot k' \times \tilde{I} + i\omega'\tilde{\sigma}'] \cdot E' = 0$$
$$[k' \times \tilde{\sigma}'^{-1} \cdot k' \times \tilde{I} + i\omega'\tilde{\mu}'] \cdot H' = 0 \qquad (10)$$

respectively. While the wave equations for E', H' are not identical, equations (10) lead to identical dispersion equations, as can be seen by first deriving from (10) the dispersion equations

$$\det\,[k' \times \tilde{\mu}'^{-1} \cdot k' \times \tilde{\sigma}'^{-1} + i\omega'\tilde{I}] = 0$$
$$\det\,[k' \times \tilde{\sigma}'^{-1} \cdot k' \times \tilde{\mu}'^{-1} + i\omega'\tilde{I}] = 0 \qquad (11)$$

respectively, and then using the rules for the determinant of a sum, and the determinant of a product of determinants. Consequently either one of the equations (11), which are identical, is generally represented by $F(K') = 0$.

Now, in moving media, it is *not* sufficient to apply (7) to (11) in order to find the dispersion equation in the laboratory system of reference. There is no a priori guarantee that this act, which can be written as $F'(K'[K] = 0$, actually yields the dispersion equation for an observer in the laboratory system, because he is only capable of measuring fields, and the dispersion equation must be derived from the relativistic transformations of (9). However, the transformation of E', H' (2), (8), introduces D, B, so that this immediately brings up the problem of relating D' to E' and B' to H'. To overcome this difficulty, we take the first equation of (10) and substitute (8), then eliminate B by using Maxwell's equation $k \times E = \omega B$ in the laboratory system, yielding

$$[k' \times \tilde{\mu}'^{-1} \cdot k' \times \tilde{I} + i\omega\tilde{\sigma}'] \cdot \tilde{V} \cdot [\tilde{I} + v \times k \times \tilde{I}/\omega] \cdot E = 0. \qquad (12)$$

Since in general $\det \tilde{V} \neq 0$, $\det\,[\tilde{I} + v \times k \times \tilde{I}/\omega] \neq 0$, the vanishing of the determinant of the first dyadic in brackets defines the dispersion equation. But this is exactly the result (10). Consequently, we have now proved that

$$0 = F'(K') = F'(K'[K]) \equiv F(K) \qquad (13)$$

i.e., the dispersion equation is invariant in the sense that the substitution of (7) into $F(K') = 0$ as represented in (13), yields the dispersion equation in the laboratory system of reference. We can define the new function $F(K)$, as in (13). It must be born in mind the F', F are different functions of their respective arguments K', K.

The fact that the dispersion equation of a moving medium, as observed from the laboratory frame of reference, can be determined without splitting of $\tilde{\sigma}'$ means that all consistent physical models, whether based on the concept of conductance or polarization current, or a combination thereof, should lead to identical dispersion equations. Tai [3] noted this for certain first order velocity effects. However, due to the invariance of the dispersion equation, as defined above, this is a relativistically exact conclusion without further qualifications.

The establishment of the invariance of the dispersion equation facilitates the general argument of velocity induced modes.

The number of possible wave modes in a given medium (in the comoving system of reference) is determined by the number of distinct roots of the dispersion equation, when the frequency is taken as a constant. Thus F' can be represented as a product, with each factor constituting the dispersion equation for one of the modes,

$$F'(K') = F_1'(K')\,F_2'(K') \cdots F_n'(K') = 0. \qquad (14)$$

If there are no multiple roots, then n is the number of modes. Some authors consider $n/2$ as the number of modes, not counting separately backward and forward going waves. Usually purely oscillatory modes, where $F_m'(\omega')$ involves ω' only, and evanescent modes are discarded, but must be retained for moving media consideration, because in the laboratory system such modes appear as propagating waves. Substituting (7) in (14), in order to derive dispersion equations for the laboratory observer, may be conducive to new velocity dependent modes. It is evident that if $F'(k', \omega')$ contains a term ω'^l, where l is larger than any power of k', then (7) will introduce a term $(v \cdot k)^l$, hence new roots of k will be present. This means that in the laboratory system of reference new velocity induced wave modes will appear. Such modes will not appear for dispersion equations where ω' is of the same or lower power, compared to k'. There is no paradox in the fact that a different number of modes will be observed in the comoving and in the laboratory systems of reference. To identify the different modes in the laboratory system, all are launched with the same frequency ω. In the comoving system some of these waves, although having different frequencies ω', will belong to identical modes.

The simplest physical example for generation of a velocity induced mode is provided by a cold, unmagnetized collisionless plasma, for which (11) becomes

$$[k'^2 - \mu_0\epsilon_0\omega'^2(1 - \omega_p'^2/\omega'^2)]^2(1 - \omega_p'^2/\omega'^2)\,\omega'^2 = 0. \qquad (15)$$

where ω_p' is the plasma frequency in the comoving frame of reference. The term in brackets (15) describes two modes, which are the forward and backward transverse electromagnetic waves in the plasma. Upon transformation of this dispersion equation into the laboratory system, using (7), no new modes will appear, due to the fact that ω'^2, k'^2 are of the same power. The rest of (15), which can be written as $\omega'^2 = \omega_p'^2$ is usually discarded on ground that this is a pure oscillation rather than a wave. However, for moving media it must be retained, because in the laboratory system, after using (7), we have

$$\gamma(\omega - v \cdot k) = \pm\omega_p' \qquad (16)$$

describing a new, velocity induced, wave mode. This is the familiar space charge wave for a drifting plasma, and (16) describes two modes, corresponding to two possible waves in the drifting plasma.

Note added in proof: Prof. Suchy kindly called my attention to references [7], [8], [9].

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A High-Speed Low-Cost Modulo P_i Multiplier with RNS Arithmetic Applications

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***Abstract*—Modulo P_i multipliers are implemented by look-up tables when P_i is small (5 bits or less) and by index calculus if P_i is larger (6 bits or more). However, index calculus only works for prime moduli P_i. In this letter, we introduce a new square-law multiplier that is useful for modulo P_i multiplication where P_i is any modulus. It is expected that this will have important applications in RNS arithmetic computing hardware.**

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