Scattering of Electromagnetic Waves by a Cylinder Moving Along Its Axis

DAN CENSOR

Abstract—The scattering of a time-harmonic, linearly polarized plane electromagnetic wave by a cylinder uniformly moving along its axis is discussed. The formalism is relativistically exact, and explicit forms are provided for first-order velocity effects. Consideration is given to both a cylinder moving in free space, using the procedure suggested by Einstein, and two refractive media; it is verified that the first case is a special case of the second one. Thin scatterers are considered and it is shown that no first-order velocity effects are present. For a moving medium, having in its rest frame the same constitutive parameters as the surrounding medium, it is shown that the velocity-independent part vanishes, but scattered fields of the first order in the velocity are still present. Moreover, these waves appear with the opposite polarization (compared to the incident wave).

I. INTRODUCTION

In this paper scattering by an axially moving cylinder is studied. From the point of view of applications, many problems suggest themselves, e.g., scattering by moving masses of air, jet exhausts, and streams of moving particles such as electrons. From the theoretical point of view it is considered worthwhile to study special cases of scattering in velocity-dependent systems, and to point out the new first-order velocity effects.

To date numerous papers are available, applying propagation and scattering in moving media and by moving objects to special cases of interest. The subsequent studies deal with the pertinent wave equation and the associated Green's functions, therefore this background is not needed here. Yeh [1], [2] and Yeh and Cassey [3] discuss transmission and reflection involving plane interfaces. Guided waves in moving media are considered by Collier and Tai [4], Du and Compton [5], Gruenberg and Daly [6], Berger and Grienmann [7], and others. Scattering by a cylinder in free space is considered by Lee and Mittra [8], but only for the far field. Scattering and multiple scattering by objects moving in free space are discussed by Censor [9], who also extends the formalism to scatterers moving in refractive media and scatterers immersed in moving media [10]. A first-order formalism, following an earlier study by Nathan and Censor [11], is applied to rotating media [12]. Scattering by a rotating circular cylinder is discussed by Censor and Nathan [13]. A simple situation corresponding to random media, and application to Doppler broadening diagnostics, are given by Censor [14].

At present we consider the problem of a time-harmonic linearly polarized plane electromagnetic wave scattered by an axially moving cylinder. (See Fig. 1.) In free space one can use the procedure suggested by Einstein [15]. Accordingly, the incident wave is transformed into the frame of reference of the object at rest, and the scattered wave is computed and then transformed back into the frame of reference of the observer. The general case of an external refractive medium is more complicated [10], but it becomes relatively simple for the present case, since the surface separating the cylinder from the external medium is time-invariant. The general solution for the electromagnetic fields in simple media, in terms of cylindrical coordinates, is transformed from the cylinder's frame of reference into that of the observer, and the boundary conditions are applied to get explicit results. Since the boundary conditions are derived from Maxwell's equations without reference to the constitutive relations for the media at hand, the conventional boundary conditions are valid, i.e., the tangential electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{H} \), respectively, must be continuous across the surface. Since the boundary surface is time-invariant, this implies that no Doppler effects are present, i.e., the time dependence of the internal and external fields is the same, referred to the same frame of reference.

Consider two frames of reference \( \Gamma, \Gamma' \) in relative uniform motion. The cylinder at rest is attached to \( \Gamma' \), the external medium is at rest with respect to \( \Gamma \). Observed from \( \Gamma \), we see \( \Gamma' \) moving in the positive \( z \)-direction, Fig. 1. The two cartesian coordinate systems \( x, y, z \) and \( x', y', z' \), corresponding to \( \Gamma, \Gamma' \), respectively, coincide at \( t=t'=0 \), hence the following Lorentz transformation applies:

![Fig. 1. Geometry for scattering by an axially moving cylinder, as observed from \( \Gamma' \), the frame of reference of the external medium at rest. The cylinder moves in the \( z \)-direction, and the incident wave propagates in direction \( \delta \), in the \( xz \) plane.](image-url)
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\[ z' = \gamma (z - vt), \]
\[ x' = x, \]
\[ y' = y, \]
\[ t' = \gamma(t - v z/c^2), \]
\[ \gamma = (1 - \beta^2)^{-1/2}, \]
where \( \mu_0, \varepsilon_0 \) are the constitutive parameters associated with free space. Application of the principle of relativity [15] to Maxwell’s equations (see Sommerfeld [16], who cites Minkowski’s original papers) yields the transformation formulas for the fields,

\[ E'_\perp = \gamma (E_\perp + \nu \times B) = \gamma (E_\perp + \mu \nu \times H), \]
\[ H'_\perp = \gamma (H_\perp - \nu \times D) = \gamma (H_\perp - \varepsilon \nu \times E), \]

where \( \parallel \) and \( \perp \) denote components parallel and perpendicular to the velocity, respectively. The simple constitutive relations \( D = \varepsilon E, B = \mu H \) are already incorporated in (2). Subsequently \( \mu, \varepsilon \) are considered to be constants, either real or complex.

In particular, consider the transformation formulas for a plane wave,

\[ \psi = fe^{i\omega t}, \quad \phi = k \cdot \mathbf{r} - \omega t, \quad \psi = E, H, D, B, \]

specified in \( \Gamma' \), in terms of \( \Gamma \) coordinates. It has been shown [9], [10] that for the corresponding plane wave in \( \Gamma' \),

\[ \psi' = f' e^{i\omega t}, \quad \phi' = k' \cdot \mathbf{r}' - \omega t', \quad \psi' = E', H', D', B', \]

the following transformation formulas apply. The amplitudes are related by

\[ \mathbf{t}' = \mathbf{\tilde{f}} \cdot \mathbf{t}, \]
\[ \mathbf{\tilde{f}} = [(1 - \gamma) \mathbf{\hat{z}} + \gamma \eta \mathbf{\hat{k}}] \mathbf{\hat{t}} = \gamma (1 - \eta \mathbf{\hat{t}} \cdot \mathbf{\hat{k}}) \mathbf{\hat{t}}, \]
\[ \eta = v/C \quad \text{for } \psi = E, H, \]
\[ \eta = v C/c \quad \text{for } \psi = D, B, \]

where \( \mathbf{\hat{C}} \) is the phase velocity associated with the medium at hand; for free space \( C = c \) and \( \eta = \beta; \mathbf{\tilde{f}} \) is a dyadic, \( \mathbf{\hat{t}} \) is the unit vector in the direction of propagation. The frequencies transform according to

\[ \omega' = \gamma \omega (1 - v \cos \alpha /c), \]

The propagation vector transforms according to

\[ \mathbf{k}' = \mathbf{k} - [(1 - \gamma) \mathbf{\hat{z}} + \gamma k \mathbf{\hat{C}} /c^2] \mathbf{\hat{t}}, \]

and the absolute value of (7) is prescribed by

\[ k' = \gamma k (1 - \beta^2 \sin^2 \alpha - 2 \beta \mathbf{\hat{C}} \cdot \cos \alpha /c + \beta^2 \mathbf{\hat{C}}^2 /c^2)^{1/2}. \]

The direction of propagation changes according to

\[ \tan \alpha' = \tan \alpha /\gamma (\cos \alpha /c \cdot \mathbf{\hat{C}} /c^2). \]

II. CYLINDER MOVING IN FREE SPACE

We consider here two problems: 1) scattering of a plane wave propagating in \( \Gamma \) in direction \( \alpha \) with respect to the \( z \)-axis, such that in \( \Gamma' \) its direction of propagation is perpendicular to the \( z \)-axis; and 2) scattering by a plane wave which in \( \Gamma' \) propagates perpendicularly to the \( z \)-axis. Without loss of generality, an arbitrary incident plane wave is resolved into two plane waves, one transverse-magnetic, the other transverse-electric, with respect to \( \mathbf{z} \), and the two cases are discussed separately.

For case 1) consider in \( \Gamma \) a plane wave (3). In view of (9), \( \alpha' = \pi/2 \) prescribes in free space

\[ \cos \alpha = \beta. \]

Therefore (6) yields for the frequency

\[ \omega' = \omega /\gamma, \]

i.e., a transverse Doppler effect. For the present free-space case \( k \) and \( \mathbf{\tilde{f}} \) also undergo the same transformation (11). The vector \( f \) is given according to (5):

\[ \mathbf{f}' = f'(\mathbf{z} - \gamma \beta \mathbf{\hat{z}}), \]

hence \( f \) makes an angle \( \alpha \) [defined by (10)] with the negative \( x \)-direction. For the circular cylinder the boundary value problem is readily solved in \( \Gamma' \). The internal field is represented by means of the nonsingular Bessel functions

\[ \Psi' = \sum_{n=-\infty}^{\infty} i^n b_n J_n(K'r) e^{i n \pi - i\phi'}, \]

where \( u' \) stands for an electric or magnetic field polarized in the \( \mathbf{z} \) direction, and where \( H_n(k'r) = H_n(0)(k'r) \) is the Hankel function of the first kind and order \( n \). According to Maxwell’s equations the associated transverse field is \( \nabla \times \mathbf{u}' \), where \( A = \omega \mu \omega - \epsilon \omega \mu, \mathbf{u}' = E', H', \mathbf{D}', \mathbf{B}', \mathbf{E}, \mathbf{H}, \)

The plane wave \( \Psi' \) can be represented in cylindrical wave functions by replacing \( a J_n(k'r) \) by \( J_n(k'r) \) in (13), where \( k' \) is the corresponding propagation constant in free space. The scattered wave is given by

\[ u' = u' e^{i \mathbf{\tilde{f}} \cdot \mathbf{r}} \]

where the primed cylindrical functions are differentiated with respect to the argument, \( b = \mu_0 /\mu, \epsilon_0 /\epsilon \) for \( E, H \), polarization, respectively. Application of the boundary conditions at the surface \( r = a \) yields [17]

\[ b_n = [J_n'(k'a)J_n(K'a) - Z J_n(k'a)J_n'(K'a)] / \Delta_n, \]

\[ Z = bK'/k' = bK/k, \]

(14)

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Similarly (2) implies \( H'_\parallel = \gamma H'_\parallel, \quad H'_\perp = \gamma H'_\perp, \quad E'_\parallel = \mu \mathbf{\epsilon} H'_\parallel, \quad E'_\perp = \mathbf{\epsilon} \mathbf{\mu} H'_\perp, \quad H'_\parallel = \mathbf{\epsilon} \mathbf{\mu} E'_\parallel, \quad H'_\perp = \mathbf{\mu} \mathbf{\epsilon} E'_\perp \). As these are first-order veloc-
ity effects, therefore they should be taken into account even for small velocities, but as far as the coefficients (15) are concerned, they involve only zero-order and second-order terms.

Now consider a plane wave in $\Gamma$, propagating in the $x$-direction, with either the $E$ or $H$ field polarized along the $z$-axis. Exchanging primed and unprimed quantities, and replacing $\beta$ by $-\beta$ in (10)-(12) yields the transformation of the incident wave into $\Gamma'$. For example, for $E$ polarization in $\Gamma'$, we obtain

$$E' = \frac{2}{\varepsilon} E'' = \frac{2}{\varepsilon} e^{ikr - i\omega t} = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n J_n(ka)e^{in\theta - \frac{k}{\varepsilon} r - i\omega t},$$

which applies to the internal field. Index $i$ denotes the internal region and $h'$ is prescribed by the incident wave, in order that the boundary conditions be satisfied. The finite field in the internal region is represented in terms of the nonsingular $J_n$ functions. The scattered field is written in terms of the Hankel functions,

$$E_{s'} = \frac{2}{\varepsilon} E_s = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n \alpha_n H_n(\lambda r)e^{in\theta - h' r - i\omega t'},$$

$$H_{s'} = \frac{2}{\varepsilon} H_s = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n \beta_n H_n(\lambda r)e^{in\theta - h' r - i\omega t'},$$

(19)

etc., similar to (18) with $\lambda'$ replacing $\lambda$, $k'$ replacing $K'$, $\mu_0$ instead of $\mu_\perp$, and index $i$ replaced by $e$.

The continuity of the tangential components of $E$ and $H$ fields across the surface $r = a$ and the orthogonality of (18) and (19) with respect to $e^{in\theta}$ yield the equations for the coefficients,

$$- nh'J_n(\lambda a)a_n/\lambda^2 a + \mu_\perp J_n'(\lambda a)a_n/\lambda^2 a + nh'H_n(\lambda a)a_n/\lambda^2 a$$

$$= - i\mu_\perp \mu_\perp H_n'(\lambda a)a_n/\lambda \mu_0^2 a,$$

$$iK'J_n(\lambda a)a_n/\lambda \mu_0^2 a + nh'H_n(\lambda a)a_n/\lambda^2 a - iK'H_n'(\lambda a)a_n/\mu_0^2 a$$

$$- nh'H_n(\lambda a)a_n/\lambda \mu_0^2 a = i\gamma J_n'(\lambda a)/\mu_0 \mu_0, J_n(\lambda a)a_n = 0$$

Correct to the first order in $\beta$, this yields,

$$a_n = \frac{1}{J_n(Ka)}[J_n'(Ka)H_n(ka)\mu_\perp K + J_n(ka)H_n'(ka)\mu_\perp K] + J_n(ka)H_n'(ka)\mu_\perp K/\mu_0 K$$

$$+ J_n'(ka)J_n(ka)H_n(ka) + J_n(ka)H_n'(ka)J_n'(ka)/J_n, \Delta = J_n(ka)J_n'(ka)H_n(ka)H_n(ka)(\mu_0 K/\mu_\perp K + \mu_0 K/\mu_\perp K)$$

$$- J_n'(ka)H_n(ka)H_n(ka), b_n = J_n(ka)[H_n'(ka)J_n(ka) - H_n(ka)J_n'(ka)](1 - K^2/\lambda^2) \sin \theta/\mu_0^2 a \Delta =$$

$$- J_n(ka)(1 - K^2/\lambda^2)2n\beta\mu_\perp K/\pi\lambda^2 \mu_0 a \Delta,$$

The general solution for the electromagnetic field in cylindrical coordinates is given (for example, see Stratton [18]) by

$$E_{s'} = \frac{2}{\varepsilon} E_s = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n \alpha_n H_n(\lambda r)e^{in\theta - h' r - i\omega t'},$$

$$H_{s'} = \frac{2}{\varepsilon} H_s = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n \beta_n H_n(\lambda r)e^{in\theta - h' r - i\omega t'},$$

(18)

thus $b_n$ is of the first order in $\beta$, while $a_n$ contains $\beta^2$ velocity effects only. Inserting (21) in (19) and finding the remaining components by inspection of (18) yields the scattered field for a $\Gamma'$ observer. Specialization of the inverse of (2) to the present case yields the scattered field as observed in $\Gamma$. To the first order in $\beta$ this yields

$$E_s = \frac{2}{\varepsilon} E_s = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n \alpha_n H_n(\lambda r)e^{in\theta - h' r - i\omega t'},$$

$$H_s = \frac{2}{\varepsilon} H_s = \frac{2}{\varepsilon} \sum_{n=-\infty}^{\infty} i^n \beta_n H_n(\lambda r)e^{in\theta - h' r - i\omega t'},$$

(22)
Consequently in Γ there are first-order velocity-dependent field components, involving $H_e$.

### III. TWO REFRACTIVE MEDIA

For free space the conventional wave equation for media at rest was valid in the external region, both in Γ and Γ′. In the present case this simplicity is lost. Since the observer is at rest with respect to Γ, the fields (18) are transformed from Γ′ to Γ. This yields

$$E_0 = E_0^{′} = \delta E_0 \neq \frac{\omega}{c} \sum_{n=-\infty}^{\infty} \int \left( J_n(\lambda r)e^{ik_0r - i\omega t} \right) \frac{d^2}{dx^2} - i\omega \frac{d}{dx} E_0^{′},$$

$$H_0 = H_0^{′} = \frac{\delta H_0}{2} \sum_{n=-\infty}^{\infty} \int \left( J_n(\lambda r)e^{ik_0r - i\omega t} \right) \frac{d^2}{dx^2} - i\omega \frac{d}{dx} H_0^{′},$$

$$E_0 = \delta (E_0^{′} + \mu e H_0^{′}) = \delta (P \frac{\partial E_0}{r} + Q \mu e \frac{\partial H_0}{r}) \gamma / \Lambda^2,$$

$$P = - \lambda + K^{′} \phi / \omega,$$

$$Q = \omega - \nu h,$$

$$E_0 = \delta (E_0^{′} - \mu e H_0^{′}) = \delta (P \frac{\partial E_0}{r} - Q \mu e \frac{\partial H_0}{r}) \gamma / \Lambda^2,$$

$$H_0 = \delta (H_0^{′} + \nu e H_0^{′}) = \delta (P \frac{\partial H_0}{r} + Q \mu e \frac{\partial H_0}{r}) \gamma / \Lambda^2.$$  

The special cases discussed in the previous section (which also follow as a special case of two refractive media) suggest that the two special cases $h^′ = 0$ and $h = 0$ be considered. The first case does not introduce the simplicity that led to (15), since $P$ and $Q$ (23) do not vanish. The second case yields

$$\omega = \omega^′ / \gamma,$$

$$h = \omega^′ / \omega = \omega^′ / \gamma,$$

$$P = \gamma \omega^′ (C_1^2 - c^{-2}),$$

$$Q = \omega / \gamma,$$

$$\Lambda^2 = \omega^′ (C_1^2 - \beta^2 / c^2),$$

and a corresponding simplification of (24), as a result of $h = 0$. The incident plane wave propagates in the $x$-direction with the E field (say) polarized along the axis,

$$E_0^{′} = \frac{\delta E_0^{′}}{2} \sum_{n=-\infty}^{\infty} \int \left( J_n(\lambda r)e^{ik_0r - i\omega t} \right) \frac{d^2}{dx^2} - i\omega \frac{d}{dx} E_0^{′},$$

$$H_0^{′} = \delta (H_0^{′} + \nu e H_0^{′}),$$

$$H_0 = \delta (H_0^{′} + \nu e H_0^{′}).$$

The equations for the coefficients follow from the orthogonality of (23), (24), and (27) with respect to $e^{i\omega t}$ and the boundary conditions at $r = a$.

### IV. SPECIAL CASES: CLEAR AIR SCATTERING AND THIN SCATTERERS

Solving (28) for $a_n$ we again obtain (21) with index $a$ replaced by $e$; for $b_n$ we obtain

$$b_n = \left[ J_n(\lambda r) \left( H_n^{(k)}(ka)(kJ_n(ka) - H_n(ka)) \right) \right] \inf \left( C_1^2 - c^{-2} \right) / \mu e A = \left[ J_n(\lambda r) \left( C_1^2 - c^{-2} \right) 2n \pi / \lambda e \csc K^2 \Delta = \right.$$

$$- J_n(\lambda r) \left( C_1^2 - c^{-2} \right) 2n \pi / \lambda e \csc K^2 \Delta,$$

where $\Delta$ is given by (21) with index $e$ replacing index $a$. For $C_1 = c$ (29) reduces to (21); when $C_1 = c$, both (21) and (29) vanish. The scattered field, correct to the first order in the velocity, is specified by (24) with inspection of (18) for the other components, (21) with the above modification.

The result (29) predicts that even when the cylinder and the external region possess the same constitutive parameters, a moving cylinder will produce scattered waves. This "clear air scattering" effect verifies that a moving object constitutes a different medium. The scattered wave depends on the velocity, and this is a first-order effect. Thus we have for the scattered wave, excited by (27),
$E^t_0 = H^t_0 = H^r_0 = 0,$

$H^t_s = i \omega \epsilon_s \sum_{n=-\infty}^{\infty} v_n b_n H_n(kr)e^{i\theta - i\omega t}$,

$E^t_s = \mu_0 H_s \delta \mu_0 \omega / k^2 \nu,$

$E^r_s = -i \delta_0 H_s \delta \mu_0 \omega / k^2$ \(s \neq 0\).

(30)

It follows from (30) that there is an inversion effect with respect to the polarization. The exciting plane wave (27) is polarized with the electric field along the $z$-axis and the magnetic field in the $xy$-plane. The scattered wave (30) has its magnetic field along the axis and the electric field in the $xy$-plane. The fields follow from (16) and (17). For thin dielectric cylinders, the monopole, dipole term, and the rest-frame constitutive parameters, and even if the boundary between them and the rest of the medium is time-invariant, will produce a first-order scattering effect. The polarization of the scattered field will be inverted, as compared to the incident wave.

**REFERENCES**


