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NON-RELATIVISTIC BOUNDARY CONDITIONS AND SCATTERING IN THE PRESENCE OF ARBITRARILY MOVING MEDIA AND OBJECTS: CYLINDRICAL PROBLEMS

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Abstract—Recently non-relativistic boundary conditions, based on the Lorentz force formulas, have been investigated. It was shown that to the first order in the relative velocity v/c the results for scattering problems are in agreement with the exact relativistic formalism. Examples for scattering by material objects moving in free space have been discussed.

Presently the feasibility of non-relativistically solving scattering problems involving arbitrary material media is investigated. For concreteness, two representative canonical problems were chosen: scattering by a uniformly moving circular cylinder, and the related problem of a cylinder at rest, comprised of a uniformly moving medium in the cylindrical cross-sectional plane.

The investigation demonstrates that solving such problems is feasible, and indicates the complexity involved in such an analysis. The main highlights are that we need to evaluate the phases and amplitudes of waves at the scatterer's surface, employing formulas based on the Lorentz force formulas and the Fresnel drag concept. The explicit solutions for the scattering problem display velocity-dependent interaction of the scattering coefficients.

1. INTRODUCTION AND PROGRAMMATIC OUTLINE

Maxwell's theory in velocity dependent systems was since long ago recognized as a programmatic tool for a profounder understanding of electrodynamics, see Sommerfeld's [1, p. 280] historical account. This attitude is emphasized by A. Einstein, entitling his monumental (1905) article "On the Electrodynamics of moving bodies". Sommerfeld cites also articles by H. Hertz (1890), E. Cohn (1901, 1902), H. A. Lorentz (1903), and H. Minkowski (1908). Minkowski, already in possession Einstein's (1905) theory of Special Relativity, for the first time correctly deals with the problem of material media in motion. Since then, for over a century, assiduous efforts have been devoted to extend our knowledge of electrodynamics, especially as concerns scattering and propagation in the presence of moving objects and moving media.

Strictly speaking, Einstein's theory applies to inertial systems only, and thus excludes spatiotemporally varying velocities. This reduces the class of pertinent problems to a mere handful of (mostly trivial) cases. This difficulty has been recognized from the early stages, and in order to remedy the situation, it was *heuristically* assumed that the theory applies to low acceleration as well [1].

At the present time it is therefore interesting to close the circle, and being in possession of all the heretofore relativistic results, ask the following question: can we re-introduce a non-relativistic analysis and to what extent will it yield commensurate results?

Recently the question of non-relativistic approximate boundary conditions for the electromagnetic field has been discussed [2]. The main tenet was that we refrain from using Einsten's Special Relativity theory and the associated Lorentz transformations [3, 4], and admit instead boundary conditions based on ideas inferred from the Lorentz force formulas. Existing results based on Special Relativity have been exploited as benchmarks for examining the new findings. It has been shown that to the first order in the relative velocity v/c the two approaches are in agreement. The analysis in [2] was limited to material objects moving in free space (vacuum), therefore the analysis was simpler: It has been assumed that the excitation and scattered waves propagate in free space, regardless if measured relative to the scatterer at rest or in motion. In a sense this is already an assumption based on the Special Relativity theory and the negative results of the Michelson-Morely experiments. Therefore calling the method "non-relativistic" does not imply that we go back to the Galilean transformations. Presently the new theory is extended to more general situations of scattering in arbitrary media. The same tenet is pursued here. With waves defined in the medium at rest, the time dependent signal at the origin of an arbitrary reference system yields a correct first order in v/c expression for the time dependent term, i.e., for the frequency. The relative phase shift to different locations requires the introduction of what essentially constitutes the Fresnel theory for moving media. Although presented long before the advent of Einstein's Special Relativity theory, this is also patently a relativistic result, as explained below. Consequently it is impossible to completely disassociate the discussion from Special Relativity theory.

The macroscopic Maxwells equations which are the fundamental "law of nature" concerning us here, see [5] for notation, are represented in the form

$$\begin{aligned} \partial_{\mathbf{x}^{(*)}} \times \mathbf{E}^{(*)} &= -\partial_{t^{(*)}} \mathbf{B}^{(*)} - \mathbf{j}_m^{(*)} \\ \partial_{\mathbf{x}^{(*)}} \times \mathbf{H}^{(*)} &= \partial_{t^{(*)}} \mathbf{D}^{(*)} + \mathbf{j}_e^{(*)} \\ \partial_{\mathbf{x}^{(*)}} \cdot \mathbf{D}^{(*)} &= \rho_e^{(*)} \\ \partial_{\mathbf{x}^{(*)}} \cdot \mathbf{B}^{(*)} &= \rho_m^{(*)} \end{aligned} \tag{1}$$

Using for a while the language of Special Relativity, (*) generically denotes that we are measuring the electromagnetic fields in the inertial (non-accelerated) system of reference $\Gamma^{(*)}$, in terms of the native coordinates $\mathbf{X}^{(*)} = (\mathbf{x}^{(*)}, ict^{(*)})$ of this system, e.g., $\mathbf{E}^{(*)} = \mathbf{E}^{(*)}(\mathbf{X}^{(*)})$. We use the Minkowski four-vector notation for the four-dimensional spatiotemporal domain. The operator $\partial_{\mathbf{x}^{(*)}}\times$ indicates the Curl, similarly $\partial_{t^{(*)}}$ indicates the partial time derivative. Indices e- (electric), or m- (magnetic) refer to electric, and (virtual) magnetic sources, respectively.

Let us drop the frame of reference notation (*) for the time being and consider the boundary and jump conditions in the electromagnetic field, within some arbitrary inertial system. There is a misconception in some textbooks who claim that the boundary and jump conditions are *derived* from Maxwell's equations. Obviously boundary conditions are *imposed* on differential equations and not *vice-versa*. The way the boundary and jump conditions are derived for the electromagnetic field, is by limiting processes which ensure that the Maxwell equations (1) stay valid as we approach, as close as we wish, the boundary surfaces. For two spatial regions, denoted "*1*" and "*2*", it is found that in general, four conditions are necessary, two scalar conditions derived from the last two equations in (1):

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \rho_{mS} \tag{2}$$
$$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_{eS} \tag{3}$$

and two vector conditions derived from the first two equations in (1):

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{j}_{mS} \tag{4}$$
$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{j}_{eS} \tag{5}$$

In (2), (3) on the boundary between the two regions, the normal components of vectors **B**, **D**, are discontinuous, the jump indicated by the magnetic and electric surface charge densities ρ_{mS}, ρ_{eS}, respectively. The unit normal vector $\hat{\mathbf{n}}$ points into region "*1*". Similarly (4), (5), indicate the discontinuity of the tangential components of vectors **E**, **H**, across the boundary, with the jump given by (the negative sign of) the magnetic, and electric, surface current densities, $-\mathbf{j}_{mS}$, $\mathbf{j}_{eS}$, respectively.

In general all the four relations (2)–(5) are needed, e.g., when dealing with electrostatics and magnetostatics, or when surface sources ρ_{eS}, ρ_{mS}, $\mathbf{j}_{eS}$, $\mathbf{j}_{mS}$ are present. However, in dynamical (time-dependent) systems, and in the absence of sources, two equations, e.g., (4), (5), are sufficient. This well known result follows from (1), multiplying the first two equations by $\hat{\mathbf{n}}\cdot$ yields

$$\hat{\mathbf{n}} \cdot (\partial_{\mathbf{x}} \times (\mathbf{E}_1 - \mathbf{E}_2)) = -\partial_t \hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) \tag{6}$$
$$\hat{\mathbf{n}} \cdot (\partial_{\mathbf{x}} \times (\mathbf{H}_1 - \mathbf{H}_2)) = -\partial_t \hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \tag{7}$$

In the absence of sources and neglecting the constant due to the time integration, the right hand side of (6), (7) vanishes according to (2), (3). The left hand side of (6), (7), due to the multiplication by $\hat{\mathbf{n}}\cdot$, involves only field components tangential to the boundary. Hence ignoring integration constants, it agrees with (4), (5), respectively. Consequently we encounter here a redundancy.

It is noted that the boundary conditions are associated with the Maxwell's equation without resorting to material properties of media, i.e., no constitutive relations are involved.

Einstein's Special Relativity theory postulates the covariance of (1) for all inertial systems, meaning that for any specific inertial system, the equations (1) possess the same functional structure, e.g., for two inertial systems denoted $\Gamma^{(1)}$, $\Gamma^{(2)}$, (1) exists with the appropriate indices $* = 1, 2$, respectively.

The spatiotemporal coordinates $\mathbf{X}^{(1)}$, $\mathbf{X}^{(2)}$, are related by the

Lorentz transformation

$$
\begin{aligned}
\mathbf{x}^{(2)} &= \tilde{\mathbf{U}}^{(2\mapsto 1)} \cdot \left(\mathbf{x}^{(1)} - \mathbf{v}^{(2\mapsto 1)} t^{(1)}\right) \\
t^{(2)} &= \gamma^{(2\mapsto 1)} \left(t^{(1)} - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{x}^{(1)}/c^2\right) \\
\gamma^{(2\mapsto 1)} &= \gamma^{(1\mapsto 2)} = \left(1 - \beta^{(2\mapsto 1)2}\right)^{-1/2} \\
\hat{\mathbf{v}}^{(2\mapsto 1)} &= -\hat{\mathbf{v}}^{(1\mapsto 2)} = \mathbf{v}^{(2\mapsto 1)}/v^{(2\mapsto 1)}, \\
\beta^{(2\mapsto 1)} &= v^{(2\mapsto 1)}/c, \quad v^{(2\mapsto 1)} = \left|\mathbf{v}^{(2\mapsto 1)}\right| \\
\tilde{\mathbf{U}}^{(2\mapsto 1)} &= \tilde{\mathbf{U}}^{(1\mapsto 2)} = \tilde{\mathbf{I}} + (\gamma^{(2\mapsto 1)} - 1)\hat{\mathbf{v}}^{(2\mapsto 1)}\hat{\mathbf{v}}^{(2\mapsto 1)},
\end{aligned} \tag{8}
$$

where in (8) the tilde denotes dyadics, $\tilde{\mathbf{I}}$ denotes the idemfactor dyadic, and $\mathbf{v}^{(2\mapsto 1)}$ is the velocity of motion of $\Gamma^{(2)}$ as observed from $\Gamma^{(1)}$. For brevity (8) can be denoted by $\mathbf{X}^{(2)} = \mathbf{X}^{(2)}[\mathbf{X}^{(1)}]$, and it is easily shown that (8) when solved for the $\Gamma^{(2)}$ coordinates, yields $\mathbf{X}^{(1)} = \mathbf{X}^{(1)}[\mathbf{X}^{(2)}]$, which has the same structure as (8) with interchanged indices, and $\mathbf{v}^{(2\mapsto 1)}$ replaced by $\mathbf{v}^{(1\mapsto 2)} = -\mathbf{v}^{(2\mapsto 1)}$. Subsequently we are going to consider non-relativistic scattering, which supposedly will be correct to the first order in v/c. Within this approximation (8) is considered with $\tilde{\mathbf{U}} = \tilde{\mathbf{I}},\ \gamma = 1$.

$$
\begin{aligned}
\mathbf{x}^{(2)} &= \mathbf{x}^{(1)} - \mathbf{x}^{(2\mapsto 1)} t^{(1)} \\
t^{(2)} &= t^{(1)} - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{x}^{(1)}/c^2
\end{aligned} \tag{9}
$$

Obviously (9) is not the classical Galilean transformation, which is characterized by $t^{(2)} = t^{(1)}$ instead of the second line (9). By taking differentials and dividing, equations (9) yield

$$
\begin{aligned}
\mathbf{u}^{(2)} &= (\mathbf{u}^{(1)} - \mathbf{v}^{(2\mapsto 1)})/(1 - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{u}^{(1)}/c^2) \\
&\approx \mathbf{u}^{(1)} - \mathbf{v}^{(2\mapsto 1)} + \mathbf{u}^{(1)}(\mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{u}^{(1)})/c^2,\ \mathbf{u}^{(*)} = d\mathbf{x}^{(*)}/dt^{(*)}
\end{aligned} \tag{10}
$$

which is the first order approximation for the relativistic formula for the addition of velocities. For $\mathbf{u}^{(2)},\ \mathbf{u}^{(1)} \ll c$ (10) reduces to the Galilean equation $\mathbf{u}^{(2)} = \mathbf{u}^{(1)} - \mathbf{v}^{(2\mapsto 1)}$, but this is not the case when $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ are of the same order of magnitude as c. The latter is the case that will concern us subsequently in relation to electromagnetic wave propagation. For the scalar case of collinear velocities (10) can be written as

$$
\begin{aligned}
u^{(2)} &= \left(u^{(1)} - v^{(2\mapsto 1)}\right) / \left(1 - v^{(2\mapsto 1)} u^{(1)}/c^2\right) \\
&\approx u^{(1)} \left(1 - (v^{(2\mapsto 1)}/u^{(1)})(1 - (u^{(1)}/c)^2)\right)
\end{aligned} \tag{11}
$$

Akin to the Lorentz transformation (9) we have the spectral (Fourier) transformation (e.g., see [5]), usually referred to as the relativistic Doppler effect

$$\begin{aligned} \mathbf{k}^{(2)} &= \tilde{\mathbf{U}}^{(2\mapsto 1)} \cdot \left(\mathbf{k}^{(1)} - \mathbf{v}^{(2\mapsto 1)}\omega^{(1)}/c^2\right) \\ \omega^{(2)} &= \gamma^{(2\mapsto 1)} \left(\omega^{(1)} - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{k}^{(1)}\right) \end{aligned} \tag{12}$$

and to the first order in v/c we have

$$\begin{aligned} \mathbf{k}^{(2)} &= \mathbf{k}^{(1)} - \mathbf{v}^{(2\mapsto 1)}\omega^{(1)}/c^2 \\ \omega^{(2)} &= \omega^{(1)} - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{k}^{(1)} \end{aligned} \tag{13}$$

Again, (13) is not the classical Galilean transformation, which is characterized by $\mathbf{k}^{(2)} = \mathbf{k}^{(1)}$ instead of the first line (13), and cannot be reduced to it. Similarly to (10), we derive from (13), to the first order in v/c, the slowness function

$$\mathbf{s}^{(2)} = \left(\mathbf{s}^{(1)} - \mathbf{v}^{(2\mapsto 1)}/c^2\right) / \left(1 - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{s}^{(1)}\right), \quad \mathbf{s}^{(*)} = \mathbf{k}^{(*)}/\omega^{(*)} \tag{14}$$

For the scalar one-dimensional case the inverse of (14) yields the transformation for the phase velocity $v_{ph}^{(*)} = 1/s^{(*)} = \omega^{(*)}/k^{(*)}$

$$\begin{aligned} v_{ph}^{(2)} &= 1/s^{(2)} = \left(1 - v^{(2\mapsto 1)}/v_{ph}^{(1)}\right) / \left(1/v_{ph}^{(1)} - v^{(2\mapsto 1)}/c^2\right) \\ &\approx v_{ph}^{(1)} \left(1 - (v^{(2\mapsto 1)}/v_{ph}^{(1)})(1 - A^{(1)})\right) \\ &\approx c/N^{(1)} - v^{(2\mapsto 1)} \left(1 - 1/(N^{(1)})^2\right), \\ N^{(1)} &= c/v_{ph}^{(1)}, \quad A^{(1)} = \left(v_{ph}^{(1)}/c\right)^2 \end{aligned} \tag{15}$$

The identical expressions in (11), (15) show that the phase velocity satisfies the same addition rule as any arbitrary velocity given in (11). On the other hand, (15) is recognized as the Fresnel formula including the "drag coefficient" (e.g., see Van Bladel [6]). It follows that the first order v/c approximation (15), describing the phase velocity in a moving medium, cannot be derived from classical Galilean considerations, but must be consistently included in our analysis.

The fields in $\Gamma^{(1)}, \Gamma^{(2)}$ are related by the field transformation formulas

$$\begin{aligned} \mathbf{E}^{(2)} &= \tilde{\mathbf{V}}^{(2\mapsto 1)} \cdot \left(\mathbf{E}^{(1)} + \mathbf{v}^{(2\mapsto 1)} \times \mathbf{B}^{(1)}\right) \\ \mathbf{B}^{(2)} &= \tilde{\mathbf{V}}^{(2\mapsto 1)} \cdot \left(\mathbf{B}^{(1)} - \mathbf{v}^{(2\mapsto 1)} \times \mathbf{E}^{(1)}/c^2\right) \end{aligned}$$

$$\begin{aligned}
\mathbf{D}^{(2)} &= \tilde{\mathbf{V}}^{(2\mapsto 1)} \cdot \left(\mathbf{D}^{(1)} + \mathbf{v}^{(2\mapsto 1)} \times \mathbf{H}^{(1)}/c^2\right) \\
\mathbf{H}^{(2)} &= \tilde{\mathbf{V}}^{(2\mapsto 1)} \cdot \left(\mathbf{H}^{(1)} - \mathbf{v}^{(2\mapsto 1)} \times \mathbf{D}^{(1)}\right) \\
\mathbf{j}^{(2)}_{e,m} &= \tilde{\mathbf{U}}^{(2\mapsto 1)} \cdot \left(\mathbf{j}^{(1)}_{e,m} - \mathbf{v}^{(2\mapsto 1)} \rho^{(1)}_{e,m}\right) \\
\rho^{(2)}_{e,m} &= \gamma^{(2\mapsto 1)} \cdot \left(\rho^{(1)}_{e,m} - \mathbf{v}^{(2\mapsto 1)} \cdot \mathbf{j}^{(1)}_{e,m}/c^2\right) \\
\tilde{\mathbf{V}}^{(2\mapsto 1)} = \tilde{\mathbf{V}}^{(1\mapsto 2)} &= \gamma^{(2\mapsto 1)}\tilde{\mathbf{I}} + (1 - \gamma^{(2\mapsto 1)})\hat{\mathbf{v}}^{(2\mapsto 1)}\hat{\mathbf{v}}^{(2\mapsto 1)}
\end{aligned} \tag{16}$$

Once again, to the first order in v/c, in (16) we take $\tilde{\mathbf{V}} = \tilde{\mathbf{I}}$, $\tilde{\mathbf{U}} = \tilde{\mathbf{I}}$, $\gamma = 1$. It is crucial to note that in (16) $\mathbf{E}^{(1)} = \mathbf{E}^{(1)}(\mathbf{X}^{(1)})$ and $\mathbf{E}^{(2)} = \mathbf{E}^{(2)}(\mathbf{X}^{(2)})$, etc., depending on their native space coordinates, and the Lorentz transformation (8) $\mathbf{X}^{(2)} = \mathbf{X}^{(2)}[\mathbf{X}^{(1)}]$ mediating between the coordinate systems. In summary — in the Special Relativity context we deal with two distinct but related spaces. In contradistinction, for the Lorentz force formulas given below, all events takes place in one coordinate system.

2. RELATIVISTIC BOUNDARY CONDITIONS

In order to introduce relativistic boundary conditions, we need first to consider a point on the boundary in its co-moving frame of reference, the frame where this point of the boundary is observed as being at rest. At this location (4), (5) apply. Obviously, right from the beginning this presents a problem when adjacent points on the boundary move at different velocities: inasmuch as (4), (5) are derived from integral considerations (by constructing the elementary "pillboxes" and "rectangles" at rest on the surface, as demonstrated in many a textbook), it is unclear to what extent (4), (5) hold when the boundary changes in time. This indicates that relativistically exact boundary conditions are adequate for a limited class of boundaries only. Moreover, strictly speaking, Special Relativity as defined above in terms of the covariance of (1), the Lorentz transformation (8), and the field transformations (16), applies only to constant velocities, otherwise the model breaks down. For example, if in (8) $\mathbf{v}^{(2\mapsto 1)}$ is not a constant in time, the inversion of $\mathbf{X}^{(2)} = \mathbf{X}^{(2)}[\mathbf{X}^{(1)}]$ as in (8) will not yield the same formulas for $\mathbf{X}^{(1)} = \mathbf{X}^{(1)}[\mathbf{X}^{(2)}]$, as claimed above.

Let us therefore assume that for the time being we are dealing with constant velocities. Consider two moving media separated by a boundary. The boundary's co-moving frame of reference is denoted by $\Gamma^{(b)}$, and the media are at rest in their respective frames of reference $\Gamma^{(1)}$, $\Gamma^{(2)}$, moving relative to the boundary with velocities $\mathbf{v}^{(1\mapsto b)}$, $\mathbf{v}^{(2\mapsto b)}$, respectively. It follows that conforming with (4), (5) we now have,

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{E}_1^{(b)} - \mathbf{E}_2^{(b)}\right) = 0 \tag{17}$$

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{H}_1^{(b)} - \mathbf{H}_2^{(b)}\right) = 0 \tag{18}$$

which according to (16) become

$$\hat{\mathbf{n}}^{(b)}\times\left(\tilde{\mathbf{V}}^{(b\mapsto 1)}\cdot(\mathbf{E}_1^{(1)}+\mathbf{v}^{(b\mapsto 1)}\times\mathbf{B}_1^{(1)})-\tilde{\mathbf{V}}^{(b\mapsto 2)}\cdot(\mathbf{E}_2^{(2)}+\mathbf{v}^{(b\mapsto 2)}\times\mathbf{B}_2^{(2)})\right)=0 \tag{19}$$

$$\hat{\mathbf{n}}^{(b)}\times\left(\tilde{\mathbf{V}}^{(b\mapsto 1)}\cdot(\mathbf{H}_1^{(1)}-\mathbf{v}^{(b\mapsto 1)}\times\mathbf{D}_1^{(1)})-\tilde{\mathbf{V}}^{(b\mapsto 2)}\cdot(\mathbf{H}_2^{(2)}-\mathbf{v}^{(b\mapsto 2)}\times\mathbf{D}_2^{(2)})\right)=0 \tag{20}$$

respectively. Problems of this kind have been discussed in the past, e.g., see [7], see also [6, 8] for many related references.

3. THE LORENTZ FORCE AND BOUNDARY CONDITIONS

To the first order in v/c, in (19), (20) the dyadics $\tilde{\mathbf{V}}^{(b\mapsto 1)} = \tilde{\mathbf{V}}^{(b\mapsto 2)} = \tilde{\mathbf{I}}$ and are therefore ignored, yielding respectively

$$\hat{\mathbf{n}}^{(b)}\times\left(\mathbf{E}_1^{(1)}+\mathbf{v}^{(b\mapsto 1)}\times\mathbf{B}_1^{(1)} - \mathbf{E}_2^{(2)} - \mathbf{v}^{(b\mapsto 2)}\times\mathbf{B}_2^{(2)}\right) = 0 \tag{21}$$

$$\hat{\mathbf{n}}^{(b)}\times\left(\mathbf{H}_1^{(1)}-\mathbf{v}^{(b\mapsto 1)}\times\mathbf{D}_1^{(1)} - \mathbf{H}_2^{(2)} + \mathbf{v}^{(b\mapsto 2)}\times\mathbf{D}_2^{(2)}\right) = 0 \tag{22}$$

At this point it is noted that the Lorentz force $\mathbf{f}_e^{(1)}$ acting on a charge (which is a relativistic invariant $q_e^{(1)} = q^{(2)} = q_e^{(b)}$) on the boundary, measured in $\Gamma^{(1)}$, *in terms of its native coordinates* $\mathbf{X}^{(1)}$, is given by

$$\mathbf{f}_e^{(1)} = q_e^{(b)}\left(\mathbf{E}^{(1)} + \mathbf{v}^{(b\mapsto 1)} \times \mathbf{B}^{(1)}\right) \tag{23}$$

The above remark that we deal in terms of one and the same coordinate system, i.e., $\mathbf{f}_e^{(1)}$, $\mathbf{E}^{(1)}$, $\mathbf{v}^{(b\mapsto 1)}$, $\mathbf{B}^{(1)}$ are all functions of $\mathbf{X}^{(1)}$, is crucial to the implementation of the Lorentz force formula and although some formulas are similar in form to the relativistic ones, the arguments are different.

It follows that if magnetic sources were existent, we would be able to measure magnetic Lorentz forces, and corresponding to (23), we would have

$$\mathbf{f}_m^{(1)} = q_m^{(b)} \left(\mathbf{H}^{(1)} - \mathbf{v}^{(b \mapsto 1)} \times \mathbf{D}^{(1)} \right) \tag{24}$$

Thus (21), (22) can be rewritten, respectively, in the form

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{f}_e^{(1)} - \mathbf{f}_e^{(2)} \right) = 0 \tag{25}$$

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{f}_m^{(1)} - \mathbf{f}_m^{(2)} \right) = 0 \tag{26}$$

The Lorentz force, and the relativistically exact boundary conditions, agree to the first order in v/c only. This indicates that one should use (25), (26) only for relatively moderate velocities. Most practical problems are of this nature. On the other hand, the Lorentz force formula does not assume a constant velocity, hence in (23), (24) $\mathbf{v}(\mathbf{X}^{(*)})$ may be any function of space and time.

A plethora of special cases of interest are offered by (19), (20) and the corresponding (21), (22). Of course the simplest case involves vanishing velocities, leading back to (10), (11). When the boundary is at rest with respect to medium $\Gamma^{(2)}$, (21), (22) become, respectively

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{E}_1^{(1)} + \mathbf{v}^{(b \mapsto 1)} \times \mathbf{B}_1^{(1)} - \mathbf{E}_2^{(2)} \right) = 0 \tag{27}$$

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{H}_1^{(1)} - \mathbf{v}^{(b \mapsto 1)} \times \mathbf{D}_1^{(1)} - \mathbf{H}_2^{(2)} \right) = 0 \tag{28}$$

This corresponds to a class of problems where objects, defined by the material they are composed of, and by the time-independent boundary surface specified in their co-moving frame, move through the external medium. Immediately the question of mechanical continuity arises: Rigorously speaking, the motion of the scatterer relative to the external medium causes some flow pattern which should be taken into account. Problems of this kind which combine the fluid-dynamic and electrodynamic effects are more difficult to solve, e.g., see [9, 10], and will not be considered here. Rather, the mechanical flow considerations will be ignored as done in [7].

The shortcoming of such an arbitrary model which ignores the fluid-dynamical interactions is obvious. However in many cases, and until more thorough solutions are available, it is considered as an acceptable approximation. This is evident in the geometry of the Fizeau experiment. For an historical background regarding the Fresnel drag coefficient and the related Fizeau experiment see for example

[6, 8]. In the classical Fizeau experiment light is passed through a fluid moving in a long tube, entering and leaving through transparent window interfaces. Of course, in the regions where the fluid is injected and drained, the fluid dynamical non-uniform flow affects the situation, but these effects have been always ignored. Problems of this kind would correspond to (21), (22) in the form

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{E}_1^{(1)} - \mathbf{E}_2^{(2)} - \mathbf{v}^{(b\mapsto 2)} \times \mathbf{B}_2^{(2)}\right) = 0 \quad (29)$$

$$\hat{\mathbf{n}}^{(b)} \times \left(\mathbf{H}_1^{(1)} - \mathbf{H}_2^{(2)} + \mathbf{v}^{(b\mapsto 2)} \times \mathbf{D}_2^{(2)}\right) = 0 \quad (30)$$

respectively, where $\mathbf{v}^{(b\mapsto 1)} = 0$ is assumed, and the velocity effects are introduced *via* $\mathbf{v}^{(b\mapsto 2)}$. Recent analytical work on such a situation is given by [11].

The rest of this study is devoted to demonstrate the feasibility of the present model. Two cylindrical problem are chosen, sufficiently complicated to show the power of the formalism. The solutions are outlined with sufficient detail to show the feasibility.

4. SCATTERING BY A LINEALLY MOVING CYLINDER

In the recent foray into the subject of scattering by moving boundaries based on the Lorentz force formulas [2], we analyzed the problem of scattering by a circular cylinder with a cross-sectional radius R, excited by a wave moving in free space perpendicularly to the cylindrical axis. As mentioned above, free space was assumed regardless whether we consider the scatterer in motion or at rest. This assumption is consistent with (15), whereby for $v_{ph}^{(1)} = c$ we obtain $v_{ph}^{(2)} = v_{ph}^{(1)} = c$. To the first order in v/c the results also agree with the relativistically exact solution [5, 12].

Presently the analysis is extended to the case of a cylinder moving through an arbitrary material medium. As explained above, the fluid-dynamical interaction of the external and internal media at the boundary is ignored.

The excitation plane wave in the external medium (indicated by "*1*") is given by

$$\begin{aligned}
&\mathbf{E}_{ex} = \hat{\mathbf{z}} E_{ex}, \quad \mathbf{H}_{ex} = -\hat{\mathbf{y}} H_{ex} \\
&E_{ex} = E_{ex0} e^{i\varphi_{ex}}, \quad H_{ex} = H_{ex0} e^{i\varphi_{ex}}, \quad \varphi_{ex} = k_{ex} x - \omega_{ex} t \\
&k_{ex}/\omega_{ex} = (\mu^{(1)}\varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)}, \quad E_{ex}/H_{ex} = (\mu^{(1)}/\varepsilon^{(1)})^{1/2} = \zeta^{(1)}
\end{aligned} \quad (31)$$

with $\mathbf{E}_{ex}$ polarized along the cylindrical z axis.

The cylinder moves according to $x = vt$, $\mathbf{v} = \hat{\mathbf{x}}v$ along the direction of propagation of the exciting wave $\hat{\mathbf{k}}_{ex}$. Accordingly we define a local coordinate system $\mathbf{r}_T$ denoted by index T in which the boundary is at rest

$$\begin{aligned} x_T &= x - vt = r\cos\theta - vt = r_T\cos\theta_T \\ y_T &= y = r\sin\theta = r_T\sin\theta_T \end{aligned} \tag{32}$$

A simple substitution of (32) into (31) amounts to a Galilean transformation and is avoided. However it is noted that in (13), to the first order in v/c, the frequency transforms identically for the Galilean and Lorentzian transformations. Therefore we consider the time signal obtained by substituting (32) into (31) for the local origin $r_T = 0$. Of course we have the freedom of shifting (32) to a different origin, which will introduce a constant factor in (32). Therefore choosing $r_T = 0$ as our reference, although this point is geometrically inside the cylinder and not accessible to the external waves, makes for a more symmetrical situation but is not mandatory. Thus we obtain

$$\begin{aligned} &\varphi_0 = \varphi_{ex}|_{r_T=0} = -\omega_T t \\ &\omega_T = \omega_{ex}(1-\beta^{(1)}), \qquad \beta^{(1)} = v/v_{ph}^{(1)} \end{aligned} \tag{33}$$

At any other location r_T we have to include an appropriate phase shift as prescribed by the first line of (13)

$$\varphi_T = k_T x_T - \omega_T t, \qquad k_T = k_{ex}\left(1 - \beta^{(1)}A^{(1)}\right) \tag{34}$$

which in free space $A = 1$ reduces to the results in [2]. Specifically, at the cylinder's rim we have a time-dependent signal $e^{i\varphi_T}$

$$\varphi_T = k_T R\cos\theta_T - \omega_T t \tag{35}$$

Observe that (34), (35) imply $\varphi_T \neq \varphi_{ex}$, unless we qualify t. This point does not invalidate the following analysis, and will be picked up again in Section 5.

In order to include the amplitude effect, (27), (28), (31) prescribe

$$\begin{aligned} &\mathbf{E}_{exT} = \mathbf{E}_{ex} + \mathbf{v}\times\mathbf{B}_{ex} = \mathbf{E}_{ex} + \mathbf{v}\times\mu^{(1)}\mathbf{H}_{ex} = \hat{\mathbf{z}}E_{exT} \\ &E_{exT} = E_{0T}e^{i\varphi_T}, \qquad E_{0T} = E_{ex0}(1-\beta^{(1)}) \end{aligned} \tag{36}$$

and similarly

$$\begin{aligned} &\mathbf{H}_{exT} = \mathbf{H}_{ex} - \mathbf{v}\times\mathbf{D}_{ex} = \mathbf{H}_{ex} - \mathbf{v}\times\varepsilon^{(1)}\mathbf{E}_{ex} = -\hat{\mathbf{y}}H_{exT} \\ &H_{exT} = H_{0T}e^{i\varphi_T} = E_{0T}e^{i\varphi_T}/\zeta^{(1)}, \qquad H_{0T} = H_{ex0}(1-\beta^{(1)}) \end{aligned} \tag{37}$$

In (36), (37) we are *not* dealing with a *wave*, rather with a time-dependent *signal* field measured at an arbitrary fixed point r_T, and the question of (36) and (37) representing a wave, satisfying a solution of the wave equation and dispersion relation does not arise.

Recasting (35)–(37) in a Fourier-Bessel integral yields

$$\begin{aligned} &\mathbf{E}_{exT} = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t}\Sigma, \quad \mathbf{H}_{exT} = -\hat{\mathbf{y}} E_{0T} e^{-i\omega_T t}\Sigma/\zeta^{(1)} \\ &\Sigma = \sum_{m=-\infty}^{\infty} i^m J_m(k_T R) e^{im\theta_T} \end{aligned} \tag{38}$$

with J_m denoting the non-singular Bessel functions. The medium inside the cylinder is at rest relative to the boundary. In a consistent manner, we define $\mathbf{E}_{in}$, the field inside the cylinder, in terms of the frequency prescribed by (33), and non-singular cylindrical functions

$$\begin{aligned} &\mathbf{E}_{in} = \hat{\mathbf{z}} E_{in} = \hat{\mathbf{z}} E_{0T} \sum_{m=-\infty}^{\infty} i^m B_m J_m(k_{in} r_T) e^{im\theta_T - i\omega_T t} \\ &k_{in}/\omega_T = (\mu^{(2)}\varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)} \end{aligned} \tag{39}$$

with index "*2*" characterizing the material in the internal domain. The computation of the coefficients B_m, subject to the boundary conditions, is carried out below. Note that in (39) we have normalized the coefficients B_m with respect to E_{0T}, the amplitude value attained by the excitation wave at the boundary. From (1) the associated field $\mathbf{H}_{in}$ is derived as

$$\mathbf{H}_{in} = \left(\hat{\mathbf{r}}_T r_T^{-1} \partial_{\theta_T} - \hat{\boldsymbol{\theta}}_T \partial_{r_T}\right) E_{in}/(i\omega_T \mu^{(2)}) \tag{40}$$

Similarly to (39), on the boundary we substitute in (39), (40) $r_T = R$. The boundary condition, as in (28) involves

$$\begin{aligned} \hat{\mathbf{r}}_T \times \mathbf{H}_{inT} &= -\hat{\mathbf{z}} k_{in} \partial_{k_{in}R} E_{inT}/(i\omega_T \mu^{(2)}) \\ &= \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t} \sum_{m=-\infty}^{\infty} i^{m+1} B_m J'_m(k_{in} R) e^{im\theta_T}/\zeta^{(2)} \\ E_{in}/H_{in} &= (\mu^{(2)}/\varepsilon^{(2)})^{1/2} = \zeta^{(2)} \end{aligned} \tag{41}$$

where the prime on J'_m denotes differentiation of the Bessel function with respect to its argument.

In order to evaluate the scattered field, we need to discuss plane waves, polarized along the cylindrical z axis and propagating in an arbitrary direction indicated by angle α with respect to the x axis

$$
\begin{aligned}
&\mathbf{E}_\alpha = \hat{\mathbf{z}} E_\alpha, \quad \mathbf{H}_\alpha = \hat{\mathbf{k}}_\alpha \times \hat{\mathbf{z}} H_\alpha, \quad E_\alpha = E_{\alpha 0} e^{i\varphi_\alpha}, \quad H_\alpha = H_{\alpha 0} e^{i\varphi_\alpha} \\
&\varphi_\alpha = \mathbf{k}_\alpha \cdot \mathbf{r} - \omega_\alpha t = k_\alpha r \cos(\theta - \alpha) - \omega_\alpha t = k_{\alpha,x} x + k_{\alpha,y} y - \omega_\alpha t \\
&k_{\alpha,x} = k_\alpha \cos\alpha, \quad k_{\alpha,y} = k_\alpha \sin\alpha, \quad k_\alpha/\omega_\alpha = (\mu^{(1)}\varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)}
\end{aligned}
\tag{42}
$$

which for $\alpha = 0$ reduce to (31). According to (32), and similarly to (33), the phase of (42) at $r_T = 0$ is given by

$$
\begin{aligned}
\varphi_{\alpha 0} &= \varphi_\alpha|_{r_T=0} = -\omega_{\alpha T} t \\
\omega_{\alpha T} &= \omega_\alpha (1 - \beta^{(1)} \cos\alpha)
\end{aligned}
\tag{43}
$$

which for $\alpha = 0$ reduce to (33).

The boundary conditions must be satisfied at all times. Consequently the scattered wave must be constructed in such a way that on the cylinder's rim the time dependence is identical to that of the incident wave there. Therefore we must satisfy

$$
\begin{aligned}
\omega_{\alpha T} &= \omega_T \\
\omega_\alpha &= \omega_{ex}(1 - \beta^{(1)})/(1 - \beta^{(1)} \cos\alpha) \\
&= \omega_{ex}\left(1 + \beta^{(1)}(\cos\alpha - 1)\right) + O\left(\beta^{(1)}\right)^2
\end{aligned}
\tag{44}
$$

where the last line (44) applies to first order in $\beta^{(1)}$, which should be consistently used due to our non-relativistic premises stated above. Incorporating (13), we get similarly to (34) and (35), to the first order in v/c

$$
\begin{aligned}
\varphi_{\alpha T} &= \mathbf{k}_{\alpha T} \cdot \mathbf{r}_T - \omega_{\alpha T} t \\
&= k_\alpha \left(\cos\alpha - \beta^{(1)} A^{(1)}\right) r_T \cos\theta_T + k_\alpha r_T \sin\alpha \sin\theta_T - \omega_T t \\
&= \mathbf{k}_\alpha \cdot \mathbf{r}_T - \omega_T t - \beta^{(1)} k_{ex} A^{(1)} r_T \cos\theta_T
\end{aligned}
\tag{45}
$$

Specifically at the cylinder's rim (45) applies with $r_T = R$. By substituting from (44) into (45) we thus get

$$
\begin{aligned}
\varphi_{\alpha T} &= K \hat{\mathbf{k}}_\alpha \cdot \hat{\mathbf{r}}_T - \omega_T t + \beta^{(1)} K B \\
B &= (C_\alpha - 1)(C_{\theta_T} C_\alpha + S_{\theta_T} S_\alpha) - C_{\theta_T} A^{(1)} \\
K &= k_{ex} R, \; C_\sigma = \cos\sigma, \; S_\sigma = \sin\sigma
\end{aligned}
\tag{46}
$$

where in (46) the term B explicitly displays the extra velocity-dependent terms. Once again note that (45), (46) imply $\varphi_{\alpha T} \neq \varphi_\alpha$, unless we qualify t.

Similarly to (36), (37), the amplitude effect is included, yielding

$$\begin{aligned}
&\mathbf{E}_{\alpha T} = \mathbf{E}_\alpha + \mathbf{v} \times \mathbf{B}_\alpha = \mathbf{E}_\alpha + \mathbf{v} \times \mu^{(1)}\mathbf{H}_\alpha = \hat{\mathbf{z}} E_{\alpha T} \\
&E_{\alpha T} = E_\alpha(1 - \beta^{(1)} C_\alpha) = E_{\alpha 0T} e^{i\varphi_{\alpha T}}, \quad E_{\alpha 0T} = E_{\alpha 0}(1 - \beta^{(1)} C_\alpha) \\
&\mathbf{H}_{\alpha T} = \mathbf{H}_\alpha - \mathbf{v} \times \mathbf{D}_\alpha = \mathbf{H}_\alpha - \mathbf{v} \times \hat{\mathbf{z}}\varepsilon^{(1)} E_\alpha = \mathbf{H}_\alpha + \hat{\mathbf{y}}\beta^{(1)} H_\alpha \\
&H_{\alpha T} = H_{\alpha 0T} e^{i\varphi_{\alpha T}} = E_{\alpha T}/\zeta^{(1)}, \quad H_{\alpha 0T} = H_{\alpha 0}(1 - C_\alpha \beta^{(1)})
\end{aligned} \tag{47}$$

For simplicity, we will first concentrate on the problem of the perfectly conducting cylinder, for which only the **E** fields matter. Inasmuch as all the **E** fields in our problem are polarized along the z axis, the problem becomes scalar and therefore less cumbersome.

The general approach towards the construction of the scattered wave is to assume a superposition (integral) of plane waves, as prescribed by (42)

$$\mathbf{E}_{sc} = \hat{\mathbf{z}} E_{0T} \frac{1}{\pi} \int_C e^{ik_\alpha r \cos(\theta - \alpha) - i\omega_\alpha t} G(\alpha) d\alpha \tag{48}$$

once again normalizing the amplitude with respect to E_{0T} as in (39). The weighting function $G(\alpha)$ depends on the angle of propagation α, and can be represented in terms of its Fourier series

$$G = G(\alpha) = \sum_{m=-\infty}^{\infty} A_m e^{im\alpha} \tag{49}$$

where for the limiting case $\beta^{(1)} = 0$, $G(\alpha)$ (49) reduces to

$$g = g(\alpha) = \sum_{m=-\infty}^{\infty} a_m e^{im\alpha} \tag{50}$$

It is noted that by judiciously shifting indices we can recast a Fourier series

$$e^{\mp in\alpha} \sum_{m=-\infty}^{\infty} a_m e^{im\alpha} = \sum_{m=-\infty}^{\infty} a_{m\pm n} e^{im\alpha} = g_{\pm n} \tag{51}$$

for finite integers n, where the notation $g_{\pm n}$, and $g_0 = g$ when there is no shift, will save us some tedious writing. The coefficients a_m in (50) are presumably known — obtained either analytically, or acquired experimentally. The contour C must yet be determined such that we get outgoing waves. This will be clarified below.

At the boundary $r_T = R$ all the plane waves in the integrand (48) possess the same time-dependence, according to (44), and at the cylinder's rim the integral in (48) becomes according to (46)

$$e^{-i\omega_T t}\frac{1}{\pi}\int_C e^{iK\cos(\theta_T-\alpha)+i\beta^{(1)}KB}G(\alpha)d\alpha \tag{52}$$

There is not much point in trying to acquire an exact representation of (52), e.g., by recasting exponentials in terms of series of Bessel functions. That would lead to hopelessly complicated expressions, and in any case any concrete computation will necessitate the truncation of the series. Usually $K = k_{ex}R$ is finite and not too large, otherwise we would treat the cylinder at the limiting case of small curvature, e.g., assume the surface as locally plane. Also $\beta^{(1)}$ is small for practical cases. Therefore only the first order in $K\beta^{(1)}$ will be retained, in the form

$$e^{-i\omega_T t}\frac{1}{\pi}\int_C e^{iK\cos(\theta_T-\alpha)}\left(G + i\beta^{(1)}KBg\right)d\alpha \tag{53}$$

where in (53) we have already used g, the zero order approximation, instead of G.

Taking into account the amplitude effect, (47), adds another factor $1-\beta^{(1)}\cos\alpha$ in the integrand, hence for the first order velocity effects we obtain

$$\begin{aligned}
&\mathbf{E}_{scT} = \hat{\mathbf{z}}E_{0T}e^{-i\omega_T t}\frac{1}{\pi}\int_C e^{iK\cos(\theta_T-\alpha)}\left(G+\beta^{(1)}gh\right)d\alpha\\
&h = iKB - C_\alpha = D_1 + D_2C_\alpha + D_3C_\alpha S_\alpha + D_4S_\alpha + D_5C_\alpha^2\\
&D_1 = iKC_{\theta_T}A^{(1)},\ D_2 = -(1+iKC_{\theta_T}),\ D_3 = iKS_{\theta_T},\\
&D_4 = -iKS_{\theta_T},\ D_5 = iKC_{\theta_T}
\end{aligned} \tag{54}$$

Nota bene in (54) that the terms $D_1 \ldots D_5$ are functions of θ_T and therefore do not participate in the integration. Raising and lowering indices according to (51), we recast the first order velocity effects in (54) as

$$\begin{aligned}
gh =& D_1g_0 + D_2(g_{+1}+g_{-1})/2 + iD_3(g_{+2}-g_{-2})/4\\
&+ iD_4(g_{+1}-g_{-1})/2 + D_5(g_{+2}+2g_0+g_{-2})/4
\end{aligned} \tag{55}$$

and we have to manipulate the integral (54), (55) to derive proper expressions for G.

For simplicity we shall first focus on the problem of scattering by a perfectly conducting cylinder. Substituting (55) back into (54) yields a sum of integrals, with the appropriate function $G(\alpha)$ or $g_{\pm n}(\alpha)$ in the integrand, the integrals being multiplied by the appropriate terms $D_1 \ldots D_5$ involving $S_{\theta_T}, C_{\theta_T}$ which are treated as coefficients and taken outside the integral sign.

Consequently we now choose C as the contour of integration appearing in the Sommerfeld integral for the Hankel functions (see for example Stratton [13]) of the first kind, $H_m^{(1)} = H_m$, which coupled with the time factor $e^{-i\omega_T t}$ produce outgoing waves

$$\int_C = \int_{\theta_T-(\pi/2)+i\infty}^{\theta_T+(\pi/2)-i\infty} \tag{56}$$

Accordingly (54), (55) is now recast in series of cylindrical functions in the form

$$\begin{aligned}\mathbf{E}_{scT} = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t} \sum_{m=-\infty}^{\infty} i^m H_m(K) e^{im\theta_T} \{A_m + \beta^{(1)} [D_1 a_m + D_2 (a_{m+1} + a_{m-1})/2 \\ + iD_3(a_{m+2} - a_{m-2})/4 + iD_4(a_{m+1} - a_{m-1})/2 + D_5(a_{m+2} + 2a_m + a_{m-2})]\}\end{aligned} \tag{57}$$

Our goal is to compute the coefficients A_m, but in order to achieve it, we need to recast (57) in a series orthogonal in $e^{im\theta_T}$. To that end we recast the trigonometric functions in $D_1 \ldots D_5$ as exponentials, and judiciously raise and lower indices in the various series. Thus we find

$$\begin{aligned}
&\mathbf{E}_{scT} = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t} \sum_{m=-\infty}^{\infty} i^m e^{im\theta_T} [A_m H_m(K) + \beta^{(1)} F_m(K)] \\
&F_m(K) = D_6 + D_7 + D_8 + D_9 + D_{10} \\
&D_6 = KA^{(1)}(a_{m+1}H_{m+1} - a_{m-1}H_{m-1})/2 \\
&D_7 = -H_m(a_{m+1} + a_{m-1})/2 + K[H_{m+1}(a_{m+2} + a_m) - H_{m-1}(a_m + a_{m-2})]/4 \\
&D_8 = K[H_{m-1}(a_{m+1} - a_{m-3}) + H_{m+1}(a_{m+3} - a_{m-1})]/8 \\
&D_9 = -K[H_{m+1}(a_{m+2} - a_m) + H_{m-1}(a_m - a_{m-2})]/4 \\
&D_{10} = K[H_{m-1}(a_{m+1} + 2a_{m-1} + a_{m-3}) - H_{m+1}(a_{m+3} + 2a_{m+1} + a_{m-1})]/8
\end{aligned} \tag{58}$$

In (58) all the terms H_m and $D_6 \ldots D_{10}$ are functions of $K = k_{ex}R$, and do not involve the angle θ_T. For the perfectly conducting cylinder the total field at the surface vanishes, i.e., in accordance with (27)

$$\hat{\mathbf{r}}_T \times (\mathbf{E}_{exT} + \mathbf{E}_{scT}) = 0 \tag{59}$$

Specifically, (38), (58) and the orthogonality with respect to $e^{im\theta_T}$ prescribe

$$A_m H_m(k_{ex}R) + \beta^{(1)} F_m(k_{ex}R) + J_m(k_T R) = 0 \tag{60}$$

which is an explicit family of equations for the needed coefficients A_m, in terms of coefficients already known from the velocity-independent solution. For $\beta^{(1)} = 0$ (60) is recognized as the equation obtained for a perfectly conducting cylinder at rest.

The boundary conditions for arbitrary media are prescribed by (27), (28), which for the present case become

$$\hat{\mathbf{r}}_T \times (\mathbf{E}_{exT} + \mathbf{E}_{scT} - \mathbf{E}_{inT}) = 0 \tag{61}$$

$$\hat{\mathbf{r}}_T \times (\mathbf{H}_{exT} + \mathbf{H}_{scT} - \mathbf{H}_{inT}) = 0 \tag{62}$$

For (61) all the ingredients are already available: instead of (60) we now need to include the internal field given by (39), yielding

$$A_m H_m(k_{ex}R) + \beta^{(1)} F_m(k_{ex}R) + J_m(k_T R) = B_m J_m(k_{in}R) \tag{63}$$

thus providing one family of equations for the unknowns A_m, B_m.

In order to evaluate (62), we start with (37) and derive

$$\begin{aligned} &\hat{\mathbf{r}}_T \times \mathbf{H}_{exT} = -\hat{\mathbf{r}}_T \times \hat{\mathbf{y}} E_{0T} e^{-i\omega_T t} \Sigma/\zeta^{(1)} \\ &= -\hat{\mathbf{z}} \cos\theta_T E_{0T} e^{-i\omega_T t} \Sigma/\zeta^{(1)} = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t} \Sigma'/\zeta^{(1)}, \\ &\Sigma' = -\cos\theta_T \Sigma = \sum_{m=-\infty}^{\infty} i^{m+1} J'_m(k_T R) e^{im\theta_T} \end{aligned} \tag{64}$$

For the internal field $\hat{\mathbf{r}}_T \times \mathbf{H}_{inT}$ is already specified by (41). For the scattered wave we start with (42), (47), now prescribing

$$\begin{aligned} &\hat{\mathbf{r}}_T \times \mathbf{H}_{\alpha T} = \hat{\mathbf{r}}_T \times \left(\mathbf{H}_\alpha + \hat{\mathbf{y}}\beta^{(1)} H_\alpha\right) = H_\alpha \hat{\mathbf{r}}_T \times \left(\hat{\mathbf{k}}_\alpha \times \hat{\mathbf{z}} + \hat{\mathbf{y}}\beta^{(1)}\right) \\ &= H_\alpha \hat{\mathbf{z}} p = E_\alpha \hat{\mathbf{z}} p/\zeta^{(1)} = \mathbf{E}_\alpha p/\zeta^{(1)} \\ &p = -\hat{\mathbf{r}}_T \cdot \hat{\mathbf{k}}_\alpha + \beta^{(1)} \left|\hat{\mathbf{r}}_T \times \hat{\mathbf{y}}_T\right| = -\cos(\theta_T - \alpha) + \beta^{(1)} \cos\theta_T \end{aligned} \tag{65}$$

The argument that started from the single wave (42), (47) leads us to (65), in terms of the associated $\mathbf{E}_\alpha$ field. The total scattered wave at the boundary was given by (54), therefore from (65) it follows that

$$\begin{aligned} &\hat{\mathbf{r}}_T \times \mathbf{H}_{SCT} = \hat{\mathbf{z}}(E_{0T}/\zeta^{(1)}) e^{-i\omega_T t} \frac{1}{\pi} \int_C e^{iK\cos(\theta_T - \alpha)} I d\alpha \\ &I = (G + \beta^{(1)} gh) p = -G\cos(\theta_T - \alpha) + \beta^{(1)} g(\cos\theta_T - h\cos(\theta_T - \alpha)) \end{aligned} \tag{66}$$

It is noted that in (66) the factor $-\cos(\theta_T - \alpha)$ is tantamount to applying a differentiation operator $i\partial_K$ to the exponential in the integral, and can be pulled outside the integral sign, yielding

$$\hat{\mathbf{r}}_T \times \mathbf{H}_{scT} = \hat{\mathbf{z}}(E_{0T}/\zeta^{(1)}) e^{-i\omega_T t} \sum_{m=-\infty}^{\infty} i^{m+1} e^{im\theta_T} (A_m H'_m + \beta^{(1)} I_m)$$
$$I_m(K) = F'_m + (a_{m+1} H_{m+1} - a_{m-1} H_{m-1})/2 \tag{67}$$

where in (67) the prime on H'_m, F'_m denotes differentiation with respect to the argument K.

We finally find an explicit expression for (62)

$$A_m H'_m(k_{ex}R) + \beta^{(1)} I_m(k_{ex}R) + J'_m(k_T R) = B_m(\zeta^{(1)}/\zeta^{(2)}) J'_m(k_{in}R) \tag{68}$$

Together, (63), (68) provide two explicit families of equations for the coefficients A_m, B_m, and thus the problem is considered as solved.

5. EVALUATION OF THE SCATTERED WAVE INTEGRAL

Thus far we have concentrated on the question of deriving appropriate equations for the scattering coefficients A_m. We need now to discuss the evaluation of the scattered wave integral (48), assuming that the scattering amplitude $G(\alpha)$ is by now a known function. We will consider two methods, suitable for various circumstances.

This goal is achieved by first effecting a coordinate transformation in (48) that will allow us to take the time factor outside the integral sign, as in (52). For $\varphi_{\alpha T} = \varphi_\alpha$ to the first order in v/c we need to define in (34), (45), (46) a modified time t_T instead of t

$$t_T = t - \mathbf{v} \cdot \mathbf{r}/c^2 \tag{69}$$

The definition (69) is recognized as the first order in v/c relativistic Lorentz transformation for the time. This also corrects a deficiency in [2], where this step was overlookedd.

Thus we obtain

$$\mathbf{E}_{sc}(\mathbf{r}, t) = \mathbf{E}_{sc}(\mathbf{r}_T, t_T) = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t_T} \frac{1}{\pi} \int_C e^{i\kappa\cos(\theta_T - \alpha) + i\beta^{(1)}\kappa B} G(\alpha) d\alpha$$
$$\kappa = k_{ex} r_T \tag{70}$$

where the contour for (70) is chosen as in (56).

Using the same technique employed above in (54) prescribes now

$$\mathbf{E}_{sc}(\mathbf{r}_T, t_T) = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t_T} \frac{1}{\pi} \int_C e^{i\kappa \cos(\theta_T - \alpha)} \left(G + i\beta^{(1)} \kappa B g \right) d\alpha \quad (71)$$

The situation in (71) is different from (54). First of all we have excluded the Lorentz force boundary considerations expressed in (54), which was expressed by $-C_\alpha$ in the term $h = iKB - C_\alpha$. Also note that this approximation holds only to the first order in $\beta^{(1)}\kappa$. If we attempt to express r_T in terms of r according to (32), it becomes clear that (71) holds only for short time intervals for which the scattered wave is measured in the vicinity of the scatterer. This is of course a severe limitation on the solution. Subject to this limitation (71) can be expressed as in (54), replacing K by κ and modifying D_2 to $D_2 = -i\kappa C_{\theta_T}$. Consequently (55) applies with this modification, and (57) follows. We could leave the modified (57) as the final solution, because there is no further need to exploit the orthogonality relation. However, it is easy to continue one step and finally recast (58) in the form

$$\begin{aligned}
&\mathbf{E}_{sc}(r_T, t_T) = \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t_T} \sum_{m=-\infty}^{\infty} i^m e^{im\theta_T} [A_m H_m(\kappa) + \beta^{(1)} L_m(\kappa)] \\
&L_m(\kappa) = D_6 + D_7 + D_8 + D_9 + D_{10} \\
&D_6 = \kappa A^{(1)} (a_{m+1} H_{m+1} - a_{m-1} H_{m-1})/2 \\
&D_7 = \kappa [H_{m+1}(a_{m+2} + a_m) - H_{m-1}(a_m + a_{m-2})]/4 \\
&D_8 = \kappa [H_{m-1}(a_{m+1} - a_{m-3}) + H_{m+1}(a_{m+3} - a_{m-1})]/8 \\
&D_9 = -\kappa [H_{m+1}(a_{m+2} - a_m) + H_{m-1}(a_m - a_{m-2})]/4 \\
&D_{10} = \kappa [H_{m-1}(a_{m+1} + 2a_{m-1} + a_{m-3}) - H_{m+1}(a_{m+3} + 2a_{m+1} + a_{m-1})]/8
\end{aligned} \quad (72)$$

where in (72) the coefficients A_m are by now already computed and known.

The result (72) is interesting because it shows the interaction of the different multipoles and the creation of new velocity-dependent multipole terms. This phenomenon was observed previously in the free space case [2, 5, 12].

Once (72) is available, the coordinates $\mathbf{r}$, t are substituted for $\mathbf{r}_T$, t_T using (32), (69). This completes the computation and provides for a solution in terms of the original coordinate system.

The shortcomings of the approximation (72) stems from the fact that the solution contains the distance factor κ. We now return to

(70) to consider the exact solution (although still to the first order in of v/c), expressed in terms of inverse powers of κ, and thus applicable to arbitrary intermediate and large distances. Combining the velocity-dependent term in the exponential with the scattering amplitude, we now rewrite (70) in the form

$$\begin{aligned}\mathbf{E}_{sc}(\mathbf{r}_T, t_T) &= \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t_T} \frac{1}{\pi} \int_C e^{i\kappa\cos(\theta_T - \alpha)} Q(\alpha,\theta_T,r_T) d\alpha \\ Q(\alpha, \theta_T, r_T) &= e^{i\beta^{(1)}\kappa B} G(\alpha) = \sum_{m=-\infty}^{\infty} q_m(\theta_T,r_T) e^{im\alpha}\end{aligned} \tag{73}$$

Note in (73) the dependence of $Q(\alpha, \theta_T, r_T)$ on θ_T, r_T. This does not affect the integration which is performed with respect to α only, therefore we allow varying coefficients $q_m(\theta_T, r_T)$.

We use a method which represents the solution in terms of a series of inverse distance powers and appropriate differential operators. This method has been devised by Twersky [14] as an asymptotic series, and was later presented as an exact series [15]. The number of terms retained before the series is truncated depends on the distance from the scatterer. Essentially, it is shown [14] that (74) can be represented asymptotically by

$$\begin{aligned}\mathbf{E}_{sc}(\mathbf{r}_T, t_T) &= \hat{\mathbf{z}} E_{0T} e^{-i\omega_T t_T} D_\alpha\{Q(\alpha, \theta_T, r_T)\} \\ D_\alpha\{Q\} &= H\left(1 + \frac{1 + 4\partial_\alpha^2}{i8\kappa} - \frac{9 + 40\partial_\alpha^2 + 16\partial_\alpha^4}{128\kappa^2} \cdots \right) Q(\alpha,\theta_T,r_T)\Bigg|_{\alpha=\theta_T} \\ &= H \sum_{\mu=0} \frac{(1+4\partial_\alpha^2)(9+4\partial_\alpha^2)\cdots([2\mu-1]^2+4\partial_\alpha^2)}{(i8\kappa)^\mu \mu!} Q(\alpha,\theta_T,r_T)\Bigg|_{\alpha=\theta_T} \\ H &= H(\kappa) = (2/(i\pi\kappa))^{1/2} e^{i\kappa}\end{aligned} \tag{74}$$

Although (74) is true to the original in [14], the present case is a little more subtle, since the expression $Q(\alpha,\theta_T,r_T)$ standing for the scattering amplitude already involves both α and θ_T. Thus $|_{\alpha=\theta_T}$ in (74) prescribes that we first perform the indicated differentiations with respect to α, and then substitute $\alpha = \theta_T$ in the results.

The exact series (in the sense of a convergent series, as opposed to (74) which is only asymptotically convergent) in [15] follow a similar scheme, and the differential operators are based on the Lommel polynomials [16], and instead of the asymptotic form $H(\kappa)$, the functions H_0, H_1 feature in the analog of (74).

According to (74), the leading term for the asymptotic approximation is

$$\begin{aligned}\mathbf{E}_{sc}(\mathbf{r}_T, t_T) &= \hat{\mathbf{z}} E_{0T} e^{i\kappa + i\beta^{(1)}\kappa B - i\omega_T t_T} \left(2/(i\pi\kappa)\right)^{1/2} G(\theta_T) \\ B &= C_{\theta_T}(1 - A^{(1)}) - 1\end{aligned} \tag{75}$$

which also follows from the steepest descent or the saddle point approximations for integrals with a complex exponential kernel. Accordingly, as $k_{ex} r_T$ becomes large, the main contribution to the integral's value is due to angles α in the vicinity of θ_T, while at other angles the phasors described by the exponential change rapidly and tend to mutually cancel. These are the same ideas leading to the Cornu spiral theory for knife edge diffraction (Fresnel diffraction at a straight edge), see for example [17].

Once again, after deriving $\mathbf{E}_{sc}(\mathbf{r}_T, t_T)$ from (73)–(75), (32), (69) are exploited in order to finally derive $\mathbf{E}_{sc}(\mathbf{r}_T, t)$ in terms of the original coordinates. Thus the problem can finally be considered as solved.

6. SCATTERING BY A MEDIUM IN MOTION

In a sense, the subsequent problem is complimentary to that of the moving cylinder analyzed above. It also constitutes a new version of the Fizeau experiment (e.g., see [6, 8]) considered here for motion of a fluid within a cylinder. Most of the ingredients for the solution we seek are already outlined by the formulas given above. Here we consider a cylindrical boundary at rest with respect to the external medium "*1*". Observed from the boundary at rest in "*1*", the interior medium "*2*" moves according to the velocity $x = -vt$, $\mathbf{v} = -\hat{\mathbf{x}} v$, i.e., in the opposite direction of $\hat{\mathbf{k}}_{ex}$ given in (31). This means that from medium "*2*", in terms of its local rest coordinate system ξ, η, the boundary is observed to move according to $\xi = vt$, $\mathbf{v} = \hat{\boldsymbol{\xi}} v = \hat{\mathbf{x}} v$. Subject to the new configuration, instead of (32) we now have

$$\begin{aligned}\xi &= x + vt = r\cos\theta + vt = \rho\cos\tau \\ \eta &= y = r\sin\theta = \rho\sin\tau\end{aligned} \tag{76}$$

where in (76) $x = x_T$, $y = y_T$ are the coordinates both for the incident and scattered wave, and for the boundary at rest, while ξ, η are the coordinates for the internal medium "*2*" at rest.

The relevant boundary conditions are given by (21), (22) with $\mathbf{v}^{(b \mapsto 1)} = 0$. Consequently for the observer in medium "*1*", at rest with respect to the boundary, no Doppler frequency shifts occur.

At the boundary the tangential fields of the excitation wave (31) prescribes, instead of (38), (64)

$$\mathbf{E}_{exT} = \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex} t} \overline{\Sigma}, \quad \hat{\mathbf{r}}_T \times \mathbf{H}_{exT} = \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex} t} \overline{\Sigma}' / \zeta^{(1)}$$
$$\overline{\Sigma} = \sum_{m=-\infty}^{\infty} i^m J_m(K) e^{im\theta}, \quad \overline{\Sigma}' = \sum_{m=-\infty}^{\infty} i^{m+1} J'_m(K) e^{im\theta}, \quad K = k_{ex} R \tag{77}$$

Note that once again we suppress the subscript T on t_T.

At the boundary, the tangential field components of the scattered wave are given by the velocity-independent expressions

$$\begin{aligned} \mathbf{E}_{scT} &= \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex} t} \sum_{m=-\infty}^{\infty} i^m A_m H_m(K) e^{im\theta} \\ \hat{\mathbf{r}}_T \times \mathbf{H}_{scT} &= \hat{\mathbf{z}} (E_{ex0}/\zeta^{(1)}) e^{-i\omega_{ex} t} \sum_{m=-\infty}^{\infty} i^{m+1} A_m H'_m(K) e^{im\theta} \end{aligned} \tag{78}$$

Thus far all the terms relevant to the external medium "*1*" were presented. In the interior domain "*2*" one has to consider the medium's motional effects too. We start with a plane wave propagating in an arbitrary direction α as in (42), with the appropriate modifications

$$\begin{gathered} \mathbf{E}_\alpha = \hat{\mathbf{z}} E_\alpha, \quad \mathbf{H}_\alpha = \hat{\mathbf{k}}_\alpha \times \hat{\mathbf{z}} H_\alpha, \quad E_\alpha = E_{\alpha 0} e^{i\varphi_\alpha}, \quad H_\alpha = H_{\alpha 0} e^{i\varphi_\alpha} \\ \varphi_\alpha = \mathbf{k}_\alpha \cdot \boldsymbol{\rho} - \omega_\alpha t = k_\alpha \rho \cos(\tau - \alpha) - \omega_\alpha t = k_{\alpha,\xi} \xi + k_{\alpha,\eta} \eta - \omega_\alpha t \\ k_{\alpha,\xi} = k_\alpha \cos\alpha, \quad k_{\alpha,\eta} = k_\alpha \sin\alpha, \quad k_\alpha/\omega_\alpha = (\mu^{(2)} \varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)} \end{gathered} \tag{79}$$

As in (43), (44), we start by using (76) to evaluate the phase of the wave (79) at the $r = 0$. The associated frequency must be identical to ω_{ex} of the incident and scattered waves

$$\begin{aligned} \varphi_\alpha |_{r=0} &= -\omega_{\alpha T} t, \; \omega_{\alpha T} = \omega_\alpha (1 - \beta^{(2)} \cos\alpha) = \omega_{ex} \\ \omega_\alpha &= \omega_{ex} (1 + \beta^{(2)} \cos\alpha) + O(\beta^{(2)})^2, \; \beta^{(2)} = v/v_{ph}^{(2)} \end{aligned} \tag{80}$$

Using (13) we obtain for the phase of the wave (79), similarly to (45)

$$\begin{aligned} \varphi_{\alpha T} &= \mathbf{k}_{\alpha T} \cdot \mathbf{r} - \omega_{\alpha T} t \\ &= k_\alpha r \left(\cos\alpha - \beta^{(2)} A^{(2)} \right) \cos\theta + k_\alpha r \sin\alpha \sin\theta - \omega_{ex} t \\ &= \mathbf{k}_\alpha \cdot \mathbf{r} - \omega_{ex} t - \beta^{(2)} k_{ex} A^{(2)} r \cos\theta, \; A^{(2)} = (v_{ph}^{(2)}/c)^2 \end{aligned} \tag{81}$$

to the first order in $\beta^{(2)}$, and therefore, at $r = R$ we obtain for the phase of the wave (79) similarly to (46)

$$\begin{aligned} \varphi_{\alpha T} &= \overline{K}\hat{\mathbf{k}}_\alpha \cdot \hat{\mathbf{r}} - \omega_{ex}t + \beta^{(2)}\overline{KB} \\ \overline{B} &= C_\alpha(C_\theta C_\alpha + S_\theta S_\alpha) - C_\theta A^{(2)},\ \overline{K} = \overline{k}R,\ \overline{k} = \omega_{ex}/v_{ph}^{(2)} \end{aligned} \tag{82}$$

the formal difference being the factor C_α in (82) instead of $(C_\alpha - 1)$ in (46).

Similarly to (47), the amplitude effect is included, yielding

$$\begin{aligned} \mathbf{E}_{\alpha T} &= \mathbf{E}_\alpha + \mathbf{v} \times \mathbf{B}_\alpha = \mathbf{E}_\alpha + \mathbf{v} \times \mu^{(2)}\mathbf{H}_\alpha = \hat{\mathbf{z}}E_{\alpha T} \\ E_{\alpha T} &= E_\alpha(1 - \beta^{(2)}C_\alpha) = E_{\alpha T0}e^{i\varphi_{\alpha T}},\ E_{\alpha T0} = E_{\alpha 0}(1 - \beta^{(2)}C_\alpha) \\ \mathbf{H}_{\alpha T} &= \mathbf{H}_\alpha - \mathbf{v} \times \mathbf{D}_\alpha = \mathbf{H}_\alpha - \mathbf{v} \times \hat{\mathbf{z}}\varepsilon^{(2)}E_\alpha = \mathbf{H}_\alpha + \hat{\mathbf{y}}\beta^{(2)}H_\alpha \\ H_{\alpha T} &= H_{\alpha T0}e^{i\varphi_{\alpha T}} = E_{\alpha T}/\zeta^{(2)}, \quad H_{\alpha T0} = H_{\alpha 0}(1 - \beta^{(2)}C_\alpha) \end{aligned} \tag{83}$$

Mimicking (48), the internal field is constructed as a superposition of waves. The contour should be chosen such that nonsingular cylindrical functions J_m be involved. The Sommerfeld contour integral for the nonsingular Bessel functions is therefore anticipated and we write

$$\begin{aligned} \mathbf{E}_{in} &= \hat{\mathbf{z}}E_{ex0}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik_\alpha\rho\cos(\tau-\alpha) - i\omega_\alpha t}\overline{G}d\alpha \\ \overline{G} &= \overline{G}(\alpha) = \sum_{m=-\infty}^{\infty} B_m e^{im\alpha} \end{aligned} \tag{84}$$

where like in (50) for the limiting case $\beta^{(2)} = 0$, $\overline{G}(\alpha)$ reduces to

$$\overline{g} = \overline{g}(\alpha) = \sum_{m=-\infty}^{\infty} b_m e^{im\alpha} \tag{85}$$

and the coefficients b_m (85) are presumably known, as were a_m in (50).

As in (52), (53), by substituting from (82), the integral in (84) becomes at the boundary

$$\begin{aligned} \mathbf{E}_{in} &= \hat{\mathbf{z}}E_{ex0}e^{-i\omega_{ex}t}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\overline{K}\cos(\theta-\alpha) + i\beta^{(2)}\overline{KB}}\overline{G}d\alpha \\ &= \hat{\mathbf{z}}E_{ex0}e^{-i\omega_{ex}t}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\overline{K}\cos(\theta-\alpha)}(\overline{G} + i\beta^{(2)}\overline{KB}\overline{g})d\alpha + O(\beta^{(2)})^2 \end{aligned} \tag{86}$$

Like in (54), the amplitude effect expressed in (83) is included, yielding

$$\begin{aligned}\mathbf{E}_{inT} &= \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex}t} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\overline{K}\cos(\theta-\alpha)} (\overline{G} + \beta^{(2)} \overline{g}\overline{h}) d\alpha \\ \overline{h} &= i\overline{KB} - C_\alpha = \overline{D}_1 + \overline{D}_2 C_\alpha + \overline{D}_3 C_\alpha S_\alpha + \overline{D}_5 C_\alpha^2 \\ \overline{D}_1 &= -i\overline{K} C_{\theta_T} A^{(2)},\ \overline{D}_2 = -1,\ \overline{D}_3 = i\overline{K} S_{\theta_T},\ \overline{D}_5 = i\overline{K} C_{\theta_T}\end{aligned} \tag{87}$$

where in (87) we have retained the enumeration of indices, in spite of the redundant $\overline{D}_4 = 0$, in order to preserve the similarity to (54). Similarly to (55), we now have

$$\begin{aligned}\overline{g}\overline{h} =& \overline{D}_1 \overline{g}_0 + \overline{D}_2 (\overline{g}_{+1} + \overline{g}_{-1})/2 \\ &+ i\overline{D}_3 (\overline{g}_{+2} - \overline{g}_{-2})/4 + \overline{D}_5 (\overline{g}_{+2} + 2\overline{g}_0 + \overline{g}_{-2})/4\end{aligned} \tag{88}$$

and as in (57), when considered within the integral (87), the expression (88) prescribes

$$\begin{aligned}\mathbf{E}_{inT} =& \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex}t} \sum_{m=-\infty}^{\infty} i^m J_m(\overline{K}) e^{im\theta} \{ B_m + \beta^{(2)} [\overline{D}_1 b_m \\ &+ \overline{D}_2 (b_{m+1} + b_{m-1})/2 + i\overline{D}_3 (b_{m+2} - b_{m-2})/4 \\ &+ \overline{D}_5 (b_{m+2} + ba_m + b_{m-2})/4] \}\end{aligned} \tag{89}$$

Similarly to (58), we find for the present case

$$\begin{aligned}&\mathbf{E}_{inT} = \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex}t} \sum_{m=-\infty}^{\infty} i^m e^{im\theta} [B_m J_m(\overline{K}) + \beta^{(2)} \overline{F}_m(\overline{K})] \\ &\overline{F}_m(\overline{K}) = \overline{D}_6 + \overline{D}_7 + \overline{D}_8 + \overline{D}_{10} \\ &\overline{D}_6 = \overline{K} A^{(2)} (b_{m+1} J_{m+1} - b_{m-1} J_{m-1})/2 \\ &\overline{D}_7 = -J_m (b_{m+1} + b_{m-1})/2 \\ &\overline{D}_8 = \overline{K} [J_{m-1} (b_{m+1} - b_{m-3}) + J_{m+1} (b_{m+3} - b_{m-1})]/8 \\ &\overline{D}_{10} = \overline{K} [J_{m-1} (b_{m+1} + 2b_{m-1} + b_{m-3}) - J_{m+1} (b_{m+3} + 2b_{m+1} + b_{m-1})]/8\end{aligned} \tag{90}$$

where in (90) $J_m = J_m(\overline{K})$ etc., and we have once again retained the redundant enumeration of indices, in spite of $\overline{D}_9 = 0$, in order to point out the similarity.

The implementation of (61) led to (63). Similarly, (77), (78), (90), yield

$$
\begin{aligned}
&A_m H_m(K) + J_m(K) = B_m J_m(\overline{K}) + \beta^{(2)} \overline{F}_m(\overline{K}) \\
&K = k_{ex} R,\ \overline{K} = \overline{k} R,\ \overline{k} = \omega_{ex}/v_{ph}^{(2)}
\end{aligned}
\tag{91}
$$

providing one family of equations for the coefficients A_m, B_m.

The magnetic field boundary condition is prescribed by (62). For vanishing velocity in the external domain, (64) becomes

$$
\begin{aligned}
&\hat{\mathbf{r}} \times \mathbf{H}_{exT} = -\hat{\mathbf{r}} \times \hat{\mathbf{y}} E_{ex0} e^{-i\omega_{ex} t} \Sigma/\zeta^{(1)} = \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex} t} i \Sigma'/\zeta^{(1)} \\
&\Sigma' = \sum_{m=-\infty}^{\infty} i^{m+1} J'_m(K) e^{im\theta}
\end{aligned}
\tag{92}
$$

Similarly for the scattered wave, (67) prescribes, for the external medium at rest with respect to the boundary

$$
\hat{\mathbf{r}} \times \mathbf{H}_{scT} = \hat{\mathbf{z}} E_{ex0} e^{-i\omega_{ex} t} \sum_{m=-\infty}^{\infty} i^{m+1} A_m H'_m(K) e^{im\theta}/\zeta^{(1)} \tag{93}
$$

For the internal field we start with the analog of (65) for the present case

$$
\begin{aligned}
\hat{\mathbf{r}} \times \mathbf{H}_{\alpha T} &= \hat{\mathbf{r}} \times \left(\mathbf{H}_\alpha + \hat{\mathbf{y}} \beta^{(2)} H_\alpha\right) = H_\alpha \hat{\mathbf{r}} \times \left(\hat{\mathbf{k}}_\alpha \times \hat{\mathbf{z}} + \hat{\mathbf{y}} \beta^{(2)}\right) \\
&= H_\alpha \hat{\mathbf{z}} \overline{p} = E_\alpha \hat{\mathbf{z}} \overline{p}/\zeta^{(2)} = \mathbf{E}_\alpha \overline{p}/\zeta^{(2)} \\
\overline{p} &= -\hat{\mathbf{r}} \cdot \hat{\mathbf{k}}_\alpha + \beta^{(2)} |\hat{\mathbf{r}} \times \hat{\mathbf{y}}| = -\cos(\theta - \alpha) + \beta^{(2)} \cos\theta
\end{aligned}
\tag{94}
$$

and similarly to (66)

$$
\begin{aligned}
&\hat{\mathbf{r}} \times \mathbf{H}_{inT} = \hat{\mathbf{z}} (E_{ex0}/\zeta^{(2)}) e^{-i\omega_{ex} t} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\overline{K} \cos(\theta - \alpha)} \overline{I} d\alpha \\
&\overline{I} = (\overline{G} + \beta^{(2)} \overline{g} \overline{h}) \overline{p} = -\overline{G} \cos(\theta - \alpha) + \beta^{(2)} \overline{g} (\cos\theta - \overline{h} \cos(\theta - \alpha))
\end{aligned}
\tag{95}
$$

Once again note in (95) the factor $-\cos(\theta - \alpha)$, corresponding to differential operator $i\partial_{\overline{K}}$ applied to the exponential in the integral, yielding as in (67)

$$
\begin{aligned}
&\hat{\mathbf{r}} \times \mathbf{H}_{inT} = \hat{\mathbf{z}} \left(E_{ex0}/\zeta^{(2)}\right) e^{-i\omega_{ex} t} \sum_{m=-\infty}^{\infty} i^{m+1} e^{im\theta} \left(B_m J'_m(\overline{K}) + \beta^{(2)} \overline{I}_m(\overline{K})\right) \\
&\overline{I}_m(\overline{K}) = \overline{F}'_m + (b_{m+1} J_{m+1} - b_{m-1} J_{m-1})/2
\end{aligned}
\tag{96}
$$

Finally, corresponding to (68), we get now

$$A_m H'_m(K) + J'_m(K) = (\zeta^{(1)}/\zeta^{(2)}) \left(B_m J'_m(\overline{K}) + \beta^{(2)} \overline{I}_m(\overline{K}) \right) \quad (97)$$

together with (91) providing the second family of equations needed to solve for A_m, B_m.

Inasmuch as the goal here is to find the scattered field in the external domain, there is no need to derive the full solution of the internal wave field, as prescribed by (84), the way it has been done for the moving cylinder in (71), (72), although this can be done. Unlike (72), here the interior of the cylinder is a finite domain, and the approximation to the first order of $k_{ex} r \beta^{(2)}$ is valid. With that the problem is considered to be solved.

7. SUMMARY AND CONCLUDING REMARKS

Time-dependent boundary conditions are encountered when moving objects and/or moving media are present. Unlike other branches of physics, e.g., elastodynamics, in electrodynamics it is not only a question of applying the time dependence to such boundary conditions — they first of all must be properly defined, as we did above. One approach of dealing with the problem was by avoiding it: Having at our disposal Einstein's Special Relativity theory, one can use what Van Bladel [6] nicknamed "frame hopping". Accordingly one transforms the problem to the co-moving frame of reference of the boundary at rest, solves the problem, then transforms the results back into the original "laboratory frame of reference". Direct solutions involving time-dependent media are also referenced in [6, 8], for example when the media are moving parallel to the boundaries, such that the Minkowski's constitutive relations can be implemented e.g., see [1, 5]. See also [2] where the equivalence of Minkowski's constitutive relations and the boundary conditions based on the Lorentz force formulas is discussed.

Armed with the arsenal of previous results, a direct approach based on the Lorentz force formulas was recently suggested, and some problems involving objects moving in free space have been discussed [2]. To the first order in v/c the two methods are in agreement.

Presently the feasibility of implementing the formalism to problems involving arbitrary media and boundaries in motion was examined. In order to present concrete and explicit results, two cylindrical problems have been chosen, which turn out to be complementary in many respects: the moving cylinder, and the cylindrically shaped moving medium. In order to demonstrate the feasibility, the computations are worked out in great detail.

Not surprisingly, problems of this kind give rise to mode interaction, whereby the solution for coefficients of some order m involves velocity-independent coefficients of other orders, e.g., $m \pm 1, m \pm 2$, etc. Thus for example, a moving thin cylinder which at rest exhibits a monopole pattern for the scattered field, will in the presence of motion involve velocity-dependent higher order monopoles. This phenomenon has been observed before [5, 12].

Future work will have to deal with various configurations and evaluate appropriate solutions analytically, and also provide simulations. For example, objects whose shapes are perturbed by periodic mechanical motion seem to be of interest for practical problems involving remote sensing of motion and vibration, from rotating and vibrating objects in the mechanical workshop to the problem acquisition of signatures of airplanes, helicopters, and various missiles, who have such motional elements. The present formalism will provide a consistent approach based on first principles.

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