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Multiple Scattering of Electromagnetic Waves by Arbitrary Configurations

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This paper extends to three-dimensional vector electromagnetic scattering problems our previous development of the scalar problems. We introduce a vector-dyadic formalism that facilitates exploiting the previous results, and derive analogous integral equations which specify the multiple-scattering amplitudes for many objects in terms of the corresponding functions for isolated scatterers. One representation is in terms of the dyadic analog of Beltrami's operator. For arbitrary configurations, the multi-scattered amplitudes are developed as series in inverse powers of the separations of scatterers (with coefficients in terms of isolated scatterer amplitudes and their derivatives); for two scatterers, we derive a corresponding closed form in terms of a differential operator. Another representation is a system of algebraic equations for the many-body multipole coefficients in terms of the isolated scatterer values. Explicit closed forms are derived for two arbitrarily spaced elementary scatterers (electric dipoles, magnetic dipoles, etc.) both by separations of variables, and by working with elementary dyadic fields.

1. INTRODUCTION

IN previous papers[1-3] we considered the two- and three-dimensional scalar problems of multiple scattering of waves by arbitrary configurations of arbitrary scatterers. In the present paper, the results are extended to the three-dimensional electromagnetic case. We parallel our previous analysis of the three-dimensional scalar case,[3] and exploit as much of that development as feasible; similarly, because recent surveys of the literature of scattering by more than one object are available[4,5] we restrict citations to explicitly related work. Conventional integral and series scalar–vector representations which are adequate (although not particularly convenient) for isolated-scatterer problems, are too cumbersome for multiple scattering problems. We therefore work with vector-dyadic representations essentially as in Morse and Feshbach,[6] and in Saxon[7,8]; we supplement these with

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[1] V. Twersky, in *Electromagnetic Waves*, R. E. Langer, Ed. (University of Wisconsin Press, Madison, Wisc., 1962), pp. 361–389.

[2] J. E. Burke, D. Censor, and V. Twersky, J. Acoust. Soc. Am. 37, 5 (1965).

[3] V. Twersky, J. Math. Phys. 3, 83 (1962).

[4] J. E. Burke and V. Twersky, Radio Sci. **68D**, 500 (1964).

[5] V. Twersky, J. Res. Natl. Bur. Std. **64D**, 715 (1960).

[6] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), particularly p. 1897 and Chap. 13.

[7] D. S. Saxon, "Scattering of Light," Scientific Report No. 9, Department of Meteorology, UCLA (1955).

[8] D. S. Saxon, Phys. Rev. 100, 1771 (1955).

results of Hansen,[9] Stratton,[10] Silver,[11] and Wilcox,[12] as well as with additional representations and theorems derived in the course of the present development (e.g., by separating variables in the vector wave equation). We use dyadic surface integral forms, complex integral dyadic plane-wave representations, inverse distance series involving the vector scattering amplitudes acted on by the dyadic analog of Beltrami's operator, series for dyadic fields in terms of dyads of vector harmonics, etc. To facilitate discussion we start with a relatively conventional vector formalism, and then switch to dyadic representations.

In the following we always indicate dyadics by using a tilde—$\tilde{g}$, $\tilde{u}$, $\tilde{\varphi}$, etc., and write vectors as $\mathbf{g}$, $\mathbf{u}$, $\boldsymbol{\varphi}$, etc.; a caret always indicates a unit vector—$\hat{g}$, $\hat{x}$, $\hat{\theta}$, etc., but we also define some special symbols ($\mathbf{o}$, $\mathbf{i}$, $\boldsymbol{\epsilon}$, $\mathbf{n}$, etc.) to represent unit vectors. For brevity, we regard the numbered equations and figures of Ref. 3 as part of the present text, and cite them as Eq. (3:8), Fig. 3:1, etc.

2. ONE SCATTERER

2.1. Vector Fields

The three-dimensional scattering of a plane electromagnetic wave (with $e^{-i\omega t}$ suppressed) is specified in the external region by a solution of

$$\nabla \times \nabla \times \boldsymbol{\psi} - k^2\boldsymbol{\psi} = 0, \quad \nabla \cdot \boldsymbol{\psi} = 0, \quad k = |k| = 2\pi/\lambda, \tag{1}$$

subject to prescribed conditions on the scatterer's surface, and subject to the condition that $\boldsymbol{\psi}$ consist of a plane wave $\boldsymbol{\varphi}$ plus a radiated wave $\mathbf{u}$. With increasing distance from the scatterer ($r \to \infty$) the function $\boldsymbol{\psi}$ (which represents either the $\mathbf{E}$ or $\mathbf{H}$ field) reduces to a plane wave

$$\boldsymbol{\varphi}(\mathbf{i}: \boldsymbol{\epsilon}) = \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} = k\mathbf{i}, \quad \mathbf{r} = r\mathbf{o}, \tag{2}$$

where $\boldsymbol{\epsilon}$, $\mathbf{i}$, and $\mathbf{o}$ are unit vectors. Because of the divergence condition $\nabla \cdot \boldsymbol{\varphi} = 0$, the "polarization vector" $\boldsymbol{\epsilon}$ is perpendicular to the direction of incidence, $\boldsymbol{\epsilon} \cdot \mathbf{i} = 0$; to make this explicit, we write

$$\boldsymbol{\varphi}(\mathbf{i}: \boldsymbol{\epsilon}) = \boldsymbol{\epsilon} \cdot (\tilde{I} - \mathbf{ii})e^{i\mathbf{k}\cdot\mathbf{r}} = \boldsymbol{\epsilon} \cdot \tilde{\varphi}(\mathbf{i}), \quad \tilde{\varphi}(\mathbf{i}) \equiv (\tilde{I} - \mathbf{ii})e^{ik\mathbf{i}\cdot\mathbf{r}}, \tag{3}$$

where $\tilde{I}$ is the unit dyadic, and $\tilde{\varphi}$ is a dyadic plane wave. The difference $\boldsymbol{\psi} - \boldsymbol{\varphi} = \mathbf{u}$, the scattered wave, may be specified by the Sommerfeld–Silver radiation condition[11,12]

$$\lim r[\mathbf{o} \times (\nabla \times \mathbf{u}) + ik\mathbf{u}] = 0, \quad \text{as } r \to \infty. \tag{4}$$

For concreteness, we may take the origin of coordinates of $\mathbf{r}$ as the center of the smallest sphere which completely encloses the scatterer; we use the same geometry as in Fig. 3:1.

From (4) and Green's theorem it follows[12] that for $r \sim \infty$,

$$\mathbf{u} \sim \mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon})h(kr), \quad h(r) \equiv h_0^{(1)}(r) = e^{ir}/ir, \tag{5}$$

where the normalized "scattering amplitude" $\mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon})$ specifies the "far-field" response in the direction of observation $\mathbf{o}$ to plane-wave excitation of direction of incidence $\mathbf{i}$ and polarization $\boldsymbol{\epsilon}$. Since $\nabla \cdot \mathbf{u} = 0$, we have $\mathbf{o} \cdot \hat{g} = 0$, and we may write

$$\mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon}) = (\tilde{I} - \mathbf{oo}) \cdot \mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon}).$$

In general, we take $\boldsymbol{\psi} = \mathbf{E}$, and $\nabla \times \boldsymbol{\psi} = \mathbf{H}i\omega\mu_0 = \mathbf{H}i\omega$. At the surface of a perfect conductor,

$$\mathbf{n} \times \boldsymbol{\psi} = \mathbf{n} \times (\boldsymbol{\varphi} + \mathbf{u}) = 0, \tag{6}$$

where $\mathbf{n}$ is the surface normal. For a scatterer specified by relative electrical constants ϵ and μ we introduce the internal field $\boldsymbol{\psi}'$ such that

$$\nabla \times \nabla \times \boldsymbol{\psi}' - k'^2\boldsymbol{\psi}' = 0, \quad \nabla \cdot \boldsymbol{\psi}' = 0, \quad k' = k(\epsilon\mu)^{\frac{1}{2}}, \tag{7}$$

and use the surface conditions

$$\mathbf{n} \times \boldsymbol{\psi} = \mathbf{n} \times \boldsymbol{\psi}', \quad \mathbf{n} \times (\nabla \times \boldsymbol{\psi}) = \mathbf{n} \times (\nabla \times \boldsymbol{\psi}'/\mu). \tag{8}$$

Surface integral representation: Introducing the free-space dyadic Green's function[13,7,6]

$$\tilde{\Gamma}(\mathbf{r}, \mathbf{r}') = \left(\tilde{I} + \frac{\nabla\nabla}{k^2}\right) \frac{kh(k\,|\mathbf{r} - \mathbf{r}'|)}{4\pi i}, \quad \nabla \times \nabla \times \tilde{\Gamma} - k^2\tilde{\Gamma} = -\tilde{I}\delta(\mathbf{r} - \mathbf{r}'), \tag{9}$$

we apply Gauss' theorem for dyadics to construct

$$\begin{aligned}\int [(\nabla \times \nabla \times \mathbf{u}) \cdot \tilde{\Gamma} - \mathbf{u} \cdot (\nabla \times \nabla \times \tilde{\Gamma})]\, dV &= \int \mathbf{n} \cdot [(\nabla \times \mathbf{u}) \times \tilde{\Gamma} + \mathbf{u} \times (\nabla \times \tilde{\Gamma})]\, dS \\ &= -\int [(\nabla \times \mathbf{u}) \cdot (\mathbf{n} \times \tilde{\Gamma}) - (\mathbf{n} \times \mathbf{u}) \cdot (\nabla \times \tilde{\Gamma})]\, dS.\end{aligned} \tag{10}$$

In the region external to the scatterer, we use (1), (9),

[9] W. W. Hansen, Phys. Rev. 47, 139 (1935); see also Physics 7, 460 (1936); J. Appl. Phys. 8, 282 (1937); W. W. Hansen and J. G. Beckerly, Physics 7, 220 (1936); Proc. IRE 24, 1594 (1936).

[10] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

[11] S. Silver, *Microwave Antenna Theory and Design* (McGraw-Hill Book Company, Inc., New York, 1949).

[12] C. H. Wilcox, Commun. Pure Appl. Math. 9, 115 (1956).

[13] H. Levine and J. Schwinger, Commun. Pure Appl. Math. 3, 355 (1950).

and (10) to reduce the dyadic form of Green's theorem (10) to

$$\mathbf{u}(\mathbf{r}) = \int [(\nabla \times \mathbf{u}) \cdot (\mathbf{n} \times \bar{\Gamma}) - (\mathbf{n} \times \mathbf{u}) \cdot (\nabla \times \bar{\Gamma})]\, dS, \quad (11)$$

where now $\mathbf{n}$ points away from the scatterer, and where the integral is over any surface enclosing the scatterer and excluding $\mathbf{r}$. We rewrite (11) as

$$\begin{aligned}\mathbf{u}(\mathbf{r}; \mathbf{i}: \boldsymbol{\epsilon}) &= \frac{k}{4\pi i} \int [(\nabla \times \mathbf{u}) \cdot (\mathbf{n} \times \bar{h}) - (\mathbf{n} \times \mathbf{u}) \cdot (\nabla \times \bar{h})]\, dS \\ &\equiv \{\bar{h}(k\,|\mathbf{r} - \mathbf{r}'|), \mathbf{u}(\mathbf{r}'; \mathbf{i}: \boldsymbol{\epsilon})\}; \\ \bar{h}(k\,|\mathbf{r} - \mathbf{r}'|) &= (\bar{I} + \nabla\nabla/k^2) h(k\,|\mathbf{r} - \mathbf{r}'|) = \bar{\Gamma} 4\pi i/k, \\ h(r) &= e^{ir}/ir. \end{aligned} \quad (12)$$

If we replace $\bar{\Gamma}$ by $\bar{\Gamma} \cdot \mathbf{e}$ in the above, where $\mathbf{e}$ is an arbitrary constant vector, then (10) reduces to the usual vector form of Green's theorem [say $(10) \cdot \mathbf{e}$] and the left-hand sides of (11) and (12) reduce to $\mathbf{u} \cdot \mathbf{e}$.

Since $\{\bar{h}(k\,|\mathbf{r} - \mathbf{r}'|), \boldsymbol{\varphi}(\mathbf{r}')\} = 0$ for $\mathbf{r}$ outside S, we may also write

$$\mathbf{u}(\mathbf{r}) = \{\bar{h}(k\,|\mathbf{r} - \mathbf{r}'|), \boldsymbol{\psi}(\mathbf{r}')\}. \quad (13)$$

From (13), (9), and (8), we obtain

$$\boldsymbol{\psi} = \boldsymbol{\varphi} - \int \left[\left(\frac{K^2}{\mu} - k^2 \right) \boldsymbol{\psi}' \cdot \bar{\Gamma} + \left(1 - \frac{1}{\mu} \right) (\nabla \times \boldsymbol{\psi}') \cdot (\nabla \times \bar{\Gamma}) \right] dV, \quad (14)$$

which also holds for an interior point, in which case $\boldsymbol{\psi} = \boldsymbol{\psi}'$ is supplied by the internal (instead of the external) singularity of $\bar{\Gamma}$. The case $\mu = 1$ is discussed in detail by Saxon,[7] and a generalization of (14) is considered in Ref. 14.

If $k\,|\mathbf{r} - \mathbf{r}'| \gg 1$ and $r \gg r'$, then

$$\bar{h}(k\,|\mathbf{r} - \mathbf{r}'|) \sim (\bar{I} - \mathbf{oo}) e^{-ik\mathbf{o}\cdot\mathbf{r}'} h(kr) = \bar{\varphi}(-\mathbf{o}) h(kr), \quad (15)$$

and (12) reduces to the far-field form (5) with

$$\mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon}) = \{(\bar{I} - \mathbf{oo}) e^{-ik\mathbf{o}\cdot\mathbf{r}'}, \mathbf{u}(\mathbf{r}'; \mathbf{i}: \boldsymbol{\epsilon})\} = \{\bar{\varphi}(-\mathbf{o}), \mathbf{u}\}. \quad (16)$$

For any unit vector $\boldsymbol{\gamma}$ perpendicular to $\mathbf{o}$ we have

$$\boldsymbol{\gamma} \cdot \mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon}) = \{\boldsymbol{\varphi}(-\mathbf{o}: \boldsymbol{\gamma}), \mathbf{u}(\mathbf{i}: \boldsymbol{\epsilon})\}, \quad (17)$$

where $\mathbf{r}'$ has been suppressed. If $\boldsymbol{\gamma} = \hat{g} = \mathbf{g}/g$, then the left side of (17) reduces to $g(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon})$.

Scattering theorems: To facilitate subsequent applications we use the present formalism to derive certain theorems which $\mathbf{g}$ fulfills. See Saxon[8] for derivation based on a tensor scattering matrix.

Consider two solutions of a scattering problem for two different incident waves, say $\boldsymbol{\psi}_1 = \boldsymbol{\varphi}_1 + \mathbf{u}_1$ and $\boldsymbol{\psi}_2 = \boldsymbol{\varphi}_2 + \mathbf{u}_2$, such that $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}(\mathbf{i}_1: \boldsymbol{\epsilon}_1)$, etc. Since $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ satisfy the same conditions at the scatterer [i.e., (6), or (7) plus (8)], we have $\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2\}_S = 0$ on its surface S, and since $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ fulfill (1) in the external region, it follows from $(10) \cdot \mathbf{e}$ that

$$\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2\} = \{(\boldsymbol{\varphi}_1 + \mathbf{u}_1), (\boldsymbol{\varphi}_2 + \mathbf{u}_2)\} = 0 \quad (18)$$

for any surface (including the surface at infinity S_∞) surrounding the scatterer. Since

$$\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\} = \{\mathbf{u}_1, \mathbf{u}_2\}_{S_\infty} = 0,$$

(18) reduces to

$$\{\boldsymbol{\varphi}_1, \mathbf{u}_2\} = -\{\mathbf{u}_1, \boldsymbol{\varphi}_2\} = \{\boldsymbol{\varphi}_2, \mathbf{u}_1\}, \quad (19)$$

where the last equality follows from the explicit form of the operator in (12). Thus since $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}(\mathbf{i}_1: \boldsymbol{\epsilon}_1) = \boldsymbol{\epsilon}_1 \cdot \bar{\varphi}(\mathbf{i}_1)$, we use (17) in (19) to obtain the reciprocity relation

$$\boldsymbol{\epsilon}_1 \cdot \mathbf{g}(-\mathbf{i}_1, \mathbf{i}_2: \boldsymbol{\epsilon}_2) = \boldsymbol{\epsilon}_2 \cdot \mathbf{g}(-\mathbf{i}_2, \mathbf{i}_1: \boldsymbol{\epsilon}_1). \quad (20)$$

This holds for the relatively weak surface condition $\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2\}_S = 0$, which includes (6), etc.

If $\boldsymbol{\psi}_1$ is replaced by its complex conjugate $\boldsymbol{\psi}_1^*$, then for lossless scatterers

$$\{\boldsymbol{\psi}_1^*, \boldsymbol{\psi}_2\} = \{(\boldsymbol{\varphi}_1^* + \mathbf{u}_1^*), (\boldsymbol{\varphi}_2 + \mathbf{u}_2)\} = 0. \quad (21)$$

We have $\{\boldsymbol{\varphi}_1^*, \boldsymbol{\varphi}_2\} = 0$, and

$$\begin{aligned}\{\mathbf{u}_1^*, \mathbf{u}_2\} &= \frac{k}{4\pi i} 2ik \int (\mathbf{o} \times \mathbf{g}_2 h) \cdot (\mathbf{o} \times \mathbf{g}_1 h)^*\, dS_\infty \\ &= \frac{1}{2\pi} \int \mathbf{g}(\mathbf{o}; \mathbf{i}_2: \boldsymbol{\epsilon}_2) \cdot \mathbf{g}^*(\mathbf{o}, \mathbf{i}_1: \boldsymbol{\epsilon}_1)\, d\Omega_o, \end{aligned} \quad (22)$$

where $d\Omega_o$ is the differential solid angle around $\mathbf{o}$, and the integration is over all angles of observation. Since $\boldsymbol{\varphi}^* = \boldsymbol{\epsilon}^* e^{-ik\mathbf{i}^*\cdot\mathbf{r}} = \boldsymbol{\varphi}(-\mathbf{i}^*: \boldsymbol{\epsilon}^*)$, we reduce (21) to

$$\boldsymbol{\epsilon}_1^* \cdot \mathbf{g}(\mathbf{i}_1^*, \mathbf{i}_2: \boldsymbol{\epsilon}_2) + \boldsymbol{\epsilon}_2^* \cdot \mathbf{g}^*(\mathbf{i}_2^*, \mathbf{i}_1: \boldsymbol{\epsilon}_1) = \frac{-1}{2\pi} \int \mathbf{g}(\mathbf{o}, \mathbf{i}_2: \boldsymbol{\epsilon}_2) \cdot \mathbf{g}^*(\mathbf{o}, \mathbf{i}_1: \boldsymbol{\epsilon}_1)\, d\Omega_o. \quad (23)$$

In particular, in the forward scattered direction, such that all i's reduce to $\mathbf{i}$, and all $\boldsymbol{\epsilon}$'s to $\boldsymbol{\epsilon}$, we obtain the "energy theorem"

$$-\mathrm{Re}\, \boldsymbol{\epsilon} \cdot \mathbf{g}(\mathbf{i}, \mathbf{i}: \boldsymbol{\epsilon}) = \frac{1}{4\pi} \int |\mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon})|^2\, d\Omega_o = \frac{k^2}{4\pi} Q(\mathbf{i}: \boldsymbol{\epsilon}), \quad (24)$$

where Q is the total scattering cross section. If the scatterer is not lossless, then $4\pi/k^2$ times the left-most form of (24) equals the sum of scattering plus absorption cross sections.

[14] V. Twersky, J. Math. Phys. 3, 716 (1962).

Plane-wave representation: If $\mathbf{u}$ is known, then (16) gives $\mathbf{g}$ by integration. An inverse relation follows from (14) by using[15]

$$h(k\,|\mathbf{r}-\mathbf{r}'|)=\frac{1}{2\pi}\int e^{ik\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\,d\Omega_p, \qquad (25)$$

where the limits of the complex paths of the angles associated with the unit vector $\mathbf{p}(\tau,\beta)$ (each path analogous to one in Sommerfeld's integral for $H_0^{(1)}$) are chosen to ensure $\operatorname{Im}\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')>0$. See additional discussion in Noether[15] and after (3:8).

Substituting the corresponding dyadic

$$\tilde{h}(k\,|\mathbf{r}-\mathbf{r}'|)=\left(\tilde{I}+\frac{\nabla\nabla}{k^2}\right)h(k\,|\mathbf{r}-\mathbf{r}'|)=\frac{1}{2\pi}\int(\tilde{I}-\mathbf{pp})e^{ik\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\,d\Omega_p \qquad (26)$$

into $\mathbf{u}$ of (12), and using definition (16), we obtain the vector analog of (3:9):

$$\mathbf{u}(\mathbf{r};\mathbf{i})=\frac{1}{2\pi}\int e^{ik\mathbf{p}\cdot\mathbf{r}}\{(\tilde{I}-\mathbf{pp})e^{-ik\mathbf{p}\cdot\mathbf{r}'},\mathbf{u}(\mathbf{r}';\mathbf{i}:\boldsymbol{\epsilon})\}\,d\Omega_p=\frac{1}{2\pi}\int e^{ik\mathbf{p}\cdot\mathbf{r}}\mathbf{g}(\mathbf{p},\mathbf{i}:\boldsymbol{\epsilon})\,d\Omega_p, \qquad (27)$$

which holds at least for $r>r'_{\max}\equiv a$. (See Ref. 3 for weaker condition.)

Cartesian representation in inverse powers of r: The asymptotic form given in (5) is the leading term of a series expansion of $\mathbf{u}$ in inverse powers of r which converges for $r>r'_{\max}=a$; see Wilcox[12] for a detailed discussion. This series, with coefficients expressed in different forms, may be obtained from (27) by various procedures, e.g., by means of

$$\frac{1}{2\pi}\int e^{ir\mathbf{p}\cdot\mathbf{o}}F(\mathbf{p})\,d\Omega_p\doteq h(r)\mathfrak{D}(r;D)F(\mathbf{o}),$$

$$\mathfrak{D}(r;D)=1+\frac{i}{2r}D+\left(\frac{i}{2r}\right)^2\frac{D(D-1\cdot2)}{2!}+\cdots+\left(\frac{i}{2r}\right)^n\frac{D(D-1\cdot2)(D-2\cdot3)\cdots(D-[n-1]n)}{n!},$$

$$D=\frac{-1}{\sin^2\theta}\,[\partial_\varphi^2+\sin\theta\partial_\theta(\sin\theta\partial_\theta)], \qquad (28)$$

where r is a parameter, $F(\mathbf{o})$ is representable as series of surface harmonics, and D is Beltrami's operator; see (3:10) to (3:16) for details. Using (28) for the Cartesian components of (27), we obtain

$$\mathbf{u}=h(kr)\mathfrak{D}(kr;D)\mathbf{g}(\mathbf{o})=h(kr)\left[\mathbf{g}+\frac{i}{2kr}D\mathbf{g}+\left(\frac{i}{2kr}\right)^2\frac{D(D-2)}{2}\mathbf{g}+\cdots\right], \qquad (29)$$

subject to $\nabla\cdot\mathbf{u}=0$ and $\nabla\times\nabla\times\mathbf{u}-k^2\mathbf{u}=0$.

Special function representations: In the following, except for normalization factors, we work with the transverse vector spherical functions introduced by Hansen,[9] and discussed by Stratton,[10] Morse and Feshbach,[6] Saxon,[7] Stein,[16] and others. We use

$$\mathbf{M}_{nm}(\mathbf{r})=h_n(kr)\mathbf{C}_n^m(\mathbf{o}),$$
$$\mathbf{C}_n^m(\mathbf{o})=-\mathbf{r}\times\nabla Y_n^m(\mathbf{o})=\left(\hat{\theta}\frac{\partial_\varphi}{\sin\theta}-\hat{\varphi}\partial_\theta\right)Y_n^m(\mathbf{o})=-\mathbf{L}(\mathbf{o})Y_n^m(\mathbf{o}),$$
$$Y_n^m(\mathbf{o})=P_n^m(\cos\theta)e^{im\varphi},$$
$$Y_n^{-m}(\mathbf{o})=(-1)^m[(n-m)!/(n+m)!]P_n^m(\cos\theta)e^{-im\varphi}. \qquad (30)$$

Here $h_n=h_n^{(1)}$ is a radiating spherical Hankel function, and P_n^m is an associated Legendre function. Similarly

$$\mathbf{N}_{nm}(\mathbf{r})=[n(n+1)h_n(kr)\mathbf{P}_n^m(\mathbf{o})+\partial_{kr}[krh_n(kr)]\mathbf{B}_n^m(\mathbf{o})]/kr,$$
$$\mathbf{P}_n^m(\mathbf{o})=\mathbf{o}Y_n^m(\mathbf{o}),\quad \mathbf{B}_n^m(\mathbf{o})=r\nabla Y_n^m(\mathbf{o})=\mathbf{o}\times\mathbf{C}_n^m(\mathbf{o}). \qquad (31)$$

The two sets are related through $k\mathbf{N}=\nabla\times\mathbf{M}$ and $k\mathbf{M}=\nabla\times\mathbf{N}$. For real directions, the corresponding even and odd vector harmonics $\mathbf{P}_{nm}^{e,o}$ and $\mathbf{C}_{nm}^{e,o}$ of Morse and Feshbach[6] (pp. 1865, 1898, 1899) are the real and imaginary parts, respectively, of the present $\mathbf{P}_n^m$ and $\mathbf{C}_n^m/[n(n+1)]^{\frac{1}{2}}$. We have

$$\mathbf{C}_n^{-m}(\mathbf{o})=(-1)^m[(n-m)!/(n+m)!][\mathbf{C}_n^m(\mathbf{o}^*)]^*.$$

We also work with $\mathbf{N}$ in an alternative form, essentially as in Morse and Feshbach[6] (p. 1866):

$$i^{n-1}\mathbf{N}_n(r)=\frac{n(n+1)}{2n+1}[h_{n-1}i^{n-1}\mathbf{E}_{n-1}-h_{n+1}i^{n+1}\mathbf{H}_{n+1}],$$
$$\mathbf{E}_{n-1}\equiv\mathbf{P}_n+\frac{\mathbf{B}_n}{n},\quad \mathbf{H}_{n+1}\equiv\mathbf{P}_n-\frac{\mathbf{B}_n}{n+1}, \qquad (32)$$

where we have dropped arguments and the index m for brevity. Henceforth, also for brevity, all four-digit page numbers we cite are to be found in Morse and Feshbach.[6]

[15] F. Noether, in *Theory of Functions*, R. Rothe, F. Ollendorf, and K. Pohlhausen, Eds. (Technology Press, Cambridge, Mass., 1948), p. 167, Eq. (7).

[16] S. Stein, Quart. Appl. Math. 19, 15 (1961).

The angular functions satisfy the following orthogonality relations:

$$\int \mathbf{C}_n^{-m} \cdot \mathbf{B}_\nu^\mu \, d\Omega = \int \mathbf{C}_n^{-m} \cdot \mathbf{P}_\nu^\mu \, d\Omega = \int \mathbf{P}_n^{-m} \cdot B_\nu^\mu \, d\Omega = 0,$$

$$\int \mathbf{C}_n^{-m} \cdot \mathbf{C}_\nu^\mu \, d\Omega = \int \mathbf{B}_n^{-m} \cdot \mathbf{B}_\nu^\mu \, d\Omega$$
$$= n(n+1) \int \mathbf{P}_n^{-m} \cdot \mathbf{P}_\nu^\mu \, d\Omega$$
$$= (-1)^m 4\pi \delta_{n\nu} \delta_{m\mu} n(n+1)/(2n+1),$$

$$\int \mathbf{E}_{n-1}^{-m} \cdot \mathbf{H}_{n+1}^\mu \, d\Omega = \int \mathbf{E}_{n-1}^{-m} \cdot \mathbf{C}_\nu^\mu \, d\Omega$$
$$= \int \mathbf{H}_{n+1}^{-m} \cdot \mathbf{C}_\nu^\mu \, d\Omega = 0,$$

$$n \int \mathbf{E}_{n-1}^{-m} \cdot \mathbf{E}_{\nu-1}^\mu \, d\Omega = (n+1) \int \mathbf{H}_{n+1}^{-m} \cdot \mathbf{H}_{\nu+1}^\mu \, d\Omega$$
$$= (-1)^m 4\pi \delta_{n\nu} \delta_{m\mu},$$
$$\mathbf{P}_n^m \cdot \mathbf{B}_n^m = \mathbf{P}_n^m \cdot \mathbf{C}_n^m = \mathbf{B}_n^m \cdot \mathbf{C}_n^m = 0,$$
$$\mathbf{C}_n^m \cdot \mathbf{C}_\nu^\mu = \mathbf{B}_n^m \cdot \mathbf{B}_\nu^\mu. \qquad (33)$$

The asymptotic forms of Hansen's functions are

$$i^n \mathbf{M}_n(kr) \sim h(kr) \mathbf{C}_n(\mathbf{o}), \quad i^{n-1} \mathbf{N}_n(kr) \sim h(kr) \mathbf{B}_n(\mathbf{o}), \qquad (34)$$

where $h = h_0^{(1)}$ as in (5).

From pp. 1782 and 1875 we may write the normalized dyadic Green's function for $r > r'$ as

$$\tilde{h}(k|\mathbf{r} - \mathbf{r}'|) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [\mathbf{M}_{nm}(\mathbf{r}) \mathbf{M}_{n,-m}^1(\mathbf{r}') + \mathbf{N}_{nm}(\mathbf{r}) \mathbf{N}_{n,-m}^1(\mathbf{r}')](-1)^m d_n,$$
$$d_n \equiv \frac{2n+1}{n(n+1)}, \qquad (35)$$

where the functions with superscript 1 are the nonsingular nonradiating functions (j type), and those without superscripts are the radiating functions ($h^{(1)}$ type). If we substitute (35) into (12), we obtain

$$\mathbf{u}(\mathbf{r}; \mathbf{i}: \boldsymbol{\epsilon}) = \sum [\mathbf{M}_{nm}(\mathbf{r}) c_{nm}(\mathbf{i}: \boldsymbol{\epsilon}) - i \mathbf{N}_{nm}(\mathbf{r}) b_{nm}(\mathbf{i}: \boldsymbol{\epsilon})] i^n,$$
$$c_{nm}(\mathbf{i}: \boldsymbol{\epsilon}) = i^{-n}(-1)^m d_n \{\mathbf{M}_{n,-m}^1(\mathbf{r}'), \mathbf{u}(\mathbf{r}'; \mathbf{i}: \boldsymbol{\epsilon})\},$$
$$b_{nm} = i^{-n+1}(-1)^m d_n \{\mathbf{N}_{n,-m}^1, \mathbf{u}\}. \qquad (36)$$

The scattering coefficients (or multipole coefficients) c and b are of the magnetic-type and electric-type, respectively. If we introduce (34) into (36), and compare with (5) we have

$$\mathbf{g}(\mathbf{o}, \mathbf{i}: \boldsymbol{\epsilon}) = \sum [\mathbf{C}_n^m(\mathbf{o}) c_{nm}(\mathbf{i}: \boldsymbol{\epsilon}) + \mathbf{B}_n^m(\mathbf{o}) b_{nm}(\mathbf{i}: \boldsymbol{\epsilon})], \qquad (37)$$

which may also be derived directly from (16) by substituting [from (35) with $r \sim \infty$, or from p. 1866].

$$\tilde{\varphi}(\mathbf{r}'; -\mathbf{o}) = (\tilde{I} - \mathbf{oo}) e^{-ik\mathbf{o}\cdot\mathbf{r}'}$$
$$= \sum [\mathbf{C}_n^m(\mathbf{o}) \mathbf{M}_{n,-m}^1(\mathbf{r}') + i \mathbf{B}_n^m(\mathbf{o}) \mathbf{N}_{n,-m}^1(\mathbf{r}')] i^{-n} (-1)^m d_n. \qquad (38)$$

We could also have obtained (36) for $\mathbf{u}$ by substituting (37) into (27). Thus

$$\mathbf{u} = \frac{1}{2\pi} \int e^{ik\mathbf{p}\cdot\mathbf{r}} \sum [\mathbf{C}_n^m(\mathbf{p}) c_{nm} + \mathbf{B}_n^m(\mathbf{p}) b_{nm}] \, d\Omega_p,$$
$$c_{nm} = \frac{(-1)^m d_n}{4\pi} \int \mathbf{C}_n^{-m}(\mathbf{o}) \cdot \mathbf{g}(\mathbf{o}) \, d\Omega_o,$$
$$b_{nm} = \frac{(-1)^m d_n}{4\pi} \int \mathbf{B}_n^{-m}(\mathbf{o}) \cdot \mathbf{g}(\mathbf{o}) \, d\Omega_o, \qquad (39)$$

which reduces to (36) on using

$$i^n \mathbf{M}_{nm}(\mathbf{r}) = \frac{1}{2\pi} \int e^{ik\mathbf{p}\cdot\mathbf{r}} \mathbf{C}_n^m(\mathbf{p}) \, d\Omega_p,$$
$$i^{n-1} \mathbf{N}_{nm}(\mathbf{r}) = \frac{1}{2\pi} \int e^{ik\mathbf{p}\cdot\mathbf{r}} \mathbf{B}_n^m(\mathbf{p}) \, d\Omega_p \qquad (40)$$

(the radiating function analogs of the forms on pp. 1865–1866).

General representation in inverse powers of r: The present series leads to an inverse-distance expansion fully analogous to (3:16). For the scalar case,[3] we substituted Hankel's polynomial form

$$h_n(r) i^n = h(r) \Big[1 + n(n+1)(i/2r) + n(n+1)[n(n+1) - 1 \cdot 2] (1/2!)(i/2r)^2 + \cdots \Big]$$
$$\equiv h(r) \mathfrak{D}(r; n[n+1]), \qquad (41)$$

into $u = \sum a_{nm}(i) h_n(kr) i^n Y_n^m(\mathbf{o})$, and then used Legendre's equation

$$n(n+1) Y_n(\mathbf{o}) = D Y_n(\mathbf{o}), \qquad (42)$$

and the scalar amplitude $g = \sum a_{nm}(\mathbf{i}) Y_n^m(\mathbf{o})$ to obtain the form $u = h(kr) \mathfrak{D}(kr; D) g(\mathbf{o})$ [implicit in the Cartesian representation (29)].

We obtain the analog of (42) for the vector spherical harmonics by separating variables in the vector wave equation (1); we write $\boldsymbol{\psi}(\mathbf{r})$ as a series of functions $R_n(r) \mathbf{F}_n(\mathbf{0})$, and obtain

$$\mathbf{F}_n(\mathbf{o})[r^2 k^2 + (1/R_n) \partial_r (r^2 \partial_r R_n)]$$
$$= r^2 [\nabla \times (\nabla \times \mathbf{F}_n) - \nabla\nabla \cdot \mathbf{F}_n] \equiv \tilde{D} \cdot \mathbf{F}_n(\mathbf{o}), \qquad (43)$$

where $\tilde{D}$ reduces to $D\tilde{I}$ in Cartesian coordinates. In polar coordinates, with $\mathbf{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_\varphi \hat{\varphi}$, $\mathbf{o} = \hat{r}$, we have

$$\tilde{D}\cdot\mathbf{F}=\hat{r}\left\{DF_r+2F_r+\frac{2}{\sin\theta}[\partial_\theta(\sin\theta F_\theta)+\partial_\varphi F_\varphi]\right\}$$
$$+\hat{\theta}\left\{DF_\theta+\frac{1}{\sin^2\theta}[F_\theta+2\cos\theta\partial_\varphi F_\varphi]-2\partial_\theta F_r\right\}$$
$$+\hat{\varphi}\left\{DF_\varphi-\frac{1}{\sin^2\theta}[-F_\varphi+2\cos\theta\partial_\varphi F_\theta]-\frac{2}{\sin\theta}\partial_\varphi F_r\right\}. \quad (44)$$

If we specialize (43) to $R\mathbf{F}=\mathbf{M}$ of (30), and use Bessel's equation

$$[k^2r^2+\partial_r(r^2\partial_r h_n)/h_n]=n(n+1), \quad (45)$$

we obtain the vector analog of (42):

$$[n(n+1)-\tilde{D}\cdot]\mathbf{C}_n(\mathbf{o})=0. \quad (46)$$

Similarly, if we specialize (43) to $R\mathbf{F}=\mathbf{N}$ of (32), apply (45) for h_{n-1} and h_{n+1} and use the orthogonality properties of $\mathbf{E}_{n-1}$, $\mathbf{H}_{n+1}$ as in (33), we obtain

$$[(n-1)n-\tilde{D}\cdot]\mathbf{E}_{n-1}=0, \quad \mathbf{E}_{n-1}=\mathbf{P}_n+\mathbf{B}_n/n, \quad (47)$$
$$[(n+1)(n+2)-\tilde{D}\cdot]\mathbf{H}_{n+1}=0,$$
$$\mathbf{H}_{n+1}=\mathbf{P}_n-\mathbf{B}_n/(n+1). \quad (48)$$

The above provides a different procedure than the usual one of synthesizing solutions of the vector wave equation from known solutions of the scalar equation: We separate variables in the vector equation to obtain (43) and work with solutions of the form $h_\nu(r)\mathbf{F}_\nu(\mathbf{o})$, where $\mathbf{F}$ represents the three sets of eigenvectors $\mathbf{E}_{n-1}$, $\mathbf{C}_n$, and $\mathbf{H}_{n+1}$ of the linear operator $\tilde{D}$.

Using (41) we rewrite $\mathbf{M}$ of (30) and $\mathbf{N}$ of (32) as

$$i_n\mathbf{M}_n=h\mathfrak{D}(n[n+1])\mathbf{C}_n,$$
$$i^{n-1}\mathbf{N}_n=h\frac{n(n+1)}{2n+1}$$
$$\times\left[\mathfrak{D}([n-1]n)\mathbf{E}_{n-1}-\mathfrak{D}([n+1][n+2])\mathbf{H}_{n+1}\right], \quad (49)$$

where the three $\mathfrak{D}$'s are polynomials in $(n[n+1])$, $([n-1]n)$, and $([n+1][n+2])$, respectively. From (46)–(48), we have

$$\mathfrak{D}(\nu[\nu+1])\mathbf{F}_\nu=\mathfrak{D}(\tilde{D}\cdot)\mathbf{F}_\nu\equiv\tilde{\mathfrak{D}}\cdot\mathbf{F}_\nu,$$
$$\tilde{\mathfrak{D}}(kr;\tilde{D})=\tilde{I}+(i/2kr)\tilde{D}$$
$$+(i/2kr)^2\tilde{D}\cdot(\tilde{D}-1\cdot 2\tilde{I})+\cdots. \quad (50)$$

Using (50) in (49), we obtain

$$i^n\mathbf{M}_n(\mathbf{r})=h(kr)\tilde{\mathfrak{D}}(kr;\tilde{D})\cdot\mathbf{C}_n(\mathbf{o}), \quad (51)$$
$$i^{n-1}\mathbf{N}_n(\mathbf{r})$$
$$=h(kr)\tilde{\mathfrak{D}}(kr;\tilde{D})\cdot[n(n+1)/(2n+1)](\mathbf{E}_{n-1}-\mathbf{H}_{n+1}). \quad (52)$$

From the definitions in (32),

$$\mathbf{E}_{n-1}-\mathbf{H}_{n+1}=\frac{2n+1}{n(n+1)}\mathbf{B}_n,$$

(52) reduces to

$$i^{n-1}\mathbf{N}_n(\mathbf{r})=h(kr)\tilde{\mathfrak{D}}(kr;\tilde{D})\cdot\mathbf{B}_n(\mathbf{o}). \quad (53)$$

We may now construct the full vector analog of the scalar solution (3:16). Substituting (51) and (53) into (36), reduces the solution to

$$\mathbf{u}=h(kr)\tilde{\mathfrak{D}}(kr;\tilde{D})\cdot\sum[\mathbf{C}_n^m(\mathbf{o})c_{nm}+\mathbf{B}_n^m(\mathbf{o})b_{nm}]$$
$$=h(kr)\tilde{\mathfrak{D}}(kr;\tilde{D})\cdot\mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon}), \quad (54)$$

where the differentiations are with respect to the angles of $\mathbf{o}$.

The longitudinal (with respect to $\mathbf{o}$) $\mathbf{P}$ terms do not appear explicitly in (54) [or in (53)]; however, except for the leading term (the far-field form $h\mathbf{g}$), components along $\mathbf{o}$ are generated by the $\tilde{D}\cdot$ operation. The polar representation of the above series form obtained by using (44) for $\tilde{D}$, with polar components of subsequent terms expressed recursively in terms of the first ($\mathbf{g}$), was derived originally by Wilcox,[12] who also showed that the series in $r^{-\nu}$ converged absolutely and uniformly in r, θ, and φ in any region $r>r'_{\max}=a$.

Since our series for the scattering amplitude $\mathbf{g}(\mathbf{o})$ of (37) is a general transverse form, we see from (27) and (54) that

$$\frac{1}{2\pi}\int e^{ik\mathbf{p}\cdot\mathbf{r}}\mathbf{F}(\mathbf{p})\,d\Omega_p=h(r)\tilde{\mathfrak{D}}(r;\tilde{D})\cdot\mathbf{F}(\mathbf{o}), \quad (55)$$

where r is a parameter, and where $\mathbf{F}(\mathbf{o})$ is representable as a series of transverse vector surface harmonics. To cover vector problems for which $\nabla\cdot\boldsymbol{\psi}\neq 0$, we generalize (55) to include nontransverse components ($\mathbf{P}_n^m$). This corresponds to fields which involve the longitudinal functions

$$\mathbf{L}_{nm}(\mathbf{r})=\partial_{kr}[h_n(kr)]\mathbf{P}_n^m(\mathbf{o})+(h_n/kr)\mathbf{B}_n^m(\mathbf{o})$$
$$=\frac{i^{-n+1}}{2n+1}[nh_{n-1}i^{n-1}\mathbf{E}_{n-1}+(n+1)h_{n+1}i^{n+1}\mathbf{H}_{n+1}]$$
$$=\frac{i^{-n+1}}{2\pi}\int e^{ik\mathbf{p}\cdot\mathbf{r}}\mathbf{P}_n^m(\mathbf{p})\,d\Omega_p, \quad (56)$$

essentially as on p. 1865, and in terms of $\mathbf{E}$ and $\mathbf{H}$ of (32). Thus if

$$\mathbf{F}(\mathbf{o})=\sum[\mathbf{C}_n^m(\mathbf{o})c_{nm}+\mathbf{B}_n^m(\mathbf{o})b_{nm}+\mathbf{P}_n^m(\mathbf{o})p_{nm}], \quad (57)$$

then substituting (57) for $\mathbf{F}(\mathbf{p})$ in (55) we obtain

$$\frac{1}{2\pi}\int e^{ik\mathbf{p}\cdot\mathbf{r}}\mathbf{F}(\mathbf{p})\,d\Omega_p$$
$$=\sum[\mathbf{M}_{nm}(\mathbf{r})c_{nm}-i\mathbf{N}_{nm}b_{nm}-i\mathbf{L}_{nm}p_{nm}]i^n\equiv\mathbf{V}. \quad (58)$$

From the second equality of (56), and from (41), (47), and (48) we obtain

$$\mathbf{L}_{nm}i^{n-1} = \frac{h}{2n+1}\Big[n\mathfrak{D}([n-1]n)\mathbf{E}_{n-1} + (n+1)\mathfrak{D}([n+1][n+2])\mathbf{H}_{n+1}\Big] = h\tilde{\mathfrak{D}}(\tilde{D})\cdot\frac{1}{2n+1}[n\mathbf{E}_{n-1}+(n+1)\mathbf{H}_{n+1}] = h\tilde{\mathfrak{D}}(\tilde{D})\cdot\mathbf{P}_n^m(\mathbf{o}), \quad (59)$$

where the final form followed from

$$n\mathbf{E}_{n-1} + (n+1)\mathbf{H}_{n+1} = (2n+1)\mathbf{P}_n.$$

Substituting (51), (53), and (59) into the series $\mathbf{V}$ of (58) gives

$$\mathbf{V} = h\tilde{\mathfrak{D}}(\tilde{D})\cdot\sum[\mathbf{C}_n^m(\mathbf{o})c_{nm} + \mathbf{B}_n^m(\mathbf{o})b_{nm} + \mathbf{P}_n^m(\mathbf{o})p_{nm}] = h(kr)\tilde{\mathfrak{D}}(kr;\tilde{D})\cdot\mathbf{F}(\mathbf{o}). \quad (60)$$

Thus (55) holds for any $\mathbf{F}(\mathbf{o})$ representable in terms of any series of vector surface harmonics.

2.2. Dyadic Fields

We may parallel the above development of the scattering problem of the vector plane wave $\boldsymbol{\varphi}(\mathbf{i}:\boldsymbol{\epsilon})$ of (2) by the analogous development for the dyadic plane wave introduced in (3):

$$\tilde{\varphi}(\mathbf{i}) = (\tilde{I} - \mathbf{ii})e^{ik\mathbf{i}\cdot\mathbf{r}} = (\tilde{I} + \nabla\nabla/k^2)e^{ik\mathbf{i}\cdot\mathbf{r}} = \nabla\times\nabla\times\tilde{I}e^{ik\mathbf{i}\cdot\mathbf{r}}/k^2. \quad (61)$$

The dyadic scattering problem, because of its higher symmetry, is often the easier one: for the vector form (2) we must specify both a direction of incidence $\mathbf{i}$ plus a direction of polarization $\boldsymbol{\epsilon}$ in the plane perpendicular to $\mathbf{i}$, but in (61) we specify only the direction of incidence $\mathbf{i} = \mathbf{k}/k$. The vector plane wave follows from $\boldsymbol{\varphi}(\mathbf{i}:\boldsymbol{\epsilon}) = \tilde{\varphi}(\mathbf{i})\cdot\boldsymbol{\epsilon}$, and we may introduce a dyadic scattering amplitude[6–8] $\tilde{g}(\mathbf{o},\mathbf{i})$, such that the vector amplitude follows from

$$\mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon}) = \tilde{g}(\mathbf{o},\mathbf{i})\cdot\boldsymbol{\epsilon}. \quad (62)$$

We may rewrite (61) as

$$\tilde{\varphi}(\mathbf{i}) = (\boldsymbol{\epsilon\epsilon} + \boldsymbol{\delta\delta})e^{ik\mathbf{i}\cdot\mathbf{r}} = \boldsymbol{\varphi}(\mathbf{i}:\boldsymbol{\epsilon})\boldsymbol{\epsilon} + \boldsymbol{\varphi}(i:\boldsymbol{\delta})\boldsymbol{\delta}, \quad (63)$$

where $\mathbf{i}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\delta}$ form an orthogonal set of unit vectors. From the superposition principle, the corresponding dyadic scattered wave is thus

$$\tilde{u}(\mathbf{r};\mathbf{i}) = \mathbf{u}(\mathbf{i}:\boldsymbol{\epsilon})\boldsymbol{\epsilon} + \mathbf{u}(\mathbf{i}:\boldsymbol{\delta})\boldsymbol{\delta}. \quad (64)$$

Asymptotically, we have

$$\tilde{u}(\mathbf{r};\mathbf{i}) \sim h(kr)[\mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon})\boldsymbol{\epsilon} + \mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\delta})\boldsymbol{\delta}] = h(kr)\tilde{g}(\mathbf{o},\mathbf{i}) \quad (65)$$

with

$$\tilde{g}(\mathbf{o},\mathbf{i}) = \mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon})\boldsymbol{\epsilon} + \mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\delta})\boldsymbol{\delta} \quad (66)$$

in accord with (62) and with Saxon's definition.[7,8]

Although we could construct the dyadic functions from the vector ones by using (64) and (66), it is somewhat simpler to consider the dyadic scattering problem systematically. In the following, (1d) means Eq. (1) in terms of $\tilde{\varphi}$, etc.

Surface integrals: If we transpose the dyadic $\tilde{\Gamma}$ terms in (10), we obtain

$$\int[\tilde{\Gamma}^{\mathrm{T}}\cdot(\nabla\times\nabla\times\mathbf{u}) - (\nabla\times\nabla\times\tilde{\Gamma})^{\mathrm{T}}\cdot\mathbf{u}]\,dV = -\int[(\mathbf{n}\times\tilde{\Gamma})^{\mathrm{T}}\cdot(\nabla\times\mathbf{u}) - (\nabla\times\tilde{\Gamma})^{\mathrm{T}}\cdot(n\times\mathbf{u})]\,dS, \quad (67)$$

where the superscript T indicates the transposed (Gibbs' conjugate) dyadic. For any dyadic solution of (1d),

$$(\nabla\times\nabla\times\tilde{F})^{\mathrm{T}} = k^2\tilde{F}^{\mathrm{T}} = \nabla\times\nabla\times\tilde{F}^{\mathrm{T}},$$
$$(\mathbf{n}\times\tilde{F})^{\mathrm{T}} = -\tilde{F}^{\mathrm{T}}\times\mathbf{n},\quad (\nabla\times\tilde{F})^{\mathrm{T}} = -\tilde{F}^{\mathrm{T}}(\times\nabla), \quad (68)$$

where $(\times\nabla)$ operating to the left on $\tilde{F}$ in the last equality means differentiate to the left on $\tilde{F}$ but leave the vector part of $\times\nabla$ on the right of $\tilde{F}$. In particular, for $\tilde{F} = \tilde{\Gamma}$ of (9), we have

$$\tilde{\Gamma}^{\mathrm{T}} = \tilde{\Gamma},\quad (n\times\tilde{\Gamma})^{\mathrm{T}} = -\tilde{\Gamma}\times\mathbf{n},$$
$$(\nabla\times\tilde{\Gamma})^{\mathrm{T}} = -\nabla\times\tilde{\Gamma} = h_1(k|\mathbf{r}-\mathbf{r}'|)k\mathbf{o}\times\tilde{I}. \quad (69)$$

From the steps leading to (10) and (67), we obtain

$$\int[\tilde{F}^{\mathrm{T}}\cdot(\nabla\times\nabla\times\tilde{u}) - (\nabla\times\nabla\times\tilde{F})^{\mathrm{T}}\cdot\tilde{u}]\,dV = -\int[(\mathbf{n}\times\tilde{F})^{\mathrm{T}}\cdot(\nabla\times\tilde{u}) - (\nabla\times\tilde{F})^{\mathrm{T}}\cdot(\mathbf{n}\times\tilde{u})]\,dS. \quad (70)$$

In the region external to the scatterer, we use (70) for $\tilde{F} = \tilde{\Gamma} = \tilde{h}k/4\pi i$ to obtain

$$\tilde{u}(\mathbf{r};\mathbf{i}) = \frac{k}{4\pi i}\int[(n\times\tilde{h})^{\mathrm{T}}\cdot(\nabla\times\tilde{u}) - (\nabla\times\tilde{h})^{\mathrm{T}}\cdot(\mathbf{n}\times\tilde{u})]\,dS \equiv \{\tilde{h},\tilde{u}\}, \quad (71)$$

where $\mathbf{n}$ points away from the scatterer. It is this definition of the brace operation for dyadics, equivalent to (12) for $\tilde{u}$ replaced by a vector $\mathbf{u}$, that we use henceforth. Since $(\tilde{A}^{\mathrm{T}}\cdot\tilde{B})^{\mathrm{T}} = \tilde{B}^{\mathrm{T}}\cdot\tilde{A}$, we have $\tilde{u}^{\mathrm{T}} = \{\tilde{h},\tilde{u}^{\mathrm{T}}\}$, and also

$$\tilde{u}^{\mathrm{T}} = \{\tilde{h},\tilde{u}\}^{\mathrm{T}} = \frac{k}{4\pi i}\int[(\nabla\times\tilde{u})^{\mathrm{T}}\cdot(\mathbf{n}\times\tilde{h}) - (\mathbf{n}\times\tilde{u})^{\mathrm{T}}\cdot(\nabla\times\tilde{h})]\,dS = -\{\tilde{u},\tilde{h}\}. \quad (72)$$

Similarly

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = \{\bar{\varphi}(-\mathbf{o}), \tilde{u}(\mathbf{r}'; \mathbf{i})\}. \tag{73}$$

From (18d), i.e., $\{\bar{\psi}_1, \bar{\psi}_2\} = 0$, we proceed as for (19) to obtain

$$\{\bar{\varphi}_1, \tilde{u}_2\} = -\{\tilde{u}_1, \bar{\varphi}_2\} = \{\bar{\varphi}_2, \tilde{u}_1\}^{\mathrm{T}}, \tag{74}$$

where the last equality follows from (72). Thus using (73) in (74), we obtain Saxon's result[8]

$$\tilde{g}(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{g}^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1), \tag{75}$$

which also follows from (20) and (62):

$$\boldsymbol{\epsilon}_1 \cdot [\tilde{g}(-\mathbf{i}_1, \mathbf{i}_2) \cdot \boldsymbol{\epsilon}_2] = \boldsymbol{\epsilon}_2 \cdot [\tilde{g}(-\mathbf{i}_2, \mathbf{i}_1) \cdot \boldsymbol{\epsilon}_1] = \boldsymbol{\epsilon}_1 \cdot \tilde{g}^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) \cdot \boldsymbol{\epsilon}_2.$$

From (74), we see that $\mathbf{i}_1 \cdot \tilde{g}(\mathbf{i}_1, \mathbf{i}_2) = \tilde{g}(\mathbf{i}_1, \mathbf{i}_2) \cdot \mathbf{i}_2 = 0$, i.e., $\tilde{g}$ is transverse both fore and aft [cf. (66)]. From (66) and (74), we have

$$\tilde{g}(\mathbf{i}_1, \mathbf{i}_2) = \tilde{g}^{\mathrm{T}}(-\mathbf{i}_2, -\mathbf{i}_1) = \boldsymbol{\epsilon}_1 \mathbf{g}(-\mathbf{i}_2, -\mathbf{i}_1 : \boldsymbol{\epsilon}_1) + \boldsymbol{\delta}_1 \mathbf{g}(-\mathbf{i}_2, -\mathbf{i}_1 : \boldsymbol{\delta}_1), \tag{76}$$

which supplements (66) in providing a vector representation for $\tilde{g}(\mathbf{i}_1, \mathbf{i}_2)$ in terms of observed instead of incident polarizations.

Similarly from (21d), i.e., $\{\bar{\psi}_1^*, \bar{\psi}_2\} = 0$, we proceed as for (21) to obtain

$$\{\bar{\varphi}_1^*, \tilde{u}_2\} + \{\tilde{u}_1^*, \bar{\varphi}_2\} + \{\tilde{u}_1^*, \tilde{u}_2\} = 0. \tag{77}$$

The first term equals $\tilde{g}(\mathbf{i}_1^*, \mathbf{i}_2)$, the second reduces to $+\{\bar{\varphi}_2^*, \tilde{u}_1\}^{\mathrm{T}*} = \tilde{g}^{\mathrm{T}*}(\mathbf{i}_2^*, \mathbf{i}_1)$, and the last equals

$$\{\tilde{u}_1^*, \tilde{u}_2\} = \frac{k}{4\pi i} 2ik \int (\mathbf{o} \times \tilde{g}_1 h)^{*\mathrm{T}} \cdot (\mathbf{o} \times \tilde{g}_2 h)\, dS = \frac{1}{2\pi} \int \tilde{g}^{\mathrm{T}*}(\mathbf{o}, \mathbf{i}_1) \cdot \tilde{g}(\mathbf{o}, \mathbf{i}_2)\, d\Omega. \tag{78}$$

Thus the dyadic analog of (23) is

$$\tilde{g}(\mathbf{i}_1^*, \mathbf{i}_2) + \tilde{g}^{\dagger}(\mathbf{i}_2^*, \mathbf{i}_1) = -\frac{1}{2\pi} \int \tilde{g}^{\dagger}(\mathbf{o}, \mathbf{i}_1) \cdot \tilde{g}(\mathbf{o}, \mathbf{i}_2)\, d\Omega, \quad \tilde{g}^{\dagger} \equiv \tilde{g}^{\mathrm{T}*}, \tag{79}$$

as obtained originally by Saxon[8] by a briefer, more abstract procedure. The symbol $\tilde{g}^{\dagger}$ represents the Hermitian adjoint of $\tilde{g}$. In the forward direction $\mathbf{i}_1^* = \mathbf{i}_2^* = \mathbf{i}_1 = \mathbf{i}_2$ we may reduce (79) to (24):

$$-\boldsymbol{\epsilon} \cdot [\tilde{g}(\mathbf{i}, \mathbf{i}) + \tilde{g}^{\dagger}(\mathbf{i}, \mathbf{i})] \cdot \boldsymbol{\epsilon} = -2\,\mathrm{Re}\,[\boldsymbol{\epsilon} \cdot \tilde{g}(\mathbf{i}, \mathbf{i}) \cdot \boldsymbol{\epsilon}] = \frac{1}{2\pi} \int |\tilde{g}(\mathbf{o}, \mathbf{i}) \cdot \boldsymbol{\epsilon}|^2\, d\Omega = \frac{k^2}{4\pi} Q(\mathbf{i} : \boldsymbol{\epsilon}). \tag{80}$$

Plane wave form: To construct the dyadic analog of (27), we use $(\bar{I} - \mathbf{pp}) \cdot (\bar{I} - \mathbf{pp}) = \bar{I} - \mathbf{pp}$, and rewrite (26) in terms of the form (61) as

$$\tilde{h}(k\,|\mathbf{r} - \mathbf{r}'|) = \frac{1}{2\pi} \int \bar{\varphi}(\mathbf{r}; \mathbf{p}) \cdot \bar{\varphi}(\mathbf{r}'; -\mathbf{p})\, d\Omega_p. \tag{81}$$

Substituting in (71) and using (73), we obtain

$$\tilde{u}(\mathbf{r}; \mathbf{i}) = \frac{1}{2\pi} \int \bar{\varphi}(\mathbf{r}; \mathbf{p}) \cdot \tilde{g}(\mathbf{p}, \mathbf{i})\, d\Omega_p = \frac{1}{2\pi} \int e^{ik\mathbf{p}\cdot\mathbf{r}} \tilde{g}(\mathbf{p}, \mathbf{i})\, d\Omega_p. \tag{82}$$

Similarly for (55d), etc.

If the scatterer is not at the origin $r = 0$, but at $\mathbf{r} = \mathbf{b}$ then we may work with

$$\tilde{u} \cdot \bar{\varphi}(\mathbf{b}; \mathbf{i}) \sim h(kr) \bar{\varphi}(\mathbf{b}; -\mathbf{o}) \cdot \tilde{g}(\mathbf{o}, \mathbf{i}) \cdot \bar{\varphi}(\mathbf{b}; \mathbf{i}) = h e^{ik\mathbf{b}\cdot(\mathbf{i}-\mathbf{o})} \tilde{g}(\mathbf{o}, \mathbf{i}). \tag{83}$$

Special function series: Corresponding to

$$\bar{\varphi}(\mathbf{i}) = \bar{\varphi}^{\mathrm{T}}(\mathbf{i}) = (\bar{I} - \mathbf{ii}) e^{ik\mathbf{i}\cdot\mathbf{r}} = \sum [\mathbf{M}_{nm}^1(\mathbf{r}) \mathbf{C}_n^{-m}(\mathbf{i}) - i\mathbf{N}_{nm}^1 \mathbf{B}_n^{-m}] i^n (-1)^m d_n,$$

$$d_n = \frac{2n+1}{n(n+1)}, \tag{84}$$

we have

$$\tilde{u}(\mathbf{r}; \mathbf{i}) = \sum_{m,n} [\mathbf{M}_{nm}(\mathbf{r}) \mathbf{c}_{nm}(\mathbf{i}) - i\mathbf{N}_{nm} \mathbf{b}_{nm}(\mathbf{i})], \tag{85}$$

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = \sum [\mathbf{C}_n^m(\mathbf{o}) \mathbf{c}_{nm}(\mathbf{i}) + \mathbf{B}_n^m \mathbf{b}_{nm}(\mathbf{i})], \tag{86}$$

where

$$\mathbf{c}_{nm}(\mathbf{i}) = \sum_{\nu,\mu} [\alpha_{mn\nu\mu} \mathbf{C}_\nu^{-\mu}(\mathbf{i}) + \beta_{mn\nu\mu} \mathbf{B}_\nu^{-\mu}(\mathbf{i})],$$

$$\mathbf{b}_{nm}(\mathbf{i}) = \sum [\gamma_{mn\nu\mu} \mathbf{C}_\nu^{-\mu}(\mathbf{i}) + \delta_{nm\nu\mu} \mathbf{B}_\nu^{-\mu}(\mathbf{i})]. \tag{87}$$

The reciprocity relation (75) gives

$$\alpha_{\nu,-\mu,n,-m} = (-1)^{n+\nu} \alpha_{nm\nu\mu},$$

and similarly for δ; for β and γ we obtain this form with $(-1)^{n+\nu+1}$.

For a spherically symmetric scatterer,

$$\tilde{u}(\mathbf{r}; \mathbf{i}) = \sum [\mathbf{M}_{nm}(\mathbf{r}) \mathbf{C}_n^{-m}(\mathbf{i}) c_n - i\mathbf{N}_{nm} \mathbf{B}_n^{-m} b_n] i^n (-1)^m, \tag{88}$$

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = \sum [\mathbf{C}_n^m(\mathbf{o}) \mathbf{C}_n^{-m}(\mathbf{i}) c_n + \mathbf{B}_n^m(\mathbf{o}) \mathbf{B}_n^{-m}(\mathbf{i}) b_n] (-1)^n, \quad \sum = \sum_{n=1}^{\infty} \sum_{m=-n}^{n}, \tag{89}$$

where b and c are independent of directions. We may rewrite (89) as

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = \sum_{n=1}^{\infty} [\bar{C}_n(\mathbf{o}, \mathbf{i}) c_n + \bar{B}_n(\mathbf{o}, \mathbf{i}) b_n],$$

$$\bar{C}_n(\mathbf{o}, \mathbf{i}) = \sum_{m=-n}^{n} \mathbf{C}_n^m(\mathbf{o}) \mathbf{C}_n^{-m}(\mathbf{i}) (-1)^m = \mathbf{L}(\mathbf{o}) L(\mathbf{i}) P_n(\mathbf{o} \cdot \mathbf{i}),$$

$$\bar{B}_n(\mathbf{o}, \mathbf{i}) = [\mathbf{o} \times \mathbf{L}(\mathbf{o})][\mathbf{i} \times \mathbf{L}(\mathbf{i})] P_n(\mathbf{o} \cdot \mathbf{i}), \tag{90}$$

where $P_n(\mathbf{o}\cdot\mathbf{i}) = \sum Y_n^m(\mathbf{o})\, Y_n^{-m}(\mathbf{i})(-1)^m$ is the Legendre polynomial, and $\mathbf{L}$ is defined in (30).

The form (90) has essentially the same symmetry as for the scalar problem: The reciprocity relation (75) reduces to

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = \tilde{g}(-\mathbf{o}, -\mathbf{i}) = \tilde{g}^{\mathrm{T}}(\mathbf{i}, \mathbf{o}). \quad (91)$$

Substituting (91) into (79) gives the simpler form

$$-\mathrm{Re}\, \tilde{g}(\mathbf{i}_1, \mathbf{i}_2) = \frac{1}{4\pi} \int \tilde{g}^*(\mathbf{i}_1, \mathbf{o}) \cdot \tilde{g}(\mathbf{o}, \mathbf{i}_2)\, d\Omega, \quad (92)$$

and using (90), and

$$\int \tilde{C}_n(\mathbf{i}_1, \mathbf{o}) \cdot \tilde{B}_\nu(\mathbf{o}, \mathbf{i}_2)\, d\Omega = 0,$$

$$\int \tilde{D}_n(\mathbf{i}_1, \mathbf{o}) \cdot \tilde{D}_\nu(\mathbf{o}, \mathbf{i}_2)\, d\Omega = 4\pi \tilde{D}_n(\mathbf{i}_1, \mathbf{i}_2)\delta_{n\nu}/d_n,$$

with $\tilde{D} = \tilde{C}$ or $\tilde{B}$, we obtain

$$-d_n \,\mathrm{Re}\, c_n = |c_n|^2, \quad -d_n \,\mathrm{Re}\, b_n = |b_n|^2. \quad (93)$$

In the forward direction, we have

$$\tilde{g}(\mathbf{i}, \mathbf{i}) = (\tilde{I} - \mathbf{ii}) \sum [\tfrac{1}{2}n(n+1)(b_n + c_n)], \quad (94)$$

and the total cross section equals $-4\pi/k^2$ times $\mathrm{Re} \sum [\ ]$.

If only the dipole terms are significant, then

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = \tilde{C}_1 c_1 + \tilde{B}_1 b_1, \quad \tilde{C}_1 \equiv \tilde{C}_1^0 + \tilde{C}_1^1, \quad \tilde{B}_1 \equiv \tilde{B}_1^0 + \tilde{B}_1^1,$$
$$\tilde{C}_1^0 = \mathbf{C}_1^0(\mathbf{o})\mathbf{C}_1^0(\mathbf{i}) = \hat{\varphi}\hat{\varphi}_i \sin\theta \sin\theta_i,$$
$$\tilde{C}_1^1 = \mathrm{Re}\, \mathbf{C}_1^1(\mathbf{o})\mathbf{C}_1^{1*}(\mathbf{i})$$
$$= (\hat{\theta}\hat{\theta}_i + \hat{\varphi}\hat{\varphi}_i \cos\theta \cos\theta_i)\cos(\varphi - \varphi_i)$$
$$+ (\hat{\theta}\hat{\varphi}_i \cos\theta_i - \hat{\varphi}\hat{\theta}_i \cos\theta)\sin(\varphi - \varphi_i),$$
$$\tilde{B}_1^0 = \mathbf{B}_1^0(\mathbf{o})\mathbf{B}_1^0(\mathbf{i}) = \hat{\theta}\hat{\theta}_i \sin\theta \sin\theta_i,$$
$$\tilde{B}_1^1 = \mathrm{Re}\, \mathbf{B}_1^1(\mathbf{o})\mathbf{B}_1^{1*}(\mathbf{i})$$
$$= (\hat{\theta}\hat{\theta}_i \cos\theta \cos\theta_i + \hat{\varphi}\hat{\varphi}_i)\cos(\varphi - \varphi_i)$$
$$+ (\hat{\theta}\hat{\varphi}_i \cos\theta - \hat{\varphi}\hat{\theta}_i \cos\theta_i)\sin(\varphi - \varphi_i). \quad (95)$$

For a homogeneous sphere of radius a, for the surface conditions (6d),

$$c_n = \frac{-j_n(x)}{h_n(x)} d_n, \quad b_n = \frac{-\partial_x[x j_n(x)]}{\partial_x[x h_n(x)]} d_n, \quad x = ka. \quad (96)$$

For conditions (7d) plus (8d), we supplement (84) and (88) with the internal field

$$\tilde{\varphi}' = \sum [\mathbf{M}_{nm}^1(k'\mathbf{r})\mathbf{C}_n^{-m}(\mathbf{i})c_n' - i\mathbf{N}_{nm}^1 \mathbf{B}_n^{-m} b_n'] i^n (-1)^m, \quad (97)$$

and obtain

$$c_n = -\frac{j_n(X)\partial_x[x j_n(x)] - j_n(x)\partial_x[X j_n(X)]/\mu}{j_n(X)\partial_x[x h_n(x)] - h_n(x)\partial_x[X j_n(X)]/\mu} d_n$$
$$= c_n(\mu), \quad b_n = c_n(\epsilon), \quad X = k'a. \quad (98)$$

See Morse and Feshbach[6] (pp. 1882ff), Stratton[10] (pp. 563ff), and Van de Hulst[17] (pp. 113ff).

Small scatterer of arbitrary shape: For an arbitrarily shaped scatterer with all dimensions very small compared to wavelength, in terms of dyadic electric ($\tilde{p}$) and magnetic ($\tilde{m}$) dipole moments (pp. 1886ff), we have

$$\tilde{u} \cdot \tilde{\varphi} = \tilde{h}(kr) \cdot \tilde{p} \cdot \tilde{\varphi} + (\nabla \times \tilde{h}) \cdot \tilde{m} \cdot (\nabla \times \tilde{\varphi}/k^2)$$
$$= [\tilde{h} \cdot \tilde{p} + (\nabla \times \tilde{h}) \cdot \tilde{m} \cdot (\mathbf{i} \times \tilde{I}) i/k] \cdot \tilde{\varphi}, \quad (99)$$

where $\tilde{p}$ arises from the E field $\tilde{\varphi}$, and $\tilde{m}$ from the associated H field proportional to $\nabla \times \tilde{\varphi} = \mathbf{i} \times \tilde{\varphi} ik = \mathbf{i} \times \tilde{I} \cdot \tilde{\varphi} ik$; both $\tilde{p}$ and $\tilde{m}$ are independent of $\mathbf{i}$ and $\mathbf{o}$. From the definition of $\tilde{h}$ in (12), we obtain

$$\tilde{h} = (\tilde{I} - \mathbf{oo})\mathcal{H} + \mathbf{oo}H = \tilde{h}^{\mathrm{T}},$$
$$\mathcal{H}(x) = \frac{\partial_x[x h_1(x)]}{x}, \quad H(x) = \frac{2h_1}{x},$$
$$\nabla \times \tilde{h} = -kh_1 \mathbf{o} \times \tilde{I} = -kh_1 \tilde{I} \times \mathbf{o} = -(\nabla \times h)^{\mathrm{T}}. \quad (100)$$

Using $\tilde{h} \sim (\tilde{I} - \mathbf{oo})h$, $\nabla \times \tilde{h} \sim i\mathbf{o} \times \tilde{I}h$ in (99) to obtain $\tilde{u} \sim \tilde{g}h$, we write

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = (\tilde{I} - \mathbf{oo}) \cdot \tilde{p} \cdot (\tilde{I} - \mathbf{ii})$$
$$- (\mathbf{o} \times \tilde{I}) \cdot \tilde{m} \cdot (\tilde{I} \times \mathbf{i}) \equiv \tilde{g}_e + \tilde{g}_m. \quad (101)$$

Here

$$\tilde{I} - \mathbf{ii} = \boldsymbol{\epsilon\epsilon} + \boldsymbol{\delta\delta} = -(\tilde{I} \times \mathbf{i}) \cdot (\tilde{I} \times \mathbf{i}),$$
$$\tilde{I} \times \mathbf{i} = \mathbf{i} \times \tilde{I} = \boldsymbol{\delta\epsilon} - \boldsymbol{\epsilon\delta}$$

are both planar dyadics; the first is symmetrical, and the second is antisymmetrical. Both annihilate components of vectors parallel to $\mathbf{i}$; the second $(\tilde{I} \times \mathbf{i})$ turns perpendicular components through 90° around $\mathbf{i}$ as an axis, and the first $[\tilde{I} - \mathbf{ii} = -(\tilde{I} \times \mathbf{i})^2]$ is the negative of a turn through 180°; see Gibbs[18] for detailed discussion of $(\tilde{I} \times \mathbf{i})^n$.

From theorem (75) applied to (101), we obtain

$$\tilde{p} = \tilde{p}^{\mathrm{T}}, \quad \tilde{m} = \tilde{m}^{\mathrm{T}}; \quad (102)$$

thus each is symmetrical and may be put in the form $\tilde{p} = p_\xi \xi\xi + p_\eta \eta\eta + p_\zeta \zeta\zeta$, where the vectors correspond to the principal axis. From theorem (79), we obtain

$$-\mathrm{Re}\, \tilde{p} = \frac{1}{4\pi} \int \tilde{p} \cdot (\tilde{I} - \mathbf{oo}) \cdot \tilde{p}^*\, d\Omega = \tfrac{2}{3} \tilde{p} \cdot \tilde{p}^*,$$
$$-\mathrm{Re}\, \tilde{m} = -\frac{1}{4\pi} \int \tilde{m} \cdot (\tilde{I} \times \mathbf{o}) \cdot (\tilde{I} \times \mathbf{o}) \cdot \tilde{m}^*\, d\Omega$$
$$= \tfrac{2}{3} \tilde{m} \cdot \tilde{m}^*. \quad (103)$$

[17] H. C. van de Hulst, *Light Scattering by Small Particles* (John Wiley & Sons, Inc., New York, 1953), Chap. 9.

[18] See J. Willard Gibbs, *Vector Analysis*, Vol. II *Collected Works, Vol. II* (Yale University Press, New Haven, Conn., 1948), pp. 61ff, for $(I \times \mathbf{i})^n$; and also E. B. Wilson, *Gibb's Vector Analysis* (Yale University Press, New Haven, Conn., 1943), pp. 299ff. More generally, the dyadic operations of this paper are based on their development, and also on C. E. Weatherburn, *Advanced Vector Analysis* (Bell and Sons, London, 1949), and on Ref. 6.

With s equal to either ξ, η, or ζ, we have $-\mathrm{Re}\, p_s = \frac{2}{3}|p_s|^2$; similarly, with s or t or r equal to either x, y, or z, we have $-\mathrm{Re}\, p_{st} = -\mathrm{Re}\, p_{ts} = \frac{2}{3}\sum p_{sr}p_{rt}^*$.

The special case (95) corresponds to $\tilde{p} = b_1\tilde{I}$, $\tilde{m} = c_1\tilde{I}$:

$$\tilde{g}(\mathbf{o}, \mathbf{i}) = b_1(\tilde{I} - \mathbf{oo})\cdot(\tilde{I} - \mathbf{ii}) - c_1(\mathbf{o}\times\tilde{I})\cdot(\tilde{I}\times\mathbf{i}) = b_1(\hat{\theta}\hat{\theta} + \hat{\varphi}\hat{\varphi})\cdot(\hat{\theta}_i\hat{\theta}_i + \hat{\varphi}_i\hat{\varphi}_i) + c_1(\hat{\varphi}\hat{\theta} - \hat{\theta}\hat{\varphi})\cdot(\hat{\varphi}_i\hat{\theta}_i - \hat{\theta}_i\hat{\varphi}_i). \quad (104)$$

For later use, we make the relations between (95) and (104) explicit by rewriting $\tilde{h}$ in terms of Hansen's functions. From (35) for $r' \to 0$ we see that all terms vanish except

$$\mathbf{N}_{10} \to \tfrac{2}{3}(\mathbf{P}_1^0 + \mathbf{B}_1^0) = \tfrac{2}{3}(\hat{r}\cos\theta - \hat{\theta}\sin\theta) = \tfrac{2}{3}\hat{z},$$
$$\mathbf{N}_{11} \to \tfrac{2}{3}(\mathbf{P}_1^1 + \mathbf{B}_1^1) = \tfrac{2}{3}[\hat{r}e^{i\varphi}\sin\theta + e^{i\varphi}(\cos\theta\hat{\theta} + i\hat{\phi})] = \tfrac{2}{3}(\hat{x} + i\hat{y}),$$
$$\mathbf{N}_{1-1} = -\tfrac{1}{2}(\mathbf{N}_{11}^*) \to -\tfrac{1}{2}\,\tfrac{2}{3}(\hat{x} - i\hat{y}), \quad (105)$$

where $\hat{z}$, $(\hat{x} + i\hat{y})/\sqrt{2}$ and $(\hat{x} - i\hat{y})/\sqrt{2}$ form a set of orthonormal vectors. Consequently,

$$\tilde{h}(kr) = h^{\mathrm{T}} = \hat{z}\mathbf{N}_{10}(kr) + \tfrac{1}{2}(\hat{x} - i\hat{y})\mathbf{N}_{11} - (\hat{x} + i\hat{y})\mathbf{N}_{1-1} \equiv \hat{z}\mathbf{N}_{10} + \hat{x}\mathbf{N}_{11e} + \hat{y}\mathbf{N}_{11o}, \quad (106)$$

Using the asymptotic forms of the left- and right-hand sides, we also have

$$\tilde{I} - \mathbf{oo} = \hat{z}\mathbf{B}_1^0(\mathbf{o}) + \tfrac{1}{2}(\hat{x} - i\hat{y})\mathbf{B}_1^1 - (\hat{x} + i\hat{y})\mathbf{B}_1^{-1} = \hat{z}\mathbf{B}_1^0(\mathbf{o}) + \hat{x}\,\mathrm{Re}\,\mathbf{B}_1^1 + \hat{y}\,\mathrm{Im}\,\mathbf{B}_1^1 \equiv \hat{z}B_z(\mathbf{o}) + \hat{x}B_x(\mathbf{o}) + \hat{y}B_y(\mathbf{o}), \quad (107)$$

where since, $(\tilde{I} - \mathbf{oo}) = (\tilde{I} - \mathbf{oo})^{\mathrm{T}}$, we may transpose the left and right members of each term. Similarly, since $\nabla\times\mathbf{N}\hat{z} = (\nabla\times\mathbf{N})\hat{z} = k\mathbf{M}\hat{z}$ etc.,

$$\nabla\times\tilde{h}/k = -(\nabla\times\tilde{h})^T/k = \mathbf{M}_{10}\hat{z} + \tfrac{1}{2}\mathbf{M}_{11}(\hat{x} - i\hat{y}) - \mathbf{M}_{1-1}(\hat{x} + i\hat{y}) = -\hat{z}\mathbf{M}_{10} - \hat{x}\mathbf{M}_{11e} - \hat{y}\mathbf{M}_{11o}, \quad (108)$$

$$\mathbf{o}\times\tilde{I} = \tilde{I}\times\mathbf{o} = -(\mathbf{o}\times\tilde{I})^T = \hat{z}\mathbf{C}_1^0 + \hat{x}\,\mathrm{Re}\,\mathbf{C}_1^1 + \hat{y}\,\mathrm{Im}\,\mathbf{C}_1^1 \equiv \hat{z}\mathbf{C}_z + \hat{x}\mathbf{C}_x + \hat{y}\mathbf{C}_y. \quad (109)$$

Since $\tilde{I}\times\mathbf{o} = (\tilde{I} - \mathbf{oo})\cdot(\tilde{I}\times\mathbf{o})$, we also have $\tilde{I}\times\mathbf{o} = \mathbf{B}_z(\mathbf{o})\mathbf{C}_z(\mathbf{o}) + \mathbf{B}_x\mathbf{C}_x + \mathbf{B}_y\mathbf{C}_y$.

Substituting (107) into the electric term of (101), and letting $\hat{s}$ and $\hat{t}$ range over $\hat{x}$, $\hat{y}$, $\hat{z}$ we may write

$$\tilde{g}_e(\mathbf{o}, \mathbf{i}) = \sum_{s,t}\mathbf{B}_s(\mathbf{o})p_{st}\mathbf{B}_t(\mathbf{i}), \quad p_{st} = \hat{s}\cdot\tilde{p}\cdot\hat{t}; \quad \hat{s}, \hat{t} = \hat{x}, \hat{y}, \hat{z}. \quad (110)$$

We may also work with the first form of (107) to obtain

$$\tilde{g}_e(\mathbf{o}, \mathbf{i}) = \sum_{m\mu}\mathbf{B}_1^m(\mathbf{o})\mathbf{B}_1^{-\mu}(\mathbf{i})p_{m\mu}; \quad m, \mu = -1, 0, +1, \quad (111)$$

where, e.g.,

$$p_{01} = \hat{z}\cdot\tilde{p}\cdot(\hat{x} + i\hat{y}) = (\mathbf{P}_1^0 + \mathbf{B}_1^0)\cdot\tilde{p}\cdot(\mathbf{P}_1^1 + \mathbf{B}_1^1).$$

If the principal axes of $\tilde{p}$ coincide with $\hat{x}$, $\hat{y}$, $\hat{z}$ then (110) reduces to

$$\tilde{g}_e(\mathbf{o}, \mathbf{i}) = \mathbf{B}_x(\mathbf{o})\mathbf{B}_x(\mathbf{i})p_x + \mathbf{B}_y(\mathbf{o})\mathbf{B}_y(\mathbf{i})p_y + \mathbf{B}_z(\mathbf{o})\mathbf{B}_z(\mathbf{i})p_z. \quad (112)$$

The analogous discussion goes through for $\mathbf{g}_m$, i.e.,

$$\mathbf{g}_m = -(\tilde{I}\times\mathbf{o})\cdot\tilde{m}\cdot(\tilde{I}\times\mathbf{i}) = (\tilde{I}\times\mathbf{o})^T\cdot\tilde{m}\cdot(\tilde{I}\times\mathbf{i}) = \sum_{st}\mathbf{C}_s(\mathbf{o})m_{st}\mathbf{C}_t(\mathbf{i}), \quad \text{etc.} \quad (113)$$

See Morse and Feshbach[6] (pp. 1886ff) for an alternative development and for illustrations of $\tilde{p}$ and $\tilde{m}$. Electric dipole dyadics are also considered by Yvon,[19] Mazur,[20] Fixman,[21] Brown,[22] and others.

3. MANY SCATTERERS

For many scatterers in the geometry of Fig. 3:1, we write the vector field as

$$\boldsymbol{\Psi} = \boldsymbol{\varphi}(\mathbf{i}:\boldsymbol{\epsilon}) + \mathfrak{U}, \quad \mathfrak{U} \sim h(kr)\mathfrak{G}(\mathbf{o}, \mathbf{i}:\boldsymbol{\epsilon}), \quad (114)$$

where $\mathfrak{U}$ and $\mathfrak{G}$ have the forms (12) and (16) with $\mathbf{u}$ replaced by $\mathfrak{U}$. The "compound amplitude" $\mathfrak{G}$ fulfills the same theorems as $\mathbf{g}$.

Proceeding as in Refs. 1 and 3, we express the total scattered field of a configuration of scatterers (whose "centers" are at $\mathbf{b}_s$) as

$$\mathfrak{U} = \sum\mathbf{U}_s(\mathbf{r} - \mathbf{b}_s)e^{i\mathbf{k}\cdot\mathbf{b}_s}, \quad \mathbf{U}_s = \{\tilde{h}(k|\mathbf{r}_s - \mathbf{r}_s'|), \mathbf{U}_s(\mathbf{r}_s')\}, \quad (115)$$

where $\mathbf{r}_s = \mathbf{r} - \mathbf{b}_s$ and $\mathbf{r}_s'$ are an observation point and surface point respectively in the local coordinates of scatterer s. For $kr_s \sim \infty$,

$$\mathbf{U}_s \sim h(kr_s)\{\tilde{\varphi}(\mathbf{r}_s'; -\mathbf{o}), \mathbf{U}_s(\mathbf{r}_s')\} \equiv h(kr_s)\mathbf{G}_s(\mathbf{o}), \quad (116)$$

where $\mathbf{G}_s$, the "multiple-scattered amplitude" of scatterer s, reduces to the single-scattered function $\mathbf{g}_s$ as the others recede to infinity. In terms of $\mathbf{G}_s$ the compound amplitude equals

$$\mathfrak{G}(\mathbf{o}, \mathbf{i}:\boldsymbol{\epsilon}) = \sum_s e^{ik(\mathbf{i}-\mathbf{o})\cdot\mathbf{b}_s}\mathbf{G}_s(\mathbf{o}, [\mathbf{i}:\boldsymbol{\epsilon}]), \quad (117)$$

where the brackets are to indicate that $\mathbf{i}:\boldsymbol{\epsilon}$ plays a less complete role in $\mathbf{G}$ than in $\mathfrak{G}$ or $\mathbf{g}$.

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[19] J. Yvon, *Actualités scientifiques et industrielles* (Hermann Cie., Paris, 1937), Nos. 542 and 543.
[20] P. Mazur and M. Mandel, Physica 22, 289 (1956).
[21] M. Fixman, J. Chem. Phys. 23, 2074 (1955).
[22] W. F. Brown, Jr., in *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 17.

Integral equations: Substituting $\tilde{h}$ of (26) into $\mathbf{U}_s$ of (115), and rewriting in terms of $\mathbf{G}_s$ of (116), we obtain

$$\mathbf{U}_s(\mathbf{r}_s) = \frac{1}{2\pi}\int e^{ik\mathbf{p}\cdot\mathbf{r}_s}\mathbf{G}_s(\mathbf{p})\, d\Omega_p, \qquad (118)$$

$$\mathfrak{U} = \sum e^{i\mathbf{k}\cdot\mathbf{b}_s}\int e^{ik\mathbf{p}\cdot(\mathbf{r}-\mathbf{b}_s)}\mathbf{G}_s(\mathbf{p})\, d\Omega_p/2\pi. \qquad (119)$$

Proceeding as in Ref. 1, we use $\mathbf{r}_s = \mathbf{r}_t + \mathbf{b}_t - \mathbf{b}_s \equiv \mathbf{r}_t + \mathbf{b}_{ts}$ to express $\boldsymbol{\varphi}$ and $\mathfrak{U}$ in the local coordinates of scatterer t, and write the total field referred to t as a set of plane waves plus one outgoing wave $\mathbf{U}_t$:

$$\begin{aligned}\Psi(\mathbf{b}_t + \mathbf{r}_t) &= e^{i\mathbf{k}\cdot\mathbf{b}_t}\Big[e^{ik\mathbf{i}\cdot\mathbf{r}_t}\boldsymbol{\epsilon} + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}e^{ik\mathbf{p}\cdot\mathbf{r}_t}\mathbf{G}_s(\mathbf{p})\, d\Omega/2\pi + \mathbf{U}_t\Big] \\ &= e^{i\mathbf{k}\cdot\mathbf{b}_t}[\boldsymbol{\Phi}_t + \mathbf{U}_t], \qquad (120)\end{aligned}$$

where $\sum'$ means sum over $s \neq t$ and where $\boldsymbol{\Phi}_t$ is the total excitation at t. Then knowing the response ($\mathbf{u}$) of the scatterer to one plane wave, we use the superposition principle to write

$$\mathbf{U}_t = \mathbf{u}_t(\mathbf{i}:\boldsymbol{\epsilon}) + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\mathbf{u}_t(\mathbf{p}:\boldsymbol{\gamma}_s)\mathbf{G}_s(\mathbf{p})\, d\Omega/2\pi,$$
$$\boldsymbol{\gamma}_s \equiv \mathbf{G}_s/G_s, \qquad (121)$$

where $\boldsymbol{\gamma}_s$ is the polarization of $\mathbf{G}_s$.

The asymptotic form of (121) for $r_t \to \infty$ gives a "self-consistent" system of integral equations for the multiple-scattering amplitude:

$$\mathbf{G}_t(\mathbf{o}) = \mathbf{g}_t(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon}) + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\mathbf{g}_t(\mathbf{o},\mathbf{p}:\boldsymbol{\gamma}_s)G_s(\mathbf{p})\, d\Omega_p/2\pi, \qquad (122)$$

where in general $\mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon})$ and $\mathbf{g}(\mathbf{o},\mathbf{p}:\boldsymbol{\gamma}_s)$ are not parallel. Forming $\mathbf{e}\cdot\mathbf{G}_t$, and using the reciprocity relation (20) to replace $\mathbf{e}\cdot\mathbf{g}_t(\mathbf{o},\mathbf{p}:\boldsymbol{\gamma}_s)$ by $\boldsymbol{\gamma}_s\cdot\mathbf{g}_t(-\mathbf{p},-\mathbf{o}:\mathbf{e})$ we see from the definition (116) for $\mathbf{G}_s$ that the integral converges if $\operatorname{Im}\mathbf{p}\cdot(\mathbf{b}_{ts} + \mathbf{r}_t' - \mathbf{r}_s') > 0$. In terms of $\mathbf{b}_{ts} = b_{ts}\hat{b}_{ts}$, we require $b_{ts} > [(\mathbf{r}_t' + \mathbf{r}_s')\cdot\hat{b}_{ts}]_{\max}$, i.e., that the sum of the scatterer's projections on $\mathbf{b}_{ts}$ do not overlap.

The integral equation (122) is essentially a "reciprocity relation" between G and g. This follows on applying Green's theorem (10) to $\boldsymbol{\psi}_1$ and $\boldsymbol{\Psi}_2$, with $\boldsymbol{\psi}_1$ as the solution for $\boldsymbol{\varphi}_1$ incident on an isolated scatterer t, and $\boldsymbol{\Psi}_2$ as the solution for $\boldsymbol{\varphi}_2$ incident on a collection of scatterers which includes t; $\boldsymbol{\psi}_1$ and $\boldsymbol{\Psi}_2$ satisfy the same conditions at t's surface and the same wave equation in its interior. Consequently, essentially as for (18), we obtain

$$\begin{aligned}0 &= \{\boldsymbol{\psi}_1, \boldsymbol{\Psi}_2\}_t \\ &= \{(\boldsymbol{\varphi}_1 + \mathbf{u}_{t1}), (\boldsymbol{\varphi}_2 + \sum{}'\mathbf{U}_{s2}e^{i\mathbf{k}\cdot\mathbf{b}_{ts}} + \mathbf{U}_{t2})\}_t, \qquad (123)\end{aligned}$$

where the subscript t indicates integration is over a surface that isolates scatterer t from the others. We have $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\} = 0$ as previously; similarly $\{\boldsymbol{\varphi}_1, \mathbf{U}_{s2}\}_t = 0$ since $\mathbf{U}_{s2}$ has no singularities inside the surface that isolates t; finally $\{\mathbf{u}_{t1}, \mathbf{U}_{t2}\}_t = \{\mathbf{u}_{t1}, \mathbf{U}_{t2}\}_\infty = 0$ follows from the asymptotic forms (5) and (116). Consequently (122) reduces to

$$\{\boldsymbol{\varphi}_1, \mathbf{U}_{t2}\} = -\{\mathbf{u}_{t1}, \boldsymbol{\varphi}_2\} - \{\mathbf{u}_{t1}, \sum{}'\mathbf{U}_{s2}e^{-i\mathbf{k}\cdot\mathbf{b}_{st}}\}. \qquad (124)$$

Using the definitions of $\mathbf{G}$ and $\mathbf{g}$ as in (116) and (16), and proceeding as for (20), we reduce (124) to

$$\begin{aligned}\boldsymbol{\epsilon}_1\cdot\mathbf{G}_t(-\mathbf{i}_1, \mathbf{i}_2:\boldsymbol{\epsilon}_2) &= \boldsymbol{\epsilon}_2\cdot\mathbf{g}_t(-\mathbf{i}_2, \mathbf{i}_1:\boldsymbol{\epsilon}_1) \\ &\quad + \{\sum{}'\mathbf{U}_{s2}e^{-i\mathbf{k}\cdot\mathbf{b}_{st}}, \mathbf{u}_{t1}\}. \qquad (125)\end{aligned}$$

Introducing the plane wave representation (118) for $\mathbf{U}_{s2}$ and the definition of $\mathbf{g}$ in the kernel, gives

$$\begin{aligned}\boldsymbol{\epsilon}_1\cdot\mathbf{G}_t(-\mathbf{i}_1) &= \boldsymbol{\epsilon}_2\cdot\mathbf{g}_t(-\mathbf{i}_2, \mathbf{i}_1:\boldsymbol{\epsilon}_1) \\ &\quad + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{r})\cdot\mathbf{b}_{ts}}\mathbf{g}_t(-\mathbf{p}, \mathbf{i}_1:\boldsymbol{\epsilon}_2)\cdot\mathbf{G}_s(\mathbf{p})\, d\Omega_p/2\pi. \qquad (126)\end{aligned}$$

Applying (20) to $\mathbf{g}_t$, and replacing $-\mathbf{i}_1$ by $\mathbf{o}$ we reduce (126) to $\boldsymbol{\epsilon}_1\cdot(122)$.

Equation (122) is a mixed vector–scalar form. The analogous mixed vector–dyadic form is obtained by introducing the dyadic isolated-scattering amplitude $\tilde{g}$ of (66). Thus since $\mathbf{g}(\mathbf{o},\mathbf{i}:\boldsymbol{\epsilon}) = \tilde{g}(\mathbf{o},\mathbf{i})\cdot\boldsymbol{\epsilon}$, and $\mathbf{g}(\mathbf{o},\mathbf{p},\boldsymbol{\gamma}_s) = \tilde{g}(\mathbf{o},\mathbf{p})\cdot\boldsymbol{\gamma}_s$, we may rewrite (122) as

$$\begin{aligned}\mathbf{G}_t(\mathbf{o}) &= \tilde{g}_t(\mathbf{o},\mathbf{i})\cdot\boldsymbol{\epsilon} \\ &\quad + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\tilde{g}_t(\mathbf{o},\mathbf{p})\cdot\mathbf{G}_s(\mathbf{p})\, d\Omega_p/2\pi. \qquad (127)\end{aligned}$$

Similarly, we obtain a complete dyadic representation by introducing a multiple-scattered dyadic amplitude $\tilde{G}$, such that

$$\mathbf{G}(\mathbf{o}) = \tilde{G}(\mathbf{o})\cdot\boldsymbol{\epsilon}, \qquad (128)$$

and dropping $\boldsymbol{\epsilon}$:

$$\tilde{G}(\mathbf{o}) = \tilde{g}_t(\mathbf{o},\mathbf{i}) + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\tilde{g}_t(\mathbf{o},\mathbf{p})\cdot\tilde{G}_s(\mathbf{p})\, d\Omega/2\pi, \qquad (129)$$

which is the complete analog of (3:34). Alternatively from $\{\tilde{\varphi}_1, \tilde{\Psi}_2\}_t = 0$ we obtain (124d), i.e.,

$$\begin{aligned}\{\tilde{\varphi}_1, \tilde{U}_{t2}\} &= -\{\tilde{u}_{t1}, \tilde{\varphi}_2\} - \{\tilde{u}_{t1}, \sum{}'\tilde{U}_{s2}e^{-i\mathbf{k}\cdot\mathbf{b}_{st}}\} \\ &= -\{\tilde{u}_{t1}, \tilde{\varphi}_2\} \\ &\quad - \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\{\tilde{u}_{t1}, \tilde{\varphi}(\mathbf{r}_t;\mathbf{p})\}\cdot\tilde{G}_s(\mathbf{p})\, d\Omega/2\pi, \qquad (130)\end{aligned}$$

where the last form followed from (118d). From (116d) and (73) and (74), we reduce (130) to

$$\begin{aligned}\tilde{G}(-\mathbf{i}_1) &= \tilde{g}^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) \\ &\quad + \sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\tilde{g}^{\mathrm{T}}(-\mathbf{p},\mathbf{i}_1)\cdot\tilde{G}_s(\mathbf{p})\, d\Omega/2\pi, \qquad (131)\end{aligned}$$

from which we obtain (129) by using the reciprocity relation (75) to convert $\tilde{g}^T$ to $\tilde{g}(-\mathbf{i}_1, \mathbf{i}_2)$ and $\tilde{g}(-\mathbf{i}_1, p)$, and then replacing $-\mathbf{i}_1$ by $\mathbf{o}$. See Appendix A for additional relations and discussion of reciprocity.

Large spacings: We obtain forms of (129) convenient for large $k|\mathbf{b}_t - \mathbf{b}_s| = kb_{ts}$ by applying (55):

$$\tilde{G}_t(\mathbf{o}, \mathbf{i}) = \tilde{g}_t(\mathbf{o}, \mathbf{i}) + \sum_s{}' \tilde{\mathcal{F}}_{ts} \cdot \tilde{g}_t(\mathbf{o}, \hat{b}_{ts}) \cdot \tilde{G}_s(\hat{b}_{ts}, \mathbf{i}),$$
$$\tilde{\mathcal{F}}_{ts} = h(kb_{ts}) e^{-i\mathbf{k}\cdot\mathbf{b}_{ts}} \tilde{\mathfrak{D}}_{ts}$$
$$\equiv \tilde{\mathcal{K}}_{ts}(b^{-1}) + \tilde{\mathcal{M}}_{ts}(b^{-2}) + \tilde{\mathcal{N}}_{ts}(b^{-3}) + \cdots, \tag{132}$$

where $\tilde{\mathfrak{D}}$ is given terms of $\tilde{D}$ in (50), and the present subscripts indicate that the differentiations of (44) in $\tilde{D}$ are to be performed with respect to the angles associated with the unit vector $\hat{b}_{ts}$. We introduced the additional factor $\mathbf{i}$ in the argument of $\tilde{G}$ to facilitate iteration. If we keep only the leading term of $\tilde{\mathfrak{D}}$ (i.e., $\tilde{I}$), then

$$\tilde{G}_t(\mathbf{o}, \mathbf{i}) \sim \tilde{g}_t(\mathbf{o}, \mathbf{i}) + \sum_s{}' \frac{e^{ikb_{ts} - i\mathbf{k}\cdot\mathbf{b}_{ts}}}{ikb_{ts}} \tilde{g}_t(\mathbf{o}, \hat{b}_{ts}) \cdot \tilde{G}_s(\hat{b}_{ts}, \mathbf{i}); \tag{133}$$

if we dot-multiply from the right by $\mathbf{e}$ we have the system of equations discussed by Saxon[7] (pp. 92–99). (The analogous equations for the scalar problem, and the iterated orders-of-scattering form are discussed by Karp, and by Twersky in the papers cited in the survey, Ref. 5.)

The leading term of (132) is the single-scattered value, or equivalently the "first-order" of scattering $\tilde{g}_t(\mathbf{o}, \mathbf{i})$. Iterating (132) starting with $\tilde{g}_t(\mathbf{o}, \mathbf{i})$ yields a series in inverse powers of kb_{ts} which involves $\tilde{g}$ and its derivatives. Thus the $(kb)^{-1}$ term [either of (132) or (133)] is the far-field multiple scattering form of the second order of scattering:

$$\sum_s{}' \tilde{\mathcal{K}}_{ts} \cdot \tilde{g}_t(\mathbf{o}, \hat{b}_{ts}) \cdot \tilde{g}_s(\hat{b}_{ts}, \mathbf{i}), \quad \tilde{\mathcal{K}}_{ts} \equiv h(kb_{ts}) e^{-i\mathbf{k}\cdot\mathbf{b}_{ts}} \tilde{I};$$

i.e., the dyadic analog of (3:37). Terms to order $(kb)^{-2}$ are given by

$$\sum_s{}' \tilde{\mathcal{K}}_{ts} \cdot \tilde{g}_t(\mathbf{o}, \hat{b}_{ts}) \cdot \sum_p{}' \tilde{\mathcal{K}}_{sp} \cdot \tilde{g}_s(\hat{b}_{ts}, \hat{b}_{sp}) \cdot \tilde{g}_p(\hat{b}_{sp}, \mathbf{i}) + \sum_s{}' \tilde{\mathcal{M}}_{ts} \cdot \tilde{g}_t(\mathbf{o}, \hat{b}_{ts}) \cdot \tilde{g}_s(\hat{b}_{ts}, \mathbf{i}),$$
$$\tilde{\mathcal{M}}_{ts} \equiv (i/2kb_{ts}) \tilde{\mathcal{K}}_{ts} \cdot \tilde{D}_{ts},$$

where the double sum corresponds to the third far-field order, and the single sum is the first "mid-field" correction to the second far-field order; this is the analog of (3:38). The next terms in the expansion of $\tilde{G}$, the terms or order $(kb)^{-3}$ are given by (3:39d), obtained from (39) of Ref. 3 by replacing g by $\tilde{g}$, the previous scalar operators $\mathcal{K}$ and $\mathcal{M}$ by the present dyadics, and the previous $\mathcal{N}$ by

$$\tilde{\mathcal{N}} = [(i/2kb_{ts})^2/2] \tilde{\mathcal{K}}_{ts} \cdot \tilde{D}_{ts} \cdot (\tilde{D}_{ts} - 2\tilde{I}).$$

Algebraic equations: If we substitute spherical harmonic representations for $\tilde{g}_t$ and $\tilde{G}_t$ in (129), i.e.,

$$\tilde{g}_t(\mathbf{o}, \mathbf{i}) = \sum [\mathbf{C}_n^m(\mathbf{o}) \mathbf{c}_{nm}^t(\mathbf{i}) + \mathbf{B}_n^m(\mathbf{o}) \mathbf{b}_{nm}^t(\mathbf{i})], \tag{134}$$
$$\tilde{G}_t(\mathbf{o}) = \sum [\mathbf{C}_n^m(\mathbf{o}) \mathfrak{C}_{nm}^t + \mathbf{B}_n^m(\mathbf{o}) \mathfrak{B}_{nm}^t], \tag{135}$$

and use the orthogonality of the $\mathbf{C}$'s and $\mathbf{B}$'s we obtain

$$\mathfrak{C}_{nm}^t = \mathbf{c}_{nm}^t(\mathbf{i}) + \sum{}' \sum \int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}} \mathbf{c}_{nm}^t(\mathbf{p}) \cdot [\mathbf{C}_r^q(\mathbf{p}) \mathfrak{C}_{rq}^s + \mathbf{B}_r^q(\mathbf{p}) \mathfrak{B}_{rq}^s] \, d\Omega_p / 2\pi,$$
$$\mathfrak{B}_{nm}^t = \mathbf{b}_{nm}^t(\mathbf{i}) + \sum{}' \sum \int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}} \mathbf{b}_{nm}^t(\mathbf{p}) \cdot [\mathbf{C}_r^q(\mathbf{p}) \mathfrak{C}_{rq}^s + \mathbf{B}_r^q(\mathbf{p}) \mathfrak{B}_{rq}^s] \, d\Omega_p / 2\pi. \tag{136}$$

If we expand the isolated scattering coefficients as series of spherical harmonics as in (87), then we may write

$$\mathfrak{C}_{nm}^t = \mathbf{c}_{nm}^t(\mathbf{i}) + \sum{}' \sum [\alpha_{nm\nu\mu}^t \mathfrak{C}_{rq}^s \mathcal{E}(st; \nu\mu, rq) + \beta \mathfrak{C} \mathcal{E}' - \alpha \mathfrak{B} \mathcal{E}' + \beta \mathfrak{B} \mathcal{E}],$$
$$\mathfrak{B}_{nm}^t = \mathbf{b}_{nm}^t(\mathbf{i}) + \sum{}' \sum [\gamma \mathfrak{C} \mathcal{E} + \delta \mathfrak{C} \mathcal{E}' - \gamma \mathfrak{B} \mathcal{E}' + \delta \mathfrak{B} \mathcal{E}], \tag{137}$$

where the scheme for the indices is shown only once, and where

$$\mathcal{E}(st; \nu\mu, rq) = \frac{e^{-i\mathbf{k}\cdot\mathbf{b}_{ts}}}{2\pi} \int e^{ik\mathbf{p}\cdot\mathbf{b}_{ts}} \mathbf{C}_\nu^{-\mu}(\mathbf{p}) \cdot \mathbf{C}_r^q(\mathbf{p}) \, d\Omega$$
$$= \mathcal{E}(\mathbf{C} \cdot \mathbf{C}) = \mathcal{E}(\mathbf{B} \cdot \mathbf{B}),$$
$$\mathcal{E}' = \mathcal{E}(\mathbf{B} \cdot \mathbf{C}) = -\mathcal{E}(\mathbf{C} \cdot \mathbf{B}). \tag{138}$$

Following the procedure used for the scalar case, we write $\mathbf{C} \cdot \mathbf{C}$ and $\mathbf{B} \cdot \mathbf{C}$ as sets of products of surface harmonics YY to reduce the present $\mathcal{E}$'s to sets of the E's of (3:42), and then use (3:43) to write $\mathcal{E}$ and $\mathcal{E}'$ in terms of h's and their derivatives times Y's. We illustrate this subsequently. (In the above, we have generated implicitly the addition theorems discussed by Stein.[16])

In particular, for spherically symmetric scatterers (137) reduces to

$$\mathfrak{C}_{nm}^t = (-1)^m c_n^t \{\mathbf{C}_n^{-m}(\mathbf{i}) + \sum{}' \sum [\mathfrak{C}_{rq}^s \mathcal{E}(st; nm, rg) - \mathfrak{B}_{rq}^s \mathcal{E}']\},$$
$$\mathfrak{B}_{nm}^t = (-1)^m b_n^t \{\mathbf{B}_n^{-m}(\mathbf{i}) + \sum{}' \sum [\mathfrak{C}_{rq}^s \mathcal{E}' + \mathfrak{B}_{rq}^s \mathcal{E}]\}, \tag{139}$$

which we apply in detail to two scatterers in the next section.

4. TWO SCATTERERS

For two scatterers, we take the primary origin ($r = 0$) as the midpoint of the line joining the centers of their circumscribed spheres. The centers are located at

$$\mathbf{b}_1 = \mathbf{b}(b, \tau, \beta) = b\mathbf{b}_+,$$
$$\mathbf{b}_2 = \mathbf{b}(b, \pi - \tau, \pi + \beta) = b\mathbf{b}_- = -\mathbf{b},$$

where b, τ, β are spherical coordinates; the local coordinates with respect to these centers are written as $\mathbf{r}_1 = \mathbf{r}_+$ and $\mathbf{r}_2 = \mathbf{r}_-$. For this case the scattered field reduces to

$$\tilde{\mathfrak{U}}(\mathbf{r}) = e^{i\delta}\tilde{U}_+(\mathbf{r}_+) + e^{-i\delta}\tilde{U}_-(\mathbf{r}_-),$$
$$\pm\delta = bk\mathbf{i}\cdot\mathbf{b}_\pm = \pm\mathbf{k}\cdot\mathbf{b}, \tag{140}$$

and the compound scattering amplitude equals

$$\mathfrak{G}(\mathbf{o}, \mathbf{i}) = e^{i(\delta-\Delta)}\tilde{G}_+(\mathbf{o}, i) + e^{-i(\delta-\Delta)}\tilde{G}_-(\mathbf{o}, \mathbf{i}),$$
$$\pm\Delta = kb\mathbf{o}\cdot\hat{b}_\pm = \pm k\mathbf{o}\cdot\mathbf{b}. \tag{141}$$

The plane wave representation yields

$$\tilde{U}_\pm = \frac{1}{2\pi}\int e^{i k\mathbf{p}\cdot\mathbf{r}_\pm}\tilde{G}_\pm(\mathbf{p})\,d\Omega_p, \tag{142}$$

where

$$\tilde{G}_\pm(\mathbf{o}) = \tilde{g}_\pm(\mathbf{o}, \mathbf{i}) + e^{\mp i2\delta}\int e^{i2kb\mathbf{b}\cdot\mathbf{p}_\pm}\tilde{g}_\pm(\mathbf{o}, \mathbf{p})\cdot\tilde{G}_\mp(\mathbf{p})d\Omega_p/2\pi. \tag{143}$$

4.1. Inverse Separation Representation

For two scatterers, (132) reduces to

$$\tilde{G}_\pm(\mathbf{o}) = \tilde{g}_\pm(\mathbf{o}, \mathbf{i}) + \tilde{\mathcal{F}}_\pm\cdot\tilde{g}_\pm(\mathbf{o}, \mathbf{b}_\pm)\cdot\tilde{G}_\mp(\mathbf{b}_\pm),$$
$$\tilde{\mathcal{F}}_\pm = h(2kb)e^{\mp i2\delta}\mathfrak{D}_\pm, \tag{144}$$

where the subscripts on $\mathfrak{D}_\pm$ etc., indicate that the differentiations are to be performed with respect to the angles associated with the unit vectors $\mathbf{b}_\pm$. Replacing $\mathbf{o}$ by $\mathbf{b}_\pm$, we solve for

$$\tilde{G}_\mp(\mathbf{b}_\mp) = [\bar{I} - \tilde{\mathcal{F}}_\mp\cdot\tilde{g}_\mp(\mathbf{b}_\pm, \mathbf{b}_\mp)\cdot\tilde{\mathcal{F}}_\pm\cdot\tilde{g}_\pm(\mathbf{b}_\mp, \mathbf{b}_\pm)]^{-1}\cdot[\tilde{g}_\mp(\mathbf{b}_\pm, \mathbf{i}) + \tilde{\mathcal{F}}_\mp\cdot\tilde{g}_\mp(\mathbf{b}_\pm, \mathbf{b}_\mp)\cdot\tilde{g}_\pm(\mathbf{b}_\mp, \mathbf{i})], \tag{145}$$

which when substituted into (144) gives a closed operational form for $\tilde{G}_\pm(\mathbf{o})$ in terms of the isolated scatterer functions $\tilde{g}_\pm$, i.e., the analog of (3:50). Since the inverse dyadic $[\bar{I} - \bar{X}]^{-1}$ equals $\bar{I} + \bar{X} + \bar{X}\cdot\bar{X} + \cdots$, we see that the expansion of the closed form in powers of b^{-1} yields the series (3:51d) to (3:54d), on replacing the previous scalars by our present functions.

4.2. Radially Symmetric Scatterers

For two spherically symmetric scatterers, we use isolated scattering amplitudes $\tilde{g}_\pm$ as in (134) in terms of $c_n^\pm$ and $b_n^\pm$, and $\tilde{G}_\pm$ as in (135) in terms of $\tilde{\mathfrak{C}}_{nm}^\pm$ and $\mathfrak{B}_{nm}^\pm$. Specializing (139), we have

$$\mathfrak{C}_{nm}^\pm = (-1)^m c_n^\pm\{\mathbf{C}_n^{-m}(\mathbf{i}) + \sum_{\nu\mu}[\mathfrak{C}_{\nu\mu}^\mp\mathcal{E}_\pm(nm, \nu\mu) - \mathfrak{B}_{\nu\mu}^\mp\mathcal{E}_\pm']\},$$
$$\mathfrak{B}_{nm}^\pm = (-1)^m b_n^\pm\{\mathbf{B}_n^{-m}(\mathbf{i}) + \sum_{\nu\mu}[\mathfrak{C}_{\nu\mu}^\pm\mathcal{E}_\pm' + \mathfrak{B}_{\nu\mu}^\pm\mathcal{E}_\pm]\}, \tag{146}$$

where

$$\mathcal{E}_\pm(nm, \nu\mu) = e^{\mp i2\delta}\int e^{i2kb\mathbf{p}\cdot\mathbf{b}_\pm}\mathbf{C}_n^{-m}(\mathbf{p})\cdot\mathbf{C}_\nu^\mu(\mathbf{p})\,d\Omega/2\pi \tag{147}$$

and similarly $\mathcal{E}'$ involves $\mathbf{B}_n^{-m}\cdot\mathbf{C}_\nu^\mu$.

To illustrate the above, we keep only the electric and magnetic dipole terms (b_1 and c_1), and suppress the arguments $2kb$ in h_n, and $\hat{b}_\pm$ in Y_n^m. We retain only the six equations of (146) involving $\mathfrak{C}_{1m}$ and $\mathfrak{B}_{1m}$ for $m = 0, \pm 1$. The integral $\mathcal{E}_\pm(10, 10)$ involves $\mathbf{C}_1^0\cdot\mathbf{C}_1^0 = \sin^2\tau = \frac{2}{3}(Y_0 - Y_2)$, and consequently, from (3:14), we have $\mathcal{E}_\pm(10, 10) = \frac{2}{3}(h_0 + Y_2h_2)$; similarly for the other integrals. Thus

$$\mathfrak{C}_{10}^\pm/c_1^\pm = \mathbf{C}_1^0(\mathbf{i}) + e^{\mp i2\delta}[\tfrac{2}{3}(h_0 + Y_2h_2)\mathfrak{C}_{10}^\mp + \tfrac{1}{3}Y_2^1h_2\mathfrak{C}_{11}^\mp + Y_2^{-1}h_2\mathfrak{C}_{1-1}^\mp - Y_1^1h_1\mathfrak{B}_{11}^\mp + Y_1^{-1}h_1\mathfrak{B}_{1-1}^\mp],$$
$$-\mathfrak{C}_{11}^\pm/c_1^\pm = \mathbf{C}_1^{-1} + e^{\mp i2\delta}[Y_2^{-1}h_2\mathfrak{C}_{10}^\mp - \tfrac{1}{3}(2h_0 - Y_2h_2)\mathfrak{C}_{11}^\mp + 2Y_2^{-2}h_2\mathfrak{C}_{1-1}^\mp - Y_1^{-1}h_1\mathfrak{B}_{10}^\mp - Y_1h_1\mathfrak{B}_{11}^\mp],$$
$$-\mathfrak{C}_{1-1}^\pm/c_1^\pm = \mathbf{C}_1^1 + e^{\mp i2\delta}[\tfrac{1}{3}Y_2^1h_2\mathfrak{C}_{10}^\mp + \tfrac{1}{3}Y_2^2h_2\mathfrak{C}_{11}^\mp - \tfrac{1}{3}(2h_0 - Y_2h_2)\mathfrak{C}_{1-1}^\mp + Y_1^1h_1\mathfrak{B}_{10}^\mp + Y_1h_1\mathfrak{B}_{1-1}^\mp \tag{148}$$

plus the analogous set for $\mathfrak{B}^\pm$ obtained by interchanging all forms of "B" and "C" in the above and replacing all Y_1^m by $-Y_1^m$.

If the axis ($\hat{b}$) of the pair of scatterers is taken along the polar axis ($\hat{z}$) [i.e., scatterers located at $z_\pm = \pm b$, $x = y = 0$] then $\tau = 0, \pi$ and $\beta = 0$. All Y's but $Y_2 = 1$ and $Y_1(\hat{b}_\pm) = \pm 1$ vanish, and we may compress the remaining terms by using

$$\tfrac{2}{3}(h_0 + h_2) = 2h_1/\rho \equiv H,$$
$$\tfrac{1}{3}(2h_0 - h_2) = (\rho h_1)'/\rho \equiv \mathcal{H},$$
$$\rho = 2kb, \quad c_1 = c, \quad b_1 = b. \tag{149}$$

Thus

$$\mathfrak{C}_{10}^\pm/c = \mathbf{C}_1^0(\mathbf{i}) + e^{\mp i2\delta}H\mathfrak{C}_{10}^\mp,$$
$$\mathfrak{C}_{11}^\pm/c = -\mathbf{C}_1^{-1}(\mathbf{i}) + e^{\mp i2\delta}(\mathcal{H}\mathfrak{C}_{11}^\mp \pm h_1\mathfrak{B}_{11}^\mp),$$
$$\mathfrak{C}_{1-1}^\pm/c = -\mathbf{C}_1^1(\mathbf{i}) + e^{\mp i2\delta}(\mathcal{H}\mathfrak{C}_{1-1}^\mp \mp h_1\mathfrak{B}_{1-1}^\mp),$$
$$\mathfrak{B}_{10}^\pm/b = \mathbf{B}_1^0(\mathbf{i}) + e^{\mp i2\delta}H\mathfrak{B}_{10}^\mp,$$
$$\mathfrak{B}_{11}^\pm/b = -\mathbf{B}_1^{-1}(\mathbf{i}) + e^{\mp i2\delta}(\mathcal{H}\mathfrak{B}_{11}^\mp \pm h_1\mathfrak{C}_{11}^\mp),$$
$$\mathfrak{B}_{1-1}^\pm/b = -\mathbf{B}_1^1(\mathbf{i}) + e^{\mp i2\delta}(\mathcal{H}\mathfrak{B}_{1-1}^\mp \pm h_1\mathfrak{C}_{1-1}^\mp). \tag{150}$$

Since $\mathbf{C}_1^0 = \hat{\phi} \sin\theta_i$ and $\mathbf{B}_1^0 = \hat{\theta}\sin\theta_i$ the components $\mathfrak{C}_{10}$ and $\mathfrak{B}_{10}$ vanish for incidence along the pair's axis ($\theta = 0$); for these "axial" components, there is no coupling between electric and magnetic moments. On the other hand, the "perpendicular" components $\mathfrak{B}_{1\pm1}$ and $\mathfrak{C}_{1\pm1}$ are coupled in general.

For the axial components, we iterate once and regroup terms to obtain

$$\mathfrak{C}_{10}^{\pm} = c^{\pm}\{\mathbf{C}_1^0(\mathbf{i}) + e^{\pm i2\delta} H c^{\mp}[\mathbf{C}_1^0(\mathbf{i}) + e^{\mp i2\delta} H \mathfrak{C}_{10}^{\mp}]\}$$
$$= \frac{c^{\pm}[1 + c^{\mp} e^{\mp i2\delta} H]}{1 - c^+c^-H^2}\mathbf{C}_1^0(\mathbf{i}) \equiv c^{\pm}A(c^{\mp})\mathbf{C}_1^0(\mathbf{i}), \tag{151}$$

$$\mathfrak{B}_{10}^{\pm} = b^{\pm}A(b^{\mp})\mathbf{B}_1^0(\mathbf{i}), \tag{152}$$

where $A(b)$ is obtained on replacing c's by b's in $A(c)$.

For the perpendicular components, we first consider the cases when only the electric or only the magnetic effect exists.

Electric dipoles: If $c^{\pm} = 0$ the multiple-scattered field is fully specified by (152), and by the simplified form of the last two equations of (150),

$$\mathfrak{B}_{11}^{\pm} = -b^{\pm}[\mathbf{B}_1^{-1}(\mathbf{i}) - e^{\mp i2\delta}\mathfrak{K}\mathfrak{B}_{11}^{\mp}], \tag{153}$$

plus the analogous equation for $\mathfrak{B}_{1-1}^{\pm}$ involving $\mathbf{B}_1^1(\mathbf{i})$. Thus

$$\mathfrak{B}_{11}^{\pm} = -b^{\pm}D(b^{\mp})\mathbf{B}_1^{-1}(\mathbf{i}), \quad \mathfrak{B}_{1-1}^{\pm} = -b^{\pm}D(b^{\mp})\mathbf{B}_1^1(\mathbf{i}),$$
$$D(b^{\mp}) \equiv (1 + b^{\mp} e^{\mp i2\delta}\mathfrak{K})/(1 - b^+b^-\mathfrak{K}^2), \tag{154}$$

where D differs from A of (151) only in that H is replaced by $\mathfrak{K}$.

For this case,

$$\tilde{g}_{\pm}(\mathbf{o}, \mathbf{i}) = \tilde{B}(\mathbf{o}, \mathbf{i})b^{\pm} = (\tilde{B}^0 + \tilde{B}^1)b^{\pm}, \tag{155}$$

where the $\tilde{B}$'s are the corresponding $\tilde{B}_1$'s of (95). Similarly,

$$\tilde{G}^{\pm}(\mathbf{o}) = \tilde{G}^{\pm}(\mathbf{o}, \mathbf{i}) = b^{\pm}A^{\pm}(b^{\mp})\tilde{B}^0 + b^{\pm}D^{\pm}(b^{\mp})\tilde{B}^1. \tag{156}$$

Thus while each isolated scattering amplitude is an electric dipole determined essentially by the direction of incidence, the corresponding multiple-scattered amplitude is a sum of two uncoupled "compound dipoles": the compound axial term bA and perpendicular term bD are each the closed form of the corresponding geometrical progression of orders of scattering.

If the direction of incidence is along the dipole axis ($\mathbf{i} = \hat{z}$), then $\tilde{B}^0 = 0$, and if the incident polarization is $\boldsymbol{\epsilon} = \hat{x}$, we have

$$\tilde{B}^1 \cdot \hat{x} = (\hat{\theta}\cos\varphi\cos\theta - \hat{\varphi}\sin\varphi) \equiv \boldsymbol{\gamma}_1,$$
$$\tilde{g}(\mathbf{o}, \hat{z}) \cdot \hat{x} = \mathbf{g}(\mathbf{o}, \hat{z}: \hat{x}) = b^{\pm}\boldsymbol{\gamma}_1,$$
$$\tilde{G} \cdot \hat{x} = \mathbf{G} = b^{\pm}D(b^{\mp})\boldsymbol{\gamma}_1. \tag{157}$$

If the direction of incidence is perpendicular to the axes ($\mathbf{i} = \hat{x}$), and if the incident field is polarized parallel to the dipole's axis ($\boldsymbol{\epsilon} = \hat{z}$), then $\tilde{B}^1 \cdot \hat{z} = 0$ and

$$\tilde{B}^0 \cdot \hat{z} = -\hat{\theta}\sin\theta \equiv \boldsymbol{\gamma}_2, \quad \mathbf{g}(\mathbf{o}, \hat{x}: \hat{z}) = b_1^{\pm}\boldsymbol{\gamma}_2,$$
$$\mathbf{G} = b^{\pm}A(b^{\mp})\boldsymbol{\gamma}_2. \tag{158}$$

On the other hand, if the polarization is perpendicular to the axis ($\boldsymbol{\epsilon} = \hat{y}$) then $\tilde{B}^0 \cdot \hat{y} = 0$, and

$$\tilde{B}^1 \cdot \hat{y} = \hat{\varphi}\cos\varphi + \hat{\theta}\sin\varphi\cos\theta \equiv \boldsymbol{\gamma}_3,$$
$$\mathbf{g}(\mathbf{o}, \hat{x}: \hat{y}) = b^{\pm}\boldsymbol{\gamma}_3, \quad \mathbf{G} = b^{\pm}D(b^{\mp})\boldsymbol{\gamma}_3. \tag{159}$$

In all the above, the forwardscattered values of $\mathbf{g}$ and $\mathbf{G}$ have the same polarization as the incident wave. The same holds for arbitrary direction of incidence $\mathbf{i} = \mathbf{o}$ for which case we have $\tilde{B}^0 = \hat{\theta}\hat{\theta}\sin^2\theta$ and $\tilde{B}^1 = \hat{\theta}\hat{\theta}\cos^2\theta + \hat{y}\hat{y}$. If $\boldsymbol{\epsilon} = \hat{y}$, then $\mathbf{g} = b\boldsymbol{\epsilon}$ and $\mathbf{G} = bD\boldsymbol{\epsilon}$; similarly if $\boldsymbol{\epsilon} = \hat{\theta}$ (perpendicular to $\mathbf{i}$), then $\mathbf{g} = b\boldsymbol{\epsilon}(\sin^2\theta + \cos^2\theta) = b\boldsymbol{\epsilon}$, and

$$\mathbf{G} = b\boldsymbol{\epsilon}(A\sin^2\theta + D\cos^2\theta).$$

Although $\tilde{g}_+$ and $\tilde{g}_-$ satisfy (75), the theorem does not apply individually to the corresponding multiple-scattering functions $\tilde{G}_+$ and $\tilde{G}_-$ for the elements of the pair: the reciprocity relation applies only to the scattering amplitude for the configuration $\tilde{\mathfrak{G}}(\mathbf{o}, \mathbf{i})$ as in (141). From (141), (152), and (154), we write

$$\tilde{\mathfrak{G}} = \tilde{F}_+ + \tilde{F}_-,$$
$$\tilde{F}_+(\mathbf{o}, \mathbf{i}) = e^{i(\delta-\Delta)}\tilde{G}_+(\mathbf{o}, \mathbf{i}) = e^{ik\mathbf{b}\cdot(\mathbf{i}-\mathbf{o})}\tilde{K}_+ + e^{-ik\mathbf{b}\cdot(\mathbf{i}+\mathbf{o})}\tilde{K},$$
$$\tilde{F}_-(\mathbf{o}, \mathbf{i}) = e^{-i(\delta-\Delta)}\tilde{G}_-(\mathbf{o}, \mathbf{i}) = e^{-ik\mathbf{b}\cdot(\mathbf{i}-\mathbf{o})}\tilde{K}_- + e^{ik\mathbf{b}\cdot(\mathbf{i}+\mathbf{o})}\tilde{K};$$
$$\tilde{K}_+ = \frac{b_1^+\tilde{B}^0(\mathbf{o}, \mathbf{i})}{1 - b^+b^-H^2} - \frac{b_1^+\tilde{B}^1(\mathbf{o}, \mathbf{i})}{1 - b^+b^-\mathfrak{K}^2}, \quad \tilde{K}_- = \tilde{K}_+\frac{b^-}{b^+};$$
$$\tilde{K} = \frac{b^+b^-H}{1 - b^+b^-H^2}\tilde{B}^0(\mathbf{o}, \mathbf{i}) - \frac{b^+b^-\mathfrak{K}}{1 - b^+b^-\mathfrak{K}^2}\tilde{B}^1(\mathbf{o}, \mathbf{i}). \tag{160}$$

From (160), we have

$$\tilde{F}_+^T(-\mathbf{i}, -\mathbf{o}) = e^{ik\mathbf{b}\cdot(\mathbf{i}-\mathbf{o})}\tilde{K}_+ + e^{ik\mathbf{b}\cdot(\mathbf{i}+\mathbf{o})}\tilde{K} \neq \tilde{F}_+(\mathbf{o}, \mathbf{i}),$$
$$\tilde{F}_-^T(-\mathbf{i}, -\mathbf{o}) = e^{-ik\mathbf{b}\cdot(\mathbf{i}-\mathbf{o})}\tilde{K}_- + e^{-ik\mathbf{b}\cdot(\mathbf{i}+\mathbf{o})}\tilde{K} \neq \tilde{F}_-(\mathbf{o}, \mathbf{i}),$$
$$\tilde{\mathfrak{G}}^T(-\mathbf{i}, -\mathbf{o}) = \tilde{F}_+^T(-\mathbf{i}, -\mathbf{o}) + \tilde{F}_-^T(-\mathbf{i}, -\mathbf{o}) = \tilde{\mathfrak{G}}(\mathbf{o}, \mathbf{i}). \tag{161}$$

Thus, although the individual functions do not satisfy theorem (75) (because the phase of the $\tilde{K}$ term is not preserved) their sum does—and this is all that is required. See more general discussion in Appendix A.

In the forwardscattered direction,

$$\tilde{\mathfrak{G}}(\mathbf{i}, \mathbf{i}) = \tilde{G}_+(\mathbf{i}, \mathbf{i}) + \tilde{G}_-(\mathbf{i}, \mathbf{i})$$
$$= \tilde{K}_+ + \tilde{K}_- + 2\tilde{K}\cos(2k\mathbf{b}\cdot\mathbf{i}), \tag{162}$$

where we may use $\tilde{B}^0 = \boldsymbol{\epsilon}_1\boldsymbol{\epsilon}_1 \sin^2\theta$ and $\tilde{B}^1 = \boldsymbol{\epsilon}_1\boldsymbol{\epsilon}_1 \cos^2\theta + \boldsymbol{\epsilon}_2\boldsymbol{\epsilon}_2$. From theorem (80) the total cross section for a pair of scatterers equals

$$\mathfrak{Q}(\mathbf{i}:\boldsymbol{\epsilon}) = -(4\pi/k^2)\,\mathrm{Re}\,(\boldsymbol{\epsilon}\cdot\tilde{\mathfrak{G}}\cdot\boldsymbol{\epsilon})$$
$$= -(4\pi/k^2)2\,\mathrm{Re}\,[\boldsymbol{\epsilon}\cdot(\tilde{K}_+ + \tilde{K}_- + 2\tilde{K}\cos 2\delta)\cdot\boldsymbol{\epsilon}]. \quad (163)$$

For the special case of a pair of identical scatterers ($b^+ = b^-$) under symmetrical excitation (i.e., $\delta = k\mathbf{b}\cdot\mathbf{i} = 2n\pi;\ n = 0, \pm 1, \cdots,$ with $n = 0$ corresponding to incidence perpendicular to the axis) we have

$$\tilde{G}_- = \tilde{G}_+ = \tilde{G} = b(A\tilde{B}^0 + D\tilde{B}^1),$$
$$A = 1/(1 - bH), \quad D = 1/(1 - b\mathcal{H}), \quad (164)$$

and the total cross section follows from

$$\mathrm{Re}\,\tilde{\mathfrak{G}}(\mathbf{i},\mathbf{i}) = 2\,\mathrm{Re}\,\tilde{G}(\mathbf{i},\mathbf{i})$$
$$= \tilde{B}^0 2\,\mathrm{Re}\,bA + \tilde{B}^1 2\,\mathrm{Re}\,bD. \quad (165)$$

For lossless scatterers,

$$\mathrm{Re}\,bA = \frac{\mathrm{Re}\,b(1 - bH)^*}{|1 - bH|^2} = \frac{\mathrm{Re}\,b - |b|^2 J}{|1 - bH|^2},\quad J = \frac{2j_1}{\rho},$$

and since the theorem for an isolated lossless dipole gives $-\mathrm{Re}\,b = \frac{2}{3}|b|^2$, we have

$$\mathrm{Re}\,bA = \frac{-\frac{2}{3}|b|^2(1 + \frac{3}{2}J)}{|1 - bH|^2} = -\tfrac{2}{3}(1 + \tfrac{3}{2}J)\,|bA|^2.$$

Similarly for Re Db we replace H, J by $\mathcal{H}$, $\mathcal{J}$ with $\mathcal{J} = \partial_\rho[\rho j_1]/\rho$. Thus

$$\mathrm{Re}\,\tilde{\mathfrak{G}}(\mathbf{i},\mathbf{i}) = -\tfrac{4}{3}(1 + \tfrac{3}{2}\mathcal{J})\,|bA|^2\,\tilde{B}^0$$
$$+ \tfrac{4}{3}(1 + \tfrac{3}{2}\mathcal{J})\,|bD|^2\,\tilde{B}^1, \quad (166)$$

from which we obtain $\mathfrak{Q}$ by dot multiplication as in (163).

For $\delta = 2n\pi$, it is simple to demonstrate that $\tilde{\mathfrak{G}}(\mathbf{i}_1, \mathbf{i}_2)$ satisfies the general theorem (79). For the present case $\tilde{\mathfrak{G}}^T(\mathbf{i}_2, \mathbf{i}_1) = \tilde{\mathfrak{G}}(\mathbf{i}_1, \mathbf{i}_2)$, and consequently, (79) reduces to

$$-\mathrm{Re}\,\tilde{\mathfrak{G}}(\mathbf{i}_1, \mathbf{i}_2) = \frac{1}{4\pi}\int \tilde{\mathfrak{G}}(\mathbf{i}_1, \mathbf{o})\cdot\tilde{\mathfrak{G}}^*(\mathbf{o}, \mathbf{i}_2)\,d\Omega_o;$$

equivalently,

$$-2\,\mathrm{Re}\,\tilde{G}(\mathbf{i}_1, \mathbf{i}_2)$$
$$= \frac{1}{4\pi}\int \tilde{G}(\mathbf{i}_1, \mathbf{o})\cdot\tilde{G}^*(\mathbf{o}, \mathbf{i}_2)\,|e^{i\Delta} + e^{-i\Delta}|^2\,d\Omega_o, \quad (167)$$

where $\Delta = kb\hat{\mathbf{b}}\cdot\hat{\mathbf{o}} = kb\cos\theta$. Since $\tilde{G}^*$ is obtained by replacing bA and bD by their complex conjugates, we may show directly by proceeding as for (148) that (167) is satisfied. For example, in the right-hand side $|bD|^2\,\mathbf{B}_1^{-1}(\mathbf{i}_1)\mathbf{B}_1^1(\mathbf{i}_2)$ is multiplied by

$$\int \mathbf{B}_1^1(\mathbf{o})\cdot\mathbf{B}_1^{-1}(\mathbf{o})[2 + e^{i2\Delta} + e^{-i2\Delta}]\,d\Omega_o$$
$$= 2\int \mathbf{B}_1^1\cdot\mathbf{B}_1^{-1}\,d\Omega - \int \frac{(2 + Y_2)}{3}(e^{i2\Delta} + e^{-i2\Delta})\,d\Omega;$$

the first term gives $-\frac{16}{3}\pi$, and the second gives $-\frac{8}{3}\pi(2j_0 - j_2) = -8\pi\mathcal{J}$. Since $-\frac{4}{3}[1 + 3\mathcal{J}/2]\,|bD|^2 = 2\,\mathrm{Re}\,bD$ [from (165) and (166)], etc., we see that both sides of (167) yield identical terms.

Magnetic dipoles: Similarly if $b_1^\pm = 0$ in (150), we use (151) and the second and third equations of (150):

$$\mathfrak{C}_{11}^\pm = -c^\pm[\mathbf{C}_1^{-1}(\mathbf{i}) - e^{\mp i2\delta}\mathfrak{C}_{11}^\pm\mathcal{H}], \quad (168)$$

plus the analogous equation for $\mathfrak{C}_{1-1}^\pm$ involving $\mathbf{C}_1^1(\mathbf{i})$. Thus, as previously,

$$\mathfrak{C}_{11}^\pm = -c^\pm D(c^\mp)\mathbf{C}_1^{-1}(\mathbf{i}), \quad \mathfrak{C}_{1-1}^\pm = -c^\pm D(c^\mp)\mathbf{C}_1^1(\mathbf{i}). \quad (169)$$

The single scattered amplitude is

$$\tilde{g}_\pm = C^\pm(\tilde{C}^0 + \tilde{C}^1), \quad (170)$$

with the $\tilde{C}$'s as in (95), and the corresponding multiple scattered values are

$$\tilde{G}^\mp(\mathbf{o}, \mathbf{i}) = c^\mp A(c^\pm)\tilde{C}^0 + c^\mp D(c^\pm)\tilde{C}^1. \quad (171)$$

The present case is completely analogous to the previous and corresponding results may be obtained by inspection.

For axial incidence $\mathbf{i} = \hat{z}$ we have $\tilde{C}^0 = 0$, and if the incident $\mathbf{E}$ is polarized parallel to $\hat{x}$, then corresponding to (157), we have

$$\tilde{C}^1\cdot\hat{x} = \hat{\theta}\cos\varphi - \hat{\phi}\sin\varphi\cos\theta \equiv \boldsymbol{\gamma}_1^1,$$
$$\tilde{g}(\mathbf{o}, \hat{z})\cdot\hat{x} = \mathbf{g}(\mathbf{o}, \hat{z}:\hat{x}) = c^\mp\boldsymbol{\gamma}_1^1, \quad \mathbf{G} = c^\mp D(c^\mp)\boldsymbol{\gamma}_1^1. \quad (172)$$

For normal incidence $\mathbf{i} = \hat{x}$, if the incident $\mathbf{E}$ is along the axis ($\hat{\boldsymbol{\epsilon}} = \hat{z}$), then $\tilde{C}^0\cdot\hat{z} = 0$ and corresponding to (158),

$$\tilde{C}^1\cdot\hat{z} \equiv \boldsymbol{\gamma}_2^1 = -\boldsymbol{\gamma}_1^1, \quad \mathbf{g}(\mathbf{o}, \hat{x}:\hat{z}) = c^\pm\boldsymbol{\gamma}_2^1,$$
$$\mathbf{G} = c^\pm D(c^\mp)\boldsymbol{\gamma}_2^1. \quad (173)$$

If the polarization is along $\hat{y}$, then $\tilde{C}^1\cdot\hat{y} = 0$, and corresponding to (159),

$$\tilde{C}^0\cdot\hat{y} = \hat{\phi}\sin\theta = \boldsymbol{\gamma}_3^1, \quad \mathbf{g}(\mathbf{o}, \hat{x}:\hat{y}) = c^\pm\boldsymbol{\gamma}_3^1,$$
$$\mathbf{G} = c^\pm A(c^\mp)\boldsymbol{\gamma}_3^1. \quad (174)$$

One electric plus one magnetic dipole: The remaining elementary situation in (150) is that in which one scatterer (+) is an electric dipole and the other (−) is a magnetic dipole. For this case we set $b^- = c^+ = 0$ so that the required functions in (151) and (152) reduce to the single-scattered values

$$\mathfrak{C}_{10}^{-} = c^{-}\mathbf{C}_1(\mathrm{i}), \quad \mathfrak{B}_{10}^{+} = b^{+}\mathbf{B}_1^0(\mathrm{i}), \tag{175}$$

which correspond to the first and fourth equations of (150). The second and fifth of (150) reduce to

$$\mathfrak{C}_{11}^{-} = -c^{-}[\mathbf{C}_1^{-1} + e^{i2\delta}\mathfrak{B}_{11}^{+}h_1],$$
$$\mathfrak{B}_{11}^{+} = -b^{+}[\mathbf{B}_1^{-1} + e^{-i2\delta}\mathfrak{C}_{11}^{-}h_1], \tag{176}$$

plus the analogous set for $\mathfrak{C}_{1-1}^{-}$ and $\mathfrak{B}_{1-1}^{+}$ in terms of $\mathbf{C}_1^1$ and $\mathbf{B}_1^1$ with h_1 replaced by $-h_1$.

Solving (176) and its analog we obtain

$$\mathfrak{C}_{11}^{-} = -c^{-}E\mathbf{C}_1^{-1}(\mathrm{i}) + e^{i2\delta}F\mathbf{B}_1^{-1}(\mathrm{i}),$$
$$\mathfrak{C}_{1-1}^{-} = -c^{-}E\mathbf{C}_1^{1}(\mathrm{i}) - e^{i2\delta}F\mathbf{B}_1^{1}(\mathrm{i}),$$
$$\mathfrak{B}_{11}^{+} = -b^{+}E\mathbf{B}_1^{-1}(\mathrm{i}) + e^{-i2\delta}F\mathbf{C}_1^{-1}(\mathrm{i}),$$
$$\mathfrak{B}_{1-1}^{+} = -b^{+}E\mathbf{B}_1^{1}(\mathrm{i}) - e^{-i2\delta}F\mathbf{C}_1^{1}(\mathrm{i}),$$
$$E \equiv 1/(1 - b^{+}c^{-}h_1^2), \quad F \equiv c^{-}b^{+}h_1E. \tag{177}$$

The single-scattered amplitudes for this case are

$$\tilde{g}_{+} = b^{+}(\tilde{B}^0 + \tilde{B}^1), \quad \tilde{g}_{-} = c^{-}(\tilde{C}^0 + \tilde{C}^1), \tag{178}$$

and the corresponding multiple-scattered amplitudes equal

$$\tilde{G}_{+} = b^{+}\tilde{B}^0 + b^{+}E\tilde{B}^1 + e^{-i2\delta}F\tilde{D},$$
$$\tilde{G}^{-} = c^{-}\tilde{C}^0 + c^{-}E\tilde{C}^1 + e^{i2\delta}F\tilde{D}^1,$$
$$\tilde{D} = \mathbf{B}_1^1(\mathrm{o})\mathbf{C}_1^{-1}(\mathrm{i}) - \mathbf{B}_1^{-1}(\mathrm{o})\mathbf{C}_1^1(\mathrm{i}) = i \operatorname{Im} \mathbf{B}_1^1(\mathrm{o})\mathbf{C}_1^{1*}(\mathrm{i})$$
$$= i(\hat{\theta}\hat{\varphi}_i \cos\theta \cos\theta_i - \hat{\varphi}\hat{\theta}_i)\sin(\varphi - \varphi_i)$$
$$+ i(\hat{\theta}\hat{\theta}_i \cos\theta + \hat{\varphi}\hat{\varphi}_i \cos\theta_i)\cos(\varphi - \varphi_i),$$
$$\tilde{D}^1 = \mathbf{C}_1^1(\mathrm{o})\mathbf{B}_1^{-1}(i) - \mathbf{C}_1^{-1}(\mathrm{o})\mathbf{B}_1^1(\mathrm{i}) = -i \operatorname{Im} \mathbf{C}_1^1(\mathrm{o})\mathbf{B}_1^{1*}(\mathrm{i})$$
$$= -i(\hat{\theta}\hat{\varphi} - \hat{\varphi}\hat{\theta}_i \cos\theta\cos\theta_i)\sin(\varphi - \varphi_i)$$
$$- i(\hat{\theta}\hat{\theta}_i \cos\theta_i + \hat{\varphi}\hat{\varphi}_i\cos\theta)\cos(\varphi - \varphi_i). \tag{179}$$

The present case is much less symmetrical than the preceding ones, and provides a simple illustration of a scatterer containing cross terms [e.g., $\mathbf{C}_1^1(\mathrm{o})\mathbf{B}_1^{-1}(\mathrm{i})$] corresponding to coupling between electric and magnetic dipoles.

For axial incidence $\mathbf{i} = \hat{z}$, we have $\tilde{B}^0 = \tilde{C}^0 = 0$. If $\boldsymbol{\epsilon} = \hat{x}$ then

$$\tilde{D}\cdot\hat{x} = i\gamma_1, \quad \tilde{D}^1\cdot\hat{x} = -i\hat{\gamma}_1^1,$$
$$\tilde{G}_{+}\cdot\hat{x} = (b^{+}E + e^{-i2\delta}Fi)\gamma_1,$$
$$\tilde{G}_{-}\cdot\hat{x} = (c^{-}E - e^{i2\delta}Fi)\gamma_1^1. \tag{180}$$

In the forward direction,

$$\mathfrak{G} = \mathbf{G}_{+} + \mathbf{G}_{-} = (b^{+} + c^{-})E\hat{x} + 2\sin(2kb)F\hat{x}. \tag{181}$$

For normal incidence $\mathbf{i} = \hat{x}$, if $\boldsymbol{\epsilon} = \hat{z}$ then $\tilde{B}^1\cdot\hat{z} = \tilde{C}^0\cdot\hat{z} = \tilde{D}^1\cdot\hat{z} = 0$ and

$$\tilde{D}\cdot\hat{z} = -i\gamma_1, \quad \mathbf{G}_{+} = b^{+}\gamma_2 - ie^{-i2\delta}F\gamma_1,$$
$$\mathbf{G}_{-} = c^{-}E\gamma_2^1. \tag{182}$$

In the forward direction,

$$\mathbf{G}_{+} = b^{+}\hat{z}, \quad \mathbf{G}_{-} = c^{-}E\hat{z},$$
$$\mathfrak{G} = \mathbf{G}_{+} + \mathbf{G}_{-} = (b^{+} + c^{-}E)\hat{z}. \tag{183}$$

If $\boldsymbol{\epsilon} = \hat{y}$ then $\tilde{B}^0\cdot\hat{y} = \tilde{C}^1\cdot\hat{y} = \tilde{D}\cdot\hat{y} = 0$ and

$$\tilde{D}^1\cdot\hat{y} = -i(\theta\sin\varphi + \varphi\cos\theta\cos\varphi) = -i\gamma_4,$$
$$\mathbf{G}_{+} = b^{+}E\gamma_3, \quad \mathbf{G}_{-} = c^{-}\gamma_3^1 - i\gamma_4 e^{i2\delta}F. \tag{184}$$

In the forward direction

$$\mathfrak{G} = \mathbf{G}_{+} + \mathbf{G}_{-} = (b^{+}E + c^{-})\hat{y}. \tag{185}$$

More generally in the forward direction we write

$$\mathbf{g}_{+} = b^{+}\boldsymbol{\epsilon}, \quad \mathbf{g}_{-} = c^{-}\boldsymbol{\epsilon}, \quad \mathbf{G}_{\pm} = G_{\pm}\boldsymbol{\epsilon}. \tag{186}$$

If $\boldsymbol{\epsilon} = \hat{y}$ then

$$G_{+} = b^{+}E + e^{-i2\delta}Fi\cos\theta,$$
$$G_{-} = c^{-}\sin^2\theta + c^{-}E\cos^2\theta - e^{i2\delta}Fi\cos\theta. \tag{187}$$

Similarly, if $\boldsymbol{\epsilon} = \hat{\theta}$ then

$$G_{+} = b^{+}\sin^2\theta + b^{+}E\cos^2\theta + e^{-i2\delta}iF\cos\theta,$$
$$G_{-} = c^{-}E - e^{i2\delta}iF\cos\theta. \tag{188}$$

Thus, in all cases, the forwardscattered values have the incident polarization.

Electric plus magnetic dipoles: For the general situation of (150), each scatterer has both electric and magnetic dipole moments. The axial components are as in (151) and the corresponding perpendicular components follow from the remaining four equations of (150). Thus eliminating $\mathfrak{C}_{11}^{\mp}$ and $\mathfrak{B}_{11}^{\mp}$ from the second and fifth, we obtain

$$\mathfrak{C}_{11}^{\pm} = -c_1^{\pm}\{\mathbf{C}_1^{-1}(\mathrm{i})R(b, c) \pm \mathbf{B}_1^{-1}(\mathrm{i})S(b, c)\}/\Delta,$$
$$\mathfrak{B}_{11}^{\pm} = -b_1^{\pm}\{\mathbf{B}_1^{-1}(\mathrm{i})R(c, b) \mp \mathbf{C}_1^{-1}(\mathrm{i})S(c, b)\}/\Delta,$$
$$R(b, c) \equiv 1 - b^{+}b^{-}\mathcal{K}^2 - c^{\mp}b^{\pm}h_1^2$$
$$+ \mathcal{K}c^{\mp}e^{\mp i2\delta}[1 - b^{+}b^{-}(\mathcal{K}^2 + h_1)],$$
$$S(b, c) \equiv h_1\mathcal{K}b^{\pm}(b^{\mp} - c^{\mp})$$
$$+ h_1b^{\mp}e^{\mp i2\delta}[1 - b^{+}b^{-}(\mathcal{K}^2 + h_1^2)],$$
$$\Delta \equiv 1 - \mathcal{K}^2(b^{+}b^{-} + c^{+}c^{-}) - h_1^2(c^{+}b^{-} + c^{-}b^{+})$$
$$+ c^{+}c^{-}b^{+}b^{-}(\mathcal{K}^2 + h_1^2)^2, \tag{189}$$

where $R(c, b)$ is obtained from $R(b, c)$ by interchanging b and c, and similarly for S. The corresponding coefficients $\mathfrak{C}_{1-1}^{\pm}$ and $\mathfrak{B}_{1-1}^{\pm}$ are obtained by replacing $\mathbf{C}_1^{-1}$ and $\mathbf{B}_1^{-1}$ by $\mathbf{C}_1^1$ and $\mathbf{B}_1^1$, and h_1 by $-h_1$.

Equations (151), (152), and (189) provide the coefficients for an explicit closed form for multiple scattering by two arbitrarily separated scatterers such that each is fully specified by its appropriate electric and magnetic dipoles when isolated. The set covers the special cases considered previously and allows us to obtain corrections, e.g., to (156) for the case where the magnetic dipoles are not negligible. The present results apply to small spheres of different radii with both ϵ and μ different from unity, to two perfectly conducting spheres (for which case the electric and magnetic coefficients b, c are of the same order of magnitude), etc. Simple forms of the present results follow not only for the cases considered as illustrations but also for the limit of separations small compared to wavelength ($kb \approx 0$) for which region we use the origin expansions of the h's, as well as for separations large compared to wavelength ($kb \gg 1$) for which case we use the asymptotic form of the h's. From symmetry considerations, specializing the above two-scatterer results to identical scatterers, enables us to write down the corresponding solutions for one scatterer near a perfectly conducting plane, two protuberances on such a plane, and one protuberance on the wall of a perfectly conducting quadrant; see analogous expressions for two cylinders given previously.[23]

4.3. Small Scatterers of Arbitrary Shape

To construct analogous closed forms for two nonspherical scatterers small compared to wavelength, we base the development on (99)ff, and its generalization to an essentially arbitrary exciting electric field Φ:

$$\bar{u}\cdot\bar{\Phi} = \bar{h}\cdot\bar{p}\cdot\Phi + (\nabla\times\bar{h})\cdot\bar{m}\cdot(\nabla\times\bar{\Phi}/k^2). \quad (190)$$

Electric dipoles: For a configuration of two arbitrary electric dipoles specified by $\bar{p}_\pm$ excited by $\bar{\varphi}$, the multiple-scattered fields $\bar{U}_\pm$ may be written

$$\bar{U}_\pm = \bar{u}_\pm\cdot\bar{\varphi}_\mp^\pm + \bar{u}_\pm\cdot\bar{U}_\mp^\pm \equiv \bar{u}_\pm\cdot\bar{\Phi}^\pm,\quad \bar{u}_\pm = \bar{h}(kr_\pm)\cdot\bar{p}_\pm, \quad (191)$$

where $\bar{\varphi}^+$ is the value of the source term at the scatterer located at $b\mathbf{b}_+ = b\hat{b}$, and where $\bar{\Phi}^+$ is the corresponding total exciting field; similarly, for brevity, $\bar{U}_-^+$ means the field of $\bar{U}_-$ evaluated at $b\mathbf{b}_+$, etc. We have

$$\bar{\Phi}^\pm = \bar{\varphi}^\pm + \bar{U}_\mp^\pm = \bar{\varphi}^\pm + \bar{u}_\mp^\pm\cdot\bar{\Phi}^\mp; \quad (192)$$

consequently,

$$\bar{\Phi}^\pm = (\bar{I} - \bar{u}_\mp^\pm\cdot\bar{u}_\pm^\mp)^{-1}\cdot(\bar{\varphi}^\pm + \bar{u}_\mp^\pm\cdot\bar{\varphi}^\mp),$$
$$\bar{\varphi}^\pm = (\bar{I}-\mathbf{ii})e^{\pm i\delta},\quad \bar{u}_\pm^\mp = \bar{h}(2kb)\cdot\bar{p}_\pm,$$
$$\bar{h} = (\bar{I}-\hat{b}\hat{b})\mathcal{H} + \hat{b}\hat{b}H, \quad (193)$$

where $\delta = \mathbf{k}\cdot\mathbf{b}$, and $\mathcal{H}$ and H are defined in (100).

We consider first the case corresponding to Sec. 4.2, for which $\bar{p}_\pm = p_\pm\bar{I}$ (small spheres, elementary model for oscillating electrons, etc.), where we have replaced the previous $\bar{b}^\pm$ by $\bar{p}_\pm$ to avoid confusion with the other b's. Since

$$\bar{u}_\pm^\mp\cdot\bar{u}_\pm^\mp = p_+p_-\bar{h}\cdot\bar{h} = p_+p_-[(\bar{I}-\hat{b}\hat{b})\mathcal{H}^2 + \hat{b}\hat{b}H^2],$$

we may write

$$\bar{I} - \bar{u}_\pm^\mp\cdot\bar{u}_\pm^\mp = (\bar{I}-\hat{b}\hat{b})(1-p_+p_-\mathcal{H}^2) + \hat{b}\hat{b}(1-p_+p_-H^2),$$

and express the reciprocal as

$$(\bar{I} - \bar{u}_\mp^\pm\cdot\bar{u}_\pm^\mp)^{-1} = \frac{\bar{I}-\hat{b}\hat{b}}{1-p_+p_-\mathcal{H}^2} + \frac{\hat{b}\hat{b}}{1-p_+p_-H^2}. \quad (194)$$

We also have

$$\bar{\varphi}^\pm - \bar{u}_\mp^\pm\cdot\bar{\varphi}^\pm = [(\bar{I}-\hat{b}\hat{b})(1+\mathcal{H}p_\mp e^{\mp i2\delta}) + \hat{b}\hat{b}(1+Hp_\mp e^{\mp i2\delta})]\cdot\bar{\varphi}^\pm. \quad (195)$$

Thus using (194) and (195), we reduce (193) to

$$\Phi^\pm = [(\bar{I}-\hat{b}\hat{b})D_\mp + \hat{b}\hat{b}A_\mp]\cdot\bar{\varphi}^\pm,$$
$$A_\mp = (1+p_\mp e^{\mp i2\delta}H)/(1-p_+p_-H^2) = A(H, p_\mp),$$
$$D_\mp = A(\mathcal{H}, p_\mp), \quad (196)$$

where A and D are essentially as defined in (151) and (154). The corresponding scattered waves from (191) are thus

$$\bar{U}_\pm = \bar{h}(r_\pm)\cdot\bar{p}_\pm\cdot[(\bar{I}-\hat{b}\hat{b})D_\mp + \hat{b}\hat{b}A_\mp]\cdot\bar{\varphi}^\pm \equiv \bar{h}(r_\pm)\cdot\bar{P}_\pm\cdot\bar{\varphi}^\pm, \quad (197)$$

where $\bar{P}$ is the multiple-scattered moment. The asymptotic form of (197) for $kr \gg 1$, $r \gg b$ is $\bar{U}_\pm \sim he^{\pm i(\delta-\Delta)}\bar{G}_\pm$, with

$$\bar{G}_\pm(\mathbf{o},\mathbf{i}) = (\bar{I}-\mathbf{oo})\cdot\bar{P}_\pm\cdot(\bar{I}-\mathbf{ii})$$
$$= (\bar{I}-\mathbf{oo})\cdot p_\pm[(\bar{I}-\hat{b}\hat{b})D_\mp + \hat{b}\hat{b}A_\mp]\cdot(\bar{I}-\mathbf{ii}). \quad (198)$$

If we take $\hat{b} = \hat{z}$ (i.e., if we measure θ from $\hat{b}$), then the multiple-scattered amplitudes $\bar{G}_\pm$ of (198) may be rewritten directly in the form (156) by using (107), i.e., for this choice of axis we have

$$\bar{P}_+ = p_\pm[(\hat{x}\hat{x} + \hat{y}\hat{y})D_\mp + \hat{z}\hat{z}A_\mp],$$

and using (107) reduces (198) to the form (112) with $p_x = p_y = p_\pm D_\pm$ and $p_z = p_\pm A_\mp$.

For a small sphere of radius a with $\mu = 1$ and dielectric constant ϵ, we have $p \approx i(ka)^3(\epsilon-1)/(\epsilon+2)$. For small spacing $\rho = 2kb \ll 1$ we may use $\mathcal{H} \approx -H/2 \approx i/\rho^3$. Thus

$$\mathcal{H}p \approx -\tfrac{1}{2}Hp \to -\left(\frac{a}{2b}\right)^3\left(\frac{\epsilon-1}{\epsilon+2}\right)$$

in the static limit $k\to 0$. For this case we may also neglect $k\delta \to 0$. For identical scatterers we then have

$$\frac{\bar{P}}{p} \to \frac{\bar{I}-\hat{b}\hat{b}}{1+R} + \frac{\hat{b}\hat{b}}{1-2R},\quad R = \left(\frac{a}{2b}\right)^3\left(\frac{\epsilon-1}{\epsilon+2}\right). \quad (199)$$

[23] V. Twersky, J. Appl. Phys. 23, 407 (1952).

On the other hand, for $\rho \gg 1$ we have $\mathcal{K} \sim h_0$ and $H \sim -2ih_0/\rho$; if we neglect H, then we get the "far-field multiple-scattering form"

$$\bar{P} \sim p[D(\bar{I} - \hat{b}\hat{b}) + \hat{b}\hat{b}], \quad \mathcal{K} \sim h_0(2kb); \quad (200)$$

more generally we use $\mathcal{K} = h[1 + (i/\rho) - 1/\rho^2]$, $H = -2h[(i/\rho) - 1/\rho^2]$ to convert $\bar{P}$ to the analog of (144) plus (145).

The above dyadic dipoles are spherically symmetric in that $\bar{p} = b\bar{I}$ means the vector dipole $\bar{p} \cdot \epsilon = p\epsilon$ has the direction of the incident polarization. We can also consider the case of a fixed vector dipole $\bar{p} = p\hat{d}\hat{d}$ (i.e., a fixed metal wire oriented along $\hat{d}$), or the general dipole $\bar{p} = \bar{p}^{\mathrm{T}}$ of (102)ff. For simplicity we assume that the principal axes of the scatterers are parallel, and take $\hat{\zeta} = \hat{b}$. Thus we may write

$$\bar{u}_\pm = \bar{h} \cdot \bar{p}_\pm = \mathcal{K}(p_{\xi\pm}\hat{\xi}\hat{\xi} + p_{\eta\pm}\hat{\eta}\hat{\eta}) + Hp_{\zeta\pm}\hat{\zeta}\hat{\zeta}, \quad (201)$$

and obtain

$$\bar{I} - \bar{u}_\mp^\pm \cdot \bar{u}_\pm^\mp = (1 - \mathcal{K}^2 p_{\xi+}p_{\xi-})\hat{\xi}\hat{\xi} + (1 - \mathcal{K}^2 p_{\eta+}p_{\eta-})\hat{\eta}\hat{\eta} + (1 - H^2 p_{\zeta+}p_{\zeta-})\hat{\zeta}\hat{\zeta},$$

$$(\bar{I} - u_\mp^\pm \cdot u_\pm^\mp)^{-1} = E_\xi(\mathcal{K}, p)\hat{\xi}\hat{\xi} + E_\eta(\mathcal{K}, p)\hat{\eta}\hat{\eta} + E_\zeta(H, p)\hat{\zeta}\hat{\zeta},$$

$$E_\xi(\mathcal{K}, p) = 1/(1 - \mathcal{K}^2 p_{\xi+}p_{\xi-}), \quad \text{etc.} \quad (202)$$

Substituting into $\bar{p} \cdot \bar{\Phi} = \bar{P} \cdot \bar{\varphi}$, we construct

$$\bar{P}_\pm = p_\pm \cdot [\bar{I} - \bar{u}_\mp^\pm \cdot \bar{u}_\pm^\mp]^{-1} \cdot [\bar{I} + \bar{u}_\mp^\pm e^{\mp i2\delta}]$$
$$= p_{\xi\pm}D(p_{\xi\mp})\hat{\xi}\hat{\xi} + p_{\eta\pm}D(p_{\eta\mp})\hat{\eta}\hat{\eta} + p_{\zeta\pm}A(p_{\zeta\mp})\hat{\zeta}\hat{\zeta}. \quad (203)$$

Magnetic dipoles: Similarly for two magnetic dipoles, say each of the form

$$\bar{v} \cdot \nabla \times \bar{\varphi} = \nabla \times \bar{h} \cdot (\bar{m}/k^2) \cdot \nabla \times \bar{\varphi}$$
$$= (-h_1 \mathbf{o} \times \bar{I}) \cdot \bar{m} \cdot (\bar{I} \times \mathbf{i}) \cdot \bar{\varphi} i, \quad (204)$$

we may work with the electric functions

$$\bar{U}_\pm = \bar{v}_\pm \cdot \nabla \times \bar{\Phi}^\pm, \quad (205)$$

$$\bar{\Phi}^\pm = \bar{\varphi}^\pm + \bar{U}_\mp^\pm = \bar{\varphi}^\pm + v_\mp^\pm \cdot \nabla \times \bar{\Phi}^\mp \quad (206)$$

to obtain

$$\nabla \times \bar{\Phi}^\pm = [\bar{I} - (\nabla \times \bar{v}_\mp^\pm) \cdot (\nabla \times \bar{v}_\pm^\mp)]^{-1} \cdot [\nabla \times \bar{\varphi}^\pm + \nabla \times \bar{v}_\mp^\pm \cdot \nabla \times \bar{\varphi}^\mp]. \quad (207)$$

Since

$$\nabla \times \bar{v} = \nabla \times \nabla \times \bar{h} \cdot \bar{m}/k^2 = \bar{h} \cdot \bar{m},$$

the function $[\ ]^{-1}$ is of the same form as for the electric case but with the previous p replaced by m.

For $\bar{m}_\pm = m_\pm \bar{I}$ corresponding to the spherically symmetric case of Sec. 4.2, we have from (207) and (196),

$$\nabla \times \bar{\Phi}^\pm = [(\bar{I} - \hat{b}\hat{b})D(m_\mp) + \hat{b}\hat{b}A(m_\mp)] \cdot \bar{\varphi}^\pm \times \mathbf{i}ik. \quad (208)$$

Thus using (208) in (205) with $\bar{v}$ as in (204), we obtain

$$\bar{U}_\pm = ih_1 m_\pm(\mathbf{o} \times \bar{I}) \cdot [(\bar{I} - \hat{b}\hat{b})D(m_\mp) + \hat{b}\hat{b}A(m_\pm)] \cdot (\bar{I} \times \mathbf{i})e^{\pm i\delta}$$
$$\equiv (\nabla \times \bar{h}_\pm) \cdot \bar{M}_\pm \cdot \nabla \times \bar{\varphi}_\pm/k^2, \quad (209)$$

where $\bar{M}$ is the multiple-scattered moment. Since

$$h_1(kr_\pm) \sim -ih(kr_\pm) \sim -ih(kr)e^{\mp i\Delta},$$

the scattering amplitudes are

$$\bar{G}_\pm = -(\mathbf{o} \times \bar{I}) \cdot \bar{M}_\pm \cdot (\bar{I} \times \mathbf{i})$$
$$= -(\mathbf{o} \times \bar{I}) \cdot m_\pm[(\bar{I} - \hat{b}\hat{b})D(m_\mp) + \hat{b}\hat{b}A(m_\mp)] \cdot (\bar{I} \times \mathbf{i}). \quad (210)$$

If we take $\hat{b} = \hat{z}$ then (210) may be rewritten directly in the form (171) by using (109) to reduce $\bar{G}$ to the form (113).

For a small sphere of radius a with $\epsilon = 1$ and permittivity μ, we replace ϵ by μ in the previous illustration. Similarly for the magnetic analog of the more general case (203) we obtain

$$\bar{M}_\pm = m_{\xi\pm}D(m_{\xi\mp})\hat{\xi}\hat{\xi} + m_{\eta\pm}D(m_{\eta\mp})\hat{\eta}\hat{\eta} + m_{\zeta\pm}A(m_{\zeta\mp})\hat{\zeta}\hat{\zeta}. \quad (211)$$

See Appendix B for analogous results for scalar problem.

Electric plus magnetic: If we are dealing with one electric ($\bar{u}_+$) and one magnetic ($\bar{v}_-$) dipole, then we may work with

$$\bar{U}_+ = \bar{u}_+ \cdot (\bar{\varphi}^+ + \bar{U}_-^+) = \bar{u}_+ \cdot \bar{\Phi}^+, \quad (212)$$

$$\bar{U}_- = \bar{v}_- \cdot \nabla \times (\bar{\varphi}^- + \bar{U}_+^-) = \bar{v}_- \cdot \nabla \times \bar{\Phi}^-, \quad (213)$$

where

$$\bar{\Phi}^+ = \bar{\varphi}^+ + \bar{v}_-^+ \cdot \nabla \times \bar{\Phi}^-, \quad (214)$$

$$\nabla \times \bar{\Phi}^- = \nabla \times \bar{\varphi}^- + (\nabla \times \bar{u}_+^-) \cdot \bar{\Phi}^+. \quad (215)$$

Solving (214) and (215) we obtain

$$\bar{\Phi}^+ = [\bar{I} - \bar{v}_-^+ \cdot \nabla \times \bar{u}_+^-]^{-1} \cdot [\bar{\varphi}^+ + \bar{v}_-^+ \cdot \nabla \times \bar{\varphi}^-], \quad (216)$$

$$\nabla \times \bar{\Phi}^- = [\bar{I} - (\nabla \times \bar{u}_+^-) \cdot \bar{v}_-^+]^{-1} \cdot [\nabla \times \bar{\varphi}^- + (\nabla \times \bar{u}_+^-) \cdot \bar{\varphi}^+]. \quad (217)$$

We have

$$\bar{v}_-^+ = \nabla \times \bar{h} \cdot \bar{m}/k^2 = -kh_1(\hat{b} \times \bar{I}) \cdot \bar{m}/k^2,$$

and similarly

$$\nabla \times \bar{u}_+^- = -kh_1(-\hat{b} \times \bar{I}) \cdot \bar{p} = kh_1(\hat{b} \times \bar{I}) \cdot \bar{p}.$$

For the case of Sec. 4.2, we have $\bar{p} = p\bar{I}$ and $\bar{m} = m\bar{I}$ and

$$\bar{v}_-^+ \cdot \nabla \times \bar{u}_+^- = -m_- p_+ h_1^2(\hat{b} \times \bar{I}) \cdot (\hat{b} \times \bar{I})$$
$$= m_- p_+ h_1^2(\bar{I} - \hat{b}\hat{b}).$$

Thus

$$(\bar{I} - \bar{v}_-^+ \cdot \nabla \times \bar{u}_+^-)^{-1} = [\bar{I} - (\bar{I} - \hat{b}\hat{b})m_-p_+h_1^2]^{-1}$$
$$= \hat{b}\hat{b} + (\bar{I} - \hat{b}\hat{b})/(1 - p_+m_-h_1^2)$$
$$\equiv \hat{b}\hat{b} + (\bar{I} - \hat{b}\hat{b})E,$$

and consequently

$$\bar{\Phi}^+ = [\hat{b}\hat{b} + (\bar{I} - \hat{b}\hat{b})E - im_-h_1(\hat{b} \times \bar{I})e^{-i2\delta}(\bar{I} \times \mathbf{i})E] \cdot \bar{\varphi}^+,$$
$$E = 1/(1 - p_+m_-h_1^2). \quad (218)$$

Similarly

$$\nabla \times \bar{\Phi}^-/ik = \{[\hat{b}\hat{b} + (\bar{I} - \hat{b}\hat{b})E] \cdot (\mathbf{i} \times \bar{I}) - ip_+h_1(\hat{b} \times \bar{I})e^{i2\delta}\} \cdot \bar{\varphi}_-. \quad (219)$$

Using (218) and (219) in (212) and (213) gives the corresponding electric dyadic fields. To obtain the scattering amplitude we use

$$\bar{u}_+ = p_+\bar{h}(hr^+) \sim p_+h_0(kr)e^{-i\Delta}(\bar{I} - \mathbf{oo}),$$

and similarly

$$\bar{v}_- = -kh_1(\mathbf{o} \times \bar{I})m_-/k^2 \sim -h_0e^{i\Delta}m_-(\mathbf{o} \times \bar{I})/ik.$$

Thus

$$\bar{G}_+ = (\bar{I} - \mathbf{oo}) \cdot \{p_+[\hat{b}\hat{b} + (\bar{I} - \hat{b}\hat{b})E] - i(\hat{b} \times \bar{I}) \cdot (\mathbf{i} \times \bar{I})Fe^{-i2\delta}\} \cdot (\bar{I} - \mathbf{ii}),$$
$$F = p_+m_-h_1E, \quad (220)$$

$$\bar{G}_- = -(\mathbf{o} \times \bar{I}) \cdot \{m[\hat{b}\hat{b} + (\bar{I} - \hat{b}\hat{b})E] + i(\hat{b} \times \bar{I}) \cdot (\bar{I} \times \mathbf{i})Fe^{i2\delta}\} \cdot (\mathbf{i} \times \bar{I}), \quad (221)$$

where we replaced $\bar{I} - \mathbf{ii}$ in the last term by $-(\bar{I} \times \mathbf{i}) \cdot (\bar{I} \times \mathbf{i})$ to stress the similarities of the form of the composite moments $\mathbf{P}_+$ and $\mathbf{M}_-$ corresponding to (220) and (221), respectively. To reduce (220) and (221) to the forms in (179), we take $\hat{b}\hat{b} = \hat{z}\hat{z}$ and use (107) and (109).

For the more general case of

$$\bar{p}_+ = \sum p_x\hat{x}\hat{x}, \quad \bar{m}_- = \sum m_x\hat{x}\hat{x}, \quad x = \xi, \eta, \zeta, \quad (222)$$

with $\hat{\zeta} = \hat{b}$, we have

$$\bar{v}_-^+ = \nabla \times \bar{h} \cdot \bar{m}/k^2 = -b_1(\hat{\zeta} \times \bar{I}) \cdot \bar{m}/k$$
$$= h_1(\hat{\xi}\hat{\eta}m_\eta - \hat{\eta}\hat{\xi}m_\xi)/k,$$
$$\nabla \times \bar{u}_+^- = kh_1\hat{\zeta} \times \bar{I} \cdot \bar{p} = -kh_1(\hat{\xi}\hat{\eta}p_\eta - \hat{\eta}\hat{\xi}p_\xi).$$

Consequently

$$\bar{v}_-^+ \cdot \nabla \times \bar{u}_+^- = h_1^2(\hat{\xi}\hat{\xi}p_\xi m_\eta + \hat{\eta}\hat{\eta}p_\eta m_\xi),$$
$$\nabla \times \bar{u}_+^- \cdot \bar{v}_-^+ = h_1^2(\hat{\xi}\hat{\xi}m_\xi p_\eta + \hat{\eta}\hat{\eta}m_\eta p_\xi).$$

Thus

$$\bar{\Phi}^+ = [\hat{\zeta}\hat{\zeta} + \hat{\xi}\hat{\xi}E(p_\xi m_\eta) + \hat{\eta}\hat{\eta}E(p_\eta m_\xi)] \cdot [\bar{I} + ih_1e^{-i2\delta}(m_\eta\hat{\xi}\hat{\eta} - m_\xi\hat{\eta}\hat{\xi})(\bar{I} \times \mathbf{i})] \cdot \bar{\varphi}^+, \quad (223)$$

$$\bar{P}_+ \equiv \bar{p}_+ \cdot \bar{\Phi}^+ \cdot (\bar{\varphi}^+)^{-1}$$
$$= p_\zeta\hat{\zeta}\hat{\zeta} + p_\xi E(p_\xi, m_\eta)\hat{\xi}\hat{\xi} + p_\eta E(p_\eta m_\xi)\hat{\eta}\hat{\eta} + ie^{-i2\delta}[F(p_\xi m_\eta)\hat{\xi}\hat{\eta} - F(p_\eta m_\xi)\hat{\eta}\hat{\xi}](\mathbf{i} \times \bar{I}),$$
$$\bar{G}_+ = (\bar{I} - \mathbf{oo}) \cdot \bar{P}_+ \cdot (\bar{I} - \mathbf{ii}). \quad (224)$$

Similarly

$$\nabla \times \bar{\Phi}_- = [\hat{\zeta}\hat{\zeta} + \hat{\xi}\hat{\xi}E(m_\xi p_\eta) + \hat{\eta}\hat{\eta}E(m_\eta p_\xi)] \cdot [\bar{I} - ih_1(\hat{\xi}\hat{\eta}p_\eta - \hat{\eta}\hat{\xi}p_\xi)e^{i2\delta}(\mathbf{i} \times \bar{I})] \cdot ik(\mathbf{i} \times \bar{\varphi}^-), \quad (225)$$

$$\bar{M}_- \equiv (\bar{m}_- \cdot \nabla \times \bar{\Phi}^-)(ik\mathbf{i} \times \boldsymbol{\varphi}^-)^{-1}$$
$$= m_\zeta\hat{\zeta}\hat{\zeta} + m_\xi E(m_\xi p_\eta)\hat{\xi}\hat{\xi} + m_\eta E(m_\eta p_\xi)\hat{\eta}\hat{\eta} - ie^{i2\delta}[F(m_\xi p_\eta)\hat{\xi}\hat{\eta} - F(m_\eta p_\xi)\hat{\eta}\hat{\xi}](\mathbf{i} \times \bar{I}),$$
$$\bar{G}_- = -(\mathbf{o} \times \bar{I}) \cdot \bar{M} \cdot (\bar{I} \times \mathbf{i}), \quad (226)$$

which differs from (224) in the interchange of m and p and the replacement of i by $-i$.

If each scatterer consists of an electric plus a magnetic dipole such that the isolated-scatterer values equal $\bar{u}_\pm + \bar{v}_\pm$ with $\bar{u}$ and $\bar{v}$ as defined in this section, then we write the scattered electric dyadics as

$$\bar{U}_\pm = \bar{u}_\pm \cdot \bar{\Phi}^\pm + \bar{v}_\pm \cdot \nabla \times \bar{\Phi}^\pm = (\bar{u}_\pm + \bar{v}_\pm \cdot \nabla \times \bar{I}) \cdot \bar{\Phi}^\pm, \quad (227)$$

with

$$\bar{\Phi}^\pm = \bar{\varphi}^\pm + (\bar{u}_\mp^\pm + \bar{v}_\mp^\pm \cdot \nabla \times \bar{I}) \cdot \bar{\Phi}^\mp, \quad (228)$$

where $\nabla \times \bar{I} \cdot \bar{\Phi}$ is a temporary expedient for $\nabla \times \bar{\Phi}$. Eliminating $\bar{\Phi}^\mp$ from the right-hand side, we obtain

$$\bar{\Phi}^\pm = (\bar{\varphi}^\pm + \bar{u}_\mp^\pm \cdot \bar{\varphi}^\mp + \bar{v}_\mp^\pm \cdot \nabla \times \bar{\varphi}^\mp)$$
$$+ (\bar{u}_\mp^\pm \cdot \bar{u}_\pm^\mp + \bar{v}_\mp^\pm \cdot \nabla \times \bar{u}_\pm^\mp) \cdot \bar{\Phi}^\pm$$
$$+ (\bar{u}_\mp^\pm \cdot \bar{v}_\pm^\mp + \bar{v}_\mp^\pm \cdot \nabla \times \bar{v}_\pm^\mp) \cdot \nabla \times \bar{\Phi}^\pm, \quad (229)$$

plus the corresponding expression for $\nabla \times \bar{\Phi}^\pm$ obtained by replacing all left-hand elements in the terms in parentheses by their curls, e.g., $\nabla \times \bar{\Phi}^\pm$ involves

$$\nabla \times \varphi^\pm + (\nabla \times \bar{u}_\mp^\pm) \cdot \varphi^\mp + (\nabla \times \bar{v}_\mp^\pm) \cdot (\nabla \times \varphi^\mp),$$

etc. Essentially as for the previous case, we may solve the simultaneous equations for $\bar{\Phi}^\pm$ and $\nabla \times \bar{\Phi}^\pm$ to reduce the above to the case considered previously by separations of variables.

APPENDIX A. RECIPROCITY RELATIONS

We should stress that $\bar{G}$ of (131) does not in general satisfy the reciprocity relation of the form (75):

$$\bar{g}_t(-\mathbf{i}_1, \mathbf{i}_2) = \bar{g}_t^T(-\mathbf{i}_2, \mathbf{i}_1). \quad (A1)$$

We may always write

$$\tilde{G}_t(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{R}_t(-\mathbf{i}_2, \mathbf{i}_1) + \tilde{L}_t(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{R}_t(-\mathbf{i}_1, \mathbf{i}_2) + \sum{}' \tilde{R}'_{ts}(-\mathbf{i}_1, \mathbf{i}_2)e^{-ik\mathbf{i}_2\cdot\mathbf{b}_{ts}}, \tag{A2}$$

where $\tilde{R}_t$ (R for "reversible") includes only those "chains" of successive scattering processes which start and end with scatterer t, and $\tilde{L}$ includes those that start with $s \neq t$ and end with t. Interchanging the directions we get

$$\tilde{G}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) = \tilde{R}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) + \sum{}' \tilde{R}'_{ts}(-\mathbf{i}_2, \mathbf{i}_1)e^{-ik\mathbf{i}_1\cdot\mathbf{b}_{ts}} = \tilde{R}_t(-\mathbf{i}_1, \mathbf{i}_2) + \sum{}' \tilde{R}'_{ts}(-\mathbf{i}_1, \mathbf{i}_2)e^{-ik\mathbf{i}_1\cdot\mathbf{b}_{ts}}, \tag{A3}$$

so that

$$\tilde{G}_t(-\mathbf{i}_1, \mathbf{i}_2) - \tilde{G}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) = \sum{}' \tilde{R}'_{ts}(-\mathbf{i}_1, \mathbf{i}_2)[e^{-ik\mathbf{i}_2\cdot\mathbf{b}_{ts}} - e^{-ik\mathbf{i}_1\cdot\mathbf{b}_{ts}}], \tag{A4}$$

is not in general zero. We illustrate this explicitly for an elementary case in (160)ff. In the present Appendix we list additional theorems for G.

The compound scattering amplitude

$$\tilde{\mathfrak{G}}(-\mathbf{i}_1, \mathbf{i}_2) = \sum e^{ik(\mathbf{i}_2+\mathbf{i}_1)\cdot\mathbf{b}_{ts}}\tilde{G}_s(-\mathbf{i}_1, \mathbf{i}_2) \tag{A5}$$

of (117) satisfies the same theorems as $\tilde{g}$. Thus using (A4) in (75), we obtain

$$\tilde{G}_t(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{G}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) - \sum{}' e^{-ik(\mathbf{i}_1+\mathbf{i}_2)\cdot\mathbf{b}_{ts}}[\tilde{G}_s(-\mathbf{i}_1, \mathbf{i}_2) - \tilde{G}_s^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1)]. \tag{A6}$$

These equations follow essentially from $\{\tilde{\Psi}_1, \tilde{\Psi}_2\} = 0$ over any surface bounding the collection. In addition, we have $\{\tilde{\psi}_1, \tilde{\Psi}_2\}_t = 0$ which led to the "reciprocity" relation of (131), i.e.,

$$\tilde{G}_t(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{g}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) + \sum{}' \int e^{ik(\mathbf{p}-\mathbf{i}_2)\cdot\mathbf{b}_{ts}}\tilde{g}_t^{\mathrm{T}}(-\mathbf{p}, \mathbf{i}_1) \cdot \tilde{G}_s(\mathbf{p}, \mathbf{i}_2)\, d\Omega/2\pi, \tag{A7}$$

as well as the result obtained by interchanging and transposition:

$$\tilde{G}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) = \tilde{g}(-\mathbf{i}_1, \mathbf{i}_2) + \sum{}' \int e^{ik(\mathbf{p}-\mathbf{i}_1)\cdot\mathbf{b}_{ts}}\tilde{G}_s^{\mathrm{T}}(\mathbf{p}, \mathbf{i}_1) \cdot \tilde{g}_t(-\mathbf{p}, \mathbf{i}_2)\, d\Omega/2\pi. \tag{A8}$$

Subtracting (A8) from (A7), we obtain

$$\tilde{G}_t(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{G}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) + \sum{}' \int e^{ik\mathbf{p}\cdot\mathbf{b}_{ts}}[e^{-ik\mathbf{i}_2\cdot\mathbf{b}_{ts}}\tilde{g}_t^{\mathrm{T}}(-\mathbf{p}, \mathbf{i}_1)\tilde{G}_s(\mathbf{p}, \mathbf{i}_2) - e^{-ik\mathbf{i}_1\cdot\mathbf{b}_{ts}}\tilde{G}_s^{\mathrm{T}}(\mathbf{p}, \mathbf{i}_1) \cdot \tilde{g}_t(-\mathbf{p}, \mathbf{i}_2)]\, d\Omega/2\pi. \tag{A9}$$

Similarly from $\{\tilde{\Psi}_1, \tilde{\Psi}_2\}_t = 0$ we obtain

$$\tilde{G}_t(-\mathbf{i}_1, \mathbf{i}_2) = \tilde{G}_t^{\mathrm{T}}(-\mathbf{i}_2, \mathbf{i}_1) + \sum{}' \int e^{ik\mathbf{p}\cdot\mathbf{b}_{ts}}[e^{-ik\mathbf{i}_2\cdot\mathbf{b}_{ts}}\tilde{G}_t^{\mathrm{T}}(-\mathbf{p}, \mathbf{i})\tilde{G}_s(\mathbf{p}, \mathbf{i}_2) - e^{-ik\mathbf{i}_1\cdot\mathbf{b}_{ts}}\tilde{G}_s^{\mathrm{T}}(\mathbf{p}, \mathbf{i}_1) \cdot \tilde{G}_t(-\mathbf{p}, \mathbf{i}_2)]\, d\Omega/2\pi, \tag{A10}$$

which we may reduce to (A9) by substituting (A7) for $\tilde{G}_t^{\mathrm{T}}(-\mathbf{p})$.

The above "reciprocity relations" follow from

$$\{\tilde{\psi}_1, \tilde{\psi}_2\}_t = \{\tilde{\Psi}_1, \tilde{\Psi}_2\}_c = \{\tilde{\psi}_1, \tilde{\Psi}_2\}_t = \{\tilde{\Psi}_1, \tilde{\Psi}_2\}_t = 0, \tag{A11}$$

where t indicates the surface that isolates scatterer t from the others, and c indicates a surface around the whole collection. We may also regard the theorems for lossless scatterers that follow from

$$\{\tilde{\psi}_1^*, \tilde{\psi}_2\}_t = \{\tilde{\Psi}_1^*, \tilde{\Psi}_2\}_c = \{\tilde{\psi}_1^*, \tilde{\Psi}_2\}_t = \{\tilde{\Psi}_1^*, \tilde{\Psi}_2\}_t = 0, \tag{A12}$$

as "reciprocity relations."

The first form of (A12) yields (79), and the second yields (79) with $\tilde{g}$ replaced by the compound amplitude $\tilde{\mathfrak{G}}$. Using (A5) in theorem (79) for $\tilde{\mathfrak{G}}$, we obtain

$$\tilde{G}_t(\mathbf{i}_1^*, \mathbf{i}_2) + \tilde{G}_t^{\dagger}(\mathbf{i}_2^*, \mathbf{i}_1) = -\sum{}' [\tilde{G}_s(\mathbf{i}_1^*, \mathbf{i}_2) - \tilde{G}_s^{\dagger}(\mathbf{i}_2, \mathbf{i}_1)]e^{-ik(\mathbf{i}_2-\mathbf{i}_1^*)\cdot\mathbf{b}_{ts}} - \frac{1}{2\pi}\int [\tilde{G}_t^{\dagger}(\mathbf{o}, \mathbf{i}_1) \cdot \tilde{G}_t(\mathbf{o}, \mathbf{i}_2) + \sum{}' \tilde{G}_s^{\dagger} \cdot \tilde{G}_s e^{-ik(\mathbf{i}_2-\mathbf{i}_1^*)\cdot\mathbf{b}_{ts}} + \sum\sum{}' \tilde{G}_m^{\dagger} \cdot \tilde{G}_n e^{ik(\mathbf{o}\cdot\mathbf{b}_{mn}+\mathbf{i}_2\cdot\mathbf{b}_{nt}+\mathbf{i}_1^*\cdot\mathbf{b}_{tm})}]\,d\Omega, \tag{A13}$$

where $\mathbf{b}_{nm} = \mathbf{b}_n - \mathbf{b}_m$, etc. In the forward direction $\mathbf{i}_1^* = \mathbf{i}_2 = \mathbf{i}_1 = \mathbf{i}_2^* = \mathbf{i}$ for the class of scatterers such that $\tilde{G}^{\mathrm{T}}(\mathbf{o}, \mathbf{i}) = \tilde{G}(\mathbf{i}, \mathbf{o})$ we have

$$-\mathrm{Re}\, \tilde{G}_t(\mathbf{i}, \mathbf{i}) = \mathrm{Re} \sum{}' \tilde{G}_s(\mathbf{i}, \mathbf{i}) + \frac{1}{4\pi}\int [\tilde{G}_t^*(\mathbf{i}, \mathbf{o}) \cdot \tilde{G}_t(\mathbf{o}, \mathbf{i}) + \sum{}' \tilde{G}_s^* \cdot \tilde{G}_s + \sum\sum{}' \tilde{G}_m^* \cdot \tilde{G}_n e^{ik(\mathbf{o}-\mathbf{i})\cdot\mathbf{b}_{mn}}]\, d\Omega. \tag{A14}$$

We consider a special case of (A14) corresponding to two simple scatterers in (163)ff, and have considered other special cases in the papers on periodic and random distributions cited in Refs. 4 and 5.

The third form in (A12) yields

$$\tilde{G}_t(\mathbf{i}_1^*, \mathbf{i}_2) + \tilde{g}_t^{\dagger}(\mathbf{i}_2, \mathbf{i}_1) = -\frac{1}{2\pi}\int \tilde{g}_t^{\dagger}(\mathbf{o}, \mathbf{i}_1) \cdot \tilde{G}_t(\mathbf{o}, \mathbf{i}_2)d\Omega_o - \frac{1}{2\pi}\sum{}' \int e^{ik(\mathbf{p}-\mathbf{i}_2)\cdot\mathbf{b}_{ts}}\, \tilde{g}_t^{\dagger}(\mathbf{p}^*, \mathbf{i}_1) \cdot \tilde{G}_s(\mathbf{p}, \mathbf{i}_2)\, d\Omega_p, \tag{A15}$$

in which we may specialize to forward scattering and use (75) for $\tilde{g}$ to eliminate Re $\tilde{g}(\mathbf{i}, \mathbf{i})$. Finally the fourth form of (A12) yields

$$\begin{aligned}\tilde{G}_t(\mathbf{i}_1^*, \mathbf{i}_2) + \tilde{G}_t^\dagger(\mathbf{i}_2, \mathbf{i}_1) &= -\frac{1}{2\pi}\int \tilde{G}_t^\dagger(\mathbf{o}, \mathbf{i}_1)\cdot\tilde{G}_t(\mathbf{o}, \mathbf{i}_2)\,d\Omega_o \\ &\quad -\frac{1}{2\pi}\sum{}'\int e^{ik(\mathbf{p}-\mathbf{i}_2)\cdot\mathbf{b}_{ts}}\tilde{G}_t^\dagger(\mathbf{p}^*, \mathbf{i}_1)\cdot\tilde{G}_s(\mathbf{p}, \mathbf{i}_2)\,d\Omega \\ &\quad -\frac{1}{2\pi}\sum{}'\left[\int e^{ik(\mathbf{p}-\mathbf{i}_2)\cdot\mathbf{b}_{ts}}\tilde{G}_t^\dagger(\mathbf{p}^*, \mathbf{i}_2)\cdot\tilde{G}_s(\mathbf{p}, \mathbf{i}_1)\,d\Omega\right]^\dagger.\end{aligned} \quad \text{(A16)}$$

For forward scattering and $\tilde{G}^T(\mathbf{o}, \mathbf{i}) = \tilde{G}(\mathbf{i}, \mathbf{o})$ we have

$$-\mathrm{Re}\,\tilde{G}_t(\mathbf{i}, \mathbf{i}) = \frac{1}{4\pi}\int \tilde{G}^*(\mathbf{i}, \mathbf{o})\cdot\tilde{G}(\mathbf{o}, \mathbf{i})\,d\Omega_o + \frac{1}{2\pi}\mathrm{Re}\sum{}'\int e^{ik(\mathbf{p}-\mathbf{i})\cdot\mathbf{b}_{ts}}\tilde{G}_t^*(\mathbf{i}, \mathbf{p}^*)\cdot\tilde{G}_s(\mathbf{p}, \mathbf{i})\,d\Omega. \quad \text{(A17)}$$

We consider special cases of (A17) in the papers on periodic and random distributions cited in Refs. 4 and 5.

APPENDIX B. SCALAR DIPOLES

In the previous developments of the analogous scalar problems (Refs. 1 and 3), the case of two different monopoles was used as the simplest illustration. For the present electromagnetic case, our discussion of two electrical dipoles as in (191)ff provides the dyadic analog of the previous results for monopoles: i.e., if we replace the dyadics by appropriate scalars we again get the earlier results. Thus the scalar version of (191)ff is

$$\begin{aligned}U_\pm &= u_\pm\Phi^\pm, \quad u_\pm = a_0^\pm h(kr_\pm), \quad \varphi = e^{i\mathbf{k}\cdot\mathbf{r}}, \\ \Phi^\pm &= \varphi^\pm + u_\mp^\pm\Phi^\mp = \frac{(1 + u_\mp^\pm e^{\mp i2\delta})\varphi^\pm}{1 - u_+u_-} \\ &= \frac{[1 + a_0^\mp e^{\mp i2\delta}h(2kb)]\varphi^\pm}{1 - a_0^+a_0^-h^2},\end{aligned} \quad \text{(B1)}$$

where $h = h_0^{(1)}$ for the three-dimensional problem [see Ref. 3, Eq. (63)], and $h = H_0^{(1)}$ for the two-dimensional problem [see Ref. 1, Eq. (71)].

The corresponding scalar problems of symmetrical dipoles,

$$u = iH_1^{(1)}(kr)a_1(2)\cos(\theta - \theta_i)\varphi, \quad \varphi = e^{i\mathbf{k}\mathbf{i}\cdot\mathbf{r}}, \quad \text{(B2)}$$

$$u = ih_1^{(1)}(kr)a_1(3)\cos(\theta - \theta_i)\varphi, \quad \text{(B3)}$$

where (B2) and (B3) correspond to two dimensions and three dimensions, respectively, were considered by separations of variables.[1,3] The normalization of the scattering coefficients $a_1(2)$ and $a_1(3)$ is here chosen so that for lossless scatterers, we have $-\mathrm{Re}\,a_1(n) = n\,|a_1(n)|^2$, e.g., for $\psi = 0$ at the surface, we have $a_1 = -2J_1/H_1$ and $a_2 = -3j_1/h_1$. The present dyadic development for two magnetic dipoles (204)ff suggests an analogous development for generalizing the scalar results to two arbitrary dipoles.

We rewrite (B2) and (B3) in the single form

$$\begin{aligned}u \equiv \mathbf{v}\cdot\nabla\varphi &= -(a/k^2)(\nabla h)\cdot(\nabla\varphi) \\ &= -(a/k^2)[kh'\mathbf{o}]\cdot[ik\varphi\mathbf{i}] \\ &= -iah'\varphi(\mathbf{i}\cdot\mathbf{o}),\end{aligned} \quad \text{(B4)}$$

where h is either $H_0^{(1)}$ or $h_0^{(1)}$, and $h' = \partial_{kr}h(kr)$ is either $-H_0^{(1)}$ or $-h_0^{(1)}$; similarly $a = a_1(2)$, $a_1(3)$. We may now proceed essentially as for (205). Thus for two dipoles we use

$$U_\pm = \mathbf{v}_\pm\cdot\nabla(\varphi^\pm + U_\mp^\pm) = \mathbf{v}_\pm\cdot\nabla\Phi^\pm, \qquad \mathbf{v} = -(a/k^2)\nabla h, \quad \text{(B5)}$$

$$\Phi^\pm = \varphi^\pm + \mathbf{v}_\mp^\pm\cdot\nabla\Phi^\mp. \quad \text{(B6)}$$

Taking the gradient of (B6), and eliminating $\nabla\Phi^\pm$ from the right-hand side, we obtain the analog of (207):

$$\nabla\Phi^\pm = [\tilde{I} - \nabla\mathbf{v}_\mp^\pm\cdot\nabla\mathbf{v}_\pm^\mp]^{-1}\cdot[\nabla\varphi^\pm + \nabla\mathbf{v}_\mp^\pm\cdot\nabla\varphi^\mp]. \quad \text{(B7)}$$

We have

$$\begin{aligned}\nabla\mathbf{v} &= -(a/k^2)\nabla\nabla h \\ &= a[\hat{b}\hat{b}h_1' + (\tilde{I} - \hat{b}\hat{b})h_1/\rho] \\ &\equiv a[\hat{b}\hat{b}\mathcal{H} + (\tilde{I} - \hat{b}\hat{b})H], \quad \rho = 2kb,\end{aligned} \quad \text{(B8)}$$

where h is either $h_0^{(1)}$ or $H_0^{(1)}$, and h_1 is either $h_1^{(1)}$ or $H_1^{(1)}$. Thus $\nabla\mathbf{v}_\mp^\pm\cdot\nabla\mathbf{v}_\pm^\mp = a_+a_-[\hat{b}\hat{b}\mathcal{H}^2 + (\tilde{I} - \hat{b}\hat{b})H^2]$, and

$$\begin{aligned}[\tilde{I} - \nabla\mathbf{v}\cdot\nabla\mathbf{v}]^{-1} &= \hat{b}\hat{b}E(\mathcal{H}) + (\tilde{I} - \hat{b}\hat{b})E(H), \\ E(\mathcal{H}) &= 1/(1 - a_+a_-\mathcal{H}^2),\end{aligned} \quad \text{(B9)}$$

$$\begin{aligned}\nabla\Phi^\pm &= [\hat{b}\hat{b}A(\mathcal{H}) + (\tilde{I} - \hat{b}\hat{b})A(H)]\cdot\nabla\varphi^\pm \\ &\equiv \tilde{P}\cdot\nabla\varphi^\pm/a_\pm, \\ A(a^\pm, \mathcal{H}) &= \frac{1 + a^\mp\mathcal{H}e^{\mp i2\delta}}{1 - a_+a_-\mathcal{H}^2},\end{aligned} \quad \text{(B10)}$$

where $\tilde{P}_\pm$ are the multiple-scattered moments.

Substituting (B10) into (B5), we obtain

$$U_\pm = \mathbf{v}_\pm\cdot\tilde{P}\cdot\nabla\varphi^\pm/a_\pm \sim h(kr)e^{\pm i(\delta-\Delta)}G_\pm, \quad \text{(B11)}$$

where the multiple-scattered amplitudes equal

$$\begin{aligned}G_\pm(\mathbf{o}, \mathbf{i}) &= \mathbf{o}\cdot\tilde{P}\cdot\mathbf{i} \\ &= \mathbf{o}\cdot[\hat{b}\hat{b}a_\pm A(a^\mp, \mathcal{H}) + (\tilde{I} - \hat{b}\hat{b})a_\pm A(a^\mp, H]\cdot\mathbf{i}.\end{aligned} \quad \text{(B12)}$$

(Note the shift in location between H and $\mathcal{H}$ type compared to vector problems in the text proper; this is in accord with the relations between, e.g., acoustic pressure dipoles and the electric functions in the text.)

If we express (B12) in terms of the appropriate two- and three-dimensional special functions, we obtain the previous results [(85) of Ref. 1 for circular cylinders, and (69) of Ref. 3 for spheres].

To generalize the above to arbitrary dipoles, we replace $\mathbf{v}$ in the above by

$$\mathbf{v} = -(1/k^2)\nabla h \cdot \bar{p}, \tag{B13}$$

so that

$$u = -(1/k^2)\nabla h \cdot \bar{p} \cdot \nabla\varphi = ih_1\varphi\mathbf{o} \cdot \bar{p} \cdot \mathbf{i}, \tag{B14}$$

where, e.g., $\bar{p}$ may be constructed from the known approximations for elliptic cylinders and ellipsoids. For the case where the principal axis of $\bar{p}_+$ and $\bar{p}_-$ are parallel (i.e., essentially as in the text) we obtain (B11) and (B12) with $\bar{P}$ replaced by

$$\bar{P}_\pm = p_{\xi\pm}A(p_{\xi\mp}, H)\hat{\xi}\hat{\xi} + p_{\eta\pm}A(p_{\eta\mp}, H)\hat{\eta}\hat{\eta} + p_{\zeta\pm}A(p_{\zeta\mp}, \mathcal{H})\hat{\zeta}\hat{\zeta}, \tag{B15}$$

where the ξ term is to be dropped for two dimensions.

We could also extend the above to all moments by working with

$$u(n) = (L_n h) \otimes p_n \otimes (L_n\varphi); \quad n = 2, 3. \tag{B16}$$

Thus for monopoles

$$L = 1, \quad p = a. \tag{B17}$$

For dipoles

$$L = \nabla/ik, \quad \otimes = \cdot, \quad p = (\mathbf{ab}). \tag{B18}$$

For quadrupoles we have

$$L_n = \left(I + n\frac{\nabla\nabla}{k^2}\right), \quad \otimes = :, \quad p_n = (\mathbf{abcd})/(n-1)n, \tag{B19}$$

which represents u as the scalar resulting from double-dotting a tetradic[6] fore and aft by dyadics.