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Exact Inverse-Separation Series for Multiple Scattering in Two Dimensions

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We consider configurations of arbitrary scatterers ($s=1,\cdots,N$) in two dimensions, such that the circles circumscribing the scatterers do not intersect. As shown previously [V. Twersky, in *Electromagnetic Waves*, R. E. Langer, Ed. (University of Wisconsin Press, Madison, 1962), pp. 361–389], the solution can be written in terms of the multiple-scattered scattering amplitudes G_s, and the G_s are specified by the presumably known farfield isolated scattering amplitudes g_s by a set of integral equations $G(g)$ (which can be converted to algebraic equations involving Hankel functions of the separations b_{st}, etc.). Among other applications, the previous paper gave the complete asymptotic series for $G(g)$ in inverse powers of the b's; this was based essentially on Hankel's asymptotic expansion for the Hankel functions H_n. The present paper derives the analogous convergent representation of $G(g)$ based on the exact representation of H_n in terms of Lommel polynomials. For N scatterers, we give the multiple-scattering solution as a series in H_0, H_1, b^{-n}, and the derivatives of g with respect to angles. For two scatterers, we give a closed form in terms of a differential operator.

INTRODUCTION

PREVIOUS papers derived integral equations for multiple scattering of waves by configurations of arbitrary scatterers in two[1] and in three[2] dimensions. These equations specify the solution of the many-body problem in terms of the presumably known farfield scattering amplitudes (say g) of the scatterers in isolation. Equivalent representations as sets of algebraic equations were derived in terms of the corresponding isolated scattering coefficients (the coefficients of the Fourier, Mathieu, or Legendre series representations of g), and several applications were made. In particular, series forms of the solutions in inverse powers of the separations of scatterers were generated in terms of derivatives of g with respect to angles. These inverse-separation series, convergent in three dimensions and asymptotic in two dimensions, were based essentially on the corresponding inverse-distance (of observation) series for an isolated scatterer in two or three dimensions, or, equivalently, on Hankel's series[3] for the outgoing radial

[1] V. Twersky, "Scattering of Waves by Two Objects," in *Electromagnetic Waves*, R. E. Langer, Ed. (University of Wisconsin Press, Madison, 1962), pp. 361–389; Sylvania EDL Rept. EDL-E60 (1961).

[2] V. Twersky, "Multiple Scattering by Arbitrary Configurations in Three Dimensions," J. Math. Phys. 3, 83–91 (1962); Sylvania EDL Rept. EDL-E61 (1961).

[3] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1958), p. 198.

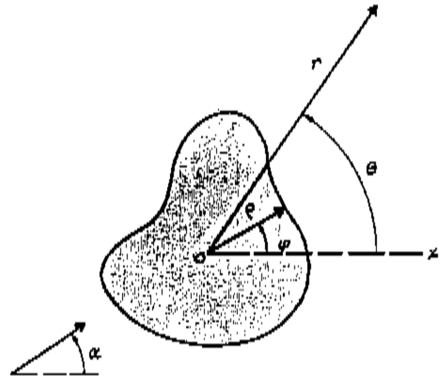

FIG. 1. Coordinates for single-body problem for case where $\boldsymbol{\varrho}$ is on the scatterer's surface. More generally, $\boldsymbol{\varrho}$ is a vector to any surface that separates the scatterer from the observation point $\mathbf{r}$. The radius of the smallest circle circumscribing the scatterer is a.

functions $H_p(r)=H_p^{(1)}(r)$ in inverse powers of r. If $p=n+\frac{1}{2}$, where n is an integer, then Hankel's series reduces to e^{ir} times a polynomial in $1/r$; however, if $p=n$, the representation is an asymptotic series; the corresponding scattering series are convergent in three dimensions and asymptotic in two dimensions. In the present paper, we derive the convergent inverse-separation series for multiple scattering in two dimensions.

For three-dimensional scattering problems, Sommerfeld[4] considered the inverse-distance series form for a single scatterer and showed that the coefficients of successive terms could be obtained recursively from the first, i.e., from the usual scattering amplitude g. Wilcox[5] showed that the series was uniformly and absolutely convergent outside the sphere circumscribing the arbitrary scatterer. Twersky[2] recast the series as a differential operator on g and used the resulting form to treat multiple scattering: for N arbitrary scatterers (such that the spheres circumscribing the scatterers do not intersect), he obtained the multiple-scattering solution and multiple-scattering amplitude (say G) as a series in the g's and the separations of the scatterers; for two, he obtained a closed form involving a differential operator for G in terms of g.

For two-dimensional scattering problems, the analog of the Sommerfeld–Wilcox development has recently been given by Karp,[6] who used essentially the exact form of $H_n(r)$ in terms of $H_0(r)$ and $H_1(r)$ times Lommel polynomials in $1/r$ (see Ref. 3, p. 297). Karp[6] showed that the representation in H_0 and H_1 and inverse powers of r converged, and described a procedure for obtaining the coefficients of successive terms recursively from two values of the scattering amplitude. In the present paper, we construct the corresponding representation in terms of two differential operators, and then derive the convergent representation in inverse powers of separations for the multiple-scattering problem.

If the Hankel asymptotic expansions of H_0 and H_1 are inserted in the present results (the exact series for N bodies and the closed operator form for 2 bodies), then we again obtain the complete asymptotic representations given previously.[1] Although the asymptotic representations are the more useful ones even for only moderately large separations, the convergent ones bring the subject in two dimensions to the same level of completeness as in three, and are expected to facilitate obtaining explicit representations for specific problems.

I. PRELIMINARY CONSIDERATIONS

We consider scattering problems such that the field outside the scatterer satisfies

$$(\nabla^2+k^2)\psi e^{-i\omega t}=0,\quad \nabla^2=\partial_x^2+\partial_y^2,\quad k=|\mathbf{k}|=2\pi/\lambda,\quad (1)$$

where ψ consists of a plane-wave source term φ and an outgoing scattered wave u:

$$\varphi=e^{i\mathbf{k}\cdot\mathbf{r}}=e^{ikr\cos(\theta-\alpha)},\qquad \mathbf{k}=k\mathbf{i},\qquad \mathbf{r}=r\mathbf{o};\quad (2)$$

$$u\sim g(\theta,\alpha)e^{ikr-i\pi/4}(2/\pi kr)^{\frac{1}{2}}\equiv g(\theta,\alpha)H(kr),\quad r\sim\infty.\quad (3)$$

In general, we take the origin of coordinates as the center of the smallest circle (radius a) circumscribing the scatterer. The scattering amplitude $g(\theta,\alpha)$ (which is independent of r) specifies the "farfield" response in the direction θ to a wave incident in the direction α. The field satisfies any of the usual boundary or transition conditions at the scatterer's surface, but we need not consider these explicitly. Although we use scattering terminology, the results apply to any radiative solution of von Helmholtz's equations, i.e., to any solution of Eq. (1) subject to Eq. (3) (e.g., g may be an antenna pattern function or a piston function).

Many different exact general representations for u and g exist: surface integrals, volume integrals, complex-integral spectral representations, infinite series of Bessel and circular functions, infinite series of Mathieu funcsions, as well as various "mixed representations" obtained by routine manipulations of basic forms. For $r>a$ (at the least), appropriate representations for u and g may be paired off as transforms, so that knowledge of either one determines the other.

In particular, if we apply Green's theorem to $u(\boldsymbol{\varrho})$ and $H_0(k|\mathbf{r}-\boldsymbol{\varrho}|)$ in the region external to the scatterer, we obtain the Helmholtz surface integral form

$$u(\mathbf{r},\alpha)=\frac{1}{4i}\int[H_0(k|\mathbf{r}-\boldsymbol{\varrho}|)\partial_n u(\boldsymbol{\varrho},\alpha)-u\partial_n H_0]dS(\boldsymbol{\varrho})$$
$$\equiv\{H_0(k|\mathbf{r}-\boldsymbol{\varrho}|),u(\boldsymbol{\varrho},\alpha)\},\quad (4)$$

where $S(\boldsymbol{\varrho})$ is any surface inclosing the scatterer and excluding $\mathbf{r}$, and ∂_n is the normal derivative outward

[4] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949), p. 192.
[5] C. H. Wilcox, "A Generalization of Theorems of Rellich and Atkinson," Proc. Am. Math. Soc. 7, 271–276 (1956).
[6] S. N. Karp, "A Convergent 'Farfield' Expansion for Two-Dimensional Radiation Functions," New York Univ. Courant Inst. Math. Sci. Rept. EM-169 (1961).

from the scatterer (see Fig. 1). The corresponding representation for g of Eq. (3) follows from Eq. (4) by substituting the asymptotic form

$$H_0(k|\mathbf{r}-\boldsymbol{\rho}|)\sim e^{ik\rho\cos(\theta-\varphi)}H(kr),\quad kr\gg 1,\quad r\gg\rho. \quad (5)$$

Thus,

$$g(\theta,\alpha)=\{e^{-ik\rho\cos(\theta-\varphi)},u(\boldsymbol{\rho},\alpha)\}, \quad (6)$$

where the integral is over any surface inclosing the scatterer.

If u is known, then Eq. (6) gives g on integration. An inverse relation follows[1] on introducing into Eq. (4) the Sommerfeld[4] representation of H_0 as a complex integral of plane waves. As shown previously,[1]

$$u(\mathbf{r},\alpha)=\frac{1}{\pi}\int e^{ikr\cos(\theta-\tau)}g(\tau,\alpha)d\tau, \quad (7)$$

where the integral is over the Sommerfeld path $\gamma+i\infty$ to $\pi+\gamma-i\infty$ with γ satisfying $r\sin(\theta-\gamma)-[\rho\sin(\varphi-\gamma)]_{\max}>0$; this path will be understood for all integrations over τ. Values of ρ on the scatterer's surface give the greatest range to γ. If we take $\gamma=\theta-\pi/2$, then we require $r>[\rho\cos(\theta-\varphi)]_{\max}$—i.e., the distance ($r$) from the "center" of the scatterer (the center of the circumscribed circle of radius a) to the observation point must be greater than the scatterer's projection on $\mathbf{r}$; this value $\gamma=\theta-\pi/2$ suffices for all r and θ if $r>a$. Thus the scattering amplitude g introduced to specify the behavior of u for $r\sim\infty$ serves to describe the field at least for all distances $r>a$.

The limiting asymptotic form $u\sim Hg$ as in Eq. (3) follows from Eq. (7) by the usual saddle-point procedure[1] for $kr\to\infty$, and this procedure may also be used to generate the complete asymptotic representation. More directly, we exploit the analogous Hankel asymptotic representation[8] for H_n:

$$i^nH_n(r)=\frac{1}{\pi}\int e^{ir\cos\tau+i\tau n}d\tau\sim H(r)\left[1+\frac{(1-4n^2)}{i8r}+\cdots\right]$$
$$=H(r)\sum_{m=0}\frac{(1-4n^2)(9-4n^2)\cdots([2m-1]^2-4n^2)}{(i8r)^m m!}$$
$$\equiv H(r)\mathfrak{R}(-n^2)\equiv\mathfrak{D}(r;-n^2). \quad (8)$$

Thus, since the Taylor series for $f(\tau)$ around a saddle point $\tau_0=0$ may be written symbolically as $f(\tau)=e^{\tau\partial}f(\tau_0)$ with $\partial\equiv\partial/\partial\tau_0$, we have the general result

$$I=\frac{1}{\pi}\int e^{ir\cos\tau}f(\tau)d\tau\sim H(r)\sum_{m=0}\frac{(1+4\partial^2)(9+4\partial^2)\cdots([2m+1]^2+4\partial^2)}{(i8r)^m m!}f(\tau_0)\equiv\mathfrak{D}(r;\partial^2)f(\tau_0), \quad (9)$$

where τ_0 is to be set equal to zero (the saddle point) after differentiation. The special case corresponding to Eq. (7) (parameter kr, and saddle point θ) is[1]

$$u\sim\mathfrak{D}(kr;\partial_\theta^2)g(\theta,\alpha)$$
$$=H(kr)\left[g+\frac{g+4\partial^2g}{i8kr}-\frac{9g+40\partial^2g+16\partial^4g}{128(kr)^2}+\cdots\right] \quad (10)$$

in terms of the scattering amplitude $g(\theta,\alpha)$ and its θ derivatives. [Terms to $(kr)^{-1}$ were derived by Karp and Zitron[7] essentially by substituting $H(kr)[g+g_1/kr]$ into the wave equation and solving for g_1.] We shortly consider the exact series analogous to Eq. (10).

An alternative exact representation[1] for u in terms of g may be constructed by using the addition theorem for $H_0(k|\mathbf{r}-\boldsymbol{\rho}|)$ in Eq. (4), or by substituting the Fourier series

$$g(\theta,\alpha)=\sum_{n=-\infty}^{\infty}a_n(\alpha)e^{in\theta}, \quad (11)$$

into (7) and isolating the usual integral representation of H_n. This leads[1] to

$$u=\sum a_n(\alpha)i^nH_n(kr)e^{in\theta},\quad r>a. \quad (12)$$

The coefficients a_n (the Fourier, scattering, or multipole coefficients) determine both u and g. We may eliminate a_n by substituting

$$a_n(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}g(\mu,\alpha)e^{-in\mu}d\mu \quad (13)$$

to obtain

$$u=\sum H_n(kr)i^ne^{in\theta}\int_0^{2\pi}g(\mu,\alpha)e^{-in\mu}d\mu/2\pi,\quad r>a. \quad (14)$$

The coefficients a_n may also be written as surface integrals by substituting Eq. (6) into (13) and replacing the plane wave by its Fourier-Bessel series; thus $a_n(\alpha)=i^{-n}\{J_n(k\rho)\,e^{-in\varphi},u(\boldsymbol{\rho},\alpha)\}$.

The complete asymptotic series (10) for u may be obtained from Eq. (12) by using Eq. (8) and the differential equation satisfied by the exponential function:

$$(\partial_\theta^2+n^2)e^{in\theta}=0. \quad (15)$$

[7] S. N. Karp and N. Zitron, "Higher Order Approximations in Multiple Scattering," J. Math. Phys. 2, 394–406 (1961).

From Eqs. (8) and (15), we have

$$i^n H_n(r)e^{in\theta}\sim \mathfrak{D}(r;\,-n^2)e^{in\theta}=\mathfrak{D}(r;\,\partial^2)e^{in\theta}, \quad (16)$$

and consequently (14) reduces asymptotically to

$$u\sim \mathfrak{D}(r;\,\partial^2)\sum_n e^{in\theta}\int_0^{2\pi} g(\mu,\alpha)e^{-in\mu}d\mu/2\pi$$
$$=\mathfrak{D}(r;\,\partial^2)g(\theta,\alpha) \quad (17)$$

as in (10). The analogous procedure for the three-dimensional problem,[2] based on Hankel's polynomial representation for $H_{n+\frac{1}{2}}$, yields u equal to e^{ikr}/ikr times an exact convergent series in inverse powers of r (a series whose coefficients involve the scattering amplitude acted on by powers of Beltrami's differential operator).

In the range $r>a$, Eqs. (7) and (14) are equivalent representations of the scattered field u (or of any radiative solution of the Helmholtz equation), and Eq. (17) is the corresponding complete asymptotic inverse-distance series. We obtained Eq. (7) from the basic form (4) and then obtained Eq. (14) from (7). We could just as well have obtained Eq. (14) from (4) (by using the addition theorem for H_0) and then obtained Eq. (7) from (14) (by using the plane-wave representation of H_n). (The sequence that we followed is ordered in terms of decreasing domain of validity in r.) Similarly, the two procedures for obtaining the complete asymptotic representation are equivalent. We mentioned both and stressed the equivalence of the plane-wave and cylindrical-wave forms to facilitate a subsequent development. Additional discussion and representations are given in Ref. 1.

As discussed by Karp,[6] a convergent inverse distance expansion for the two-dimensional field has essentially the form

$$u(\mathbf{r})=H_0(kr)\sum_{\nu=0}^{\infty}\frac{P_\nu(\theta)}{\nu!(2kr)^\nu}+H_1(kr)\sum_{\nu=0}^{\infty}\frac{Q_\nu(\theta)}{\nu!(2kr)^\nu}, \quad r>a. \quad (18)$$

Karp showed that the series converge uniformly and absolutely for $r>a$, and considered their analytical properties in detail. He pointed out that substituting Eq. (18) into the differential equation (1)—differentiating and rearranging the results as a linear combination of H_0 and H_1 and using the fact that the coefficients of the Hankel functions H_0 and H_1 must vanish—leads to P_n and Q_n recursively in terms of P_0 and Q_0; the zeroth-order coefficients were expressed in terms of g by comparison with asymptotic forms. As examples, Karp gave[6]

$$P_0(\theta)=[g(\theta)+g(\theta+\pi)]/2,\quad Q_0(\theta)=[g(\theta)-g(\theta+\pi)]i/2,$$
$$P_1=-(\partial^2Q_0+Q_0),\qquad Q_1=\partial^2P_0. \quad (19)$$

Thus, Karp[6] has provided the parallel development for two dimensions of that followed by Sommerfeld[4] and Wilcox[5] for three dimensions. We now parallel Twersky's development[2] for the three-dimensional problem and recast u explicitly in terms of ∂^2 and g; i.e., we obtain the convergent analog of Eq. (10).

II. EXACT OPERATIONAL FORM

As discussed in Watson,[3] the Hankel function may be written exactly as

$$H_n(r)=H_0(r)R_{n,0}(r)+H_1(r)R_{n-1,1}(r), \quad (20)$$

where the general form of a Lommel polynomial is[3]

$$R_{n,s}=\sum_{m=0}^{\leq n/2}\frac{(-1)^m(n-m)!\Gamma(s+n-m)}{m!(n-2m)!\Gamma(s+m)}\left(\frac{r}{2}\right)^{2m-n}. \quad (21)$$

From this, we may represent the R's of Eq. (20) as polynomials in n^2.

Corresponding to Eq. (20), and as determined by whether n is even or odd, we distinguish four cases of Eq. (21):

$$R_{2n,0}=\sum_{\nu=0}^{n-1}C_{n\nu}=\sum_{\nu=0}^{n-1}\frac{(-1)^{n+\nu}}{(2\nu)!(2r)^{2\nu}}\times\left[2^{4\nu}\frac{(n+\nu)!(n+\nu-1)!}{(n-\nu)!(n-\nu-1)!}\right],$$
$$R_{2n-1,1}=-\sum_{\nu=0}^{n-1}C_{n\nu}\frac{2}{r}\frac{(n+\nu)(n-\nu)}{(2\nu+1)}, \quad (22)$$
$$R_{2n+1,0}=\sum_{\nu=0}^{n-1}C_{n\nu}\frac{2}{r}\frac{(n+\nu+1)}{(2\nu+1)}(n+\nu),$$
$$R_{2n,1}=\sum_{\nu=0}^{n}C_{n\nu}\frac{(n+\nu)}{(n-\nu)}.$$

In the quotient of factorials in brackets in $R_{2n,0}$, we divide through by the denominator and pair off factors $(n+m)(n-m)=n^2-m^2$ to write initially

$$2^{4\nu}[(n+\nu)\cdots n\cdots(n-\nu+1)][(n+\nu-1)\cdots n\cdots(n-\nu)]$$
$$=2^{4\nu}[n^2-\nu^2]\cdots n^2\cdots[n^2-(\nu-1)^2];$$

doubling up once more and multiplying through by $2^{4\nu}$, we thus obtain

$$(2n)^2\{(2n)^2-2^2][(2n)^2-4^2]\cdots[(2n)^2-(2\nu-2)^2]\}^2\times[(2n)^2-(2\nu)^2]\equiv M_\nu(-[2n]^2), \quad (23)$$

with $M_0=1$. Equation (23) reduces the first two sets of R's of Eq. (22) to polynomials in $(2n)^2$. For $R_{2n,1}$, we deal with the bracketed quotient of factorials times $(n+\nu)/(n-\nu)$ and pair off factors $4(n+m)(n-m+1)$

$=(2n+1)^2-(2m-1)^2$; doubling up, we obtain

$$\{[(2n+1)^2-1][(2n+1)^2-9]\cdots[(2n+1)^2-(2\nu-1)^2]\}^2 \equiv N_\nu(-[2n+1]^2), \quad (24)$$

with $N_0=1$. Equation (24) reduces the last two sets of R's of Eq. (22) to polynomials in $(2n+1)^2$.

In terms of Eqs. (23), (24), and

$$L_\nu \equiv (-1)^\nu/(2r)^{2\nu}(2\nu)!, \quad (25)$$

we write the required coefficients as the polynomials

$$\begin{aligned}
R_e(-[2n]^2)&=(-1)^n R_{2n,0}=\sum_{\nu=0}^{\infty} L_\nu M_\nu(-[2n]^2),\\
R_e'(-[2n]^2)&=(-1)^n R_{2n-1,1}=-\sum_{\nu=0}^{\infty} L_\nu M_\nu(-[2n]^2)\frac{[(2n)^2-(2\nu)^2]}{2r(2\nu+1)},\\
R_o'(-[2n+1]^2)&=i(-1)^n R_{2n+1,0}=i\sum_{\nu=0}^{\infty} L_\nu N_\nu(-[2n+1]^2)\frac{[(2n+1)^2-(2\nu+1)^2]}{2r(2\nu+1)},\\
R_o(-[2n+1]^2)&=i(-1)^n R_{2n,1}=i\sum_{\nu=0}^{\infty} L_\nu N_\nu(-[2n+1]^2),
\end{aligned} \quad (26)$$

where the upper limits were replaced by ∞ since [from Eqs. (23) and (24)] all new terms are identically zero. Thus, from Eqs. (20) and (26), we have

$$\begin{aligned}
(-1)^n H_{2n}&\equiv H_0R_e(-[2n]^2)+H_1R_e'(-[2n]^2)\equiv D_e(r;-[2n]^2),\\
i(-1)^n H_{2n+1}&=H_0R_o'(-[2n+1]^2)+H_1R_o(-[2n+1]^2)\equiv D_o(r;-[2n+1]^2).
\end{aligned} \quad (27)$$

Note the factor of i incorporated in the odd values.

The essential feature of the above is that we have reduced H_{2n} and H_{2n+1} to polynomials in $(2n)^2$ and $(2n+1)^2$, respectively. Consequently, because of Eq. (15), we may replace the polynomials in n^2 by ones in $\partial^2=\partial_\theta^2$ in the product forms; thus

$$\begin{aligned}
(-1)^n H_{2n}e^{i2n\theta}&=D_e(-[2n]^2)e^{i2n\theta}=D_e(\partial^2)e^{i2n\theta},\\
(-1)^n iH_{2n+1}e^{i(2n+1)\theta}&=D_o(-[2n+1]^2)e^{i(2n+1)\theta}=D_o(\partial^2)e^{i(2n+1)\theta}.
\end{aligned} \quad (28)$$

Thus, the Hankel–Fourier series (12) may be rewritten

$$\begin{aligned}
u&=\sum a_{2n}H_{2n}(-1)^n e^{i2n\theta}+\sum a_{2n+1}H_{2n+1}(-1)^n ie^{i(2n+1)\theta}\\
&=D_e(\partial^2)\sum a_{2n}e^{i2n\theta}+D_o(\partial^2)\sum a_{2n+1}e^{i(2n+1)\theta}.
\end{aligned} \quad (29)$$

Equivalently, since

$$\begin{aligned}
\sum a_{2n}e^{i2n\theta}&=\tfrac{1}{2}\sum a_ne^{in\theta}(1+e^{in\pi})=\tfrac{1}{2}[g(\theta)+g(\pi+\theta)]\equiv g_e(\theta),\\
\sum a_{2n+1}e^{i(2n+1)\theta}&=\tfrac{1}{2}\sum a_ne^{in\theta}(1-e^{in\pi})=\tfrac{1}{2}[g(\theta)-g(\pi+\theta)]\equiv g_o(\theta),
\end{aligned} \quad (30)$$

we have thus reduced Eq. (12) to the form

$$u(\mathbf{r};\alpha)=D_e(r;\partial_\theta^2)g_e(\theta,\alpha)+D_o(r;\partial_\theta^2)g_o(\theta,\alpha). \quad (31)$$

Here,

$$\begin{aligned}
D_e&=H_0R_e+H_1R_e',\\
R_e&=1-\frac{\partial^2(\partial^2+4)}{2(2r)^2}+\frac{\partial^2(\partial^2+4)^2(\partial^2+16)}{4!(2r)^4}+\cdots,\\
R_e'&=\frac{\partial^2}{2r}-\frac{\partial^2(\partial^2+4)^2}{3!(2r)^3}+\frac{\partial^2(\partial^2+4)^2(\partial^2+16)^2}{5!(2r)^5}+\cdots,
\end{aligned} \quad (32)$$

and

$$D_o=H_0R_o'+H_1R_o,$$
$$R_o'/i=-\frac{\partial^2+1}{2r}+\frac{(\partial^2+9)(\partial^2+1)^2}{3!(2r)^3}-\frac{(\partial^2+25)(\partial^2+9)^2(\partial^2+1)^2}{5!(2r)^5}+\cdots, \tag{33}$$
$$R_o/i=1-\frac{(\partial^2+1)^2}{2(2r)^2}+\frac{(\partial^2+1)^2(\partial^2+9)^2}{4!(2r)^4}+\cdots.$$

The functions g_e and g_o are the components of the scattering amplitude $g(\theta,\alpha)$ that are symmetrical and antisymmetrical with respect to reflection in the plane $\theta+\pi/2$ (i.e., the plane perpendicular to the direction of observation).

If the scatterer is a monopole $g(\theta,\alpha)=a_0$, then $g_e=a_0$ and $g_o=0$; u reduces to D_eg_e, and since all terms but the first of R_e involve ∂^2 we obtain simply $u=H_0a_0$ as required. If the scatterer is a monopole plus dipole $g(\theta,\alpha)=a_0+2a_1\cos\theta$, then $g_e=a_0$ and $g_o=2a_1\cos\theta$; the D_eg_e term again gives H_0a_0, and for D_og_o [since all terms but the first involve $(\partial^2+1)\cos\theta=0$] we obtain $H_1i2a_1\cos\theta$. Thus $u=H_0a_0+H_1i2a_1\cos\theta$ as required.

Isolating H_0 and H_1 in Eq. (31), we have

$$u=H_0(R_eg_e+R_o'g_o)+H_1(R_og_o+R_e'g_e), \tag{34}$$

which is the same form as Eq. (18) obtained by Karp.[6] We now have the coefficients of Eq. (18) explicitly in terms of g and its θ derivatives:

$$P_{2\nu}=(-1)^\nu M_\nu(\partial_\theta^2)g_e(\theta),\quad P_{2\nu+1}=-i(-1)^\nu[\partial_\theta^2+(2\nu+1)^2]N_\nu(\partial_\theta^2)g_o(\theta),$$
$$Q_{2\nu}=i(-1)^\nu N_\nu(\partial_\theta^2)g_o(\theta),\quad Q_{2\nu+1}=(-1)^\nu[\partial_\theta^2+(2\nu)^2]M_\nu(\partial_\theta^2)g_e(\theta), \tag{35}$$

where M and N are the polynomials of Eqs. (23) and (24). Thus,

$$P_0=g_e=\tfrac{1}{2}[g(\theta)+g(\pi+\theta)],\quad P_1=-i(\partial_\theta^2+1)g_o(\theta),\quad P_2=-\partial_\theta^2(\partial_\theta^2+4)g_e(\theta),\quad P_3=+i(\partial_\theta^2+9)(\partial_\theta^2+1)^2g_o(\theta),$$
$$Q_0=ig_o=\frac{i}{2}[g(\theta)-g(\pi+\theta)],\quad Q_1=\partial_\theta^2g_e(\theta),\quad Q_2=-i(\partial_\theta^2+1)^2g_o(\theta),\quad Q_3=-\partial_\theta^2(\partial_\theta^2+4)^2g_e(\theta), \tag{36}$$

etc. The coefficients for 0 and 1 are Karp's results as in the present Eq. (19), and the other terms may also be obtained by the method that he describes.[6] From Eqs. (18) and (36), we have

$$u=H_0\left\{g_e-\frac{i(\partial^2+1)}{2kr}g_o-\frac{\partial^2(\partial^2+4)g_e}{2(2kr)^2}+i\frac{(\partial^2+9)(\partial^2+1)^2g_o}{3!(2kr)^3}+\cdots\right\}$$
$$+H_1\left\{ig_o+\frac{\partial^2g_e}{2kr}-\frac{i(\partial^2+1)^2g_o}{2(2kr)^2}-\frac{\partial^2(\partial^2+4)^2}{3!(2kr)^3}g_e+\cdots\right\}. \tag{37}$$

For subsequent applications, we substitute the definitions of g_e and g_o of Eq. (30) and rewrite Eq. (31) as

$$u=D(r;\partial_\theta^2)g(\theta,\alpha)+D'(r;\partial_\theta^2)g(\pi+\theta,\alpha),\quad D\equiv(D_e+D_o)/2,\quad D'\equiv(D_e-D_o)/2. \tag{38}$$

If we replace H_0 and H_1 in Eq. (38) by their Hankel asymptotic series as in Eq. (8), then $D\sim\mathfrak{D}$ as in Eq. (9) and $\mathfrak{D}'\sim 0$. Thus Eq. (38) is a convenient form for assessing the corrections to the complete asymptotic representation (10).

Since the special function series (12) and the complex integral representation (7) are equivalent forms for $r>a$, Eq. (31) is, of course, the exact series representation of Eq. (8) for $r>a$, as well as of any other form of any solution of Eq. (1) subject to Eq. (3). Since Eq. (7) is general in form and since we may always write $f(\tau)=[f(\tau)+f(\pi+\tau)]/2+[f(\tau)-f(\pi+\tau)]/2=f_e(\tau)+f_o(\tau)$, the above steps have shown that

$$\frac{1}{\pi}\int e^{ir\cos(\tau-\theta)}f(\tau)d\tau=\frac{1}{\pi}\int e^{ir\cos(\tau-\theta)}[f_e(\tau)+f_o(\tau)]d\tau$$
$$=D_e(r;\partial_\theta^2)f_e(\theta)+D_o(r;\partial_\theta^2)f_o(\theta)=D(r;\partial_\theta^2)f(\theta)+D'(r;\partial_\theta^2)f(\pi+\theta), \tag{39}$$

where [as in Eq. (7)] the limits are chosen to ensure convergence of the integral. Eq. (39) [as well as the previous Eq. (9)] is a general result in terms of the parameter r and an analytic function $f(\tau)$ such that $f(\theta)$ is representable as a Fourier series. [The differential operator form of Eq. (39) may also be obtained by writing $f(\tau)$ as a series in $e^{in\tau}$, expressing $e^{in\tau}$ as polynomials in $\sin\tau$, using $e^{ir\cos\tau}\sin\tau=(i/r)\partial_r e^{ir\cos\tau}$, etc., and integrating by parts.] In Sec. III, we apply Eq. (39) to the special case where f is a product of scattering amplitudes.

III. MANY SCATTERERS

For a plane wave $\varphi=e^{ikr\cos(\theta-\alpha)}$ incident on many ($t=1, 2,\cdots, N$) arbitrary scatterers located at $\mathbf{b}_t$ in the geometry of Fig. 2, it was shown previously[1] that the scattering solution of Eq. (1) could be written as

$$\Psi=\varphi+\mathfrak{U},\quad \mathfrak{U}=\sum U_t(\mathbf{r}-\mathbf{b}_t),$$

$$U_t(\mathbf{r}_t)=\{H_0(k_0|\mathbf{r}_t-\boldsymbol{\rho}_t|),\, U_t(\boldsymbol{\rho}_t,\alpha)\}$$
$$=\int e^{ikr_t\cos(\theta_t-\tau)}G_t(\tau,\alpha)\frac{d\tau}{\pi} \quad (40)$$
$$=\sum H_n(kr_t)i^n e^{in\theta_t}\int_0^{2\pi} G_t(\mu,\alpha)e^{-in\mu}\frac{d\mu}{2\pi},$$

where $\mathbf{r}_t(r_t,\theta_t)=\mathbf{r}-\mathbf{b}_t$ is the vector from the "center" of scatterer t, and where U_t is the multiple-scattered wave of scatterer t, and G_t is its corresponding scattering amplitude. As the neighbors of scatterer t recede to infinity, U_t and G_t reduce to $\varphi_t U_t$ and $\varphi_t g_t$, where $\varphi_t=e^{i\mathbf{k}\cdot\mathbf{b}_t}=e^{ikb_t\cos(\beta_t-\alpha)}$ and where u_t and g_t are the appropriate functions for scatterer t in isolation. If $kr_t\sim\infty$, then $U_t\sim H(kr_t)G_t(\theta_t,\alpha)\sim H(kr)G_t(\theta,\alpha)\times e^{-ikb_t\cos(\theta-\beta_t)}$; we may also work[1] with $G_t'=G_t/\varphi_t$ in order to make the phase of φ more explicit. The provinces of the different representations for U_t are essentially as discussed for u in Sec. I.

If we use the plane-wave form of Eq. (40) for the excitation at t arising from all neighbors (i.e., for $\sum' U_s$, where the prime means $s\neq t$) and the superposition principle for the total field scattered by t in response to a set of plane waves, we obtain[1] the basic set of self-consistent integral equations

$$G_t(\theta,\alpha)=g_t(\theta,\alpha)\varphi_t + \sum{}' \int e^{ikb_{ts}\cos(\beta_{ts}-\tau)}g_t(\theta,\tau)G_s(\tau,\alpha)d\tau/\pi, \quad (41)$$

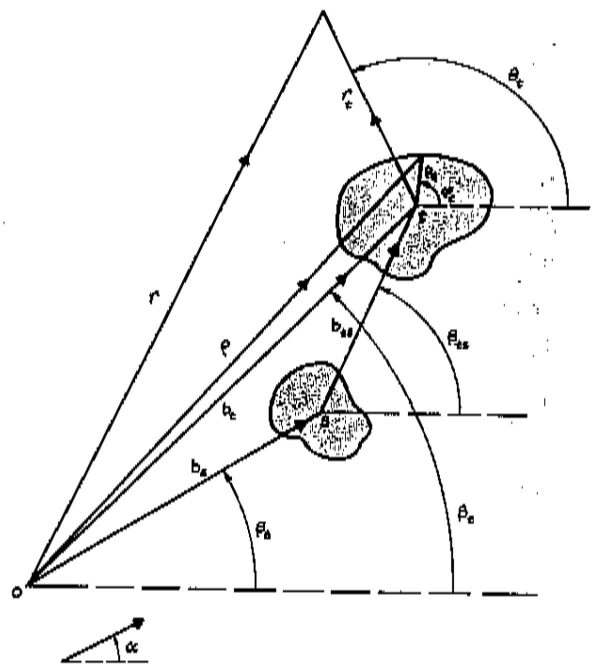

FIG. 2. Coordinates for many-body problem. Expansions hold at least for cases where $b_{ts}>a_t+a_s$, where a_t and a_s are the radii of circles circumscribing scatterers t and s.

where $\mathbf{b}_{ts}(b_{ts},\beta_{ts})=\mathbf{b}_t-\mathbf{b}_s$. The limits on the integrals, essentially as in Eq. (7), require[1] $b_{ts}\sin(\gamma-\beta_{ts})+\rho_t\sin(\gamma-\varphi_t)-\rho_s\sin(\gamma-\varphi_s)<0$; if ρ_t and ρ_s are on the appropriate scatterers' surfaces and if we take $\gamma=\beta_{ts}-\pi/2$, then we require[1] that the separation of scatterer "centers" (b_{ts}) be greater than the sum of the scatterer's projections on $\mathbf{b}_{ts}$. For $b_{ts}>a_t+a_s$, essentially as for Eq. (14), we may substitute a series $\sum B_n e^{in\tau}$ for $g_t G_s$ and isolate the integral representation of H_n to obtain

$$G_t(\theta,\alpha)=g_t(\theta,\alpha)\varphi_t+\sum{}_s' \sum{}_n H_n(kb_{ts})i^n e^{in\beta_{ts}} \times\int_0^{2\pi} g_t(\theta,\mu)G_s(\mu,\alpha)e^{-in\mu}d\mu/2\pi. \quad (42)$$

Equation (42) could also have been constructed by using the addition theorem for $H_n(k|\mathbf{r}_t+\mathbf{b}_{ts}|)$ in (40) and taking the plane-wave form of the resulting J's as the excitation—i.e.,

$$U_s(\mathbf{r}_s)=U_s(\mathbf{r}_t+\mathbf{b})=\sum{}_n H_n(kb)i^n e^{in\beta}\sum{}_\nu i^{\nu-n}J_{\nu-n}(kr_t)e^{i(\nu-n)\theta_t}\int_0^{2\pi} e^{-i\nu\mu}G_s(\mu,\alpha)d\mu/2\pi$$
$$=\sum{}_n H_n(kb)i^n e^{in\beta}\int_0^{2\pi} e^{ikr_t\cos(\theta_t-\mu)-in\mu}G_s(\mu,\alpha)d\mu/2\pi; \quad (43)$$

however, instead of working with such mixed forms (cylindrical waves to a given point and plane waves in the neighborhood of the point), it is, in general, more convenient to work with a pure plane-wave form, such as

$$U_s(\mathbf{r}_t+\mathbf{b})=\int e^{ikb\cos(\beta-\tau)+ikr_t\cos(\theta_t-\tau)}G_s(\tau,\alpha)d\tau/\pi, \quad (44)$$

which was used[1] for Eq. (41), and then transform the result. See Ref. 1 for additional discussion, for various sets of algebraic equations derived directly from Eq. (41) by substituting Fourier-series forms, or Mathieu-series for the amplitudes, and for various applications.

In particular, the complete asymptotic representation for spacings $kb_{ts}\gg 1$ is obtained[1] on applying Eq. (9) to the integrals in (41), or equivalently on applying Eq. (16) to the sum over n in (42) and proceeding as for (17):

$$G_t(\theta,\alpha)\sim g_t(\theta,\alpha)\varphi_t+\sum_s{}'\,\mathfrak{D}_{ts}g_t(\theta,\beta_{ts})G_s(\beta_{ts},\alpha), \quad (45)$$

where $\mathfrak{D}_{ts}=\mathfrak{D}(kb_{ts};\,\partial^2/\partial\beta^2_{ts})$ is defined in Eq. (9). The form (45) is most convenient for generating expansions for even moderately large spacings, and has also been applied[1] to obtain a closed operational form in terms of $\mathfrak{D}$ for two scatterers. It also holds for combinations of scatterers and sources.

The analog of Eq. (45) in terms of the exact representation Eq. (39) is

$$G_t(\theta,\alpha)=g_t(\theta,\alpha)\varphi_t+\sum_s{}'\,D_{ts}\cdot g_t(\theta,\beta_{ts})G_s(\beta_{ts},\alpha), \quad (46)$$

where $D_{ts}\cdot f(\beta_{ts})=D(kb_{ts};\,\partial^2/\partial\beta^2_{ts})\cdot f(\beta_{ts})$ stands for either of the two differential operator forms of Eq. (39). Thus, we may write the summand of Eq. (46) essentially as

$$D_\beta g(\theta,\beta)G(\beta,\alpha)+D_\beta' g(\theta,\pi+\beta)G(\pi+\beta,\alpha) \quad (47)$$

or, equivalently, as

$$D_g[g(\theta,\beta)G(\beta,\alpha)\pm g(\theta,\pi+\beta)G(\pi+\beta,\alpha)]2 = D_g[g(\theta,\beta_e)G(\beta_g,\alpha)+g(\theta,\beta_0)G(\beta_g,\alpha)], \quad (48)$$

where $g(\theta,\beta_g)=[g(\theta,\beta)\pm g(\theta,\pi+\beta)]$, etc. In general, it is more convenient to work with representation (47). The matric form

$$\begin{bmatrix} G_t(\theta,\alpha) \\ G_t(\pi+\theta,\alpha)\end{bmatrix}=\begin{bmatrix} g_t(\theta,\alpha) \\ g_t(\pi+\theta,\alpha)\end{bmatrix}+\sum{}'\begin{bmatrix} D_{ts}g_t(\theta,\beta_{ts}) & D_{ts}'g_t(\theta,\pi+\beta_{ts}) \\ D_{ts}g(\pi+\theta,\beta_{ts}) & D_{ts}'g_t(\pi+\theta,\pi+\beta_{ts})\end{bmatrix}\begin{bmatrix} G_s(\beta_{ts},\alpha) \\ G_s(\pi+\beta_{ts},\alpha)\end{bmatrix} \quad (49)$$

is particularly convenient. The two elements of the column vector $[G_t(\theta,\alpha)]$ are required to represent U_t in the form Eq. (38), and Eq. (49) can be iterated directly.

Iterating Eq. (49), we obtain

$$[G_t(\theta,\alpha)]=[g_t(\theta,\alpha)\varphi_t]+\sum{}'[D_{ts}g_t(\theta,\beta_{ts})][g_s(\beta_{ts},\alpha)] +\sum_s{}'[D_{ts}g_t(\theta,\beta_{ts})]\sum_m{}'[D_{sm}g_s(\beta_{ts},\beta_{sm})] \times[g_m(\beta_{sm},\alpha)\varphi_m]+\cdots, \quad (50)$$

where $[Dg]$ is the square matrix of Eq. (49), and the other terms are column vectors. The convergence properties of the iterated series of Eq. (50) (the "orders of scattering" series) must be based primarily on $|g|$. Only if $kb\gg 1$ (as for the asymptotic form) would we regroup[1] as an expansion in inverse powers of b; using the asymptotic forms of H_0 and H_1, we have $D\sim\mathfrak{D}$, $D'\sim 0$, and the representations for G_t of Eqs. (49) and (50) go over directly to (45) and to its iterations[1,7] discussed previously.

IV. TWO SCATTERERS

For two scatterers, we can eliminate G_s from the right-hand side of Eq. (46) and obtain $G(g)$ in closed operator form. We take the primary reference origin ($r=0$) as the midpoint of the line joining the centers of the circles circumscribing the scatterers and locate the centers by $\mathbf{b}_+(b,\beta_+)=\mathbf{b}_+(b,\beta)$, $\mathbf{b}_-(b,\beta_-)=\mathbf{b}_-(b,\pi+\beta)$. In terms of the local coordinates $\mathbf{r}_+$ and $\mathbf{r}_-$ (the vectors from the centers), we have $\mathfrak{U}=U_+(\mathbf{r}_+)+U_-(\mathbf{r}_-)$. For $kr_\pm\gg 1$ and $r_\pm\gg b$, we have

$$\mathfrak{U}\sim H(kr)\mathcal{G}(\theta,\alpha),$$

$$\mathcal{G}(\theta,\alpha)=e^{-i\Delta}G_+(\theta,\alpha)+e^{i\Delta}G_-(\theta,\alpha),\quad \Delta=kb\cos(\theta-\beta). \quad (51)$$

As in Ref. 1, we may also factor $\varphi_\pm=e^{\pm ikb\cos(\alpha-\beta)}$ from $G_\pm$.

From Eq. (49), we have

$$[G_\pm(\theta,\alpha)]=[g_\pm(\theta,\alpha)\varphi_\pm]+[D_\pm g_\pm(\theta,\beta_\pm)][G_\mp(\beta_\pm,\alpha)], \quad D_\pm=D(2kb;\,\partial^2/\partial\beta^2_\pm), \quad (52)$$

whose asymptotic form was discussed previously[1] in detail. As in Eq. (49), $[Dg]$ is a 2×2 matrix and the other terms are column vectors. In order to reduce Eq. (52) to a closed form for $G_\pm(\theta)$ in terms of g, we must express $G_\mp(\beta_\pm,\alpha)$ solely in terms of g and operators.

Substituting for θ in Eq. (52), we obtain initially

$$[G_\mp(\beta_\pm,\alpha)]=[g_\mp(\beta_\pm,\alpha)\varphi_\mp] +[D_\mp g_\mp(\beta_\pm,\beta_\mp)][G_\pm(\beta_\mp,\alpha)]. \quad (53)$$

Iterating once yields

$$[G_\mp(\beta_\pm,\alpha)]=[g_\mp(\beta_\pm,\alpha)\varphi_\mp] +[D_\mp g_\mp(\beta_\pm,\beta_\mp)][g_\pm(\beta_\mp,\alpha)\varphi_\pm] +[D_\mp g_\mp(\beta_\pm,\beta_\mp)][D_\pm g_\pm(\beta_\mp,\beta_\pm)][G_\mp(\beta_\pm,\alpha)]. \quad (54)$$

Thus, eliminating $[G_{\mp}(\beta_{\pm},\alpha)]$ from the right-hand side, we obtain

$$[G_{\mp}(\beta_{\pm},\alpha)]=[I-[D_{\mp}g_{\mp}(\beta_{\pm},\beta_{\mp})][D_{\pm}g_{\pm}(\beta_{\mp},\beta_{\pm})]]^{-1} \times[[g_{\mp}(\beta_{\pm},\alpha)\varphi_{\pm}]+[D_{\mp}g_{\mp}(\beta_{\pm},\beta_{\mp})][g_{\pm}(\beta_{\mp},\alpha)\varphi_{\pm}]]. \quad (55)$$

The inverse matrix $[I-AB]^{-1}$ expands as $I+AB+ABAB$, we work from right to left in performing the D operations in the generated "chains," and the phase factors $\varphi_{\pm}$ are not operated on.

Substituting Eq. (55) into (52) yields the sought for closed operational form $[G(g)]$, which, together with Eq. (38), provides the complete representation for the multiple-scattered functions $U_{\pm}$ in terms of H_0, H_1 and the isolated scattering amplitudes $g_{\pm}$. If $D\sim\mathfrak{D}$, then the present result reduces to the previous[1] asymptotic one. If we specialize the present g's to those corresponding to different pairs of circular cylindrical monopoles, dipoles, or monopoles plus dipoles, then the present result goes over to the special closed forms in terms of H_0 and H_1 [with H_0+H_2 replaced by $H_1(2kb)/kb$] given previously.[1] For the corresponding pairs of multipoles of the elliptic cylinder, the previous closed forms in terms of the radial Mathieu functions are now developed as series in H_0 and H_1.