

## Velocity-dependent multiple scattering by two thin cylinders

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A relativistically exact formulation is presented for multiple scattering of electromagnetic waves by arbitrary, uniformly moving objects. The formulation is based both on spectral representations which are easily transformed from one frame of reference into another and on a successive scattering scheme which iteratively satisfies the boundary conditions on the surface of the obstacles. Explicit expressions and computational results are given for scattering by two thin dielectric cylinders.

### INTRODUCTION

Comprehensive reviews concerning multiple scattering by configurations of scatterers at rest are given by *Twersky* [1960] and *Burke and Twersky* [1964]. For configurations at rest, the two-dimensional case is discussed by *Twersky* [1962a] and *Burke et al.* [1965]. The three-dimensional scalar and vector problems are given by *Twersky* [1962b] and *Twersky* [1967], respectively. Scattering of electromagnetic waves by objects moving in free space is discussed by *Censor* [1967, 1971, 1972].

We consider the problem of multiple scattering by a configuration of arbitrary obstacles moving with respect to each other and with respect to the observer. The formulation is based on an iterative scheme of successive scattering. A certain wave, derived in the frame of reference of an object at rest, is recast in terms of a plane-wave integral. The relativistic transformation formulas are applied to the plane-wave integrand of this spectral representation, in order to express the result measured by the observer. Another transformation brings us into the frame of reference of a different obstacle, where the scattered wave is computed. The details of the scattering process are explained below. The resulting integrals are complicated, and a considerable amount of manipulation is necessary in order to derive explicit expressions, in terms of well-known special functions. The formulation is demonstrated in detail, and computational results are derived for the relatively simple case of velocity-dependent scattering involving two thin dielectric cylinders.

### GENERAL THEORY

The geometry of the problem is given in Figure 1. In a surrounding medium of free space a system of  $s$  ( $s = 1, 2, \dots, i, \dots, j, \dots, n$ ) objects is given. Each object is specified by means of its surface and the boundary conditions on it, as seen in the proper frame of reference in which the object is at rest. The incident electromagnetic wave is specified in frame of reference  $\Gamma(x, y, z, t)$  in which the observer is situated. Object  $s = i$ , as observed from  $\Gamma$ , is moving with a constant velocity  $\mathbf{v}_i$ , i.e., the motion is uniform and purely translatory. In  $\Gamma$  a frame of coordinates  $x_i, y_i, z_i$ , is constructed by a rotation of  $x, y, z$ , such that  $\mathbf{v}_i = \hat{\mathbf{x}}_i v_i$ , where  $\hat{\mathbf{x}}_i$  designates a unit vector. Object  $i$  is at rest in its proper frame of reference  $\Gamma^{(i)}(x_i', y_i', z_i', t_i')$  of which the origin  $\mathbf{r}_i' = 0$  moves with velocity  $\mathbf{v}_i$ , as observed from  $\Gamma$ . At  $t_i' = t = 0$ , the two systems of coordinates  $x_i, y_i, z_i$  and  $x_i', y_i', z_i'$  coincide. In  $\Gamma^{(i)}$  the object is located by a position vector  $\mathbf{b}_i$ , the tip of which defines the origin of a local system of coordinates, to be introduced subsequently. (Of course, during the period of observation no collisions are allowed since such collisions cannot be accounted for by the present special relativistic model.)

The incident plane wave (its  $\mathbf{E}$  field, say) is defined by

$$\begin{aligned} \psi_i &= \mathbf{A} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) \\ &= \mathbf{A} \exp(i\mathbf{k} \cdot \mathbf{r}_i - i\omega t) \equiv \mathbf{A} \exp(i\phi) \end{aligned} \quad (1)$$

where

$$\mathbf{r}_i = \mathbf{r}_i(x_i, y_i, z_i)$$

and where  $\mathbf{A}$  is the amplitude,  $\mathbf{k}$  is the propagation

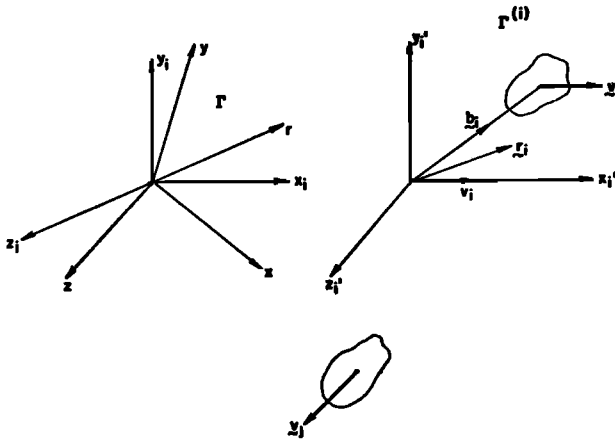


Fig. 1. Geometry of the general problem. The incident wave and the observer are situated in an inertial system of reference  $\Gamma$ . Given a configuration of  $s$  objects moving at velocities  $\mathbf{v}_i$ ,  $i = 1, \dots, s$ ,  $\Gamma^{(i)}$  is chosen such that object  $i$  is at rest. At  $t_i = t = 0$  the systems of coordinates  $x_i, y_i, z_i$  and  $x', y', z'$  coincide.

vector, and  $\omega$  is the angular frequency (henceforth, the frequency). Inasmuch as the objects are moving in free space, *Einstein's* [1905] prescription can be used. The waves are transformed into the frame of reference where the scatterer is at rest, the scattered wave is derived, and the result can be transformed again into another frame, and so on. Thus in order to find the scattered field produced by object  $i$ ,  $\psi_i$  is transformed into  $\Gamma^{(i)}$ , yielding  $\psi_i'$ .

$$\begin{aligned} \psi_i' &= \tilde{F}_i \cdot \psi_i = \mathbf{A}_i' \exp (i\mathbf{k}_i' \cdot \mathbf{r}_i' - i\omega_i' t_i') \\ &\equiv \mathbf{A}_i' \exp (i\phi_i') \end{aligned} \quad (2)$$

where

$$\begin{aligned} x_i' &= \gamma_i(x_i - v_i t) \\ y_i' &= y_i \\ z_i' &= z_i \\ t_i &= \gamma_i(t_i - v_i x_i / c^2) \\ \tilde{F}_i &= \tilde{F}_i(\mathbf{k}_i, \mathbf{v}_i) = [(1 - \gamma_i)\hat{\mathbf{v}}_i + \gamma_i \beta_i \hat{\mathbf{k}}_i] \hat{\mathbf{v}}_i \\ &\quad + \gamma_i(1 - \beta_i \cos \alpha_i) \tilde{I} \end{aligned}$$

where

$$\begin{aligned} \beta_i &= v_i / c \\ c &= (\mu_0 \epsilon_0)^{-1/2} \\ \gamma_i &= (1 - \beta_i^2)^{-1/2} \\ \cos \alpha_i &= \hat{\mathbf{k}}_i \cdot \hat{\mathbf{v}}_i \end{aligned}$$

where  $\tilde{F}_i$  is a dyadic,  $\tilde{I}$  is the idemfactor dyadic, and the Lorentz transformation and  $\phi_i = \phi_i'$  specify the transformations for  $\mathbf{k}_i$ ,  $\omega_i$  details are given by *Censor* [1969a]. In  $\Gamma^{(i)}$  the wave (2) is shifted to the local coordinate system of object  $i$ ,  $\xi_i' = \mathbf{r}_i' - \mathbf{b}_i$ . Now, it is assumed that the single scattering problem for the object at hand is known. The scattered wave  $\mathbf{u}_i'$  is recast in a spectral representation

$$\begin{aligned} \mathbf{u}_i' &= |\mathbf{A}_i'| \exp (i\mathbf{k}_i' \cdot \mathbf{b}_i) \\ &\quad \cdot \int_C \exp [i\mathbf{k}_{ip}' \cdot (\mathbf{r}_i' - \mathbf{b}_i) - i\omega_i' t_i'] \mathbf{g}_i \, d\Omega_{p'} \end{aligned} \quad (3)$$

i.e. (3) is represented as a superposition (integral) of plane waves propagating in a complex direction  $\hat{\mathbf{k}}_{ip}'$ , as specified by the appropriate contours  $C$  in the complex plane (or planes). Each such plane wave has an amplitude  $\mathbf{g}_i = \mathbf{g}_i(\hat{\mathbf{A}}_i', \hat{\mathbf{k}}_i', \hat{\mathbf{k}}_{ip}')$  which depends on  $\hat{\mathbf{A}}_i'$ , the direction of polarization of the excitation wave, its direction of propagation  $\hat{\mathbf{k}}_i'$ , and the direction of propagation  $\hat{\mathbf{k}}_{ip}'$ . This function is proportional to the scattering amplitude for the object in question. In a manner similar to that used in (2),  $\mathbf{u}_i'$  can be transformed into  $\Gamma$  by applying the inverse transformation to the plane wave in the integrand (3),

$$\begin{aligned} \mathbf{u}_i &= |\mathbf{A}_i'| \exp (i\mathbf{k}_i' \cdot \mathbf{b}_i) \\ &\quad \cdot \int_C \exp (-i\mathbf{k}_{ip}' \cdot \mathbf{b}_i + \phi_{ip}') \tilde{F}_i' \cdot \mathbf{g}_i \, d\Omega_{p'} \end{aligned} \quad (4)$$

where

$$\begin{aligned} \phi_{ip}' &= \mathbf{k}_{ip}' \cdot \mathbf{r}_i' - \omega_i' t_i' \\ \tilde{F}_i' &= \tilde{F}_i'(\mathbf{k}_{ip}', -\mathbf{v}_i) \end{aligned}$$

Thus far we have formulated the single-scattering velocity-dependent problem. The multiple scattering case follows quite naturally, by considering  $\exp (i\phi_{ip}') \tilde{F}_i' \cdot \mathbf{g}_i$  in (4) to be the excitation wave for object  $j$ . This successive scattering procedure is repeated as many times as necessary for a desired accuracy, with respect to distances and velocities. The formulation by itself is of little value as long as the integrals are not recast in terms of known computable special functions, therefore in the next section a special case is considered.

#### MULTIPLE SCATTERING BY TWO THIN DIELECTRIC CYLINDERS

In order to keep the mathematical manipulations as simple as possible, the following special case is chosen. Consider a configuration of two cylinders of circular cross section, oriented parallel to the

$\hat{z}$  axis. The propagation vector of the incident wave given below and the velocity lie in the  $xy$  plane, this ensures a scalar formulation, in contradistinction to other cases, see *Censor* [1969 b].

The geometry of the problem, Figure 2, is chosen such that one object is at rest, attached to  $\Gamma$ , and the other one is attached to the origin of  $\Gamma'(x', y', z')$ , moving along the  $x$  direction; i.e.,  $b_i = 0$ , and the objects overlap at  $t = t' = 0$  at the origin  $x = x' = 0$ . This must be excluded, since it has no physical meaning and leads mathematically to singular results.

The incident wave (an  $\mathbf{E} = E\hat{z}$  field, say) is chosen as

$$\exp(ikx - i\omega t) \quad (5)$$

The single-scattered wave produced by the object at rest in  $\Gamma$  is given by

$$\begin{aligned} u_0 &= \sum_{m=-\infty}^{\infty} i^m a_m H_m(kr) \exp(im\theta - i\omega t) \\ &= \int_C \exp[ikr \cos(\theta - \tau) - i\omega t] g(\tau) d\tau/\pi \\ g(\theta) &= \sum_{m=-\infty}^{\infty} a_m \exp(im\theta) \end{aligned} \quad (6)$$

where  $H_m = H_m^{(1)}$  is the Hankel function of the first kind, the contour  $C$  is a Sommerfeld path, and in the present case  $\tau$  extends from  $\theta - \pi/2 + i\infty$  to  $\theta + \pi/2 - i\infty$  in the complex plane  $\tau$ .

In order to derive the single scattered wave produced by the moving object, (6) is transformed into  $\Gamma'$  according to (2), yielding

$$\begin{aligned} A' \exp(ik'x' - i\omega't') \\ A'/1 = k'/k = \omega'/\omega = [(1 - \beta)/(1 + \beta)]^{1/2} \end{aligned} \quad (7)$$

The scattering problem in  $\Gamma'$  is solved, in a manner similar to that used in (6). According to (4), the wave is transformed into  $\Gamma$  by multiplying the integrand by  $\gamma(1 + \beta \cos \tau')$ , yielding

$$\begin{aligned} u_1 &= A'\gamma \int \exp[ik'r' \cos(\theta' - \tau') - i\omega't'] \\ &\quad \cdot (1 + \beta \cos \tau') g'(\tau') d\tau'/\pi \\ &= A'\gamma \sum_{m=-\infty}^{\infty} i^m b_m H_m(k'r') \exp(im\theta' - i\omega't') \end{aligned} \quad (8)$$

where

$$\begin{aligned} b_m &= a_m' + \beta(a_{m+1}' + a_{m-1}')/2 \\ g'(\theta') &= \sum_{m=-\infty}^{\infty} a_m' \exp(im\theta') \end{aligned}$$

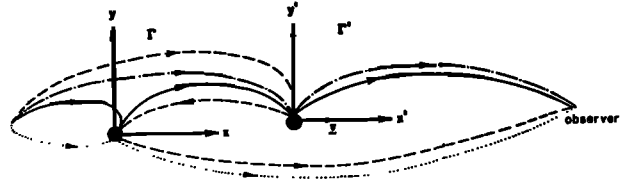


Fig. 2. Geometry and scattering modes for the special problem of two thin cylinders. One object is at rest in  $\Gamma$ , the other one is attached to  $\Gamma'$ , which moves in the  $x$ -direction. The observer is situated to the right of both, on the  $x$ -axis. Dotted and dash-dotted lines describe the single scattering modes, solid and dashed lines describe the multiple scattering modes.

By making the suitable substitutions in the Fourier-Hankel series representation, (8) may be expressed in terms of  $x, y, z, t$  as measured by an observer in  $\Gamma$ . Note that  $g'(\theta')$  in  $\Gamma'$  is excited by  $\omega'$ . This terminates the single-scattering part of the problem.

Consider now the first-order multiple-scattering mode produced by  $u_1$  when scattered from the object at rest in  $\Gamma$ . One must consider the two possible cases, whether the moving object is to the right or to the left of the object at rest in  $\Gamma$ . In the first case in (8),  $\theta' = \pi$ , which makes the limits of the integral  $\pi/2 + i\infty$  to  $3\pi/2 - i\infty$ . In order to have the Sommerfeld limits for the Hankel functions  $H_m^{(1)}$ , the variable of integration  $\tau'$  must be modified. Similar considerations apply to the other mode of multiple scattering taken into account in the following section. Since the phase is invariant, in (8) the exponent is written as

$$\begin{aligned} i[k_r r \cos(\theta - \tau) - \omega_r t] \\ k_r/k' = \omega_r/\omega' = \gamma(1 + \beta \cos \tau') \end{aligned} \quad (9)$$

The object in  $\Gamma$  is therefore excited by a complex plane wave propagating in direction  $\tau$ , according to (9). The result is  $u_{0\tau}$  for the plane-wave response, and the multiple scattered wave is given by

$$u_{1,0} = A'\gamma \int u_{0\tau} (1 + \beta \cos \tau') g'(\tau') d\tau'/\pi \quad (10)$$

Obviously the object in  $\Gamma$  is now excited by a complex frequency that depends on  $\tau$  and takes into account the velocity effects and the fact that the distance between the objects varies. Hence  $u_{0\tau}$  cannot be taken outside the integral sign. Without much loss of generality the observer is situated in the far field, hence  $u_{0\tau}$  in (10) can be written in terms of the asymptotic representation

$$u_{0r} = (2/i\pi k_r r)^{1/2} \exp(ik_r r - i\omega_r t)g(\theta) \quad (11)$$

The last restriction can be waived by adding correction terms, see *Twersky* [1962a].

Similarly the plane wave in (6) is transformed into  $\Gamma'$  by multiplying the integrand by  $\gamma(1 - \beta \cos \tau)$ , and the phase in the integrand (6) is recast as

$$k_r' r' \cos(\theta' - \tau') - i\omega_r' t' \\ k_r' / k = \omega_r' / \omega = \gamma(1 - \beta \cos \tau) \quad (12)$$

This wave excites the object in  $\Gamma'$ , producing  $u_{1\tau'}$  which is transformed back into  $\Gamma$  according to (8), yielding the analog of (10),

$$u_{0,1} = \gamma \int u_{1\tau}(1 - \beta \cos \tau)g(\tau) d\tau/\pi \quad (13)$$

where in the far field  $u_{1\tau}$  is given by

$$u_{1\tau} = (2/i\pi k_r' r')^{1/2} \cdot \exp(ik_r' r' - i\omega_r' t')g'(\theta')\gamma(1 + \beta \cos \theta') \quad (14)$$

Again care must be taken concerning the limits of the integral (13), depending on the relative position of the objects with respect to the observer.

For a dielectric cylinder with  $E$  polarized along the axis, it is well-known that the boundary value problem yields

$$a_m = [k J_m(Ka) J_m'(ka) - K J_m(ka) J_m'(Ka)] / D_m \quad (15)$$

where

$$D_m = -k J_m(Ka) H_m'(ka) + K H_m(ka) J_m'(Ka)$$

and where  $K$  is the propagation constant inside the cylinder,  $J_m$  are the nonsingular Bessel functions, the prime indicates differentiation with respect to the argument, and  $a$  is the radius. For thin cylinders,  $ka \ll 1$ , the cylindrical functions are expanded near the origin; this yields

$$a_0 = i\pi(ka/2)^2(-1 + K^2/k^2) \\ a_1 = i\pi(ka)^4(-1 + K^2/k^2)/32 \\ a_m \propto (ka)^{2m+2} \quad m = 1, 2, \dots \quad (16)$$

Hence there exists a hierarchy, which prescribes that for small  $ka$  the single scattering processes should include  $a_0$  and  $a_1$ , while the multiple scattering processes as described in Figure 2 should include  $a_0$  terms only. If the cylinders are thin enough, higher terms and higher multiple scattering modes can be neglected. Note that  $a_0$  and  $a_1$  are proportional to the square and fourth power of the excitation frequency, respectively.

Inserting these considerations in (10), (11), (13), and (14) and making the appropriate substitutions yields

$$u_{1,0} = B \int \exp[i\gamma\beta(k'r - \omega't) \cos \tau'] \cdot (1 + \beta \cos \tau')^{5/2} d\tau'/\pi \\ u_{0,1} = D \int \exp[-i\gamma\beta(kr' - \omega t') \cos \tau] \cdot (1 - \beta \cos \tau)^{5/2} d\tau/\pi \quad (17)$$

where

$$B = A' a_0(\omega) a_0'(\omega) \gamma^3 (2/i\pi k' \gamma r)^{1/2} \exp[i\gamma(k'r - \omega't)] \\ D = (1 + \beta) a_0(\omega) a_0'(\omega) \gamma^4 (2/i\pi k' \gamma r')^{1/2} \cdot \exp[i\gamma(kr' - \omega t')]$$

$a_0(\omega) = a_0'(\omega)$  are given by (16)

In order to make a computation feasible, it is necessary to evaluate the integrals in (17). To the order  $\beta^4$ , the first integral yields for  $\theta' = 0$ ,

$$\alpha_0 H_0 + 2\alpha_1 i H_1 - 2A_2 H_2 - 2i\alpha_3 H_3 + 2\alpha_4 H_4 \\ H_m = H_m[\gamma\beta(k'r - \omega't)] \\ \alpha_0 = 1 + (15/16)\beta^2 - (15/1024)\beta^4 \\ \alpha_1 = (5/4)\beta + (15/128)\beta^3 \\ \alpha_2 = (15/32)\beta^2 - (5/512)\beta^4 \\ \alpha_3 = (5/128)\beta^3 \\ \alpha_4 = -(5/2048)\beta^4 \quad (18)$$

The second integral yields the same structure, with the argument of the Hankel functions being  $-\alpha\beta(kr' - \omega t')$  and  $\alpha_1, \alpha_3$  multiplied by  $(-1)$ . Thus the total field can be computed, and explicit results may be compared with the velocity independent case given previously by *Twersky* [1962c].

#### DISCUSSION OF RESULTS

In the velocity-independent case [*Twersky*, 1962c] the amplitude and phase of the total scattering amplitude shows resonance effects, as a function of the separation between the objects. This is due to the multiply scattered waves that interfere with the single-scattered waves, reinforcing or partly cancelling them. In the velocity-dependent case, at least for one of the modes, there is a significant Doppler shift in frequency and wavelength. Consequently, beats are produced, which is a new phenomenon typical of the



Fig. 3. (a) absolute value of total field, and (b) phase of total field both normalized with respect to the field of the object at rest. The velocity is  $\beta = 0.1$ . The center  $kvt = 100$  is the retardation time of the observer with respect to the origin  $x = 0$ .

present problem. The observer is situated at a large distance  $kx = 1000$ , so that even at large velocities, the moving object does not reach him during the duration of observation. It takes a retardation time  $\Delta t = x/c$  for a phenomenon happening at the origin to reach the observer. At this time the moving object has traversed a distance  $v\Delta t = \beta x$ . The results are therefore taken for the range  $kvt$  from  $\beta kx - 10$  to  $\beta kx + 10$ . The results look similar for different velocities because of the normalization,  $tvk$ , of the time axis, but as far as the separation  $vt$  is concerned, this is a consistent representation.

In Figure 3 results are given for  $\beta = 0.1$ , i.e., for an object moving at  $30,000 \text{ km sec}^{-1}$ . The beats are clearly seen. To the left of  $kvt = 100$  the amplitude of the beats is larger, because when the object is moving towards the object at rest, loosely speaking, there is a Doppler effect  $(1 + \beta)/(1 - \beta)$  in the amplitude as well as the frequency. It is also

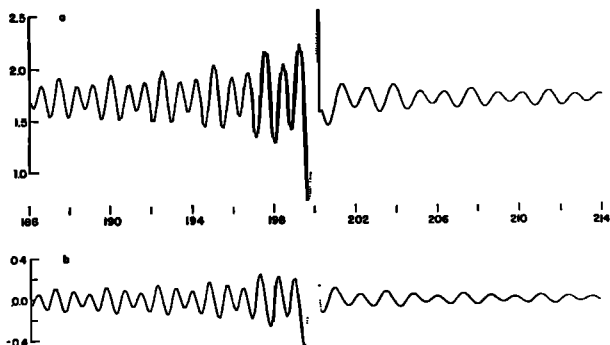


Fig. 4. Same as Figure 3, but  $\beta = 0.2$ .

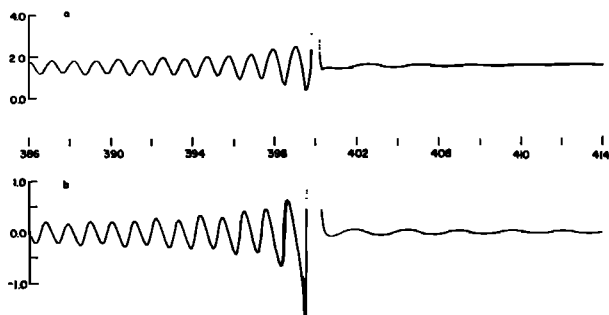


Fig. 5. Same as Figure 3, but  $\beta = 0.4$ . Here the beat phenomenon is not apparent for the given separation.

apparent that the number of beats is larger when the objects converge (to the left of  $kvt = 100$ ) compared to the case in which they diverge (for  $kvt > 100$ ). The reason is the following. When the objects diverge, the Doppler shift is  $(1 - \beta)/(1 + \beta)$ , and when they converge the signs must be inverted, so that the shift is larger for the latter case; thus,

$$\frac{1 + \beta}{1 - \beta} - \frac{1 - \beta}{1 + \beta} = \frac{4\beta}{1 - \beta^2} \quad (19)$$

which is a first-order effect in the velocity. Hence, when the shift is larger, the amplitude of the beats oscillates faster. The effects are more pronounced in Figure 4, where  $\beta = 0.2$ . For higher velocities, e.g.,  $\beta = 0.4$  as in Figure 5, the beat phenomenon is not apparent for the separations considered here.

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