

Scattering in velocity-dependent systems

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(Received September 21, 1970; revised January 7, 1971.)

Scattering of electromagnetic waves by arbitrary objects moving in free space is discussed. The objects move at constant velocities. The formalism is relativistically exact and applies to arbitrary distances from the scatterer. The results are recast in terms of the dominant (far-field) frequency; this is useful for low velocities and near-field frequencies. Numerical results are given for scattering by two-dimensional moving monopoles and dipoles.

INTRODUCTION AND STATEMENT OF THE PROBLEM

For scattering problems involving objects uniformly moving in free space, one may use the method prescribed by *Einstein* [1905]: the incident electromagnetic wave is transformed into the frame of reference where the object is at rest; the scattering problem can supposedly be solved in this frame of reference. The resulting scattered wave is now transformed back into the original frame of reference in which the observer is situated at rest. The method is illustrated by *Einstein* for the case of a plane wave scattered by a perfectly reflecting mirror.

Presently we consider the corresponding two- and three-dimensional problems. Particular examples are provided by the circular cylinder and the sphere, respectively. Results are given in terms of the well-known special functions arising from solution of the corresponding velocity-independent problems. The formalism is relativistically exact and applies to arbitrary distances. Therefore previous studies, concerned mainly with the far zone, are considered to be special cases, e.g., *Lee and Mittra* [1967] and *Restrick* [1968]. Computational results are presented, and approximations are given for low velocities and near zone in terms of the dominant (far-field) Doppler frequency.

TWO DIMENSIONS

Let Γ , Γ' be two inertial frames of reference, in which we define times t , t' and Cartesian systems of coordinates x , y , z and x' , y' , z' , respectively. Observed from Γ , the origin of Γ' is moving with constant

velocity $\mathbf{v} = v\hat{x}$. The source wave $\phi = \mathbf{E}, \mathbf{H}$ is specified in Γ as a harmonic plane wave:

$$\phi = f \exp (ik \cdot r' - i\omega t) \quad (1)$$

in the usual notation, e.g., *Censor* [1969]. The corresponding wave in Γ' is given by

$$\phi' = f' \exp (ik' \cdot r' - i\omega' t') \quad (2)$$

where r' , t' are prescribed by the Lorentz transformation, and the detailed transformation formulas for k' , ω' , are given elsewhere [*Censor*, 1969.] Consider a cylinder of arbitrary cross section whose axis coincides with y' in Γ' . Hence the motion, as observed from Γ , is perpendicular to the axis. The incident wave (equation 1) is polarized along the axis. This implies a scalar formalism.

Starting with the Hankel-Fourier series representation and exploiting the Sommerfeld integral representation for the cylindrical functions, the scattered field u' can be written as

$$\begin{aligned} u'(r', t') &= f' \sum_{m=-\infty}^{\infty} i^m a_m H_m(k'r') \exp (im\theta' - i\omega' t') \\ &= \frac{f'}{\pi} \int_{\theta' - (\pi/2) + i\epsilon}^{\theta' + (\pi/2) - i\epsilon} \exp [ik'r' \cos (\theta' - \tau') \\ &\quad - i\omega' t'] g(\tau') d\tau' \quad (3) \end{aligned}$$

Here $H_m = H_m^{(1)}$ is the Hankel function of order m of the first kind, r' is the distance from the axis, and θ' is the angle with respect to direction x , measured in Γ' . The scattering amplitude

$$g(\theta') = \sum_{m=-\infty}^{\infty} a_m \exp (im\theta')$$

is evaluated at the excitation frequency ω' . In ac-

cordance with the transformations for plane waves [Censor, 1969], we find:

$$\begin{aligned}
 u(r', t') &= \frac{\gamma f'}{\pi} \int \exp [i\psi'(r')](1 + \beta \cos \tau')g(\tau') d\tau' \\
 &= \gamma f' \sum_{m=-\infty}^{\infty} i^m b_m H_m(k'r') \exp (im\theta' - i\omega' t') \\
 b_m &= a_m + [a_{m-1} + a_{m+1}]\beta/2 \\
 \gamma &= (1 - \beta^2)^{-1/2} \quad \beta = v/c \quad (4)
 \end{aligned}$$

where c is the speed of light in free space.

Inasmuch as the structure of u in (4) is the same as u' in (3), except that a_m is replaced by b_m , other representations for the scattered field may be used. Formally one can find $u[r'(r, t), t'(r, t)]$ in terms of Γ coordinates r, t by substitution. Evidently this leads to cumbersome expressions which are not more trans-

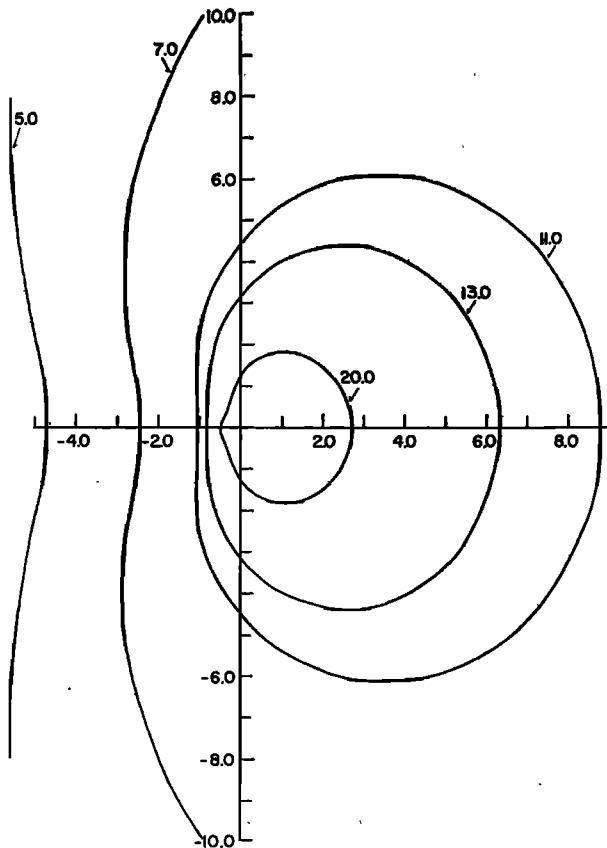


Fig. 1. Polar plot for equiamplitude contours of the scattered field produced by a moving monopole. The speed of the object, moving from left to right, is $\beta = 0.5$. The object is situated at the origin at the time $t = 0$ when the contours are computed. Distance from origin is measured in $k'r'$ units. The amplitude is arbitrarily normalized.

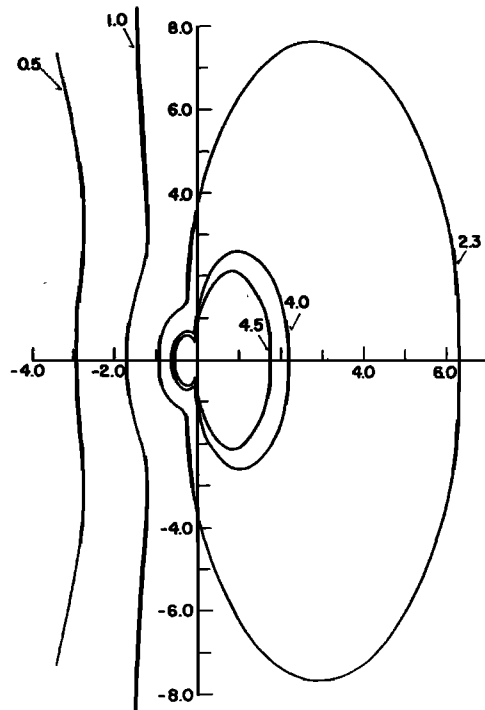


Fig. 2. Scattering by a monopole as in Figure 1, but with $\beta = 0.9$. The effect of the velocity-dependent dipole term is seen near the origin.

parent. In the far field, u behaves in many respects as a plane wave; and Doppler effects in the frequency, propagation constant, and amplitude may be identified. Far-field equiamplitude and equiphase contours have been considered by Lee and Mittra [1967]; Censor and Damenstein [1970] computed the field of a monopole and a dipole for arbitrary distances, according to (4). In the far field, results are obtained which are similar to those given by Lee and Mittra. In the near field, results show novel features.

According to (4), the field of a moving monopole is given by

$$u(r', t') = \gamma f' e^{-i\omega' t'} a_0 [H_0(k'r') + i\beta H_1(k'r') \cos \theta'] \quad (5)$$

Similarly for a dipole, assuming that $a_1 = a_{-1}$

$$\begin{aligned}
 u(r', t') &= f' \gamma e^{-i\omega' t'} a_1 \{ 2iH_1(k'r') \cos \theta' \\
 &\quad + \beta [H_0(k'r') - H_2(k'r') \cos 2\theta'] \} \quad (6)
 \end{aligned}$$

The field (5) has the formal structure of a monopole plus a velocity-dependent dipole term; similarly, (6) has the structure of a dipole plus velocity-dependent monopole and quadrupole terms. The following figures are taken from Censor and Damenstein [1970].

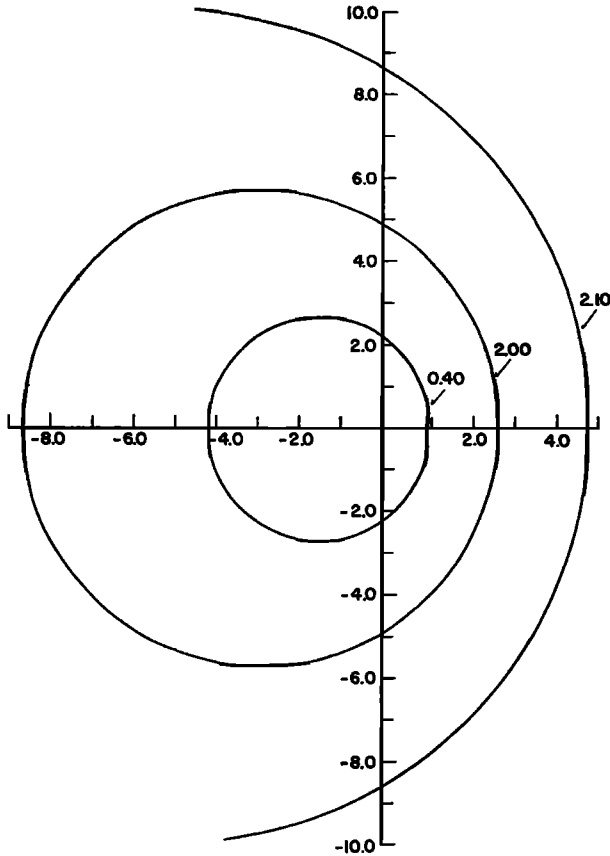


Fig. 3. Scattering by a monopole. Equiphase contours corresponding to the monopole of Figure 1, $\beta = 0.5$.

Figures 1 and 2 show the polar plot of equiamplitude surfaces for a moving monopole at $t = 0$. At large distances they resemble the results obtained by Lee and Mittra. The present results show some additional detail where the contours come closer to the origin. The effects are more pronounced for Figure 2, where $\beta = 0.9$, compared to $\beta = 0.5$ in Figure 1. The amplitude increases from outer to inner surfaces, and the numbers refer to an arbitrary normalization. The distance from the origin is measured in $k'r'$ units. Similarly, Figure 3 for $\beta = 0.5$ and Figure 4 for $\beta = 0.9$ show equiphase contours and resemble the circles found in the far-field theory. Closer to the origin there are new features characteristic of the near field as described by (5) and (6).

Figure 5 shows equiamplitude contours in the near field for a monopole moving at a speed $\beta = 0.5$. The effect of the dipole term is apparent. This is more emphasized in Figure 6, where the speed is higher: $\beta = 0.9$. In Figure 7, the far-field equiamplitude contours for a moving dipole are shown, for $\beta = 0.5$.

In the near field, Figure 8, the quadrupole term has a pronounced effect. At higher velocity $\beta = 0.9$, this effect is seen even in the far field, Figure 9, and much more emphasized in the near field, as seen in Figure 10.

THREE DIMENSIONS

In spherical coordinates we have for the scattered wave $u'(r', t')$

$$\begin{aligned}
 u'(r', t') &= \frac{f'}{2\pi} \int_{-\pi}^{\pi} d\beta' \int_0^{\pi/2-i\infty} [dr' \sin \tau' \\
 &\quad \cdot \exp(ik'\hat{p}' \cdot \mathbf{r}' - i\omega' t')] g(\hat{p}') \\
 &= f' \int d\Omega_{p'} \exp(i\psi_{p'}) g(\hat{p}') \\
 &= f' e^{-i\omega' t'} \sum_{n,m} [M_{nm}(r')c_{nm} - iN_{nm}(r')b_{nm}] i^n
 \end{aligned}$$

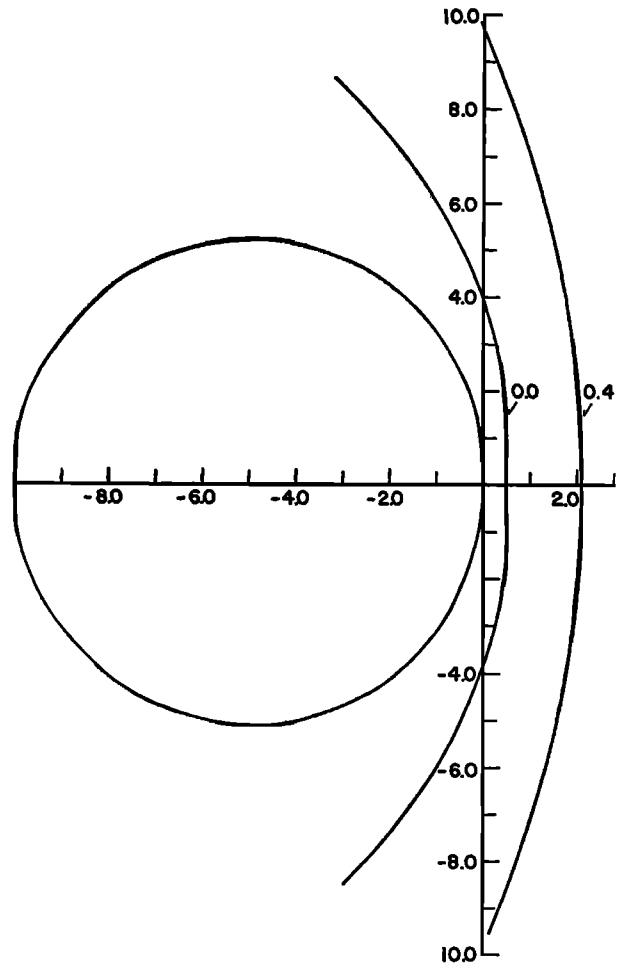


Fig. 4. Scattering by a monopole. Equiphase contours corresponding to the monopole of Figure 2, $\beta = 0.9$.

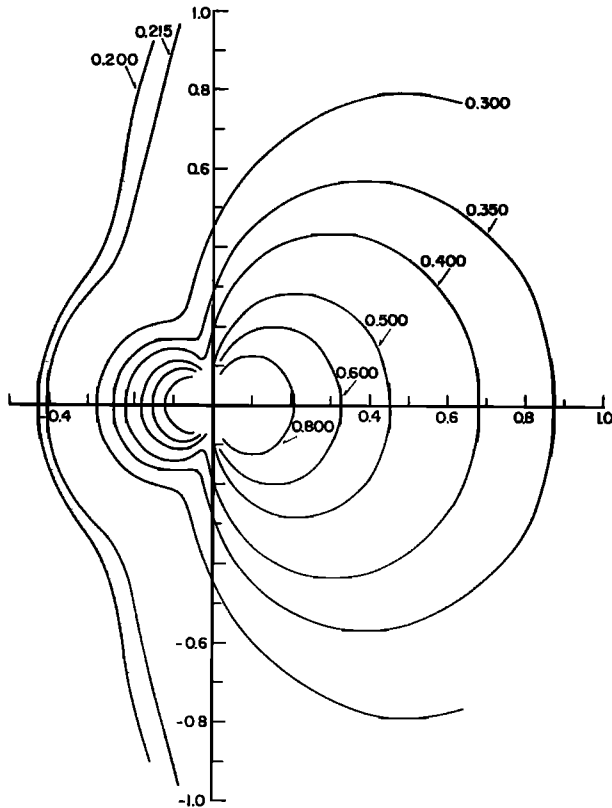


Fig. 5. Scattering by a monopole as in Figure 1. Equiamplitude contours in the near field, $\beta = 0.5$.

$$\begin{aligned}
 \mathbf{M}_{nm}(\mathbf{r}') &= h_n(k'r')\mathbf{C}_n^m(\hat{\mathbf{r}}') \\
 \mathbf{C}_n^m(\hat{\mathbf{r}}') &= -\mathbf{r}' \times \nabla Y_n^m(\hat{\mathbf{r}}') \\
 \mathbf{N}_{nm}(\mathbf{r}') &= \{n(n+1)h_n(k'r')\mathbf{P}_n^m(\hat{\mathbf{r}}') \\
 &\quad + \partial_{k'r'}[k'r'h_n(k'r')]\mathbf{B}_n^m(\hat{\mathbf{r}}')\}/k'r' \quad (7) \\
 \mathbf{P}_n^m(\hat{\mathbf{r}}') &= \hat{\mathbf{r}}' Y_n^m(\hat{\mathbf{r}}') \\
 \mathbf{B}_n^m(\hat{\mathbf{r}}') &= \hat{\mathbf{r}}' \times \mathbf{C}_n^m(\hat{\mathbf{r}}') \\
 Y_n^m(\hat{\mathbf{r}}') &= P_n^m(\cos \theta')e^{im\phi'}
 \end{aligned}$$

where $\hat{\mathbf{p}}'(\beta', \tau')$ is a complex unit vector defined by β', τ' , the azimuthal and polar angles, respectively; h_n denotes the spherical Hankel functions of the first kind, and P_n^m is the associated Legendre function. Applying the transformation formulas [Censor, 1969] to the plane waves in the integrand (7), with \mathbf{v} taken along the polar axis $\theta = 0$, we have

$$\begin{aligned}
 \mathbf{u}(\mathbf{r}'t') &= \int' d\Omega_{p'} \exp(i\psi_{p'}) \mathfrak{F}' \cdot \mathbf{g}(\hat{\mathbf{p}}') \\
 &= \int' \left\{ \mathfrak{U} \cdot \mathbf{u}'(\mathbf{r}', t') + \gamma\beta \int d\Omega_{p'} \right.
 \end{aligned}$$

$$\left. \cdot \exp(i\psi_{p'}) [\cos \tau' g(\hat{\mathbf{p}}') - \hat{\mathbf{p}}'(\mathbf{g} \cdot \hat{\mathbf{v}})] \right\} \quad (8)$$

$$\mathfrak{U} = (1 - \gamma)\hat{\mathbf{v}} + \gamma\mathcal{G}$$

This yields [Censor, 1967]

$$\begin{aligned}
 \mathfrak{F}' \cdot \mathbf{C}_n^m &= \mathfrak{U} \cdot \mathbf{C}_n^m + \gamma\beta(\alpha_1 \mathbf{P}_{n-1}^m + \alpha_2 \mathbf{P}_{n+1}^m \\
 &\quad + \alpha_3 \mathbf{C}_{n+1}^m + \alpha_4 \mathbf{C}_{n-1}^m + \alpha_5 \mathbf{B}_n^m)
 \end{aligned}$$

$$\begin{aligned}
 \mathfrak{F}' \cdot \mathbf{B}_n^m &= \mathfrak{U} \cdot \mathbf{B}_n^m \\
 &\quad + \gamma\beta(\alpha_6 \mathbf{P}_n^m + \alpha_3 \mathbf{B}_{n+1}^m + \alpha_4 \mathbf{B}_{n-1}^m + \alpha_5 \mathbf{C}_n^m)
 \end{aligned}$$

$$\alpha_1(m, n) = -\frac{(n+m)(n+1)}{2n+1}$$

$$\alpha_2(m, n) = \frac{n(n-m+1)}{2n+1}$$

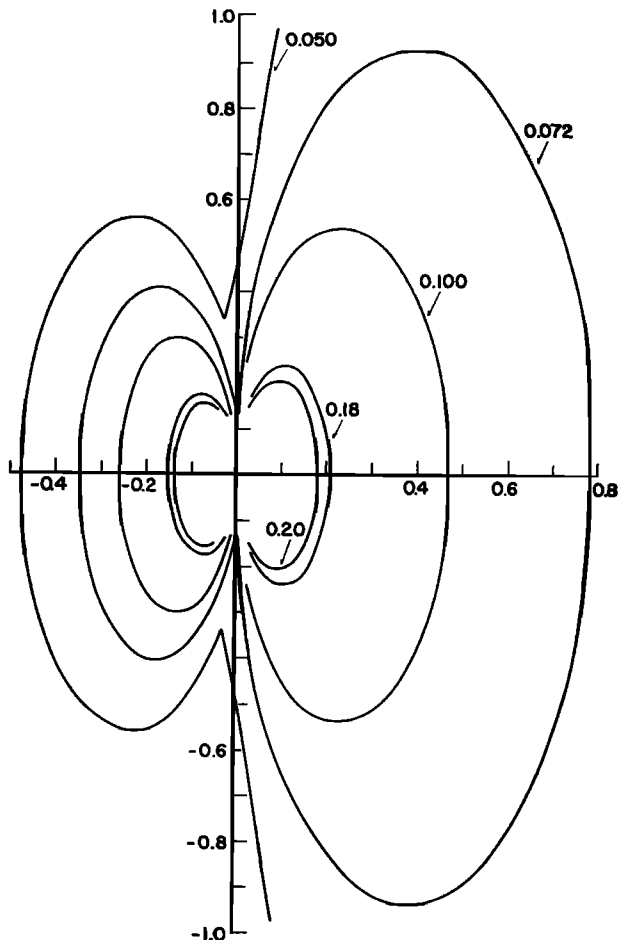


Fig. 6. Scattering by a monopole as in Figure 2. Equiamplitude contours in the near field, $\beta = 0.9$.

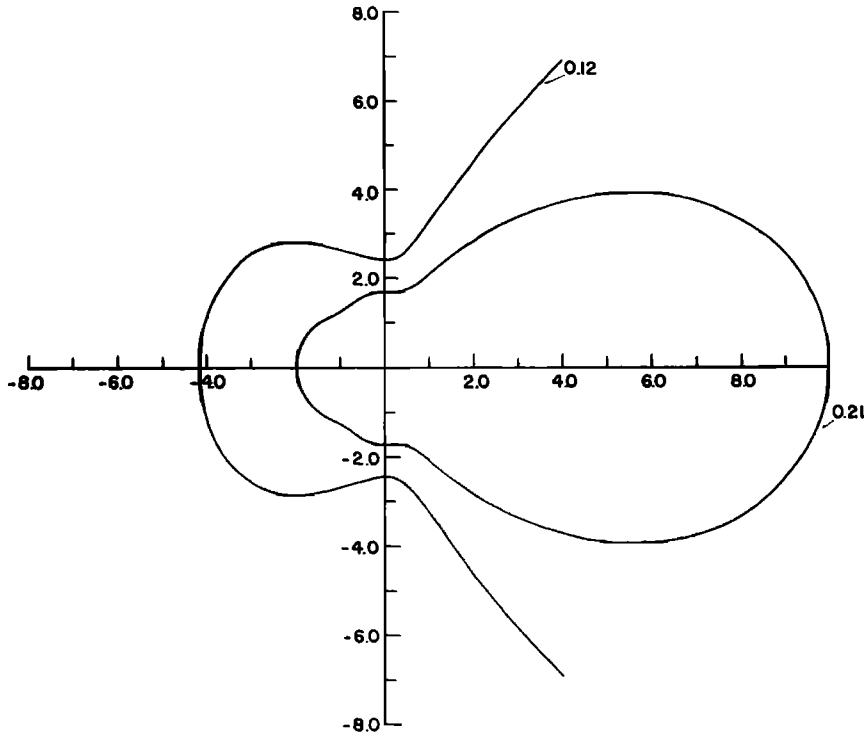


Fig. 7. Scattering by a moving dipole. Equiamplitude contours in the far field, $\beta = 0.5$.

$$\alpha_3(m, n) = \frac{n(n - m + 1)}{(2n + 1)(n + 1)}$$

$$\alpha_4(m, n) = \frac{(n + m)(n + 1)}{n(2n + 1)}$$

$$\alpha_5(m, n) = \frac{im}{n(n + 1)}$$

$$\alpha_6(m) = im$$

Hence,

$$\mathfrak{F}' \cdot \mathbf{g} = \mathfrak{U} \cdot \mathbf{g} + \gamma\beta \sum_{n,m} (d_{nm} \mathbf{C}_n^m + e_{nm} \mathbf{B}_n^m + f_{nm} \mathbf{P}_n^m)$$

$$\mathbf{g}(\hat{\mathbf{r}}') = \sum_{n=1}^{\infty} \sum_{m=-n}^n [\mathbf{C}_n^m(\hat{\mathbf{r}}')c_{nm} + \mathbf{B}_n^m(\hat{\mathbf{r}}')b_{nm}]$$

$$d_{nm} = \alpha_5(m, n)b_{nm} + \alpha_3(m, n - 1)c_{n-1,m} + \alpha_4(m, n + 1)c_{n+1,m}$$

$$e_{nm} = \alpha_5(m, n)c_{n,m} + \alpha_3(m, n - 1)b_{n-1,m} + \alpha_4(m, n + 1)b_{n+1,m}$$

$$f_{nm} = \alpha_6(m)b_{nm} + \alpha_1(m, n + 1)c_{n+1,m} + \alpha_2(m, n - 1)c_{n-1,m}$$

From (10) it follows that

$$(9) \quad \mathbf{u}(\mathbf{r}', t') = f'e^{-i\omega't'} [\mathfrak{U} \cdot \mathbf{u}'(\mathbf{r}', t') + \gamma\beta \sum_{n,m} i^n (d_{nm} \mathbf{M}_{nm} - ie_{nm} \mathbf{N}_{nm} - if_{nm} \mathbf{L}_{nm})] \quad (11)$$

$$\begin{aligned} & \exp(-i\omega't') i^{n-1} \mathbf{L}_{nm}(\hat{\mathbf{r}}') \\ &= \int d\Omega_p \exp(i\psi_p) \mathbf{P}_n^m(\hat{\mathbf{p}}') \\ &= \exp(-i\omega't') i^{n-1} \{ \partial_{k'r'} [h_n(k'r')] \mathbf{P}_n^m(\hat{\mathbf{r}}') + h_n(k'r') \mathbf{B}_n^m(\hat{\mathbf{r}}') / k'r' \} \end{aligned}$$

which is the analog of (4). It is noted that again coupling is limited to adjacent coefficients.

Of course, in free space, plane waves are transversal in all frames of reference; and $\mathfrak{F}' \cdot \mathbf{g}$, if expressed in terms of Γ coordinates \mathbf{r}, t , will have no longitudinal components.

DOMINANT FREQUENCY APPROXIMATION

Consider a ray propagating in Γ in direction α , and reflected by a system of mirrors in Γ' , such that the reflected wave propagates in direction θ in Γ . The observed frequency ω_r is related to the original fre-

quency ω by

$$\frac{\omega_r}{\omega} = \frac{1 - \beta \cos \alpha}{1 - \beta \cos \theta} \equiv V \tag{12}$$

For arbitrary waves, (12) describes the far-field frequency, henceforth called ‘dominant frequency,’ measured by an observer in Γ in the vicinity of $t = 0$. This motivated *Censor* [1967] to express the field in terms of ωV , kV and in particular to investigate first-order effects.

In the case of a moving source the time in Γ must enter in a way that changes the parameters, i.e., amplitude, frequency, propagation constant, because change of position places the object nearer or farther with respect to the observer, in a direction θ which is also time dependent. It is expected, therefore, that the change with time will depend on the expression vt/r , the ratio of the distance traversed by the object in the period t of observation and the distance of the observer r . In extreme cases of vanishingly small vt/r the period of observation might be very long and the effect will depend on the velocity in a stationary way. Thus in the light from the far stars, Doppler shifts

in the frequency are apparent, but the rate of change of the frequency and amplitude as a function of time is negligible.

To recast (5), or (4) in the general case, in terms of the dominant frequency, we exploit the addition theorem for cylindrical functions

$$H_n(\rho_1)e^{in\delta_1} = \sum_{m=-\infty}^{\infty} J_m(\rho_2)H_{n+m}(\rho)e^{im\delta_2}, \tag{13}$$

where $0 < \rho_2 < \rho$; ρ, ρ_1, ρ_2 form a triangle; δ_2 is the angle between ρ and ρ_2 ; and δ_1 , the angle between ρ and ρ_1 , is acute. In the present case, we add and subtract krV in the arguments of the cylindrical functions, and similarly add and subtract ωtV in the exponent. In (13) this prescribes $\delta_1 = \delta_2 = 0$, therefore (5) becomes

$$u = \gamma f' a_0 e^{-i\omega tV} \sum_{m=-\infty}^{\infty} C_m [H_m(krV) + i\beta \cos \theta' H_{m+1}(krV)] \tag{14}$$

$C_m = \exp(i\omega tV - i\omega' t') J_m(krV - k'r')$ which is a relativistically exact expression thus far.

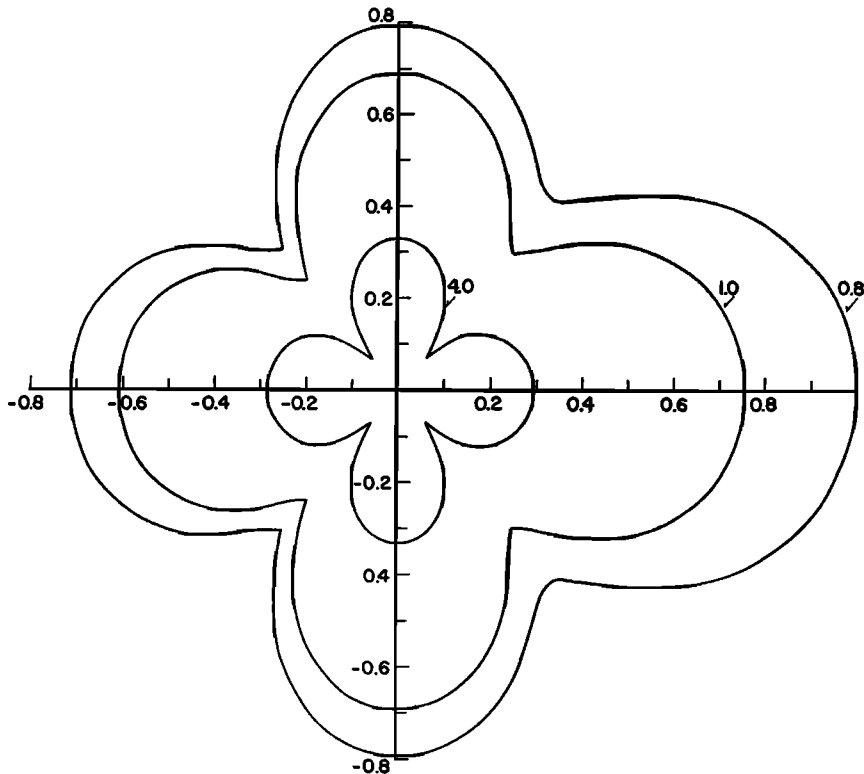


Fig. 8. Scattering by a moving dipole, as in Figure 7. Equiamplitude contours in the near field. The effect of the quadrupole term is apparent.

The representation (14) is especially convenient in describing low velocities. Thus, if only first-order effects in β are retained, this leads to

$$\begin{aligned}
 u &= f' a_0 \exp(-i\omega t V) \{ C_0 [H_0(kr V) \\
 &\quad + i\beta \cos \theta H_1(kr V)] + 2C_1 H_1(kr V) \} \\
 kr V - k'r' &= kr\beta \cos \theta (1 + ct/r) \\
 &\quad + \mathcal{O}(\beta^2) \equiv N + \mathcal{O}(\beta^2) \\
 \omega t V - \omega't' &= -kr\beta \cos \theta (1 + ct/r) \\
 &\quad + \mathcal{O}(\beta^2) \equiv -N + \mathcal{O}(\beta^2) \\
 C_0 &= \exp(iN) J_0(N) = 1 + iN + \mathcal{O}(\beta^2) \\
 C_1 &= \exp(iN) J_1(N) = N/2 + \mathcal{O}(\beta^2) \\
 C_{-1} &= -C_1.
 \end{aligned}
 \tag{15}$$

For short times of observation, N is small, having no zero order in β term; hence only $m = 0, 1, -1$ is significant in (14), and in (15) C_0, C_1, C_{-1} are simple forms.

A comparison of u in (15) and the exact form (5) has been carried out for $t = 0$ by *Censor and Damenstein* [1970], indicating that in the near field

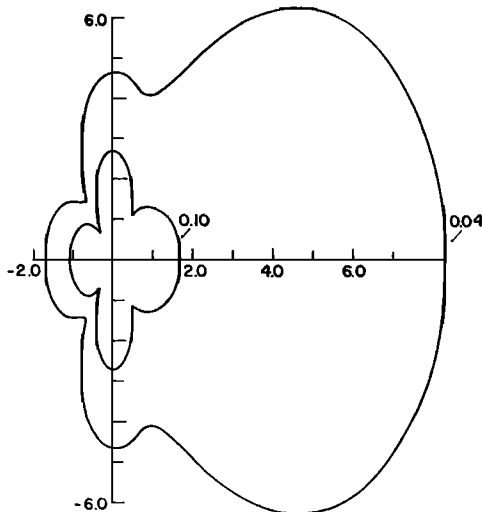


Fig. 9. Scattering by a moving dipole. Equiamplitude contours in the far field, $\beta = 0.9$. Also compare with Figure 7.

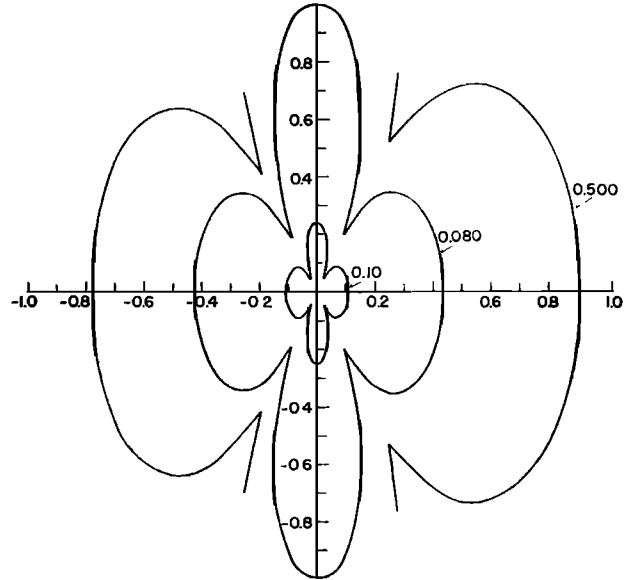


Fig. 10. Scattering by a moving dipole, as in Figure 9. Equiamplitude contours in the near field. Also compare with Figure 8.

($kr < 5$, say), the accuracy of the approximate results is within a fraction β of the exact solution.

The same technique can be applied to three-dimensional problems, using the addition theorem for spherical Hankel functions.

Acknowledgment. Work supported in part by NSF GP 8734.

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